A GROTHENDIECK TOPOS OF GENERALIZED FUNCTIONS II: ODE

LORENZO LUPERI BAGLINI AND PAOLO GIORDANO

Abstract. to write...

Contents

P. Giordano has been supported by grants P30407, P33538, P34113 and P30233 of the Austrian Science Fund FWF.

L. Luperi Baglini has been supported by grants P25311 and M1876-N35 of the Austrian Science Fund FWF.

1. Introduction

After the presentation in [20] of the basic theory of generalized smooth functions (GSF), in the present paper, we deal with Cauchy problems for ODE in the so-called *normal* form, i.e.

$$\begin{cases} y^{(n)} = F(t, y, y', \dots, y^{(n-1)}) \\ y^{(k)}(t_0) = c_k \qquad\qquad k = 0, \dots, n-1 \end{cases}, \tag{1.1}$$

where F is a GSF (in particular $F \in \mathcal{D}'(\Omega)$ is an arbitrary Schwartz distributions or $F \in \mathcal{G}^{\mathrm{s}}(\Omega)$ is a Colombeau generalized function). Clearly, the classical treatment of ODE within spaces of Sobolev-Schwartz distributions is limited to linear systems by the very fact that distributions themselves are restricted to linear operations. For a more general distribution F, there is no valid solution concept for (1.1) in a classical space of distributions because, without any further regularity assumptions, the composition on the right hand side is not defined (see e.g. [2, 1, 28]).

Features of our approach can be listed as follows: to update...

- We are going to solve problem (1.1) both for an arbitrary GSF F, and also for singular initial conditions $c_k \in {}^{\rho}\widetilde{\mathbb{R}}$, such as e.g. $y^{(k)}(t_0) = \delta^k(0)$.
- We generalize the Banach fixed point theorem, and the consequent Picard-Lindelöf theorem for Cauchy problems of the form (1.1).
- We generalize the Picard-Lindelöf theorem to the case of an infinite number $N \in {}^{\rho}\widetilde{\mathbb{N}}$ (see [20, Sec. 7.2]) of iterations.
- In most cases, as in [20], the proofs of the aforementioned theorems are essentially identical to the classical ones, but using ${}^{\rho}\widetilde{\mathbb{R}}$ instead of $\mathbb{R}$ as ring of scalars. This allows the reader to have an easier approach to this new theory of generalized functions.
- We prove classical results such as uniqueness, continuous dependence on initial conditions, maximal set of existence, Gronwall inequalities and flow properties.
- Using suitable characterizations of distributions among GSF, we also analyze when the generalized solution y is a distribution or not. We prove that the GSF solution coincides with the unique smooth one in case of an ordinary smooth ODE with standard conditions $c_k \in \mathbb{R}^d$.
- We present several non-linear examples, including a local analysis where F is an arbitrary ${}^{\rho}\widetilde{\mathbb{R}}$-polynomial, Bernoulli's ODE with generalized smooth coefficients, and non-linear examples appearing in applications in impulsive physical systems.
- We give a complete approach to linear singular ODE with GSF as coefficients. In particular, we also give a full account of the relations with classical distributional solutions, in case of ODE with classical smooth coefficients.

Conceptual schema of the paper. We can summarize the main idea of the paper by saying that we want to exploit at the highest level the classical idea of regularizing the differential problem with a net of smooth differential problems.

Anyway, this is realized with a final formalism that resemble a lot that of classical smooth functions, not with notations full of unhandy regularizing parameters.
The idea to study GSF using a countable family of norms (each one corresponding to an order of derivatives) could be considered a natural one. However, classically one of the best version of the Picard-Lindelöf theorem results by considering complete spaces of only continuous functions. Based on these motivations, in Sec. 2 we introduce and study generalized functions of class $\mathcal{C}^k$.
In Sec. 3, we recall (see [20, Def. 52]) and more extensively study the notion of functionally compact set $K \subseteq {}^\rho\widetilde{\mathbb{R}}^n$. We study ${}^\rho\widetilde{\mathbb{R}}$-Fréchet spaces ${}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}}^d)$ of generalized functions defined on functionally compact sets and prove their completeness. Even if our generalized functions always attain a maximum and a minimum value on this type of domains, for each open set $\Omega \subseteq \mathbb{R}^n$ we can suitably choose K so that these Fréchet space contains all the distributions, i.e. $\mathcal{D}'(\Omega) \subseteq {}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}}^d)$.
In Sec. 4, we study uniformly continuous generalized functions, a notion that is useful in studying maximal intervals of existence of solutions of ODE.
In Sec. 5, we generalize the Banach fixed point theorem for ${}^\rho\widetilde{\mathbb{R}}$-Fréchet spaces, in particular for generalized functions of class $\mathcal{C}^0$.
In Sec. 5.2, we prove the Picard-Lindelöf theorem for ODE with an arbitrary GSF as right hand side and arbitrary initial conditions in ${}^\rho\widetilde{\mathbb{R}}$. In general, the solution obtained in this way is defined in an infinitesimal neighborhood of the initial condition, and we present simple examples where a larger domain is not possible.
In Sec. 5.3, we study the Picard-Lindelöf theorem with an infinite number of iterations. This allows us to prove a general sufficient condition for the existence of a solution in a finite and non-infinitesimal interval.
Starting from Sec. 7.3, we prove analogous of classical theorems like uniqueness results, continuum dependence on initial data, maximal set of existence, Gronwall inequalities and flow properties.
In Sec. 8, we consider sufficient conditions to get a distributional solution, or a classical $\mathcal{C}^k$, $1 \le k \le +\infty$ solution, starting from the GSF one.
We list lots of non-linear examples in Sec. 6, and in Sec. 9 we present a full treatment of solutions of singular linear ODE. Through these examples, we present a first understanding of the differences between GSF solutions and classical or distributional solutions in case of singular ODE. The paper needs only [20] as a prerequisite.

