

Asymptotic approximations of finitely generated groups

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Abstract. We initiate the study of asymptotic approximations of finitely generated groups. We show that many groups admit such approximations and provide the first examples of infinite simple asymptotically residually finite (resp. asymptotically amenable) groups with Kazhdan's property (T).

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1. Introduction

The concept of approximation is ubiquitous in mathematics. A classical idea is to approximate objects of interest by ones simpler to investigate, and which have the required characteristics in order to reflect properties and behavior of the elusive objects we started with.

Looking for approximation in geometric group theory, first we adapt this fundamental approach. We discuss both its well-established appearance in *residual properties* of groups and its recent manifestation via *metric approximations* of groups such as sofic and hyperlinear approximations. We focus on approximations of Gromov hyperbolic groups, comment open problems, and suggest a conjecture in this setting. Then we turn over this classical way and initiate the study of approximations by groups usually known as being not so elementary to investigate. This allows to see that many interesting groups (still unknown to have algebraic or metric approximations) admit this new type of approximation which we call *asymptotic approximations*. We give many examples of asymptotically sofic / hyperlinear groups, as well as of asymptotically non-residually finite groups. In particular, we provide

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the first examples of infinite simple asymptotically residually finite (resp. asymptotically amenable) groups with Kazhdan's property (T).

2. Classical idea: approximate “complicated” groups by “easy” ones

Let G be a group and S a finite set of generators of G . We denote by $|\cdot|_S$ the word length on G induced by S and by $B_S(n) = \{g \in G: |g|_S \leq n\}$ the ball of radius n centered at the identity of G .

Let $\mathcal{F} = \{ \text{some groups} \}$ be a given family of groups.

Definition 2.1. A group G is *approximated by \mathcal{F}* if for each $n \in \mathbb{N}$ there exists a map $i: B_S(n) \rightarrow F \in \mathcal{F}$ such that

- (*) $i(g)i(h) = i(gh)$ for all $g, h, gh \in B_S(n)$;
- (**) $i(g) \neq i(h)$ for all $g \neq h$ elements of $B_S(n)$.

In other words, G is approximated by \mathcal{F} if the algebraic structures of G and of a group $F \in \mathcal{F}$ coincide whenever we focus on a ball of a given radius in G and its image in F . The map i does depend on n in general but we omit the indexing. We call such a map i an *algebraic approximation*. Assumption (*) is termed to be a *homomorphism on the ball* and (**) an *injectivity on the ball*.

By varying the groups constituting family \mathcal{F} and the choices of the map i , we recognize many famous intensively studied classes of groups. Here are some fundamental examples.

Examples 2.2 (Algebraic approximations).

- Residually finite groups (RF) are groups approximated by family $\mathcal{F} = \{ \text{finite groups} \}$ with i a homomorphism $G \rightarrow F \in \mathcal{F}$;
- Locally embeddable into finite ones (LEF) are groups approximated by family $\mathcal{F} = \{ \text{finite groups} \}$ [VG97];
- Residually amenable groups (RA) are groups approximated by family $\mathcal{F} = \{ \text{amenable groups} \}$ with i a homomorphism $G \rightarrow F \in \mathcal{F}$;
- Initially subamenable groups (ISA) are groups approximated by family $\mathcal{F} = \{ \text{amenable groups} \}$ [Gro99];
- If G is a fully residually- \mathcal{F} group (FR \mathcal{F}), then G is approximated by \mathcal{F} ;
- If G is a limit of groups from \mathcal{F} ($\lim \mathcal{F}$) in the space of marked groups [CG05], then G is approximated by \mathcal{F} .
- If G is a limit of groups approximated by \mathcal{F} ($\lim A\mathcal{F}$) in the space of marked groups, then G is approximated by \mathcal{F} .

The relationships between the above classes and concrete (non)examples of groups are known. For instance, (RF) \subsetneq (LEF) \subsetneq (ISA), (RF) \subsetneq (RA) \subsetneq (ISA) and (FR \mathcal{F}) \subseteq ($\lim \mathcal{F}$) \subseteq ($\lim A\mathcal{F}$). The free group $\mathbb{F} = \mathbb{F}(S)$ belongs to all classes, whenever it is residually- \mathcal{F} , for the last three examples. Thompson's groups T and V are not (ISA), hence not (RA), etc.

