A GROTHENDIECK TOPOS OF GENERALIZED FUNCTIONS III: NORMAL PDE

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Abstract. bla bla bla

1. Spaces of

We shall use the assumptions on K and (K_ε) given in this theorem to introduce a new notion of "compact subset" which behaves better than the usual classical notion of compactness in the sharp topology.

Definition 1. A subset K of ${}^\rho\widetilde{\mathbb{R}}^n$ is called *functionally compact*, denoted by $K \Subset_{\text{f}} {}^\rho\widetilde{\mathbb{R}}^n$, if there exists a net (K_ε) such that

1. $K = [K_\varepsilon] \subseteq {}^\rho\widetilde{\mathbb{R}}^n$;
2. (K_ε) is sharply bounded;
3. $\forall \varepsilon \in I : \ K_\varepsilon \Subset \mathbb{R}^n$.

If, in addition, $K \subseteq U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ then we write $K \Subset_{\text{f}} U$. Finally, we write $[K_\varepsilon] \Subset_{\text{f}} U$ if 2, 3 and $[K_\varepsilon] \subseteq U$ hold.

We note that in 3 it suffices to ask that K_ε is closed since it is bounded by 2, at least for ε small. The name *functionally compact subset* is motivated by showing, as it will be done in Theorem 3, that on this type of subsets, GSFs have properties very similar to those that ordinary smooth functions have on standard compact sets.

Remark 2.

1. By Thm. **??**, any internal set $K = [K_\varepsilon]$ is closed in the sharp topology. In particular, the open interval $(0, 1) \subseteq {}^\rho\widetilde{\mathbb{R}}$ is not functionally compact since it is not closed.
2. If $H \Subset \mathbb{R}^n$ is a non-empty ordinary compact set, then the internal set $[H]$ is functionally compact. In particular, $[0, 1] = [[0, 1]_{\mathbb{R}}]$ is functionally compact.
3. The empty set $\emptyset = [\emptyset] \Subset_{\text{f}} {}^\rho\widetilde{\mathbb{R}}$.
4. ${}^\rho\widetilde{\mathbb{R}}^n$ is not functionally compact since it is not sharply bounded.
5. The set of compactly supported points $\text{c}(\mathbb{R})$ is not functionally compact because the GSF $f(x) = x$ does not satisfy the conclusion (**??**) of Cor. **??**.

For functionally compact sets it is easy to prove the following generalizations of theorems from classical analysis:

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Theorem 3.

(1) *Let* $K\subseteq X\subseteq{}^{\rho}\widetilde{\mathbb{R}}^{n}$, $f\in{}^{\rho}\mathcal{GC}^{\infty}(X,{}^{\rho}\widetilde{\mathbb{R}}^{d})$. *Then* $K\Subset_{f}{}^{\rho}\widetilde{\mathbb{R}}^{n}$ *implies* $f(K)\Subset_{f}{}^{\rho}\widetilde{\mathbb{R}}^{d}$.

(2) *If* $a,b\in{}^{\rho}\widetilde{\mathbb{R}}$ *and* $a\le b$, *then* $[a,b]\Subset_{f}{}^{\rho}\widetilde{\mathbb{R}}$. *Let us note explicitly that* $a,b\in{}^{\rho}\widetilde{\mathbb{R}}$ *can also be infinite numbers, e.g.* $a=-\mathrm{d}\rho^{-N}$, $b=\mathrm{d}\rho^{-M}$ *or* $a=\mathrm{d}\rho^{-N}$, $b=\mathrm{d}\rho^{-M}$ *with* $M>N$.

(3) *Let* $K,H\Subset_{f}{}^{\rho}\widetilde{\mathbb{R}}^{n}$. *If* $K\cup H$ *is internal, then it is functionally compact. If* $K\cap H$ *is internal, then it is functionally compact.*

(4) *Let* $H\subseteq K\Subset_{f}{}^{\rho}\widetilde{\mathbb{R}}^{n}$, *then* H *internal implies* $H\Subset_{f}{}^{\rho}\widetilde{\mathbb{R}}^{n}$.

(5) *Let* $K\Subset_{f}{}^{\rho}\widetilde{\mathbb{R}}^{n}$ *and* $H\Subset_{f}{}^{\rho}\widetilde{\mathbb{R}}^{d}$, *then* $K\times H\Subset_{f}{}^{\rho}\widetilde{\mathbb{R}}^{n+d}$. *In particular, if* $a_i\le b_i$ *for* $i=1,\dots,n$, *then* $\prod_{i=1}^{n}[a_i,b_i]\Subset_{f}{}^{\rho}\widetilde{\mathbb{R}}^{n}$.

Both in the Banach fixed point theorem and in the Picard-Lindelöf theorem, we want to consider spaces of GSFs of the type $K\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}^{d}$, where $K\Subset_{\mathrm{f}}{}^{\rho}\widetilde{\mathbb{R}}^{n}$. In order to set natural ${}^{\rho}\widetilde{\mathbb{R}}$-valued norms in these spaces, we need to talk of partial derivatives $\partial^{\alpha}f(x)$ at every $x\in K$. This cannot be performed using only the Fermat-Reyes Thm. **??** since it requires the point x to be an internal one. For this reason, we consider only those K that satisfy the following

Definition 4. We say that K is a *solid set in* ${}^{\rho}\widetilde{\mathbb{R}}^{n}$ if $\mathrm{int}(K)$ is dense in K (in the sharp topology).

For example, Lem. **??** and Thm. 3 2 show that each interval $[a,b]$, where $a<b$, is a solid functionally compact set. Therefore, Thm. 3 5 gives that also n-dimensional intervals are solid sets. Trivially, every sharply open set is solid.

For this type of sets we have:

Theorem 5. *Let* K *be a solid set in* ${}^{\rho}\widetilde{\mathbb{R}}^{n}$, *and* $f\in{}^{\rho}\mathcal{GC}^{\infty}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$ *be a GSF. Then for all* $\alpha\in\mathbb{N}^{n}$ *and all* $x\in K$ *the following limit exists in the sharp topology*

$$\lim_{\substack{y\to x\\ y\in\mathrm{int}(K)}}\partial^{\alpha}f(y)=:\partial^{\alpha}f(x).$$

Moreover, if the net $f_{\varepsilon}\in\mathcal{C}^{\infty}(\Omega_{\varepsilon},\mathbb{R}^{d})$ *defines* f, *then* $\partial^{\alpha}f(x)=[\partial^{\alpha}f_{\varepsilon}(x_{\varepsilon})]$ *and hence* $\partial^{\alpha}f\in{}^{\rho}\mathcal{GC}^{\infty}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$.

Proof. We have

$$\lim_{\substack{y\to x\\ y\in\mathrm{int}(K)}}\partial^{\alpha}f(y)=\lim_{\substack{y\to x\\ y\in\mathrm{int}(K)}}[\partial^{\alpha}f_{\varepsilon}(y_{\varepsilon})]=[\partial^{\alpha}f_{\varepsilon}(x_{\varepsilon})]$$

the last equality following by the sharp continuity of the GSF $[\partial^{\alpha}f_{\varepsilon}(-)]$ at every point $x\in K\subseteq\langle\Omega_{\varepsilon}\rangle$ (see Thm. **??** **??**). □

From the extreme value property, Lem. **??**, it is natural to expect that the following generalized numbers could serve as non-Archimedean ${}^{\rho}\widetilde{\mathbb{R}}$-valued norms.

Definition 6. Let $\emptyset\ne K\Subset_{\mathrm{f}}{}^{\rho}\widetilde{\mathbb{R}}^{n}$ be a solid set. Let $m\in\mathbb{N}$ and $f\in{}^{\rho}\mathcal{GC}^{\infty}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$. Then

$$\|f\|_{m}:=\max_{\substack{|\alpha|\le m\\ 1\le i\le d}}\max\left(\left|\partial^{\alpha}f^{i}(M_{\alpha i})\right|,\left|\partial^{\alpha}f^{i}(m_{\alpha i})\right|\right)\in{}^{\rho}\widetilde{\mathbb{R}},$$

where $m_{\alpha i},M_{\alpha i}\in K$ satisfy

$$\forall x\in K:\ \left|\partial^{\alpha}f^{i}(m_{\alpha i})\right|\le\left|\partial^{\alpha}f^{i}(x)\right|\le\left|\partial^{\alpha}f^{i}(M_{\alpha i})\right|.$$

Note that the notation $\|f\|_m$ depends on K through the function f since K is its domain.

The following result allows to calculate the (generalized) norm $\|f\|_m$ using any net (f_ε) that defines f.

Theorem 7. *Under the assumptions of Def. 6, let the set $K = [K_\varepsilon] \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$. If the net (f_ε) defines f, then*

$$\|f\|_m = \left[\max_{\substack{|\alpha|\le m\\ 1\le i\le d}} \sup_{x\in K_\varepsilon} \left|\partial^\alpha f^i_\varepsilon(x)\right|\right] \in {}^\rho\widetilde{\mathbb{R}}. \tag{1.1}$$

Proof. In proving (1.1), we will also prove that the norm $\|f\|_m$ is well-defined, i.e. it does not depend on the particular choice of points $m_{\alpha i}$, $M_{\alpha i}$ as in Def. 6. As in the proof of Lem. **??**, we get the existence of $\bar{m}_{\alpha i\varepsilon}$, $\bar{M}_{\alpha i\varepsilon} \in K_\varepsilon$ such that

$$\forall x\in K_\varepsilon : \ \left|\partial^\alpha f^i_\varepsilon(\bar{m}_{\alpha i\varepsilon})\right| \le \left|\partial^\alpha f^i_\varepsilon(x)\right| \le \left|\partial^\alpha f^i_\varepsilon(\bar{M}_{\alpha i\varepsilon})\right|.$$

Hence $\left|\partial^\alpha f^i_\varepsilon(x)\right| \le \max\left(\left|\partial^\alpha f^i_\varepsilon(\bar{m}_{\alpha i\varepsilon})\right|, \left|\partial^\alpha f^i_\varepsilon(\bar{M}_{\alpha i\varepsilon})\right|\right)$. Thus

$$\max_{\substack{|\alpha|\le m\\ 1\le i\le d}} \sup_{x\in K_\varepsilon} \left|\partial^\alpha f^i_\varepsilon(x)\right| \le \max_{\substack{|\alpha|\le m\\ 1\le i\le d}} \max\left(\left|\partial^\alpha f^i_\varepsilon(\bar{m}_{\alpha i\varepsilon})\right|, \left|\partial^\alpha f^i_\varepsilon(\bar{M}_{\alpha i\varepsilon})\right|\right).$$

But $\bar{m}_{\alpha i\varepsilon}$, $\bar{M}_{\alpha i\varepsilon} \in K_\varepsilon$, so

$$\left[\max_{\substack{|\alpha|\le m\\ 1\le i\le d}} \sup_{x\in K_\varepsilon} \left|\partial^\alpha f^i_\varepsilon(x)\right|\right] = \left[\max_{\substack{|\alpha|\le m\\ 1\le i\le d}} \max\left(\left|\partial^\alpha f^i_\varepsilon(\bar{m}_{\alpha i\varepsilon})\right|, \left|\partial^\alpha f^i_\varepsilon(\bar{M}_{\alpha i\varepsilon})\right|\right)\right] =$$

$$= \max_{\substack{|\alpha|\le m\\ 1\le i\le d}} \max\left(\left|\partial^\alpha f^i(\bar{M}_{\alpha i})\right|, \left|\partial^\alpha f^i(\bar{m}_{\alpha i})\right|\right).$$

From this, both the fact that the norm $\|f\|_m$ is well-defined and claim (1.1) follow. □

Even though $\|f\|_m \in {}^\rho\widetilde{\mathbb{R}}$, using an innocuous abuse of language, in the following we will simply call $\|f\|_m$ a norm. This use of the term “norm” is justified by the following

Theorem 8. *Let $\emptyset \ne K \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$ be a solid set. Let f, $g \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$ and $m\in\mathbb{N}$. Then*

1. $\|f\|_m \ge 0$;
2. $\|f\|_m = 0$ *if and only if* $f = 0$;
3. $\forall c\in {}^\rho\widetilde{\mathbb{R}} : \ \|c\cdot f\|_m = |c|\cdot\|f\|_m$;
4. $\|f+g\|_m \le \|f\|_m + \|g\|_m$;
5. $\|f\cdot g\|_m \le 2^m\cdot\|f\|_m\cdot\|g\|_m$.