Currently, one of the most successful approach in finding solutions of nonlinear ODE in spaces of generalized functions which embed Schwartz distributions and having conceptual analogies with our approach is Colombeau's theory. See e.g. [13, 25, 39, 4] and references therein. However, in Colombeau's approach the composition of generalized functions is only partially possible. Moreover, this theory is limited to polynomial growth in the regularizing parameter, whereas GSF do not have this limitation. We also refer to [38, 41, 42, 43] for solutions of (nonlinear) ODE with delta function terms through a regularizing process.

2. Generalized $\mathcal{C}^k$ functions and their calculus

As we already mentioned in the introduction, since GSF are infinitely differentiable, it is natural to consider spaces X of these functions augmented with a countable family $(\|-\|_i)_{i\in\mathbb{N}}$ of (${}^\rho\widetilde{\mathbb{R}}$-valued) norms, one for each order of derivatives $i \in \mathbb{N}$. On the other hand, this would lead to a notion of contraction corresponding to the standard one in locally convex spaces (see e.g. [2]), i.e.:

$$\|P(u)-P(v)\|_i \le \alpha_i \cdot \|u-v\|_i \quad \forall i \in \mathbb{N}\, \forall u, v \in X,$$

and hence to consider a countable family of contraction constants. On the contrary, one of the key features of the usual Picard-Lindelöf for ordinary smooth functions is that it needs only one Lipschitz constant because the Banach fixed point theorem used in its proof is a space of only continuous functions. It is the normal form (1.1) of the ODE (in its equivalent integral form) that yields the necessary smoothness of the solution if the initial condition has the same regularity. To understand better this step, see [23] where in case of normal PDE each Picard iteration P^m necessarily sees a loss $L \in \mathbb{N}$ of derivatives $\|u-v\|_{i+mL}$ and we are hence forced to consider a countable family of norms. From the technical point of view, a non trivial problem in considering even the ordinary norms $\|f\|_i = \max_{\substack{x\in[0,\alpha]\\ h\le i}} \left|f^{(h)}(x)\right| \in \mathbb{R}$, is that in general they do *not* satisfy $\left\|\int_0^{(-)} f(s)\,\mathrm{d}s\right\|_i \le \int_0^\alpha \|f\|_i \,\mathrm{d}s$ if $i \ge 1$. Since this is an important step in the proof of the Picard-Lindelöf theorem, we are also mathematically motivated to consider spaces of only continuous functions.

3. Functionally compact sets and spaces of GSF

In order to prove a general Banach fixed point theorem suitable for singular ODE, a convenient notion of compact domain and of norm of generalized functions is crucial. In our non-Archimedean setting, an important problem is that intervals $[a,b] \subseteq {}^\rho\widetilde{\mathbb{R}}$, even for a, $b \in \mathbb{R}$, are neither compact in the sharp nor in the Fermat topology. This has been formally proved in [21, Thm. 25] for the case $\rho_\varepsilon = \varepsilon$, but it is already intuitively clear: using a finite number of infinitesimal balls we cannot cover the entire interval $[0,1]$ (more generally, no infinite standard set $U \subseteq \mathbb{R}^n$ is compact in the sharp topology). Moreover, since our "dynamical" generalized numbers include also scalars that can discontinuously jump among a finite number of open sets, the interval $[0,1]$ is also not closed in the Fermat topology. Once again, these are necessary general results. Indeed, we already argued that set-theoretical functions having infinite derivatives can be continuous only in topologies containing infinitesimal neighbourhoods (see [20, Sec. 2.1]). Moreover, discontinuous jumping representatives of generalized numbers must necessarily be considered if we want to have a general mean value theorem (see [20, Sec. 6]).

The notion of *functionally compact set* ([20, Def. 52]), i.e. sets over which our generalized functions satisfy an extreme value theorem (see [20, Cor. 51]), solves these problems. For simplicity, we recall it here:

Definition 1. A subset K of ${}^\rho\widetilde{\mathbb{R}}^n$ is called *functionally compact*, denoted by $K \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}^n$, if there exists a net (K_ε) such that

(i) $K = [K_\varepsilon] \subseteq {}^\rho\widetilde{\mathbb{R}}^n$;
(ii) (K_ε) is sharply bounded;
(iii) $\forall \varepsilon \in I:\ K_\varepsilon \Subset \mathbb{R}^n$.

If, in addition, $K \subseteq U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ then we write $K \Subset_{\mathrm{f}} U$. Finally, we write $[K_\varepsilon] \Subset_{\mathrm{f}} U$ if (ii), (iii) and $[K_\varepsilon] \subseteq U$ hold.

We note that in (iii) it suffices to ask that K_ε is closed since it is bounded by (ii), at least for ε small. The name *functionally compact subset* is motivated by showing, as it will be done e.g. in Theorem 3, that on this type of subsets, GSF have properties very close to those that ordinary smooth functions have on standard compact sets.

Remark 2.

(i) By [20, Thm. 10], any internal set $K = [K_\varepsilon]$ is closed in the sharp topology. In particular, the open interval $(0,1) \subseteq {}^\rho\widetilde{\mathbb{R}}$ is not functionally compact since it is not closed.

(ii) If $H \Subset \mathbb{R}^n$ is a non-empty ordinary compact set, then the internal set $[H]$ is functionally compact. In particular, $[0,1] = [[0,1]_{\mathbb{R}}]$ is functionally compact.

(iii) The empty set $\emptyset = [\emptyset] \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}$.

(iv) ${}^\rho\widetilde{\mathbb{R}}^n$ is not functionally compact since it is not sharply bounded.

(v) The set of compactly supported points $\mathrm{c}(\mathbb{R})$ is not functionally compact because the GSF $f(x) = x$ does not satisfy the conclusion of the extreme value theorem [20, Cor. 51].

For functionally compact sets, it is easy to prove the following generalizations of theorems from classical analysis:

Theorem 3.