The origin of algebraic approximations goes back to foundational papers of Malcev and those of Marshall Hall (who seem to come independently to the general concept of residual finiteness; their results have predecessors in works of Schreier and Levi, see [Cam89] and references therein). The development of the topic in group theory has been fruitfully interwoven with that of algebraic topology and spectral geometry.

Nowadays, a major open problem in geometric group theory is the following.

Question 2.3. *Are all Gromov hyperbolic groups residually finite?*

We will be back to this question below. Let us now consider more general approximations.

Let $\mathcal{F}_{\text{dist}} = \{ \text{some groups with distance} \}$ be a given family of groups, each of which is equipped with a bi-invariant distance (assumed to be normalized, for simplicity).

Definition 2.4. A group G is *metrically approximated* by $\mathcal{F}_{\text{dist}}$ if for each $n \in \mathbb{N}$ there exists a map $\pi: B_S(n) \rightarrow (F, \text{dist}) \in \mathcal{F}_{\text{dist}}$ such that

- (*) $\text{dist}(\pi(g)\pi(h), \pi(gh)) < 1/n$ for all $g, h, gh \in B_S(n)$;
- (**) $\text{dist}(\pi(g), \pi(h)) > 1 - 1/n$ for all $g \neq h$ elements of $B_S(n)$.

In other words, G is metrically approximated by $\mathcal{F}_{\text{dist}}$ if the algebraic structures of G and of a group $(F, \text{dist}) \in \mathcal{F}_{\text{dist}}$ almost coincide (that is, they differ by a dist–small quantity) whenever we focus on a ball of a given radius in G and its image in (F, dist) . The map π does depend on n in general but we omit the indexing. We call such a map π a *metric approximation*. Assumption (*) is termed to be an *almost homomorphism on the ball* and (**) a *uniform injectivity*.

By varying the groups constituting family $\mathcal{F}_{\text{dist}}$ and the choices of bi-invariant metrics, we obtain many interesting recently emerged classes of groups.

Examples 2.5 (Metric approximations).

- Sofic groups (S) are groups metrically approximated by family $\mathcal{F}_{\text{dist}} = \{ \text{Sym}(n), d_{\text{Ham}} \mid n \in \mathbb{N} \}$, where each symmetric group of finite degree is equipped with the normalized Hamming distance d_{Ham} ;
- Linear sofic groups (LS) are groups metrically approximated by family $\mathcal{F}_{\text{dist}} = \{ GL_n(\mathbb{C}), d_{\text{rank}} \mid n \in \mathbb{N} \}$, where the general linear groups are equipped with the normalized rank distance [AP12];
- Weakly sofic groups (WS) are groups metrically approximated by family $\mathcal{F}_{\text{dist}} = \{ \text{finite groups with distance} \}$, where each finite group is endowed with a normalized bi-invariant metric [GR08];
- Hyperlinear groups (H) are groups metrically approximated by family $\mathcal{F}_{\text{dist}} = \{ U(n), d_{\text{HS}} \mid n \in \mathbb{N} \}$, where each unitary group of finite rank is equipped with the normalized Hilbert-Schmidt distance;
- Every group algebraically approximated by $\mathcal{F}_{\text{dist}}$ is metrically approximated by $\mathcal{F}_{\text{dist}}$, provided assumption (**) of Definition 2.4 holds for the given distance;

- If G is a limit of groups metrically approximated by $\mathcal{F}_{\text{dist}}$ ($\lim \mathcal{F}_{\text{dist}}$) in the space of marked groups, then G is metrically approximated by $\mathcal{F}_{\text{dist}}$.

There is much less known on the relationship between the above classes of metrically approximated groups and on the possibility of concrete non-examples. For instance, each of the classes $(\lim \text{RF})$, $(\lim \text{RA})$, $(\lim \text{LEF})$, $(\lim \text{ISA})$ is clearly contained in $(\text{S}) \subseteq (\text{WS})$. Also, $(\text{S}) \subseteq (\text{H})$ as $d_{\text{Ham}}(\sigma, \tau) = d_{\text{HS}}(A_\sigma, A_\tau)^2/2$ for all $\sigma, \tau \in \text{Sym}(n)$ and the corresponding permutation matrices $A_\sigma, A_\tau \in U(n)$. Recently, we have proved that $(\text{S}) \subseteq (\text{LS}) \subseteq (\text{WS})$, see [AP12].