Proof. 1, 3 and 4 follow directly from Thm. 7, as does 5, using the Leibniz rule. The ‘only if’-part of property 2 follows from (1.1). □

Using our ${}^\rho\widetilde{\mathbb{R}}$-valued norms, it is now natural to define

Definition 9. Let $\emptyset \ne K \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$ be a solid set. Let $f \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$, $m\in\mathbb{N}$, $r\in{}^\rho\widetilde{\mathbb{R}}_{>0}$, then

(1) ${}^{\rho}\mathcal{GF}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d}):=\left({}^{\rho}\mathcal{GC}^{\infty}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d}),(\|-\|_{m})_{m\in\mathbb{N}}\right)$. We write $f\in{}^{\rho}\mathcal{GF}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$ to denote $f\in{}^{\rho}\mathcal{GC}^{\infty}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$.

(2) $B_{r}^{m}(f):=\left\{ g\in{}^{\rho}\mathcal{GC}^{\infty}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})\mid\|f-g\|_{m}<r\right\}$.

(3) If $V\subseteq{}^{\rho}\mathcal{GC}^{\infty}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$, then we say that V is a *sharply open set in* ${}^{\rho}\mathcal{GF}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$ if

$$\forall v\in V\,\exists m\in\mathbb{N}\,\exists r\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}:\ B_{r}^{m}(v)\subseteq V.$$

Moreover, we say that V is *a large (or Fermat) open set in* ${}^{\rho}\mathcal{GF}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$ if

$$\forall v\in V\,\exists m\in\mathbb{N}\,\exists r\in\mathbb{R}_{>0}:\ B_{r}^{m}(v)\subseteq V.$$

A trivial generalization of the classical proofs, though using Cor. **??**, shows that

Theorem 10. *Let $\emptyset\ne K\Subset_{f}{}^{\rho}\widetilde{\mathbb{R}}$ be a solid set. Then we have:*

(1) *Sharply open sets as well as large open sets* in ${}^{\rho}\mathcal{GF}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$ *form topologies on* ${}^{\rho}\mathcal{GC}^{\infty}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$.

(2) *Pointwise addition and multiplication by ${}^{\rho}\widetilde{\mathbb{R}}$-scalar in ${}^{\rho}\mathcal{GF}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$ are continuous in the sharp topology. Therefore, ${}^{\rho}\mathcal{GF}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$ is a topological ${}^{\rho}\widetilde{\mathbb{R}}$-module and ${}^{\rho}\mathcal{GF}(K,{}^{\rho}\widetilde{\mathbb{R}})$ is an ${}^{\rho}\widetilde{\mathbb{R}}$-algebra.*

(3) *${}^{\rho}\mathcal{GF}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$ with the sharp topology is separated.*

(4) *If f, $g\in B_{r}^{m}(0)$ and $t\in[0,1]$, then $tf+(1-t)g\in B_{r}^{m}(0)$. We can therefore say that every ball $B_{r}^{m}(0)$ is ${}^{\rho}\widetilde{\mathbb{R}}$-convex.*

(5) *If $t\in{}^{\rho}\widetilde{\mathbb{R}}$ and $|t|\le1$, then $t\cdot B_{r}^{m}(0)\subseteq B_{r}^{m}(0)$. We can therefore say that every ball $B_{r}^{m}(0)$ is ${}^{\rho}\widetilde{\mathbb{R}}$-balanced.*

(6) *For all $f\in{}^{\rho}\mathcal{GC}^{\infty}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$ there exists $t\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$ such that $f\in t\cdot B_{r}^{m}(0|_{K})$. We can therefore say that every ball $B_{r}^{m}(0|_{K})$ is ${}^{\rho}\widetilde{\mathbb{R}}$-absorbent.*

Because of these properties, we will call the space ${}^{\rho}\mathcal{GF}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$ an ${}^{\rho}\widetilde{\mathbb{R}}$-*Fréchet module*. It is worth noting that the natural properties stated in the previous theorem do not hold if we take the large topology instead of the sharp one, or if we consider the field $\mathbb{R}$ instead of the ring ${}^{\rho}\widetilde{\mathbb{R}}$. For example, since there exist GSFs having infinite norms $\|f\|_{m}\in{}^{\rho}\widetilde{\mathbb{R}}$, the multiplication by standard real scalar $(r,f)\in\mathbb{R}\times{}^{\rho}\mathcal{GC}^{\infty}(K,{}^{\rho}\widetilde{\mathbb{R}})\mapsto r\cdot f\in{}^{\rho}\mathcal{GC}^{\infty}(K,{}^{\rho}\widetilde{\mathbb{R}})$ is clearly not continuous with respect to the standard Euclidean topology on $\mathbb{R}$ because $r\cdot\|f\|\not\to0$ if $r\to0$ in this topology. See [19, Sec. 5.1] for general abstract theorems corresponding to this necessity of using a non-Archimedean topology in dealing with generalized functions.

The spaces ${}^{\rho}\mathcal{GF}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$ are very rich of examples and convenient properties which are well fitted for the aims of the present work. For example, let $\varphi\in\mathcal{D}_{K}(\Omega)$, $K\Subset\Omega\subseteq\mathbb{R}^{n}$, be an ordinary compactly supported smooth function; we can consider $K_{\varepsilon}:=K$ and $f_{\varepsilon}(x):=\varphi(x)$ if $x\in\Omega$ and $f_{\varepsilon}(x):=0$ otherwise to have that $\varphi|_{K}\in{}^{\rho}\mathcal{GF}(K,{}^{\rho}\widetilde{\mathbb{R}})$. Moreover, Thm. 7 implies that $\|\varphi|_{K}\|_{m}=\|\varphi\|_{m}\in\mathbb{R}$ is the usual m-norm of φ.

The following result allows to include infinite meaningful examples and to understand that every $f\in{}^{\rho}\mathcal{GF}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$ can be extended to the whole ${}^{\rho}\widetilde{\mathbb{R}}^{n}$:

Theorem 11. *Let $\emptyset\ne K=[K_{\varepsilon}]\Subset_{f}{}^{\rho}\widetilde{\mathbb{R}}^{n}$ be a solid set, then*

$$\forall f\in{}^{\rho}\mathcal{GC}^{\infty}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})\,\exists\bar{f}\in{}^{\rho}\mathcal{GC}^{\infty}({}^{\rho}\widetilde{\mathbb{R}}^{n},{}^{\rho}\widetilde{\mathbb{R}}^{d}):\ \bar{f}|_{K}=f.\tag{1.2}$$

Moreover, let Ω *be an open subset of* $\mathbb{R}^n$ *and* $J = [J_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$ *be an infinite generalized number. Set* $K_\varepsilon := \{x \in \Omega \mid |x| \le J_\varepsilon\}$ *and* $K := [K_\varepsilon]$. *Then for all* $f \in {}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}}^d)$ *(in particular, if* f *is the embedding of a Schwartz distribution) there exists* $\bar{f} \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$ *defined by* $(\bar{f}_\varepsilon)$ *such that* $\bar{f}|_{\mathrm{c}(\Omega)} = f$, $\bar{f}_\varepsilon|_{\mathbb{R}^n \setminus K_\varepsilon} = 0$ *for all* ε.

Proof. We start to prove the second conclusion. We set $V_\varepsilon := \{x \in \Omega \mid |x| < \frac{1}{2} J_\varepsilon\}$ so that $V_\varepsilon \subseteq K_\varepsilon$ for ε small. Let $\chi_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ be such that $\chi|_{V_\varepsilon} = 1$ and $\mathrm{supp}(\chi_\varepsilon) \subseteq K_\varepsilon$. Let $f \in {}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}}^d)$ be represented by (f_ε), with $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$, and set $\bar{f}_\varepsilon := \chi_\varepsilon \cdot f_\varepsilon$. Then each $\bar{f}_\varepsilon$ is compactly supported in K_ε and any $x = [x_\varepsilon] \in \mathrm{c}(\Omega)$ satisfies $x_\varepsilon \in V_\varepsilon$ for ε small because $\lim_{\varepsilon \to 0^+} J_\varepsilon = +\infty$. Therefore $\bar{f} := [\bar{f}_\varepsilon(-)]|_K \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$, and if $x_\varepsilon \in V_\varepsilon$ then $\bar{f}_\varepsilon(x_\varepsilon) = f_\varepsilon(x_\varepsilon)$, so $\bar{f}|_{\mathrm{c}(\Omega)} = f$. To prove (1.2), we can proceed similarly by considering $\chi_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ such that $\chi_\varepsilon|_{K_\varepsilon} = 1$ and $\mathrm{supp}(\chi_\varepsilon) \subseteq \bigcup_{x \in K_\varepsilon} B_1^{\mathrm{E}}(x)$. □

Theorem **??** and Thm. **??** yield an infinity of non-trivial examples of GSFs in spaces of the type ${}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}}^d)$. In fact, even though $\bar{f}$ depends on the fixed infinite number $J \in {}^\rho\widetilde{\mathbb{R}}$, each such $\bar{f}$ contains all the information of the original generalized function f because $\bar{f}|_{\mathrm{c}(\Omega)} = f$. Finally, note that (1.2) trivially yields $\left\|\bar{f}|_K\right\|_m = \|f\|_m$ for all $m \in \mathbb{N}$ because the norm $\|-\|_m$ is well defined (Thm. 7). Ultimately, this is a consequence of the Fermat-Reyes Thm. **??** and of Thm. 5, which state that every partial derivative depends only on the values of the generalized function f at interior points of the solid set K.