(i) *Let $K \subseteq X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$, $f \in {}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}}^d)$. Then $K \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$ implies $f(K) \Subset_f {}^\rho\widetilde{\mathbb{R}}^d$.*

(ii) *If $a, b \in {}^\rho\widetilde{\mathbb{R}}$ and $a \le b$, then $[a,b] \Subset_f {}^\rho\widetilde{\mathbb{R}}$. Let us note explicitly that $a, b \in {}^\rho\widetilde{\mathbb{R}}$ can also be infinite numbers, e.g. $a = -\mathrm{d}\rho^{-N}$, $b = \mathrm{d}\rho^{-M}$ or $a = \mathrm{d}\rho^{-N}$, $b = \mathrm{d}\rho^{-M}$ with $M > N$.*

(iii) *Let $K, H \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$. If $K \cup H$ is an internal set, then it is a functionally compact set. If $K \cap H$ is an internal set, then it is a functionally compact set.*

(iv) *Let $H \subseteq K \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$, then if H is an internal set, then $H \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$.*

(v) *Let $K \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$ and $H \Subset_f {}^\rho\widetilde{\mathbb{R}}^d$, then $K \times H \Subset_f {}^\rho\widetilde{\mathbb{R}}^{n+d}$. In particular, if $a_i \le b_i$ for $i = 1, \dots, n$, then $\prod_{i=1}^n [a_i, b_i] \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$.*

Both in the Banach fixed point theorem and in the Picard-Lindelöf theorem, we want to consider spaces of GSF of the type $K \longrightarrow {}^\rho\widetilde{\mathbb{R}}^d$, where $K \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}^n$. In order to set natural ${}^\rho\widetilde{\mathbb{R}}$-valued norms in these spaces, we need to talk of partial derivatives $\partial^\alpha f(x)$ at every $x \in K$. This cannot be performed using only the Fermat-Reyes [20, Thm. 33], since it requires the point x to be an internal one. For this reason, we consider only sets K that satisfy the following

Definition 4. We say that K is a *solid set in* ${}^\rho\widetilde{\mathbb{R}}^n$ if $\mathrm{int}(K)$ is dense in K (in the sharp topology).

For example, [20, Lem. 38] and Thm. 3.(v) show that n-dimensional intervals are solid functionally compact sets. Trivially, every sharply open set is solid.

For this type of sets we have:

Lemma 5. *Let K be a solid set in ${}^\rho\widetilde{\mathbb{R}}^n$, and $f \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$ be a GSF. Then for all $\alpha \in \mathbb{N}^n$ and all $x \in K$ the following limit exists in the sharp topology*

$$\lim_{\substack{y \to x \\ y \in \mathrm{int}(K)}} \partial^\alpha f(y) =: \partial^\alpha f(x).$$

Moreover, if the net $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ defines f, then $\partial^\alpha f(x) = [\partial^\alpha f_\varepsilon(x_\varepsilon)]$ and hence $\partial^\alpha f \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$.

Proof. We have

$$\lim_{\substack{y\to x\\ y\in\mathrm{int}(K)}}\partial^{\alpha}f(y)=\lim_{\substack{y\to x\\ y\in\mathrm{int}(K)}}\left[\partial^{\alpha}f_{\varepsilon}(y_{\varepsilon})\right]=\left[\partial^{\alpha}f_{\varepsilon}(x_{\varepsilon})\right]$$

the last equality following by the sharp continuity of the GSF $[\partial^{\alpha}f_{\varepsilon}(-)]$ at every point $x\in K\subseteq\langle\Omega_{\varepsilon}\rangle$ (see [20, Thm. 17.3]). □

From the extreme value property, it is natural to expect that the following generalized numbers could serve as non-Archimedean ${}^{\rho}\widetilde{\mathbb{R}}$-valued norms.

Definition 6. Let $\emptyset\neq K\Subset_{\mathrm{f}}{}^{\rho}\widetilde{\mathbb{R}}^{n}$ be a solid set. Let $m\in\mathbb{N}$ and $f\in{}^{\rho}\mathcal{GC}^{\infty}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$. Then

$$\|f\|_{m}:=\max_{\substack{|\alpha|\le m\\ 1\le i\le d}}\left(\left|\partial^{\alpha}f^{i}(M_{\alpha i})\right|\right)\in{}^{\rho}\widetilde{\mathbb{R}},$$

where $M_{\alpha i}\in K$ satisfy

$$\forall x\in K:\ \left|\partial^{\alpha}f^{i}(x)\right|\le\left|\partial^{\alpha}f^{i}(M_{\alpha i})\right|.$$

Note that the notation $\|f\|_{m}$ depends on K through the function f since K is its domain.

The following result allows the calculation of the (generalized) norm $\|f\|_{m}$ using any net (f_{ε}) that defines f.

Theorem 7. *Under the assumptions of Def. 6, let the set $K=[K_{\varepsilon}]\Subset_{f}{}^{\rho}\widetilde{\mathbb{R}}^{n}$. If the net (f_{ε}) defines f, then*

$$\|f\|_{m}=\left[\max_{\substack{|\alpha|\le m\\ 1\le i\le d}}\sup_{x\in K_{\varepsilon}}\left|\partial^{\alpha}f_{\varepsilon}^{i}(x)\right|\right]\in{}^{\rho}\widetilde{\mathbb{R}}.\tag{3.1}$$

Proof. In proving (3.1), we will also prove that the norm $\|f\|_{m}$ is well-defined, i.e. it does not depend on the particular choice of point $M_{\alpha i}$ as in Def. 6. As in the proof of the extreme value theorem [20, Lem. 50], we get the existence of $\bar{M}_{\alpha i\varepsilon}\in K_{\varepsilon}$ such that

$$\forall x\in K_{\varepsilon}:\ \left|\partial^{\alpha}f_{\varepsilon}^{i}(x)\right|\le\left|\partial^{\alpha}f_{\varepsilon}^{i}(\bar{M}_{\alpha i\varepsilon})\right|.$$

Thus

$$\max_{\substack{|\alpha|\le m\\ 1\le i\le d}}\sup_{x\in K_{\varepsilon}}\left|\partial^{\alpha}f_{\varepsilon}^{i}(x)\right|\le\max_{\substack{|\alpha|\le m\\ 1\le i\le d}}\left|\partial^{\alpha}f_{\varepsilon}^{i}(\bar{M}_{\alpha i\varepsilon})\right|.$$