The study of metric approximations is motivated by major open problems in dynamics and operator algebra. Hyperlinear groups appeared in the context of Alain Connes' embedding conjecture (1976) in operator algebra and were introduced by Florin Rădulescu. Sofic groups were introduced by Misha Gromov in his study of symbolic algebraic varieties in relation to Gottschalk's surjunctivity conjecture (1973) in topological dynamics. They were called sofic by Benjamin Weiss. They are known to satisfy Kaplansky's direct finiteness conjecture (1969) by a result of Elek and Szabo. We refer the reader to nice surveys [Pes08, PK09] for more information on sofic and hyperlinear groups.

A major open problem in the theory of metric approximations is the following.

Question 2.6. *Are all groups sofic / linear sofic / weakly sofic / hyperlinear?*

We expect a negative answer to both Question 2.3 and Question 2.6. In this context, we have the following curious result.

Properties (5), (6), and (7) below come from the theory of operator algebras, where they are crucial in relation to Alain Connes' embedding conjecture. Here \mathcal{S} denotes the full group of the hyperfinite aperiodic ergodic measure-preserving equivalence relation and $\mathcal{U} = U(R)$ the unitary group of the hyperfinite factor R of type II_1 equipped with the ultraweak topology. It is known that a group G is sofic (resp. hyperlinear) if and only if it embeds into a metric ultrapower of \mathcal{S} (resp. of \mathcal{U}), see [PK09, Kir94] for precise definitions.

Proposition 2.7. *The following are equivalent.*

1. *All hyperbolic groups are (RF);*
2. *All hyperbolic groups are (LEF);*
3. *All hyperbolic groups are (RA);*
4. *All hyperbolic groups are (ISA);*
5. *All hyperbolic groups have Kirchberg's factorization property (KFP);*
6. *All hyperbolic groups can be embedded into \mathcal{S} ($\hookrightarrow \mathcal{S}$);*
7. *All hyperbolic groups can be embedded into \mathcal{U} ($\hookrightarrow \mathcal{U}$).*

Proof. Since hyperbolic groups are finitely presented, the equivalences (1) \iff (2) and (3) \iff (4) are immediate.

Let us show (1) \iff (3). Finite groups are amenable. Therefore, it suffices to check (1) \impliedby (3). Assume that all hyperbolic groups are (RA) but there exists a hyperbolic group G_0 which is not (RF). Then, by a result of Ol'shanskii [Ol'00] and, independently, of Kapovich and Wise [KW00], there exists a non-elementary hyperbolic group G with no proper subgroups of finite index. On the other hand, there exists a hyperbolic group G_T which has Kazhdan's property (T). One can take, for instance, a co-compact lattice in $Sp(n, 1)$, $n \geq 2$. Such a group is clearly non-elementary. By another result of Ol'shanskii [Ol'93], G and G_T has a common quotient Q which is a non-elementary hyperbolic group. Such a group Q has Kazhdan's property (T) by construction and it is (RA) by assumption. Hence, every amenable quotient of Q is finite. This yields a contradiction as Q is infinite and Q has no proper finite index subgroups, by construction.

Implications (1) \implies (5), (6), (7), as well as (6) \implies (7), and (5) \implies (7) are immediate from definitions [PK09, Kir94]. Let us check (7) \implies (1). Assume that all hyperbolic groups can be embedded into \mathcal{U} but there exists a hyperbolic group G_0 which is not (RF). Proceeding as above, we find a non-elementary hyperbolic group Q which has Kazhdan's property (T), has no proper subgroups of finite index, and which can be embedded into \mathcal{U} . By a result of Kirchberg [Kir94], for a group with Kazhdan's property (T), we have (RF) \iff (KFP) \iff ($\hookrightarrow \mathcal{S}$) \iff ($\hookrightarrow \mathcal{U}$). Thus, Q is (RF). This is a contradiction as Q is infinite and has no proper finite index subgroups, by construction. \square

The preceding proposition can be viewed as a first step to the following equivalence.

Conjecture 2.8. *All hyperbolic groups are residually finite \iff all hyperbolic groups are sofic.*

It is commonly believed that a non-residually finite hyperbolic group does exist. If established, this equivalence indicates a difficulty to find such a counterexample which would answer both Question 2.3 and Question 2.6 in the negative.

3. New idea: approximate “easy” groups by “complicated” ones

The area of metric approximations of discrete groups is very attractive as many natural questions on metrically approximated groups and their applications are open. Various well-studied groups and classes of groups are still not known “to be, or not to be” metrically approximated. In particular, the following is unknown.