In the following result, we prove a fact that will be very important in Section **??**, namely that the generalized Fréchet space ${}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}}^d)$ is complete with respect to the sharp topology.

Theorem 12. *Let* $\emptyset \ne K \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$ *be a solid set. Then*

1. *The space* ${}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}}^d)$ *with the sharp topology is Cauchy complete, in the sense that any Cauchy sequence* $(u_n)_{n\in\mathbb{N}}$ *in this topology, i.e. which satisfies*
$$\forall i \in \mathbb{N}\, \forall q \in \mathbb{R}_{>0}\, \exists N \in \mathbb{N}\, \forall m, n \ge N : \ \|u_n - u_m\|_i < \mathrm{d}\rho^q \tag{1.3}$$
converges in ${}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}}^d)$ *in the sharp topology.*
2. *Any sharply closed subset of* ${}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}}^d)$ *is also Cauchy complete.*
3. *If* $H \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ *is a sharply closed set, then* $\left\{f \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d) \mid f(K) \subseteq H\right\}$ *is sharply closed in* ${}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}}^d)$.

Proof. It is only essential to prove the case $d = 1$. To show 1, let us consider a Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ in the sharp topology, i.e. we assume (1.3). Setting $i = q = k \in \mathbb{N}_{>0}$, this implies the existence of a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\|u_{n_{k+1}} - u_{n_k}\|_k < \mathrm{d}\rho^k$. Hence, picking any representative $(u_{n\varepsilon})$ of u_n as in Lem. **??**, we have
$$\left[\max_{|\alpha|\le k} \sup_{x\in K_\varepsilon} \left|\partial^\alpha u_{n_{k+1},\varepsilon}(x) - \partial^\alpha u_{n_k,\varepsilon}(x)\right|\right] < \left[\rho_\varepsilon^k\right] \quad \forall k \in \mathbb{N}_{>0}.$$
By Lemma **??**, this yields that for each $k \in \mathbb{N}_{>0}$ there exists an ε_k such that $\varepsilon_k \searrow 0$ and
$$\forall \varepsilon \in (0, \varepsilon_k) : \ \max_{|\alpha|\le k} \sup_{x\in K_\varepsilon} \left|\partial^\alpha u_{n_{k+1},\varepsilon}(x) - \partial^\alpha u_{n_k,\varepsilon}(x)\right| < \rho_\varepsilon^k. \tag{1.4}$$

Now set

$$h_{k,\varepsilon} := \begin{cases} u_{n_{k+1},\varepsilon} - u_{n_k,\varepsilon} \in \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}) & \text{if } \varepsilon \in (0,\varepsilon_k) \\ 0 \in \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}) & \text{if } \varepsilon \in [\varepsilon_k, 1) \end{cases} \tag{1.5}$$

$$u_\varepsilon := u_{n_0,\varepsilon} + \sum_{k=0}^{\infty} h_{k,\varepsilon} \quad \forall \varepsilon \in I.$$

Since $\varepsilon_k \searrow 0$, for all $\varepsilon \in I$ there exists a sufficiently big k such that we have $\varepsilon \notin (0,\varepsilon_k)$ for all $k \ge \bar{k}$. Therefore, $u_\varepsilon = u_{n_{\bar{k}+1},\varepsilon} \in \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R})$. In order to prove that (u_ε) defines a GSF of the type $K \to {}^\rho\widetilde{\mathbb{R}}$, take $[x_\varepsilon] \in K$ and $\alpha \in \mathbb{N}$. We claim that $(\partial^\alpha u_\varepsilon(x_\varepsilon)) \in \mathbb{R}_\rho$. Now, for all $p \in \mathbb{N}$ and for any $x \in \mathbb{R}^n$ we have that, for $\varepsilon \le \varepsilon_p$

$$|\partial^\alpha u_\varepsilon(x)| \le \left|\partial^\alpha u_{n_{p+1},\varepsilon}(x)\right| + \sum_{k=p+1}^{\infty} |\partial^\alpha h_{k,\varepsilon}(x)|\,.$$

If p satisfies $|\alpha| \le p$, then from (1.4) and (1.5), we get that $|\partial^\alpha h_{k,\varepsilon}(x)| \le \rho_\varepsilon^k$ for all $k \ge p+1$, $x \in K_\varepsilon$ and all $\varepsilon \in (0,1]$. Hence for $\varepsilon \in (0,\varepsilon_p)$, $|\alpha| \le p$ and all $x \in K_\varepsilon$, we obtain

$$|\partial^\alpha u_\varepsilon(x)| \le \left|\partial^\alpha u_{n_{p+1},\varepsilon}(x)\right| + \frac{\rho_\varepsilon^{p+1}}{1-\rho_\varepsilon}. \tag{1.6}$$

Inserting $x = x_\varepsilon$ and noting that $(\partial^\alpha u_{n_{p+1},\varepsilon}(x_\varepsilon)) \in \mathbb{R}_\rho$ proves our claim.

Moreover, $\|u - u_{n_p}\|_i < \mathrm{d}\rho^{p-1}$ for all $p \in \mathbb{N}_{>1}$ and all $i \le p$. This yields that $(u_{n_k})_k$ tends to u in the sharp topology, and hence so does (u_n).

If $C \subseteq {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}})$ is closed in the sharp topology and $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of C, then it converges to a point $u \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}})$. We cannot have $u \in C^c$ because otherwise $u_n \in B_r^m(u) \subseteq C^c$ for some $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, $m \in \mathbb{N}$, and for all $n \in \mathbb{N}$ sufficiently big, which is a contradiction. This shows 2.

Finally, let $(u_n)_{n\in\mathbb{N}}$ be a convergent sequence of ${}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$ such that $u_n(K) \subseteq H$ for all $n \in \mathbb{N}$. Set $u := \lim_{n\to+\infty} u_n \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$, then $\|u_n - u\|_0 = \left[\sup_{x\in K_\varepsilon} |u_{n,\varepsilon}(x) - u_\varepsilon(x)|\right] \to 0$ in the sharp topology. If $x \in K = [K_\varepsilon]$, then $x_\varepsilon \in K_\varepsilon$ for some representative $[x_\varepsilon] = x$ and for ε small. Therefore, $|u_n(x) - u(x)| \le \|u_n - u\|_0$ and hence the sequence $(u_n(x))_{n\in\mathbb{N}}$ of H tends to $u(x)$ in the sharp topology. Hence $u(x) \in H$ because we assumed that H is sharply closed. □

For a complete theory of (functionally) compactly supported GSFs in the case $\rho_\varepsilon = \varepsilon$, see [19]. In the same particular case, for an Archimedean theory of ${}^\rho\widetilde{\mathbb{R}}$-modules, see [15, 16, 17].

2. Fixed point methods for PDE

2.1. **Banach fixed point theorem for PDE.** We want to study PDE of the following form:

$$\begin{cases} \partial_t^k y(t,x) = G\left[t, x, (\partial_x^a y)_{|a|\le h}\right], \\ \partial_t^j y(t_0,x) = y_{0,j}(x) & 0 \le j < k. \end{cases} \tag{2.1}$$

We want to develop an approach based on fixed point methods. The basic concept that we will use is the following:

Definition 13. Let $\emptyset \neq K \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}^{n}$ be a solid set, and let $y_0 \in X \subseteq {}^{\rho}\mathcal{GC}^{\infty}\left(K, {}^{\rho}\widetilde{\mathbb{R}}^{d}\right)$. We say that P is a *finite sharp contraction on X with loss of derivatives L starting from y_0* if

(1) $P : X \longrightarrow X$ is a set-theoretical map.
(2) $\forall i \in \mathbb{N}\, \exists \alpha_i \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}\, \forall u, v \in X : \ \|P(u) - P(v)\|_i \leq \alpha_i \cdot \|u - v\|_{i+L}$.
(3) For all $i \in \mathbb{N}$, we have

$$\lim_{n,m\to+\infty, n\leq m} \alpha_{i+mL}^{n} \cdot \|P(y_0) - y_0\|_{i+mL} = 0,$$

where the limit is taken in the sharp topology.

Moreover, we say that P is a *finite sharp contraction on X with loss of derivatives L* if the above conditions hold for every $y_0 \in X$.

In Def. 13 it is possible to assume without loss of generality that $\forall i \in \mathbb{N}\, \alpha_i \leq \alpha_{i+1}$. As this property is helpful in many proofs, we will assume it from now on.

Let us notice that the above definitions resembles almost perfectly the definition of contraction that we gave in [24]. Our aim is to extend the methods introduced in [24] to treat also equations like 2.1. Let us start by proving an analogue of the Banach fixed point theorem:

Theorem 14 (BFPT with loss of derivatives)**.** *Let K, X, y_0, L, P be given as in Def. 13. Assume that X is a Cauchy complete set. Then:*

(1) *P is sharply continuous;*
(2) *$\exists y \in X$ such that $\lim_{n\to+\infty} P^n(y_0) = y$;*
(3) *$P(y) = y$.*

Proof. Proof of (1): let $B_r^i(P(u)) \cap X$ be an open neighborhood of $P(u)$ in X. For every $v \in K$ by condition (2) in Definition 13 we have that

$$\|P(v) - P(u)\|_i \leq \alpha_i \cdot \|v - u\|_{i+L},$$

hence $P\left(B^{i+L}_{\frac{\alpha_i}{r}}(u) \cap K\right) \subseteq B_r^i(P(u)) \cap X$, which shows that P is sharply continuous.

Proof of (2): By induction it is immediate to prove that $\forall n \in \mathbb{N}$

$$\left\|P^{n+1}(y_0) - P^n(y_0)\right\|_i \leq \alpha_i \cdot \ldots \cdot \alpha_{i+nL} \cdot \|P(y_0) - y_0\|_{i+nL} \leq \alpha_{i+nL}^{n} \|P^n(y_0) - y_0\|_{i+nL}.$$

Now for every $n, m \in \mathbb{N}, n < m$ we have

$$\|P^m(y_0) - P^n(y_0)\|_i \leq \left\|P^m(y_0) - P^{m-1}(y_0)\right\|_i + \ldots + \left\|P^{n+1}(y_0) - P^n(y_0)\right\|_i \leq$$

$$\alpha_{i+(m-1)L}^{m-1} \|P(y_0) - y_0\|_{i+(m-1)L} + \ldots + \alpha_{i+nL}^{n} \cdot \|P(y_0) - y_0\|_{i+nL} \leq \text{(as } \alpha_k \leq \alpha_{k+1}\ \forall k \in \mathbb{N})$$

$$\alpha_{i+(m-1)L}^{m-1} \|P(y_0) - y_0\|_{i+(m-1)L} + \ldots + \alpha_{i+(m-1)L}^{n} \|P(y_0) - y_0\|_{i+(m-1)L} \leq$$

$$\alpha_{i+(m-1)L}^{n} \|P(y_0) - y_0\|_{i+(m-1)L} \cdot \sum_{j=0}^{m-1-n} \alpha_{i+(m-1)L}^{j} =$$

$$\frac{\alpha_{i+(m-1)L}^{n} - \alpha_{i+(m-1)L}^{m}}{1 - \alpha_{i+(m-1)L}} \|P(y_0) - y_0\|_{i+(m-1)L}$$

and this goes to 0 by assumption. This shows that $\{P^n(y_0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X, which is a Cauchy complete set, henceforth it must have a limit $y \in X$.