But $\bar{M}_{\alpha i\varepsilon}\in K_{\varepsilon}$, so

$$\left[\max_{\substack{|\alpha|\le m\\ 1\le i\le d}}\sup_{x\in K_{\varepsilon}}\left|\partial^{\alpha}f_{\varepsilon}^{i}(x)\right|\right]=\left[\max_{\substack{|\alpha|\le m\\ 1\le i\le d}}\left|\partial^{\alpha}f_{\varepsilon}^{i}(\bar{M}_{\alpha i\varepsilon})\right|\right]=$$
$$=\max_{\substack{|\alpha|\le m\\ 1\le i\le d}}\left|\partial^{\alpha}f^{i}(\bar{M}_{\alpha i})\right|.$$

From this, both the fact that the norm $\|f\|_{m}$ is well-defined and claim (3.1) follow. □

Even though $\|f\|_{m}\in{}^{\rho}\widetilde{\mathbb{R}}$, using an innocuous abuse of language, in the following we will simply call $\|f\|_{m}$ a norm. This use of the term "norm" is justified by the following

Theorem 8. *Let $\emptyset \neq K \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$ be a solid set. Let $f, g \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$ and $m \in \mathbb{N}$. Then*

(i) $\|f\|_m \geq 0$;
(ii) $\|f\|_m = 0$ *if and only if* $f = 0$;
(iii) $\forall c \in {}^\rho\widetilde{\mathbb{R}} : \ \|c \cdot f\|_m = |c| \cdot \|f\|_m$;
(iv) $\|f + g\|_m \leq \|f\|_m + \|g\|_m$;
(v) $\|f \cdot g\|_m \leq 2^m \cdot \|f\|_m \cdot \|g\|_m$.

Proof. (i), (iii) and (iv) follow directly from Thm. 7, as does (v), using the Leibniz rule. The 'only if'-part of property (ii) follows from (3.1). □

Using our ${}^\rho\widetilde{\mathbb{R}}$-valued norms, it is now natural to define

Definition 9. Let $\emptyset \neq K \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$ be a solid set. Let $f \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$, $m \in \mathbb{N}$, $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, then

(i) ${}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}}^d) := \left({}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d), (\| - \|_m)_{m \in \mathbb{N}} \right)$. We write $f \in {}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}}^d)$ to denote $f \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$.

(ii) $B^m_r(f) := \left\{ g \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d) \mid \|f - g\|_m < r \right\}$.

(iii) If $V \subseteq {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$, then we say that V is a *sharply open set in* ${}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}}^d)$ if
$$\forall v \in V \, \exists m \in \mathbb{N} \, \exists r \in {}^\rho\widetilde{\mathbb{R}}_{>0} : \ B^m_r(v) \subseteq V.$$
Moreover, we say that V is *a large (or Fermat) open set in* ${}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}}^d)$ if
$$\forall v \in V \, \exists m \in \mathbb{N} \, \exists r \in \mathbb{R}_{>0} : \ B^m_r(v) \subseteq V.$$

A trivial generalization of the classical proofs, though using [20, Cor. 51], shows that

Theorem 10. *Let $\emptyset \neq K \Subset_f {}^\rho\widetilde{\mathbb{R}}$ be a solid set. Then we have:*

(i) *Sharply open sets, as well as large open sets* in ${}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}}^d)$ *form topologies on* ${}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$.
(ii) *Pointwise addition and multiplication by* ${}^\rho\widetilde{\mathbb{R}}$*-scalar in* ${}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}}^d)$ *are continuous in the sharp topology. Therefore,* ${}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}}^d)$ *is a topological* ${}^\rho\widetilde{\mathbb{R}}$*-module and* ${}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}})$ *is an* ${}^\rho\widetilde{\mathbb{R}}$*-algebra.*
(iii) ${}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}}^d)$ *with the sharp topology is separated.*
(iv) *If $f, g \in B^m_r(0)$ and $t \in [0, 1]$, then $tf + (1 - t)g \in B^m_r(0)$. We can therefore say that every ball $B^m_r(0)$ is* ${}^\rho\widetilde{\mathbb{R}}$*-convex.*
(v) *If $t \in {}^\rho\widetilde{\mathbb{R}}$ and $|t| \leq 1$, then $t \cdot B^m_r(0) \subseteq B^m_r(0)$. We can therefore say that every ball $B^m_r(0)$ is* ${}^\rho\widetilde{\mathbb{R}}$*-balanced.*
(vi) *For all $f \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$ there exists $t \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that $f \in t \cdot B^m_r(0|_K)$. We can therefore say that every ball $B^m_r(0|_K)$ is* ${}^\rho\widetilde{\mathbb{R}}$*-absorbent.*

Because of these properties, we will call the space ${}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}}^d)$ an ${}^\rho\widetilde{\mathbb{R}}$-*Fréchet module*. It is worth noting that the natural properties stated in the previous theorem do not hold if we take the large topology instead of the sharp one, or if we consider the field $\mathbb{R}$ instead of the ring ${}^\rho\widetilde{\mathbb{R}}$. For example, since there exist GSF having infinite norms $\|f\|_m \in {}^\rho\widetilde{\mathbb{R}}$, the multiplication by standard real scalar $(r, f) \in \mathbb{R} \times {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}) \mapsto r \cdot f \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}})$ is clearly not continuous with respect to the standard Euclidean topology on $\mathbb{R}$ because $r \cdot \|f\| \not\to 0$ if $r \to 0$ in this topology. See [18, Sec. 5.1] for general abstract theorems corresponding to this necessity of using a non-Archimedean topology in dealing with generalized functions.

The spaces ${}^{\rho}\mathcal{GF}(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$ are very rich of examples and convenient properties which are well fitted for the aims of the present work. For example, let $\varphi \in \mathcal{D}_K(\Omega)$, $K \Subset \Omega \subseteq \mathbb{R}^n$, be an ordinary compactly supported smooth function; we can consider $K_\varepsilon := K$ and $f_\varepsilon(x) := \varphi(x)$ if $x \in \Omega$ and $f_\varepsilon(x) := 0$ otherwise to have that $\varphi|_K \in {}^{\rho}\mathcal{GF}(K, {}^{\rho}\widetilde{\mathbb{R}})$. Moreover, Thm. 7 implies that $\|\varphi|_K\|_m = \|\varphi\|_m \in \mathbb{R}$ is the usual m-norm of φ.