- Are the following groups sofic / linear sofic / weakly sofic / hyperlinear?
 - Hyperbolic groups and their subgroups;

- Weakly amenable groups (= groups that are not uniformly non-amenable [ABL⁺05]);
- One-relator groups;
- Mapping class groups and outer automorphism groups $Out(\mathbb{F}_n)$ of free groups for $n \geq 3$;
- Thompson's groups F, T, V .

— Does there exist an infinite simple sofic group with Kazhdan's property (T)?

We approach these classes of groups by introducing a new way to approximate finitely generated groups.

Definition 3.1. A group G is *asymptotically approximated* by $\mathcal{F}_{\text{dist}}$ if for each $n \in \mathbb{N}$ there exist a finite generating set S_n of G and a map $\pi: B_{S_n}(n) \rightarrow (F, \text{dist}) \in \mathcal{F}_{\text{dist}}$ such that

- (*) $\text{dist}(\pi(g)\pi(h), \pi(gh)) < 1/n$ for all $g, h, gh \in B_{S_n}(n)$;
- (**) $\text{dist}(\pi(g), \pi(h)) > 1 - 1/n$ for all $g \neq h$ elements of $B_{S_n}(n)$.

In other words, G is asymptotically approximated by $\mathcal{F}_{\text{dist}}$ if there exists a sequence $(S_n)_{n \in \mathbb{N}}$ of finite generating sets of G such that the algebraic structures of G and of a group $(F, \text{dist}) \in \mathcal{F}_{\text{dist}}$ almost coincide (that is, they differ by a dist-small quantity) whenever we focus on a ball with respect to S_n of a given radius in G and its image in (F, dist) . We call such a map π an *asymptotic approximation*.

Note a reverse order of asymptotic approximation in comparison with a general idea of approximation: a group G is asymptotically approximated by a family of groups $\mathcal{F}_{\text{dist}}$ if the family $\{(G, S_n)\}_{n \in \mathbb{N}}$ approaches, in the above sense, groups from $\mathcal{F}_{\text{dist}}$.

By varying the groups constituting family $\mathcal{F}_{\text{dist}}$ and the choices of bi-invariant metrics, we get the concepts of *asymptotic residual finiteness / residual amenability / soficity / hyperlinearity, etc.* For example, a group asymptotically approximated by a family $\mathcal{F}_{\text{dist}}$ consisting of residually finite groups is called asymptotically residually finite (with respect to the given metrics). In our discussion below, we omit mentioning metrics explicitly as the choice is often rather obvious. For example, here is a *basic choice*: the length of every non-trivial group element is assigned to be 1; relative to the induced metric, assumption (*) means that π is a homomorphism on $B_{S_n}(n)$ and (**) obviously holds.

Examples 3.2 (Asymptotic approximations).

- All groups metrically approximated by a family $\mathcal{F}_{\text{dist}}$ are asymptotically approximated by $\mathcal{F}_{\text{dist}}$. For instance, residually finite groups are asymptotically residually finite, sofic groups are asymptotically sofic, both classes are asymptotically finite, etc.;
- All weakly amenable groups are asymptotically amenable (hence, asymptotically sofic). Indeed, if G is weakly amenable [ABL⁺05], then there exists a sequence $(S_n)_{n \in \mathbb{N}}$ of finite generating sets of G such that the

family $\{(G, S_n)\}_{n \in \mathbb{N}}$ approaches an amenable group in the sense of Definition 3.1.

There are numerous examples of non-amenable weakly amenable groups, many of them are not known to be sofic [ABL⁺05]. Such groups can be made to have rather unusual extreme properties. For instance, there exists a non-amenable weakly amenable simple periodic (hence, with no non-abelian free subgroups) group Q with Kazhdan's property (T); Alternatively, given an arbitrary countable group C , there exists a non-amenable weakly amenable simple group Q with Kazhdan's property (T) such that Q contains an isomorphic copy of C . Other properties can be added to Q (which is a common quotient of all non-elementary hyperbolic groups), see more details on the construction of such a group Q in [ABJ⁺09, Prop. 2.6 and 2.8, Rem. 2.9] and [ABL⁺05, Osi02] on their weak amenability.