Proof of (3): By the sharp continuity of P we have

$$P(y) = P\left(\lim_{n\to+\infty} P^n(y_0)\right) = \lim_{n\to+\infty} P^{n+1}(y_0) = y.$$

□

2.2. **Picard-Lindelöf theorem with loss of derivatives.** In this section we want to arrive to a formulation of Picard-Lindelöf theorem in the context of generalized PDE with loss of derivatives. Let us start with a definition.

Definition 15. Let $\emptyset \neq T \Subset_f {}^\rho\widetilde{\mathbb{R}}$, $\emptyset \neq S \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$ be solid sets, let $Y \subseteq {}^\rho\mathcal{GC}^\infty\left(T \times S, {}^\rho\widetilde{\mathbb{R}}^d\right)$ and let $L \in \mathbb{N}$. Then we say that F *is uniformly Lipschitz on* Y *with constants* $(\Lambda_i)_{i\in\mathbb{N}}$ *and loss of derivatives* L if:

1. $F : T \times S \to {}^\rho\widetilde{\mathbb{R}}^d$ is a set-theoretical map;
2. $\forall y \in Y\ F(-,-,y) := F(y)(-,-) \in {}^\rho\mathcal{GC}^\infty\left(T \times S, {}^\rho\widetilde{\mathbb{R}}^d\right)$;
3. $\forall i \in \mathbb{N}\, \forall u, v \in Y$
$$\|F(t,x,u) - F(t,x,v)\|_i \le \Lambda_i \cdot \|u - v\|_{i+L}\,.$$

Notice that in the previous definition we let $\|F(t,x,u) - F(t,x,v)\|_i$ be the norm in the sense of ${}^\rho\mathcal{GC}^\infty\left(T \times S, {}^\rho\widetilde{\mathbb{R}}^d\right)$, see also [24, Def TOT]. Moreover, notice that w.l.o.g. we can always assume that $\Lambda_i \le \Lambda_{i+1}\ \forall i \in \mathbb{N}$. We will always make this assumption.

Definition 15 might seem very restrictive; however, the following Theorem shows that every GSF of the form $G\left[t, x, \partial_x^a y(t,x)_{|a|\le L}\right]$ is uniformly Lipschitz:

Theorem 16. *Let* $\emptyset \neq T \Subset_f {}^\rho\widetilde{\mathbb{R}}$, $\emptyset \neq S \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$ *be solid sets, let* $L \in \mathbb{N}$, $\hat{L} := |\{a \in \mathbb{N}^n \mid |a| \le L\}|$, $G \in {}^\rho\mathcal{GC}^\infty\left(T \times {}^\rho\widetilde{\mathbb{R}}^n \times {}^\rho\widetilde{\mathbb{R}}^{d\cdot\hat{L}}, {}^\rho\widetilde{\mathbb{R}}^d\right)$, $y_0 \in {}^\rho\mathcal{GC}^\infty\left(S, {}^\rho\widetilde{\mathbb{R}}^d\right)$, $\|y_0\|_i \le s_i \in {}^\rho\widetilde{\mathbb{R}}_{>0}\ \forall i \in \mathbb{N}$. *Then for all* $H \Subset_f {}^\rho\widetilde{\mathbb{R}}^d$ *the function*

$$(t,x,y) \in T \times S \times Y \longmapsto G\left[t, x, \partial_x^a y(t,x)_{|a|\le L}\right] \in {}^\rho\widetilde{\mathbb{R}}^d$$

is uniformly Lipschitz with loss of derivatives L *on*

$$Y := y \in \left\{{}^\rho\mathcal{GC}^\infty\left(T \times S, H\right) \mid \|y - y_0\|_i \le r_i\ \forall i \in \mathbb{N}\right\}.$$

Proof. r_i nell'enunciato non è detto cosa siano; dimostrazione da sistemare una volta che sia stata sistemata quella delle formule di Di Faa nel paper delle ODE. □

We are now ready to prove the main result of this section:

Theorem 17. *Let* $t_0 \in {}^\rho\widetilde{\mathbb{R}}$, *let* $\alpha, r_i \in {}^\rho\widetilde{\mathbb{R}}_{>0}\ \forall i \in \mathbb{N}$, *let* $T_\alpha = [-\alpha, \alpha]$. *Let* ${}^\rho\widetilde{\mathbb{R}}^d \supseteq H$ *be a sharply closed set, let* $S \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$. *Let* $y_0(x), \dots, y_k(x) \in {}^\rho\mathcal{GC}^\infty\left(S, {}^\rho\widetilde{\mathbb{R}}\right)$, *let* $P_{\overrightarrow{y(x)}}(t) := \sum_{i=0}^{k-1} \frac{y_i(x)}{i!} t^i$ *and let* $\overline{B_{r_0}\left(P_{\overrightarrow{y(x)}}(t)\right)} \subseteq H\ \forall (t,x) \in T_\alpha \times S$. *Set*

$$Y_\alpha := \left\{y \in {}^\rho\mathcal{GC}^\infty\left(T_\alpha \times S, H\right) \mid \left\|y - P_{\overrightarrow{y(x)}}(t)\right\|_i \le r_i\ \forall i \in \mathbb{N}\right\}$$

and assume that F *is uniformly Lipschitz on* Y_α *with constants* $(\Lambda_i)_{i\in\mathbb{N}}$ *and loss of derivatives* L. *Finally assume that*

1. $\|F(-,-,y)\|_i \le M_i(y)$ *with* $\frac{\alpha^k}{k!} \cdot M_i(y) \le r_i$ *for all* $y \in Y_\alpha$;
2. $\lim_{n,m\to+\infty, n\le m} \left(\frac{\alpha^k}{k!} \cdot \Lambda_{i+mL}\right)^n \left\|F(-,-,P_{\overrightarrow{y}})\right\|_{i+mL} = 0$ *for all* $i \in \mathbb{N}$.

Then there exists a solution $y \in Y_\alpha$ of the Cauchy initial value problem

$$\begin{cases} \partial_t^k y(t,x) = F\left[t, x, (\partial_x^\alpha y)_{|\alpha|\le L}\right], \\ \partial_t^j y(t_0,x) = y_j(x) & 0 \le j < k. \end{cases}$$

Proof. The proof that ${}^\rho\mathcal{GC}^\infty(T_\alpha \times S, H)$ is closed is identical to that of [24, CERCARE]. Let us notice that $Y_\alpha \neq \emptyset$ as $P_{\overrightarrow{y(x)}}(t) \in Y_\alpha$. We now let $T : Y_\alpha \to {}^\rho\mathcal{GC}^\infty(T_\alpha \times S, H)$ be the function

$$T(y)(t,x) := P_{\overrightarrow{y(x)}}(t) + \int_0^t \int_0^{s_k} \left(\dots \left(\int_0^{s_2} F(s_1, x, y) ds_1 \right) \dots d_{s_{k-1}} \right) ds_k.$$

Notice that $T(y)(-,-) \in {}^\rho\mathcal{GC}^\infty\left(T_\alpha \times S, {}^\rho\widetilde{\mathbb{R}}^d\right)$ because of Condition (2) in Definition **??**.

Now, for every $y \in Y_\alpha$ and for every $i \in \mathbb{N}$ we have

$$\left\|T(y) - P_{\overrightarrow{y}}\right\|_i = \left\| \int_0^t \int_0^{s_k} \left(\dots \left(\int_0^{s_2} F(s_1, x, y) ds_1 \right) \dots d_{s_{k-1}} \right) ds_k \right\|_i \le$$
$$\int_0^t \int_0^{s_k} \left(\dots \left(\int_0^{s_2} \|F(s_1, x, y)\|_i \, ds_1 \right) \dots d_{s_{k-1}} \right) ds_k \le \frac{\alpha^k}{k!} M_i(y) \le r_i.$$

Moreover, $\forall (t,x) \in T_\alpha \times S$ we have $|T(y)(t,x) - P_{\overrightarrow{y(x)}}(t)| \le \left\|T(y) - P_{\overrightarrow{y}}\right\|_0 \le r_0$, hence $T : Y_\alpha \to Y_\alpha$.

It remains to prove that T is a finite contraction on Y_α with loss of derivatives L starting from $P_{\overrightarrow{y(x)}}(t)$: first of all, let us notice that

$$\left\| T\left(P_{\overrightarrow{y(x)}}(t)\right) - P_{\overrightarrow{y(x)}}(t) \right\|_i = \left\| \int_0^t \int_0^{s_k} \left(\dots \left(\int_0^{s_2} F(s_1, x, P_{\overrightarrow{y(x)}}(t)) ds_1 \right) \dots d_{s_{k-1}} \right) ds_k \right\|_i \le$$
$$\frac{\alpha^k}{k!} \cdot \left\| F\left(-,-,P_{\overrightarrow{y(x)}}(t)\right) \right\|_i.$$

Finally, for every $u, v \in Y_\alpha$ we have

$$\|T(u) - T(v)\|_i = \left\| \int_0^t \int_0^{s_k} \left(\dots \left(\int_0^{s_2} F(s_1, x, u) - F(s_1, x, v) ds_1 \right) \dots d_{s_{k-1}} \right) ds_k \right\|_i \le$$
$$\int_0^t \int_0^{s_k} \left(\dots \left(\int_0^{s_2} \|F(s_1, x, u) - F(s_1, x, v)\|_i \, ds_1 \right) \dots d_{s_{k-1}} \right) ds_k \le$$
$$\int_0^t \int_0^{s_k} \left(\dots \left(\int_0^{s_2} \Lambda_i \cdot \|u - v\|_{i+L} \, ds_1 \right) \dots d_{s_{k-1}} \right) ds_k = \frac{\alpha^k}{k!} \cdot \Lambda_i \cdot \|u - v\|_{i+L}.$$

Henceforth our candidate contraction constants are $\frac{\alpha^k}{k!} \cdot \Lambda_i$. To finish the proof, we need to evaluate $\lim_{n,m\to+\infty, n\le m} \left(\frac{\alpha^k}{k!} \cdot \Lambda_{i+mL}\right)^n \left\| T\left(P_{\overrightarrow{y(x)}}(t)\right) - P_{\overrightarrow{y(x)}}(t) \right\|_{i+mL}$:

$$\lim_{n,m\to+\infty, n\le m} \left(\frac{\alpha^k}{k!} \cdot \Lambda_{i+mL}\right)^n \left\| T\left(P_{\overrightarrow{y(x)}}(t)\right) - P_{\overrightarrow{y(x)}}(t) \right\|_{i+mL} \le$$
$$\lim_{n,m\to+\infty, n\le m} \left(\frac{\alpha^k}{k!} \cdot \Lambda_{i+mL}\right)^n \cdot \frac{\alpha^k}{k!} \cdot \left\| F\left(-,-,P_{\overrightarrow{y(x)}}(t)\right) \right\|_{i+mL} =$$
$$\frac{\alpha^k}{k!} \cdot \lim_{n,m\to+\infty, n\le m} \left(\frac{\alpha^k}{k!} \cdot \Lambda_{i+mL}\right)^n \cdot \left\| F\left(-,-,P_{\overrightarrow{y(x)}}(t)\right) \right\|_{i+mL} = 0$$

by our hypothesis (2). This concludes the proof. □

Remark 18.