The following result allows the inclusion of infinite meaningful examples and to understand that every $f \in {}^{\rho}\mathcal{GF}(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$ can be extended to the whole ${}^{\rho}\widetilde{\mathbb{R}}^n$:

Theorem 11. *Let $\emptyset \neq K = [K_\varepsilon] \Subset_f {}^{\rho}\widetilde{\mathbb{R}}^n$ be a solid set, then*

$$\forall f \in {}^{\rho}\mathcal{GC}^\infty(K, {}^{\rho}\widetilde{\mathbb{R}}^d)\, \exists \bar{f} \in {}^{\rho}\mathcal{GC}^\infty({}^{\rho}\widetilde{\mathbb{R}}^n, {}^{\rho}\widetilde{\mathbb{R}}^d) : \ \bar{f}|_K = f. \tag{3.2}$$

Moreover, let Ω be an open subset of $\mathbb{R}^n$ and $J = [J_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}$ be a positive infinite generalized number. Set $K_\varepsilon := \{x \in \Omega \mid |x| \le J_\varepsilon\}$ and $K := [K_\varepsilon]$. Then for all $f \in {}^{\rho}\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^{\rho}\widetilde{\mathbb{R}}^d)$ (in particular, if f is the embedding of a Schwartz distribution) there exists $\bar{f} \in {}^{\rho}\mathcal{GC}^\infty(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$ defined by $(\bar{f}_\varepsilon)$ such that $\bar{f}|_{\mathrm{c}(\Omega)} = f$, $\bar{f}_\varepsilon|_{\mathbb{R}^n \setminus K_\varepsilon} = 0$ for all ε.

Proof. We start to prove the second conclusion. We set $V_\varepsilon := \{x \in \Omega \mid |x| < \frac{1}{2}J_\varepsilon\}$ so that $V_\varepsilon \subseteq K_\varepsilon$ for ε small. Let $\chi_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ be such that $\chi|_{V_\varepsilon} = 1$ and $\mathrm{supp}(\chi_\varepsilon) \subseteq K_\varepsilon$. Let $f \in {}^{\rho}\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^{\rho}\widetilde{\mathbb{R}}^d)$ be represented by (f_ε), with $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$, and set $\bar{f}_\varepsilon := \chi_\varepsilon \cdot f_\varepsilon$. Then each $\bar{f}_\varepsilon$ is compactly supported in K_ε and any $x = [x_\varepsilon] \in \mathrm{c}(\Omega)$ satisfies $x_\varepsilon \in V_\varepsilon$ for ε small because $\lim_{\varepsilon \to 0^+} J_\varepsilon = +\infty$. Therefore $\bar{f} := [\bar{f}_\varepsilon(-)]|_K \in {}^{\rho}\mathcal{GC}^\infty(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$, and if $x_\varepsilon \in V_\varepsilon$ then $\bar{f}_\varepsilon(x_\varepsilon) = f_\varepsilon(x_\varepsilon)$, so $\bar{f}|_{\mathrm{c}(\Omega)} = f$. To prove (3.2), we can proceed similarly by considering $\chi_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ such that $\chi_\varepsilon|_{K_\varepsilon} = 1$ and $\mathrm{supp}(\chi_\varepsilon) \subseteq \bigcup_{x \in K_\varepsilon} B_1^{\mathrm{E}}(x)$. $\square$

We recall that ${}^{\rho}\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^{\rho}\widetilde{\mathbb{R}}^d)$ can be identified with the space $\mathcal{G}^s(\Omega) \supseteq \mathcal{D}'(\Omega)$ of Colombeau generalized functions on Ω (see [20, Rem. 26.5]). Therefore, Thm. 11 yields an infinity of non-trivial examples of GSF in spaces of the type ${}^{\rho}\mathcal{GF}(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$. In fact, even though $\bar{f}$ depends on the fixed infinite number $J \in {}^{\rho}\widetilde{\mathbb{R}}$, each such $\bar{f}$ contains all the information of the original generalized function f because $\bar{f}|_{\mathrm{c}(\Omega)} = f$. Finally, note that (3.2) trivially yields $\left\|\bar{f}|_K\right\|_m = \|f\|_m$ for all $m \in \mathbb{N}$ because the norm $\|-\|_m$ is well defined (Thm. 7). Ultimately, this is a consequence of the Fermat-Reyes [20, Thm. 33] and of Thm. 5, which state that every partial derivative depends only on the values of the generalized function f at interior points of the solid set K.

In the following result, we prove that the generalized Fréchet space ${}^{\rho}\mathcal{GF}(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$ is complete with respect to the sharp topology.

Theorem 12. *Let $\emptyset \neq K \Subset_f {}^{\rho}\widetilde{\mathbb{R}}^n$ be a solid set. Then*

(i) The space ${}^{\rho}\mathcal{GF}(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$ with the sharp topology is Cauchy complete, in the sense that any Cauchy sequence $(u_n)_{n \in \mathbb{N}}$ in this topology, i.e. which satisfies

$$\forall i \in \mathbb{N}\, \forall q \in \mathbb{R}_{>0}\, \exists N \in \mathbb{N}\, \forall m, n \ge N : \ \|u_n - u_m\|_i < \mathrm{d}\rho^q \tag{3.3}$$

converges in ${}^{\rho}\mathcal{GF}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$ *in the sharp topology.*

(ii) *Any sharply closed subset of* ${}^{\rho}\mathcal{GF}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$ *is also Cauchy complete.*

(iii) *If* $H\subseteq{}^{\rho}\widetilde{\mathbb{R}}^{d}$ *is a sharply closed set, then* $\left\{f\in{}^{\rho}\mathcal{GC}^{\infty}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})\mid f(K)\subseteq H\right\}$ *is sharply closed in* ${}^{\rho}\mathcal{GF}(K,{}^{\rho}\widetilde{\mathbb{R}}^{d})$.