- Free Burnside groups $B(m, n)$ with $m \geq 2$ and odd exponent $n \geq 1003$ are not asymptotically residually finite. Indeed, by the celebrated Novikov-Adyan solution of the bounded Burnside Problem such a group is infinite and by the famous Zelmanov solution of the restricted Burnside Problem it cannot be residually finite. By a result of Atabekyan [Ata09], there is a number $L < (400n)^3$ such that for an arbitrary set K generating a noncyclic subgroup $\langle K \rangle$ of $B(m, n)$, there are elements $u, v \in \langle K \rangle$ for which the pair $\{u, v\}$ is a basis of a free Burnside subgroup of exponent n (which is not residually finite), and the lengths of the elements u and v with respect to the generating set K satisfy the inequalities $|u|_K < L$ and $|v|_K < L$. Thus, if a family $\mathcal{F}_{\text{dist}}$ asymptotically approximates such a $B(m, n)$, then $\mathcal{F}_{\text{dist}}$ cannot consist of residually finite groups only.

The next result provides numerous examples of asymptotically sofic groups and answers, in this new context of asymptotic approximations, questions from the beginning of this Section.

Recall that a girth of a graph is the length of shortest non-trivial loop and the *girth of a group* is the supremum of the girths of all of its Cayley graphs with respect to finite sets of generators [Sch00, Akh03].

Theorem 3.3. *All groups of infinite girth are asymptotically residually finite. In particular, the following groups are asymptotically residually finite (hence, asymptotically sofic):*

- Hyperbolic groups and their finitely generated subgroups;
- One-relator groups;
- Thompson's group F ; more generally, finitely generated subgroups of $PL_o(I)$;
- Finitely generated subgroups of convergence groups;
- Finitely generated subgroups of a mapping class group;
- Outer automorphism groups $\text{Out}(\mathbb{F}_n)$ and its subgroups with an (iwip) element;

In addition, there exists an infinite simple asymptotically residually finite (hence, asymptotically sofic) group with Kazhdan's property (T).

Proof. By definition, groups of infinite girth are asymptotically approximated by a family consisting of a free non-abelian group \mathbb{F} of finite rank. Therefore, they are asymptotically residually finite (or adapting our terminology, *asymptotically free*). The groups above are indeed asymptotically residually finite as they are known to be either of infinite girth or residually finite.

Hyperbolic groups and their finitely generated subgroups are of infinite girth [Akh03] whenever they are not virtually cyclic (otherwise, they are obviously residually finite).

One-relator groups are of infinite girth [Akh03] if and only if they are not solvable. Virtually solvable subgroups of one-relator groups consist of: cyclic group of finite or infinite order, free abelian group of rank 2, fundamental group of a Klein bottle, and the Baumslag-Solitar group $BS(1, n) = \langle a, b \mid aba^{-1} = b^n \rangle$ with $|n| \geq 2$. All these groups are residually finite.

Thompson's group F is of infinite girth [AST11, Bri10], therefore it is asymptotically free (hence, asymptotically residually finite). For finitely generated subgroups of $PL_o(I)$, the group of orientation preserving piecewise linear homeomorphisms of the closed interval, see [Akh11], where non-solvable subgroups are proven to be of infinite girth. Solvable subgroups of $PL_o(I)$ are characterized in [Ble08], they are all residually finite.

The results on the convergence (resp. the mapping class) groups follow from [Nak11, Yam11] as such subgroups are of infinite girth whenever they are not virtually cyclic (resp. virtually abelian). Similarly, for subgroups of $Out(\mathbb{F}_n)$ [Nak08] having a so-called (iwip) element. The group itself $Out(\mathbb{F}_n)$ has infinite girth in an obvious way as it surjects onto $GL_n(\mathbb{Z})$ which is of infinite girth [Akh03].

By a result of Ol'shanskii, there exists a torsion-free Tarski monster group, that is an infinite non-abelian group all whose proper subgroups are infinite cyclic [Ch. 9, §28.1][Ol'91]. Moreover, there exists such a group which does not satisfy any non-trivial identity, hence of infinite girth, see Ol'shanskii's argument and the proof that such a group is of infinite girth in [Zus09]. Clearly, Tarski monster is simple. In addition, it can be made to have Kazhdan's property (T): it suffices to build such a torsion-free Tarski monster starting from a torsion-free hyperbolic group with property (T), for instance, a co-compact lattice in $Sp(n, 1)$, $n \geq 2$ [Ol'93]. \square

We do not know any example of a group which is not asymptotically sofic / hyperlinear (e.g. for a basic choice of the metrics).

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