(1) Uniqueness cannot be proved as we did for the ODE case in [24, CHECK]. In fact, if $y(x,t), z(x,t)$ are two solution of equation 2.1 then

$$|y(t,x)-z(t,x)| \le \max_{t\in T_\alpha, x\in S}\int_{t_0}^{t}\|F(-,-,y)-F(-,-,z)\|_0 \le \alpha\cdot\Lambda_0\cdot\|y-z\|_L$$

which does not entail that $y=z$, in genere.

(2) Notice that in Thm. 17 we can take $H=\overline{B_s(P_{\overrightarrow{y}}(0))}$ where $s=r_0+\left\|P_{\overrightarrow{y}}\right\|_0$. In fact, for every $y\in\overline{B_{r_0}(P_{\overrightarrow{y}}(x))}$ we have

$$|y|\le|y-P_{\overrightarrow{y}}(x)|+|P_{\overrightarrow{y}}(x)|<r_0+\left\|P_{\overrightarrow{y}}\right\|_0,$$

and so $y\in H$.

Theorems 16 and 17 can be combined fo give simpler conditions under which an interval of existence of solutions of equation 2.1 exists.

Corollary 19. *Let $t_0\in{}^\rho\widetilde{\mathbb{R}}$, $\beta, r_i\in{}^\rho\widetilde{\mathbb{R}}_{>0}\,\forall i\in\mathbb{N}$, $T:=[t_0-\beta,t_0+\beta]$, $L\in\mathbb{N}$ and $\hat{L}=|\{a\in\mathbb{N}^n \mid |a|\le L\}|$. Let $G\in{}^\rho\mathcal{GC}^\infty\left(T\times{}^\rho\widetilde{\mathbb{R}}^n\times{}^\rho\widetilde{\mathbb{R}}^{d\cdot\hat{L}},{}^\rho\widetilde{\mathbb{R}}^d\right)$, $\emptyset\ne S\Subset_f{}^\rho\widetilde{\mathbb{R}}^n$, $y_0(x),\dots,y_k(x)\in{}^\rho\mathcal{GC}^\infty\left(S,{}^\rho\widetilde{\mathbb{R}}\right)$, and let $P_{\overrightarrow{y(x)}}(t):=\sum_{i=0}^{k-1}\frac{y_i(x)}{i!}t^i$ with $\left\|P_{\overrightarrow{y(x)}}\right\|_i\le s_i\in{}^\rho\widetilde{\mathbb{R}}_{>0}\forall i\le\hat{L}$.*

Set $H:=\overline{B_{r_0+s_0}(0)}\subseteq{}^\rho\widetilde{\mathbb{R}}_{>0}$, $D:=\prod_{i=0}^{\hat{L}}\overline{B_{r_i+s_i}(0)}$, $M_i:=\|G|_{T\times S\times D}\|_i$. Let $(\Lambda_i)_{i\in\mathbb{N}}$ be the Lipschitz constants for G as stated in Theorem 16 (which depend on $H,T,S,(r_i)_i,(s_i)_i$). Finally, assume that $\alpha\in(0,\beta]$ is such that

(1) $\exists R\in{}^\rho\widetilde{\mathbb{R}}\,\forall i\in\mathbb{N}\,\Lambda_i\le R$;

(2) $\exists a\in\mathbb{R}_{>0}$ such that:
   - *(a) $\frac{\alpha^k}{k!}\le\min\left(\frac{d\rho^a}{R},\frac{r_i}{M_i}\right)$;*
   - *(b) $\lim_{n,j\to+\infty}d\rho^{na}\left\|G\left(-,-,\left(\partial_x^a P_{\overrightarrow{y(x)}}\right)_{|a|\le L}\right)\right\|_j=0.$*

Then there exists a solution $y\in Y_\alpha$ of equation 2.1.

Proof. Set $T_\alpha:=[t_0-\alpha,t_0+\alpha]$ and let $M_{i,\alpha}(y):=M_{i,\alpha}:=\|G|_{T_\alpha\times S\times D}\|_i\ \forall y\in Y_\alpha$. Let $(\Lambda_{i,\alpha})_{i\in\mathbb{N}}$ be the Lipschitz constants of

$$(t,x,y)\in T\times S\times Y_\alpha\to G\left[t,x,\left(\partial_x^a y(t,x)\right)_{|a|\le L}\right]\in{}^\rho\widetilde{\mathbb{R}}$$

on Y_α as in Thm. 16. We have the following facts:

- $\forall x\in S\,|P_{\overrightarrow{y(x)}}|\le\left\|P_{\overrightarrow{y(x)}}\right\|_0\le s_0<r_0+s_0$, hence $y_0\in{}^\rho\mathcal{GC}^\infty(S,H)$;
- $\overline{B_{r_0}(P_{\overrightarrow{y(x)}})}\subseteq H\,\forall x\in S$ by Remark 18.2;
- $\Lambda_{i,\alpha}\le\Lambda_{i+1,\alpha}\,\forall i\in\mathbb{N}$ by Remark TO ADD IN THE PROOF OF LIPSCHITZ;
- If we set $\forall y\in Y_\alpha\ F(t,x,y):=G\left(x,y,\left(\partial_x^a P_{\overrightarrow{y(x)}}\right)_{|a|\le L}\right)$, we have that $\|F(-,-,y)\|_i\le\|G|_{T_\alpha\times S\times D}\|_i=M_{i,\alpha}(y)$;

To conclude the proof we show that we fulfil the hypotheses of Theorem 17. In fact:

- $\frac{\alpha^k}{k!} \cdot M_{i,\alpha}(y) = \frac{\alpha^k}{k!} \cdot M_{i,\alpha} \le \frac{\alpha^k}{k!} \cdot M_i$ as $T_\alpha \subseteq T$, and $\frac{\alpha^k}{k!} \cdot M_i \le r_i$ by hypothesis (2a);
- As $T_\alpha \subseteq T$,

$$\left(\frac{\alpha^k}{k!} \cdot \Lambda_{i+mL,\alpha}\right)^n \cdot \left\| F\left(-,-,P_{\overrightarrow{y(x)}}\right)\right\|_{i+mL} \le \left(\frac{\alpha^k}{k!} \cdot \Lambda_{i+mL}\right)^n \cdot \left\| F\left(-,-,P_{\overrightarrow{y(x)}}\right)\right\|_{i+mL}.$$

As $\Lambda_{i+mL} \le R$ and $\frac{\alpha^k}{k!} \cdot R \le d\rho^a$, we get that

$$\left(\frac{\alpha^k}{k!} \cdot \Lambda_{i+mL}\right)^n \cdot \left\| F\left(-,-,P_{\overrightarrow{y(x)}}\right)\right\|_{i+mL} \le d\rho^{na} \cdot \left\| F\left(-,-,P_{\overrightarrow{y(x)}}\right)\right\|_{i+mL}$$

and we conclude as $\lim_{n,j\to+\infty} d\rho^{na} \left\| G\left(-,-,\left(\partial_x^a P_{\overrightarrow{y(x)}}\right)_{|a|\le L}\right)\right\|_j = 0$ by hypothesis (2b).

□

Corollary 20. *Let us assume the same hypotheses of Cor. 19 except the hypothesis (2). If there exists $M, r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that*

(1) $\forall i \in \mathbb{N}\, 0 < M_i \le M$ *and* $r \le r_i$;
(2) $\exists a \in \mathbb{R}_{>0}$ *such that* $\frac{\alpha^k}{k!} \le \min\left(\frac{d\rho^a}{R}, \frac{r}{M}\right)$

then there exists a solution $y \in Y_\alpha$ of equation 2.1.

Proof. We have that $\frac{\alpha^k}{k!} \le \frac{r}{M} \le \frac{r_i}{M_i}$ for every $i \in \mathbb{N}$ and $\left\| G\left(-,-,\left(\partial_x^a P_{\overrightarrow{y(x)}}\right)_{|a|\le L}\right)\right\|_j \le \left\| G|_{T\times S\times D}\right\|_j = M_j \le M$. Therefore $\lim_{n,j\to+\infty} d\rho^{na} \left\| G\left(-,-,\left(\partial_x^a P_{\overrightarrow{y(x)}}\right)_{|a|\le L}\right)\right\|_j \le \lim_{n,j\to+\infty} d\rho^{na} \cdot M = 0$. □

TO ADD: Remarks of Paolo + a very long list of examples.

3. Boundary conditions for differential problems in the GSF setting

In this section we want to adress an issue that, as far as we know, has not been discussed yet in GSF theory nor in Colombeau theory. The issue is the following: which kind of "boundary conditions" are we allowed to consider in the GSF setting when we deal with differential equations? As the issue has nothing to do with the particular set of indeces I that one fixes to build the scalars, in this section we let $I = (0, 1]$ (the general treatment would be completely analogous).

This issue originates by a simple observation: boundaries in the sharp topology are badly behaving objects. To explain what we mean, let us consider $[0,1]_{{}^\rho\widetilde{\mathbb{R}}}$. Whilst the interior of $[0,1]$ is the sharply open set $(0,1)_{{}^\rho\widetilde{\mathbb{R}}}$, the boundary of $[0.1]_{{}^\rho\widetilde{\mathbb{R}}}$ is

$$\partial\left([0,1]_{{}^\rho\widetilde{\mathbb{R}}}\right) = \left\{x \in [0,1]_{{}^\rho\widetilde{\mathbb{R}}} \mid x \text{ is not invertible or } 1-x \text{ is not invertible}\right\},$$

which contains $0, 1$ (as expected) but also many bad behaving objects like e.g. points $x = [x_\varepsilon]$ where $x \in [0,1]_{{}^\rho\widetilde{\mathbb{R}}}$ and $x_\varepsilon = 0$ for every $\varepsilon \in J \subseteq I$ with $\inf J = 0$. This bad behaviour has strong consequences also on GSF:

Remark 21. Let $f, g \in {}^\rho\mathcal{GC}^\infty\left([0,1]_{{}^\rho\widetilde{\mathbb{R}}}, {}^\rho\widetilde{\mathbb{R}}^n\right)$. If $f = g$ on $\partial\left([0,1]_{{}^\rho\widetilde{\mathbb{R}}}\right)$ then $f = g$ on $[0,1]_{{}^\rho\widetilde{\mathbb{R}}}$.