Proof. It is only essential to prove the case $d=1$. To show (i), let us consider a Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ in the sharp topology, i.e. we assume (3.3). Setting $i=q=k\in\mathbb{N}_{>0}$, this implies the existence of a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\|u_{n_{k+1}}-u_{n_k}\|_k<\mathrm{d}\rho^k$. Hence, picking any representative $(u_{n\varepsilon})$ of u_n by Thm. 7 we have

$$\left[\max_{|\alpha|\le k}\sup_{x\in K_\varepsilon}\left|\partial^\alpha u_{n_{k+1},\varepsilon}(x)-\partial^\alpha u_{n_k,\varepsilon}(x)\right|\right]<\left[\rho_\varepsilon^k\right]\quad\forall k\in\mathbb{N}_{>0}.$$

By [20, Lem. 8], this yields that for each $k\in\mathbb{N}_{>0}$ there exists an ε_k such that $\varepsilon_k\downarrow 0$ and

$$\forall\varepsilon\in(0,\varepsilon_k):\ \max_{|\alpha|\le k}\sup_{x\in K_\varepsilon}\left|\partial^\alpha u_{n_{k+1},\varepsilon}(x)-\partial^\alpha u_{n_k,\varepsilon}(x)\right|<\rho_\varepsilon^k.\tag{3.4}$$

Now set

$$h_{k,\varepsilon}:=\begin{cases}u_{n_{k+1},\varepsilon}-u_{n_k,\varepsilon}\in\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}) & \text{if } \varepsilon\in(0,\varepsilon_k)\\ 0\in\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}) & \text{if } \varepsilon\in[\varepsilon_k,1)\end{cases}\tag{3.5}$$

$$u_\varepsilon:=u_{n_0,\varepsilon}+\sum_{k=0}^{\infty}h_{k,\varepsilon}\quad\forall\varepsilon\in I.$$

Since $\varepsilon_k\downarrow 0$, for all $\varepsilon\in I$ there exists a sufficiently large k such that we have $\varepsilon\notin(0,\varepsilon_k)$ for all $k\ge\bar{k}$. Therefore, $u_\varepsilon=u_{n_{\bar{k}+1},\varepsilon}\in\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R})$. In order to prove that (u_ε) defines a GSF of the type $K\to{}^{\rho}\widetilde{\mathbb{R}}$, take $[x_\varepsilon]\in K$ and $\alpha\in\mathbb{N}$. We claim that $(\partial^\alpha u_\varepsilon(x_\varepsilon))\in\mathbb{R}_\rho$. Now, for all $p\in\mathbb{N}$ and for any $x\in\mathbb{R}^n$ we have that, for $\varepsilon\le\varepsilon_p$

$$\left|\partial^\alpha u_\varepsilon(x)\right|\le\left|\partial^\alpha u_{n_{p+1},\varepsilon}(x)\right|+\sum_{k=p+1}^{\infty}\left|\partial^\alpha h_{k,\varepsilon}(x)\right|.$$

If p satisfies $|\alpha|\le p$, then from (3.4) and (3.5), we get that $|\partial^\alpha h_{k,\varepsilon}(x)|\le\rho_\varepsilon^k$ for all $k\ge p+1$, $x\in K_\varepsilon$ and all $\varepsilon\in(0,1]$. Hence for $\varepsilon\in(0,\varepsilon_p)$, $|\alpha|\le p$ and all $x\in K_\varepsilon$, we obtain

$$\left|\partial^\alpha u_\varepsilon(x)\right|\le\left|\partial^\alpha u_{n_{p+1},\varepsilon}(x)\right|+\frac{\rho_\varepsilon^{p+1}}{1-\rho_\varepsilon}.\tag{3.6}$$

Inserting $x=x_\varepsilon$ and noting that $(\partial^\alpha u_{n_{p+1},\varepsilon}(x_\varepsilon))\in\mathbb{R}_\rho$ proves our claim.

Moreover, $\|u-u_{n_p}\|_i<\mathrm{d}\rho^{p-1}$ for all $p\in\mathbb{N}_{>1}$ and all $i\le p$. This yields that $(u_{n_k})_k$ tends to u in the sharp topology, and hence so does (u_n).

If $C\subseteq{}^{\rho}\mathcal{GC}^\infty(K,{}^{\rho}\widetilde{\mathbb{R}})$ is closed in the sharp topology and $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of C, then it converges to a function $u\in{}^{\rho}\mathcal{GC}^\infty(K,{}^{\rho}\widetilde{\mathbb{R}})$. We cannot have $u\in C^c$ because otherwise $u_n\in B_r^m(u)\subseteq C^c$ for some $r\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$, $m\in\mathbb{N}$, and for all $n\in\mathbb{N}$ sufficiently big, which is a contradiction. This shows (ii).

Finally, let $(u_n)_{n\in\mathbb{N}}$ be a convergent sequence of ${}^{\rho}\mathcal{GC}^\infty(K,{}^{\rho}\widetilde{\mathbb{R}}^d)$ such that $u_n(K)\subseteq H$ for all $n\in\mathbb{N}$. Set $u:=\lim_{n\to+\infty}u_n\in{}^{\rho}\mathcal{GC}^\infty(K,{}^{\rho}\widetilde{\mathbb{R}}^d)$, then $\|u_n-u\|_0=\left[\sup_{x\in K_\varepsilon}|u_{n,\varepsilon}(x)-u_\varepsilon(x)|\right]\to 0$ in the sharp topology. If $x\in K=[K_\varepsilon]$, then $x_\varepsilon\in K_\varepsilon$ for some representative $[x_\varepsilon]=x$ and for ε small. Therefore, $|u_n(x)-u(x)|\le$

$\|u_n - u\|_0$ and hence the sequence $(u_n(x))_{n\in\mathbb{N}}$ of H tends to $u(x)$ in the sharp topology. Hence $u(x) \in H$ because we assumed that H is sharply closed. $\square$

For a complete theory of (functionally) compactly supported GSF in the case $\rho_\varepsilon = \varepsilon$, see [18]. In the same particular case, for an Archimedean theory of ${}^\rho\widetilde{\mathbb{R}}$-modules, see [14, 15, 16].