Proof. Let $x=[x_\varepsilon]\in[0,1]_{\rho\widetilde{\mathbb{R}}}$. Let $I_1,I_2\subseteq(0,1]$ be sets with $I_1\cap I_2=\emptyset$ and $\inf I_1=\inf I_2=0$. For $i=1,2$ let $\chi_i(\varepsilon)$ be the characteristic function of I_i and let $\chi_i:=[\chi_i(\varepsilon)]$. If $f=[f_\varepsilon],g=[g_\varepsilon]$, we have

$$f(x)=[f_\varepsilon(x_\varepsilon)]=[f_\varepsilon(\chi_1(\varepsilon)x_\varepsilon+\chi_2(\varepsilon)x_\varepsilon)]=$$

$$[\chi_1(\varepsilon)f_\varepsilon(x_\varepsilon)+\chi_2(\varepsilon)f_\varepsilon(x_\varepsilon)]=\chi_1 f(\chi_1 x)+\chi_2 f(\chi_2 x).$$

We conclude by observing that $\chi_1 x,\chi_2 x\in\partial\left([0,1]_{\rho\widetilde{\mathbb{R}}}\right)$, hence $\chi_1 f(\chi_1 x)+\chi_2 f(\chi_2 x)=\chi_1 g(\chi_1 x)+\chi_2 g(\chi_2 x)=g(x)$ by analogous computations. □

As an obvious consequence, we get that letting "boundary conditions" mean prefixed conditions on arbitrary subsets of the boundary is not a valid choice in the GSF setting. To understand what the meaning of "boundary conditions" in the GSF setting should be, we use the notion of subpoint introduced in [REFERENZA], in particular the following trivial result that we recall for clarity:

Lemma 22. *Let $X\subseteq{}^\rho\widetilde{\mathbb{R}}^n$, $Y\subseteq{}^\rho\widetilde{\mathbb{R}}^d$, $f\in{}^\rho\mathcal{GC}^\infty(X,Y)$, $x\in X$. If x' is a subpoint of x and $J=dom(x')$ then $f(x)|_J=f|_J(x|_J)$.*

We recall that an m-ple $(x_i')_{i=1}^m$ is said to cover a given point x if all x_i''s are subpoints of x and $\bigcup_{i=1}^m dom(x_i')=dom(x)$.

Lemma 23. *Let $x\in{}^\rho\widetilde{\mathbb{R}}^n$, let $(x_i')_{i=1}^m$ be a cover of x and let $f,g\in{}^\rho\mathcal{GC}^\infty\left({}^\rho\widetilde{\mathbb{R}}^n,{}^\rho\widetilde{\mathbb{R}}^k\right)$ be such that $\forall i\le m\ f(x_i')|_{dom(x_i')}=g(x_i')|_{dom(x_i')}$. Then $f(x)=g(x)$.*

Proof. This follows in a straightforward way from Lemma 22 and the fact that generalized smooth functions can be characterized ε-wise. □

Notice that Remark 21 is a particular case of the previous Theorem, as every point $x\in[0,1]_{\rho\widetilde{\mathbb{R}}}$ can be covered by a pair x_1',x_2' of subpoints of points in $\partial([0,1])_{\rho\widetilde{\mathbb{R}}}$. This shows that our problem with boundary conditions is due to the fact that, for many interesting examples of domains Ω, internal points of Ω can be covered by m-ples of points on the boundary $\partial\Omega$ which, as a consequence of Lemma 23, fixes the value on the interior of Ω independently of the differential equation considered.

To solve the above problem, we introduce the following definition:

Definition 24. Let $X\subseteq{}^\rho\widetilde{\mathbb{R}}^n$, let $x\in\partial X$. We say that x is a strict border point of X if for every $y\in X$ we have that x,y have no common subpoint. We call strict border of X the set

$$\widehat{\partial X}:=\{x\in\partial X\mid x\text{ is a strict border point of }X\}.$$

Contrary to the general border, the strict border has the following property, which in some sense shows that this is the "right notion" to consider when dealing with boundary conditions for differential properties as, roughly speaking, it means that fixing the value of a generalized smooth function on the strict border leaves completely free the values it can attain on the interior:

Theorem 25. *Let $X\subseteq{}^\rho\widetilde{\mathbb{R}}^n$, let $f\in{}^\rho\mathcal{GC}^\infty\left(\overline{X},{}^\rho\widetilde{\mathbb{R}}^m\right)$, let y be an interior point of X and let $\alpha\in{}^\rho\widetilde{\mathbb{R}}^m$. Then there exists $g\in{}^\rho\mathcal{GC}^\infty\left(\overline{X},{}^\rho\widetilde{\mathbb{R}}^m\right)$ such that*

1. $f|_{\widehat{\partial X}}=g|_{\widehat{\partial X}}$;
2. $g(y)=\alpha$.

Proof. By substituting g with $g - f$ (and consequently α with $\alpha - f(y)$), we can assume that $f = 0$ on $\widehat{\partial X}$. Then this result holds as a consequence of the sheaf properties of generalized smooth functions, as METTERE REFERENZA PRECISA. □

Whilst at first sight that of strict border point might seem a strange notion, what happens is that actually it formalizes precisely most simple examples that one has in mind:

- $0, 1$ are the only strict border points of $[0, 1]$;
- more in general, the strict border of an open ball $B_r(x) \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ is
$$\left\{ y \in {}^\rho\widetilde{\mathbb{R}}^n \mid d(x, y) = r \right\};$$
- 0 is the only strict border point of ${}^\rho\widetilde{\mathbb{R}}_{>0}$;
- more in general, for every $i \le n$ the strict border of the open semispace $\left\{ (x_1, \dots, x_n) \in {}^\rho\widetilde{\mathbb{R}}^n \mid x_i > 0 \right\}$ is $\left\{ (x_1, \dots, x_n) \in {}^\rho\widetilde{\mathbb{R}}^n \mid x_i = 0 \right\}$,

and so on.

4. Characteristics for generalized PDE

The goal of this section is to show that the classical theory of characteristics can be extended to generalized PDE of the form

$$\begin{cases} F\left(Du, u, x\right) = 0, & x \in \Omega; \\ u = g & \text{on } \Gamma, \end{cases} \tag{4.1}$$

where Ω is an open set, Γ is a subset of the strict border of Ω and $g \in {}^\rho\mathcal{GC}^\infty\left(\Gamma, {}^\rho\widetilde{\mathbb{R}}\right)$. As we are going to show, this extension can be done in an almost strightforward way. To underscore this fact, we will follow closely the presentation of characteristics (in the classical theory) given by Evans in [14], concentrating only on the modifications we need to do to adapt the theory to our setting.

4.1. **Flat boundaries.**

SEZIONE DA RIGUARDARE UNA VOLTA FISSATI I RISULTATI DEGLI ARTICOLI PRECEDENTI, si può sicuramente scrivere meglio di così

In what follows, we will be interested in the following kind of open domains:

Definition 26. Let $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ be a sharp open set and let $x_0 \in \partial U$. We say that U can be flattened at x_0 if there exists a sharp neighborhood V of x_0, a sharp neighborhood W of 0 and an isomorphism $\Phi : V \cap U \to W \cap \{(x_1, \dots, x_n) \mid x_n \ge 0\}$. We say that the boundary of U can be flattened if U can be flattened at x_0 for every $x_0 \in \partial U$.

As in the classical case, when the boundary of U can be flattened, all remaining computations are easier to carry on. The only nontrivial fact to be checked in the GSF setting is the following (RIGUARDARE SE SI SEMPLIFICA LA SCRITTURA con altri risultati):

Lemma 27. *Let K be a solid subset with no isolated points, let $\Omega = int(K)$ and let $\Psi : K \to {}^\rho\widetilde{\mathbb{R}}^m$ be such that $\Psi|_{int(K)} : \Omega \to U$ is a isomorphism. Then:*

1. *$\overline{U}$ is a solid set;*

(2) $\Psi\left(K \setminus int(K)\right) = \overline{U} \setminus U$;
(3) $\Psi\left(\widehat{\partial\Omega}\right) = \widehat{\partial U}$.

Proof. (1) Let A be an open (nonempty) subset of $\overline{U}$. Let $B = \Psi^{-1}(A)$. Then $\exists x \in \Omega \cap B$. So $\Psi(x) \in int(U) \cap A$. This shows that $U = int(\overline{U})$ is dense in $\overline{U}$.

(2) This holds in general for isomorphisms.

(3) Let us show $\subseteq$, the other inclusion can be proven similarly considering Ψ^{-1}. Let $x \in \widehat{\partial\Omega}$ and assume that $\Psi(x) \notin \widehat{\partial U}$. By (2), necessarily $\Psi(x) \in \partial U$. So $\Psi(x) \in \partial U \setminus \widehat{\partial U}$, hence there exists $y \in U$ so that $\Psi(x)$ and y have a subpoint in common. But then $\Psi^{-1}\left(\Psi(x)\right) = x$ and $\Psi^{-1}(y) \in int(K)$ have a subpoint in common, which is absurd as $x \in \widehat{\partial\Omega}$. □

Now, as in the classical case, let us see what happens to Problem (4.1) when we flat out locally the boundary of our set. As in the classical case, assume that we are in a neighborhood of $x_0 \in \Gamma$ and let Φ be the isomorphism flattening a neighborhood U of x_0 in K.

If $u \in {}^{\rho}\mathcal{GC}^{\infty}\left(K, {}^{\rho}\widetilde{\mathbb{R}}^{m}\right)$ we let $v(y) := u\left(\Psi(y)\right)$ for $y \in V$. If $\Phi = \Psi^{-1}$, we have $u(x) = v\left(\Psi(x)\right)$ for $x \in U$. If $u(x)$ solves Problem (4.1), we then have for $i = 1, \ldots, n$

$$u_{x_i} = \sum_{k=1}^{n} v_{y_k}\left(\Phi(x)\right)\left(\Phi_k\right)_{x_i}(x),$$

namely $Du(x) = Dv(y)D\Phi(x)$, where we have used composition and chain rules for GSF. By substituting in $F(Du, u, x) = 0$ we get $0 = F\left(Dv(y)D\Phi(\Psi(y)), v(y), \Psi(y)\right)$, which is an expression of the form $G\left(Dv, v, y\right) = 0$ in V, as expected. Moreover, notice that $h(y) := g\left(\Psi(y)\right)$, which is defined on $\Phi(\Gamma)$ that is included in the strict border of $\Phi(V)$. Namely, even in the GSF setting, straightening the boundary near x_0 converts our original problem (4.1) in a problem with the same formal expression, but defined on a domain with a straight strict boundary, so with boundary conditions defined on a subset of the flat set $\{(x_1, \ldots, x_n) \mid x_n = 0\}$.