4. Uniformly continuous GSF

In this section, we will present a few basic results about uniformly continuous GSF that will be later used in Section 7.3. Our goal is to study the possibility of extending GSF from open intervals (a,b) (for $a < b \in {}^\rho\widetilde{\mathbb{R}}$) to their closure $\overline{(a,b)} = [a,b]$. We start by noting that if $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ is a solid set, then

$$f|_{\mathrm{int}(X)} = g|_{\mathrm{int}(X)} \quad \Rightarrow \quad f = g \tag{4.1}$$

because $\mathrm{int}(X)$ is dense in X and GSF are sharply continuous.

The definition of uniformly continuous GSF is as one might expect:

Definition 13. Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$, $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^m$ and let $f \in {}^\rho\mathcal{GC}^\infty(X,Y)$. We say that f is *uniformly continuous* on X if for every $\eta \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ there exists $\delta \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that for every x, $y \in X$

$$|x-y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \eta.$$

All the basic properties of uniformly continuous functions that we will need in this paper are listed in the following theorem. Notably, almost all the proofs are identical to their classical counterparts:

Theorem 14. *Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^i$, $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^j$, $Z \subseteq {}^\rho\widetilde{\mathbb{R}}^k$. Then:*

(i) *If $f \in {}^\rho\mathcal{GC}^\infty(X,Y)$ and $g \in {}^\rho\mathcal{GC}^\infty(Y,Z)$ are uniformly continuous then $g\circ f \in {}^\rho\mathcal{GC}^\infty(X,Z)$ is uniformly continuous;*

(ii) *If $f \in {}^\rho\mathcal{GC}^\infty(X,Y)$ is uniformly continuous then f maps Cauchy sequences in X (in the sharp topology) into Cauchy sequences in Y;*

(iii) *Let X be a solid set and let $f \in {}^\rho\mathcal{GC}^\infty(X,Y)$. Assume that for every multi-index $\alpha \in \mathbb{N}^i$ the partial derivative $\partial^\alpha f \in {}^\rho\mathcal{GC}^\infty(X,Y_\alpha)$ is uniformly continuous on X, where $Y_\alpha \subseteq {}^\rho\widetilde{\mathbb{R}}^j$ is a sharply closed set. Then f can be extended to the sharp closure $\overline{X}$ in a unique way, i.e. there exist a unique GSF $\bar{f} \in {}^\rho\mathcal{GC}^\infty\left(\overline{X},\overline{Y}\right)$ such that $\bar{f}|_X = f$. Moreover, this extension operator $\overline{(-)}$ preserves partial derivative, i.e. $\overline{\partial^\alpha f} = \partial^\alpha \bar{f}$ for all $\alpha \in \mathbb{N}^i$.*

Finally, if $i = 1$ and X is a sharply open set such that for every a, $b \in X$ we have $[a, a\vee b] \subseteq X$ (e.g. if X is an interval; we recall that $[x_\varepsilon]\vee[y_\varepsilon] := [\max(x_\varepsilon, y_\varepsilon)]$, see [20]) then:

(iv) *If $f \in {}^\rho\mathcal{GC}^\infty(X,Y)$ and f' is sharply bounded on X then f is uniformly continuous on X;*

(v) *If $\overline{X}$ is functionally compact, then $f^{(n)}$ is uniformly continuous on X for all $n \in \mathbb{N}$.*

Proof. The proofs of (i) and (ii) are identical to the classical one for metric spaces. Proof of (iii): Let $\bar{x} \in \overline{X}$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $\bar{x} = \lim_{n\in\mathbb{N}} x_n$ in the sharp topology. Set

$$\bar{f}(\bar{x}) := \lim_{n\in\mathbb{N}} f(x_n). \tag{4.2}$$

The existence of this limit follows from the fact that $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y_α by (ii), and Y_α is closed by assumption. Moreover, the uniform continuity of f on X guarantees that (4.2) does not depend on the choice of the net $(x_n)_{n\in\mathbb{N}}$. In particular, this shows that

$$\overline{\partial^\alpha f}(\bar{x}) = \lim_{\substack{x\to\bar{x}\\ x\in X}} \partial^\alpha f(x) = \lim_{\substack{x\to\bar{x}\\ x\in\mathrm{int}(X)}} \partial^\alpha f(x). \tag{4.3}$$

Hence, from Lem. 5 we deduce that every $\overline{\partial^\alpha f} \in {}^\rho\mathcal{GC}^\infty\left(\overline{X}, Y\right)$. Moreover, since every GSF is continuous, we notice that from Lem. 5, we also get that

$$\forall\alpha\in\mathbb{N}^i:\ \partial^\alpha\bar{f} = \overline{\partial^\alpha f}.$$

In fact, for every $x\in\overline{X}$ we have

$$\partial^\alpha\bar{f}(x) = \lim_{\substack{y\to x\\ y\in\mathrm{int}(\overline{X})}} \partial^\alpha f(y) = \lim_{\substack{y\to x\\ y\in X}} \partial^\alpha f(y) = \overline{\partial^\alpha f(x)}.$$

Proof of (iv): Let us assume that $|f'(x)| < M$ for every $x\in X$, where $M\in{}^\rho\widetilde{\mathbb{R}}_{>0}$. Let $\eta\in{}^\rho\widetilde{\mathbb{R}}_{>0}$, and set $\delta := \frac{\eta}{4M}$. Let $a, b\in X$ be such that $|a-b|<\delta$. There are a few cases to consider:

(a) If $a<b$, by the mean value theorem (that can be applied as, by assumption, $[a,b]=[a,a\vee b]\subseteq X$) there exists $c\in[a,b]$ such that $|f(a)-f(b)| = |f'(c)||a-b| < M\cdot\delta = \frac{\eta}{4} < \eta$. The situation is similar if $a>b$.