4.2. **Derivation of characteristic ODEs.** Let $F : {}^{\rho}\widetilde{\mathbb{R}}^{n} \times {}^{\rho}\widetilde{\mathbb{R}} \times K \to {}^{\rho}\widetilde{\mathbb{R}}$. As in the classical case, the goal is to find $u \in {}^{\rho}\mathcal{GC}^{\infty}\left(K, {}^{\rho}\widetilde{\mathbb{R}}\right)$ such that

$$F\left(Du, u, x\right) = 0,$$

subjected to the boundary condition

$$u = g \text{ on } \Gamma,$$

where $g \in {}^{\rho}\mathcal{GC}^{\infty}\left(\Gamma, {}^{\rho}\widetilde{\mathbb{R}}\right)$ is given.

Let $I = [0, a] \subseteq {}^{\rho}\widetilde{\mathbb{R}}$ be an interval that we use to parametrize the desired generalized curve $x(s) \in {}^{\rho}\mathcal{GC}^{\infty}\left(I, {}^{\rho}\widetilde{\mathbb{R}}^{n}\right)$. Set $z(s) := u(x(s)) \in {}^{\rho}\mathcal{GC}^{\infty}\left(I, {}^{\rho}\widetilde{\mathbb{R}}\right)$ and $p(s) = D\left(u(x(s))\right) := D\left(u(x(s))\right) = \left(u_{x_1}(x(s)), \ldots, u_{x_n}(x(s))\right) \in {}^{\rho}\mathcal{GC}^{\infty}\left(I, {}^{\rho}\widetilde{\mathbb{R}}^{n}\right)$. Using the fact that composition and derivation of GSF follows the same rules of smooth functions, we can easily prove the following:

Theorem 28. *Let* $u \in {}^{\rho}\mathcal{GC}^{\infty}\left(\Omega, {}^{\rho}\widetilde{\mathbb{R}}\right)$ *solve the nonlinear, first order PDE (4.1) in* Ω*. Let* $I = [0, a] \subseteq {}^{\rho}\widetilde{\mathbb{R}}$ *and assume that* $x \in {}^{\rho}\mathcal{GC}^{\infty}\left(I, {}^{\rho}\widetilde{\mathbb{R}}^{n}\right)$ *solves the equation*

$$\dot{x}(s) = D_p F\left(p(s), z(s), x(s)\right),$$

where $\dot{(\cdot)}$ *denotes* $\frac{d}{ds}$*. Then for every* $s \in I$ *with* $x(s) \in \Omega$ *we have*

$$\begin{cases} \dot{p}(s) = -D_x F\left(p(s), z(s), x(s)\right) - D_z F\left(p(s), z(s), x(s)\right) p(s), \\ \dot{z}(s) = D_p F\left(p(s), z(s), x(s)\right) \cdot p(s). \end{cases}$$

Proof. As the classical proof uses only the chain rule for derivation, we can easily repeat it: since $p(s) = D\left(u(x(s))\right)$, by deriving with respect to s we get

$$\dot{p}^{i}(s) = \sum_{j=1}^{n} u_{x_i x_j}\left(x(s)\right) \dot{x}^{j}(s), \tag{4.2}$$

whilst deriving $F\left(Du, u, x\right)$ with respect to x_i we get

$$\sum_{j=1}^{n} F_{p_j}\left(Du, u, x\right) u_{x_i} u_{x_j} + F_z\left(Du, u, x\right) u_{x_i} + F_{x_i}\left(Du, u, x\right) = 0. \tag{4.3}$$

As in the classical case, set

$$\dot{x}(s) = F_{p_j}\left(p(s), z(s), x(s)\right)$$

for $j = 1, \ldots, n$. Then we can cancel all the second derivatives appearing in equation (4.2), and by substituting $x(s)$ to x in equation (4.3) we get the desired expression for $\dot{p}(s)$. The expression for $z(s)$ is obtained deriving $z(s) = u(x(s))$ with respect to s. □

4.3. **Compatibility conditions on boundary data.** As in the classical case, we now want to characterize which kind of boundary conditions can be given so that the characteristic ODEs can be solved. Let us fix $x_0 \in \Gamma$ and let us solve this problem locally. The values of $x(0), p(0), z(0)$ are trivially fixed by copying the classical approach: as the curve $x(s)$ starts from x_0, we have to fix $x(0) = x_0$, which forces to set $z(0) = u(x(0)) = g(x(0)) = g\left(x_0\right)$, as $u = g$ on Γ. To fix the value of $p(0)$ we notice that, as $u\left(x_1, \ldots, x_{n-1}, 0\right) = g\left(x_1, \ldots, x_{n-1}\right)$ on Γ near x_0, differentiation with respect to x_i (for $i = 1, \ldots, n-1$) of this equality forces the necessary conditions $p_i(0) = u_{x_i}\left(x(0)\right) = g_{x_i}(x(0))$. Finally, as $(p(0), z(0), x(0))$ must solve the equation $F\left(p(0), z(0), x(0)\right) = 0$, we are led to impose the following n equations (called compatibility conditions) for $p(0) = \left(p_1(0), \ldots, p_n(0)\right)$:

$$\begin{cases} p_i(0) = g_{x_i}(x(0)), & i = 1, \ldots, n-1 \\ F\left(p(0), z(0), x(0)\right) = 0, \end{cases}$$

which are formally identical to the classical ones; notice that, as in the classical case, the value $p(0)$ is not uniquely determined, since the above system may not have a solution, or might not have a unique solution.

The only technical part where we have a slight difference with respect to the classical case comes now. We have fixed the values of $p(0), x(0), z(0)$ for the characteristic starting from x_0, but to be able to solve our PDE in a neighborhood of x_0 we need to give proper initial conditions for all $y \in \Gamma \cap N$, where N is a sharp neighborhood of x_0 to be found, meaning that we have to find a GSF

$q(\cdot) = (q_1(\cdot), \ldots, q_n(\cdot))$ so that for every point $y = (y_1, \ldots, y_{n-1}, 0) \in \Gamma \cap N$ we can solve the characteristic ODE with initial conditions

$$p(0) = q(y), z(0) = g(y), x(0) = y.$$

Of course, $q(\cdot)$ must be chosen in such a way that the compatibility conditions are fulfilled for every $y \in \Gamma \cap N$. The classical way to adress this problem is to make use of the implicit function theorem; in the GSF setting, this theorem has the following form, as proven in [34]:

Theorem 29 (Implicit function theorem)**.** *Let $U \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$, $V \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ be sharply open sets. Let $F \in {}^{\rho}\mathcal{GC}^{\infty}\left(U \times V, {}^{\rho}\widetilde{\mathbb{R}}^d\right)$ and $(x_0, y_0) \in U \times V$. If $\partial_2 F(x_0, y_0)$ is invertible in $L({}^{\rho}\widetilde{\mathbb{R}}^d, {}^{\rho}\widetilde{\mathbb{R}}^d)$, then there exists a sharply open neighbourhood $U_1 \times V_1 \subseteq U \times V$ of (x_0, y_0) such that*

$$\forall x \in U_1 \, \exists! y_x \in V_1 : \; F(x, y_x) = F(x_0, y_0). \tag{4.4}$$

Moreover, the function $f(x) := y_x$ for all $x \in U_1$ is a GSF $f \in {}^{\rho}\mathcal{GC}^{\infty}(U_1, V_1)$ and satisfies

$$Df(x) = -\left(\partial_2 F(x, f(x))\right)^{-1} \circ \partial_1 F(x, f(x)). \tag{4.5}$$

We can now prove that noncharacteristic boundary conditions in the GSF setting are formally identical to those in the classical case.

Theorem 30 (Noncharacteristic boundary conditions)**.** *There exists a sharp neighborhood N of x_0 and an unique $q(\cdot) \in {}^{\rho}\mathcal{GC}^{\infty}\left(N, {}^{\rho}\widetilde{\mathbb{R}}^n\right)$ such that $q(x_0) = p(0)$ and for every $y \in N \cap \Gamma$ the compatibility conditions*

$$\begin{cases} p_i(y) = g_{x_i}(y), & i = 1, \ldots, n-1 \\ F(q(y), g(y), y) = 0 \end{cases}$$

are fulfilled, provided that $F_{p_n}(p(0), z(0), x(0))$ is invertible[1].

Proof. We can follow the lines of the classical proof. Let $G : {}^{\rho}\widetilde{\mathbb{R}}^n \times {}^{\rho}\widetilde{\mathbb{R}}^n \to {}^{\rho}\widetilde{\mathbb{R}}^n$ be the map $G(p, y) = (G_1(p, y), \ldots, G_n(p, y))$, where

$$\begin{cases} G_i(p, y) = p_i - g_{x_i}(y), & i = 1, \ldots, n-1; \\ G_n(p, y) = F(p, g(y), y). \end{cases}$$

Then $G(p(0), x(0)) = 0$, and

$$D_p G(p(0), x(0)) =$$

$$\begin{pmatrix} 1 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 \\ F_{p_1}(p(0), z(0), x(0)) & \ldots & F_{p_{n-1}}(p(0), z(0), x(0)) & F_{p_n}(p(0), z(0), x(0)) \end{pmatrix}.$$

Thus $det D_p G(p(0), x(0)) = F_{p_n}(p(0), z(0), x(0))$, which is invertible by our assumption. So we can apply Theorem 29 and we deduce that there exists a neighborhood N of x_0 and an unique ultrafunction $g(\cdot)$ so that $G(q(y), y) = 0$ for every $y \in \Gamma \cap N$. □

[1] As usual, the classical condition $F_{p_n}(p(0), z(0), x(0)) \neq 0$ is substituted, in the GSF setting, by the condition $F_{p_n}(p(0), z(0), x(0))$ is invertible.

Remark 31. Most problems we will be interested in will have the form $\frac{\partial u}{\partial t} = H\left(x, u, \frac{\partial u}{\partial x}\right)$ with strict boundaries of the form $\{(x_1, \dots, x_{n-1}, t) \mid t = 0\}$. In these cases, the above theorem can always be applied.

4.4. **Local solutions.** Let us explicitly fix some assumptions that we will use through this section:

- $K \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ is a closed solid set (not necessarily functionally compact);
- $\Omega = int(K)$;
- Γ is a subset of the strict boundary of Ω;
- x_0 is a point on the boundary Γ;
- Ω can be locally straightened in Γ : we assume for every $x_0 \in \Gamma$, Γ can be rewritten in a neighborhood of x_0 as a subset of the flat set $\{(x_1, \dots, x_n) \mid x_n = 0\}$;
- $(p(0), z(0), x(0))$ is assumed to be admissible and noncharacteristic;
- $q(\cdot)$ is a function defined on a neighborhood $N \cap \Gamma$ of x_0 so that
  - $p(0) = q(x_0)$;
  - $(q(y), g(y), y)$ is admissible for all $y \in \Gamma$ close to x_0.

Under the above notations, assume to have solved the characteristic ODE for every initial condition given by $y \in \Gamma \cap N$. We will write $p(y, s), z(y, s), x(y, s)$ to explicitly denote the dependence of the values on the starting point y of the characteristic and on the parameter s that parametrizes the generalized curve.

The first result that we have is that even in the GSF setting there exists a small sharp neighborhood V of x_0 such that all points in V belong to some (projected) characteristic:

Theorem 32 (Local Invertibility)**.** *Let us assume that the noncharacteristic condition $F_{p_n}(p_0, z_0, x_0)$ invertible is fulfilled. Then there exists an open interval $(-a, a) \subseteq {}^{\rho}\widetilde{\mathbb{R}}$, a neighborhood W of x_0 in Γ and a neighborhood V of x_0 in ${}^{\rho}\widetilde{\mathbb{R}}$ such that $\forall x \in V\ \exists! s \in (-a, a), y \in W$ such that $x = x(y, s)$. Moreover, the map $x \in V \to (s, y) \in I \times W$ is a GSF.*

Proof. By definition, it must be $x(x_0, 0) = x_0$. Also, for every y we have $x(y, 0) = y$, so that $\forall i, j \le n-1$ we have

$$\frac{\partial x_j}{\partial y_i}(x_0, 0) = \begin{cases} 1, & \text{if } i = j \le n-1 \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, the characteristic ODE for $\dot{x}(s)$ implies that

$$\frac{\partial x_j}{\partial s}(x_0, 0) = F_{p_j}(p(0), z(0), x_0),$$

which for $j = n$ is invertible by hypothesis. Thus

$$\begin{pmatrix} 1 & \dots & 0 & F_{p_1}(p(0), z(0), x_0) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & F_{p_{n-1}}(p(0), z(0), x_0) \\ 0 & \dots & 0 & F_{p_n}(p(0), z(0), x_0) \end{pmatrix},$$

which is a matrix with an invertible determinant. The thesis then follows by a direct application of Theorem 29. $\square$

Let V be the sharp open set given in Theorem 32. In V the quantities y and s can be written in function of x, therefore also u and p can be written, via composition (which is allowed for GSF) as functions of x for $x \in V$:

$$\begin{cases} u(x) := z\left(y(x), s(x)\right), \\ p(x) := p\left(y(x), s(x)\right). \end{cases} \tag{4.6}$$

Our main result of this section is Theorem 34; following its classical proof, at a certain point we need to use the simple classical fact that the only solution of a linear ODE with initial condition 0 is the constant function 0. Unfortunately, in the GSF setting the growt conditions forces us to have a somehow weakened form of this classical result, that has been proven in [34, Theorem 55] and that we recall here for completness:

Theorem 33 (Solution of first order linear ODE)**.** *Let* $A \in {}^{\rho}\mathcal{GC}^{\infty}\left([a,b], {}^{\rho}\widetilde{\mathbb{R}}^{d\times d}\right)$, *where* a, $b \in {}^{\rho}\widetilde{\mathbb{R}}$, $a < b$, *and* $t_0 \in [a,b]$, $y_0 \in {}^{\rho}\widetilde{\mathbb{R}}^{d}$. *Assume that*

$$\left|\int_{t_0}^{t} A(s)\,\mathrm{d}s\right| \le -C \cdot \log \mathrm{d}\rho \quad \forall t \in [a,b], \tag{4.7}$$

where $C \in \mathbb{R}_{>0}$. *Then there exists one and only one* $y \in {}^{\rho}\mathcal{GC}^{\infty}\left([a,b], {}^{\rho}\widetilde{\mathbb{R}}^{d}\right)$ *such that*

$$\begin{cases} y'(t) = A(t)\cdot y(t) & \text{if } t \in [a,b] \\ y(t_0) = y_0 \end{cases} \tag{4.8}$$

Moreover, this y *is given by* $y(t) = \exp\left(\int_{t_0}^{t} A(s)\,\mathrm{d}s\right)\cdot y_0$ *for all* $t \in [a,b]$.

We now have all the ingredients we need to prove the main theorem of this section:

Theorem 34. *Let* $I = (-a,a)$ *and let* W *be the neighborhood of* x_0 *in* Γ *given by Theorem 32. Let us assume that* $\forall y \in W, \forall s \in (0,a)$ *we have that*

$$\left|\int_0^s \frac{\partial F}{\partial z}(y,\xi)\,d\xi\right| \le -C \log d\rho$$

for some $C \in \mathbb{R}_{>0}$.

Then the GSF u *given in 4.6 solves the partial differential equation* $F\left(Du(x), u(x), x\right) = 0$ *for every* $x \in V$, *with boundary condition* $u(x) = g(x)$ *for* $x \in \Gamma \cap V$.

Proof. Let $y \in \Gamma$ be a point close to x_0 so that we can solve the characteristic ODE for $p(s) = p(y,s), z(s) = z(y,s), x(s) = x(y,s)$. We claim that if $y \in \Gamma$ is sufficiently close to x_0 (in a precise sense that will be specified in a moment) then

$$f(y,x) := F\left(p(y,s), z(y,s), x(y,s)\right) = 0.$$

In fact, $f(y,0) = F\left(p(y,0), z(y,0), x(y,0)\right) = F\left(q(y), g(y), y\right) = 0$ and (using the characteristic ODE)

$$\frac{\partial f}{\partial s}(y,s) = \sum_{j=1}^{n} \frac{\partial F}{\partial p_j}\dot{p}_j + \frac{\partial F}{\partial z}\dot{z} + \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}\dot{x}_j =$$

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}\left(-\frac{\partial F}{\partial x_j} - \frac{\partial F}{\partial z}p_j\right) + \frac{\partial F}{\partial z}\left(\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}p_j\right) + \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}\left(\frac{\partial F}{\partial p_j}\right) = 0.$$

Then $f(y,0)=0$, $\frac{\partial f}{\partial s}(y,s)=0$, hence $f(y,s)=0$ thanks to the mean value theorem for GSF (see [21]).

Now, as $f(y,s)=0$, we have that $F\left(p(y,s),z(y,s),x(y,s)\right)=0$, hence $F\left(p(x),u(x),x\right)=0\,\forall x\in V$. Therefore, if one shows that $p(x)=Du(x)$, we are done. Let us prove this fact: let $s\in I,y\in W$. First of all, we have

$$\frac{\partial z}{\partial s}\left(y,s\right)=\sum_{j=1}^{n}p_{j}\left(y,s\right)\frac{\partial x_{j}}{\partial s}\left(y,s\right),$$

as from the characteristic ODEs we have

$$\dot{z}=\dot{x}\cdot p=\sum_{j=1}^{n}p_{j}(y,s)\frac{\partial x_{j}}{\partial s}\left(y,s\right).$$

Moreover, we have the equalities

$$\frac{\partial z}{\partial y_{i}}(y,s)=\sum_{j=1}^{n}p_{j}(y,s)\frac{\partial x_{j}}{\partial y_{i}}(y,s)\ \ (i=1,\dots,n-1).$$

In fact, fix $y\in\Gamma$ and $i\le n-1$. Set

$$r_{i}(s):=\frac{\partial z}{\partial y_{i}}(y,s)-\sum_{j=1}^{n}p_{j}(y,s)\frac{\partial x_{j}}{\partial y_{i}}(y,s);$$

our goal is to show that $\dot{r}_{i}(s)=-\frac{\partial F}{\partial z}r_{i}(s)$ which, as $r_{i}(0)=g_{x_{i}}(y)-q_{i}(y)=0$ by the compatibility conditions, proves that $r_{i}(s)=0$ for every $s\in I$ thanks to our hypothesis on $\frac{\partial F}{\partial z}$ and Theorem 33). To show that $\dot{r}_{i}(s)=-\frac{\partial F}{\partial z}r_{i}(s)$, we proceed as follows: we first notice, with a direct computation, that

$$\dot{r}_{i}(s)=\frac{\partial^{2}z}{\partial y_{i}\partial s}-\sum_{j=1}^{n}\left[\frac{\partial p_{j}}{\partial s}\cdot\frac{\partial x_{j}}{\partial y_{i}}+p_{j}\cdot\frac{\partial^{2}x_{j}}{\partial y_{i}\partial s}\right].\tag{4.9}$$

Then, as $\dot{z}=\sum_{j=1}^{n}p_{j}(y,s)\frac{\partial x_{j}}{\partial s}(y,s)$, by differentiating with respect to y_{i} we get

$$\frac{\partial^{2}z}{\partial s\partial y_{i}}=\sum_{j=1}^{n}\left[\frac{\partial p_{j}}{\partial y_{i}}\frac{\partial x_{j}}{\partial s}+p_{j}\frac{\partial^{2}x_{j}}{\partial s\partial y_{i}}\right]$$

which can be substituted in equation 4.9 to get

$$\dot{r_{i}}(s)=\sum_{j=1}^{n}\left[\frac{\partial p_{j}}{\partial y_{i}}\frac{\partial x_{j}}{\partial s}-\frac{\partial p_{j}}{\partial s}\frac{\partial x_{j}}{\partial y_{i}}\right].\tag{4.10}$$

Hence by the characteristic ODE for $\dot{p}$ we get

$$\dot{r}_{i}(s)=\sum_{j=1}^{n}\left[\frac{\partial p_{j}}{\partial y_{i}}\left(\frac{\partial F}{\partial p_{j}}\right)-\left(-\frac{\partial F}{\partial z}p_{j}\right)\frac{\partial x_{j}}{\partial x_{i}}\right].$$

Now differentiating $F\left(p(y,s),z(y,s),x(y,s)\right)=0$ with respect to y_{i} we get

$$\sum_{j=1}^{n}\frac{\partial F}{\partial p_{j}}\frac{\partial p_{j}}{\partial y_{i}}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial y_{i}}+\sum_{j=1}^{n}\frac{\partial F}{\partial x_{j}}\frac{\partial x_{j}}{\partial y_{i}}=0.$$

We substite in equation 4.10 to finally get that

$$\dot{r}(s) = \frac{\partial F}{\partial z} \cdot \left(\sum_{j=1}^{n} p_j \frac{\partial x_j}{\partial y_i} - \frac{\partial z}{\partial y_i} \right) = -\frac{\partial F}{\partial z} r^i(s).$$

We have now all the ingredients to prove the claim that for all $x \in V$ $p(x) = Du(x)$: in fact $\forall j \leq n$

$$\frac{\partial u}{\partial x_j} = \frac{\partial z}{\partial s}\frac{\partial s}{\partial x_j} + \sum_{j=1}^{n-1} \frac{\partial z}{\partial y_i}\frac{\partial y_i}{\partial x_j} =$$

$$\left(\sum_{k=1}^{n} p_k \frac{\partial x_k}{\partial s} \right) \frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1} \left(\sum_{k=1}^{n} p_k \frac{\partial x_k}{\partial y_i} \right) \frac{\partial y_i}{\partial x_j} =$$

$$\sum_{k=1}^{n} p_k \left(\frac{\partial x_k}{\partial s}\frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1} \frac{\partial x_k}{\partial y_i}\frac{\partial y_i}{\partial x_j} \right) = \sum_{k=1}^{n} p_k \frac{\partial x_k}{\partial x_j} = p_j.$$

□

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