(b) If $a\le b$, let $(a_n)_{n\in\mathbb{N}}$ be a sequence of the sharply open set $(-\infty,a)\cap X$ that converges to a. As $|a-b|<\delta$, there exists $n\in\mathbb{N}$ such that $|b-a_m|<\delta$ for all $m>n$. Arguing as in the previous point, we can show that $|f(a_m)-f(b)| < M\cdot\delta = \frac{\eta}{4}$ for all $m>n$, namely $f(a_m)\in B_{M\cdot\delta}(f(b))$. Hence

$$f(a) = \lim{}_{m\in\mathbb{N}} f(a_m) \in \overline{B_{M\cdot\delta}(f(b))} \subset B_{\frac{\eta}{2}}(f(b)),$$

as $\frac{\eta}{2} > M\cdot\delta$. The case $b\ge a$ can be handled similarly.

(c) It remains to study the case when $|a-b|<\delta$ but a,b are incomparable, i.e. if none of the previous cases hold. Let us consider $a\vee b$. As $|a-b|<\delta$, we have that $|a-a\vee b|<\delta$ and $|b-a\vee b|<\delta$ (in fact, e.g. $|a-a\vee b|\le|a-b|$). Notice that $a\le a\vee b$ and $b\le a\vee b$, hence by the previous point we get that $|f(a)-f(a\vee b)|<\frac{\eta}{2}$ and $|f(b)-f(a\vee b)|<\frac{\eta}{2}$. We hence conclude as follows

$$|f(a)-f(b)| = |f(a)-f(a\vee b)+f(a\vee b)-f(b)| \le$$

(d)

$$|f(a)-f(a\vee b)| + |f(a\vee b)-f(b)| < \frac{\eta}{2}+\frac{\eta}{2} = \eta.$$

Proof of (v): If $\overline{X}$ functionally compact, by the extreme value theorem [20, Cor. 51], we get that $\partial^n f$ is sharply bounded on X for every $n\in\mathbb{N}$, and we can conclude by (iv). □

5. Banach fixed point and Picard-Lindelöf theorems for GSF

In this section, we introduce finite sharp contractions for GSF and we prove a corresponding Banach-like fixed point theorem in spaces of GSF. To motivate our general approach in this section, let us consider the following simple example. Let $0\le\alpha<1$, $\alpha\in{}^\rho\widetilde{\mathbb{R}}$, and let $T_\alpha : {}^\rho\widetilde{\mathbb{R}} \to {}^\rho\widetilde{\mathbb{R}}$ be such that

$$\forall x\in{}^\rho\widetilde{\mathbb{R}}:\ T_\alpha(x) = \alpha\cdot x.$$

As $0 \le \alpha < 1$, one might expect T_α to be a contraction on ${}^\rho\widetilde{\mathbb{R}}$, since for every x, $y \in {}^\rho\widetilde{\mathbb{R}}$ and $n \in \mathbb{N}$ we have that

$$\left|T_\alpha^n(x) - T_\alpha^n(y)\right| = \alpha^n \left|x - y\right| < \left|x - y\right|. \tag{5.1}$$

However, the set of radii of the sharp topology is ${}^\rho\widetilde{\mathbb{R}}_{>0}$, so for the left hand side of (5.1) to go to zero in this topology as $n \to +\infty$, we need the property

$$\forall r \in {}^\rho\widetilde{\mathbb{R}}_{>0}\, \exists n \in \mathbb{N} : \ \alpha^n < r. \tag{5.2}$$

If $\alpha < \mathrm{d}\rho^k$ for some $k \in \mathbb{N}_{>0}$, this property holds, and a similar property will be used in Section 5.1 to develop a first theory of contractions for GSF which is very close to the classical one. However, in many natural cases this property does not hold. For example, if we want that the function $T_{\frac{1}{2}}(x)$ respects our intuition of contraction in the sharp topology, we need to generalize our approach, because $\left(\frac{1}{2}\right)^n > \mathrm{d}\rho$ for all $n \in \mathbb{N}$. This will be solved in Sec. 5.3 by exploiting the idea of iterating the function T_α an infinite amount of times, namely of considering objects like T_α^N where N is a *hyperfinite* number in ${}^\rho\widetilde{\mathbb{R}}$ (see [20, Sec. 7.2]):

$$N \in {}^\rho\widetilde{\mathbb{N}} = \left\{[n_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}} \mid n_\varepsilon \in \mathbb{N} \quad \forall \varepsilon\right\}.$$

5.1. **Banach fixed point theorem for finite contractions.** The notion of finite contraction we are going to introduce corresponds to the standard idea of contraction in locally convex spaces, see e.g. [2].

Definition 15. Let $\emptyset \ne K \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}^n$ be a solid set, and let $X \subseteq {}^\rho\mathcal{GC}^\infty\left(K, {}^\rho\widetilde{\mathbb{R}}^d\right)$. We say that T is a *finite (sharp) contraction on* X if

(i) $T : X \longrightarrow X$ is a set-theoretical map.
(ii) $\forall i \in \mathbb{N}\, \exists \alpha_i \in {}^\rho\widetilde{\mathbb{R}}_{>0}\, \forall u, v \in X : \ \left\|T(u) - T(v)\right\|_i \le \alpha_i \cdot \left\|u - v\right\|_i.$
(iii) For all $i \in \mathbb{N}$, we have $\lim_{n\to+\infty} \alpha_i^n = 0$, where the limit is taken in the sharp topology.

For every $i \in \mathbb{N}$, such an $\alpha_i \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ will be called an i-th *contraction constant* for T.

The adjective *finite* is motivated by condition (iii), where we consider the limit for finite $n \in \mathbb{N}$ and not the hyperlimit for infinite $n \in {}^\rho\widetilde{\mathbb{N}}$. This implies that for all i, $k \in \mathbb{N}$ we must have $\alpha_i^n < \mathrm{d}\rho^k$ for $n \in \mathbb{N}$ sufficiently large, and hence $\alpha_i < \mathrm{d}\rho^a$ for some $a \in \mathbb{R}_{>0}$, which is stronger than $\alpha_i \approx 0$ (and it is equivalent to (5.2)).

For finite sharp contractions, we can proceed as in the classical case, as we are now going to show. The proof of the following lemma is a straightforward generalization of the classical one.

Lemma 16. *Let $\emptyset \ne K \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$ be a solid set, and let $X \subseteq {}^\rho\mathcal{GC}^\infty\left(K, {}^\rho\widetilde{\mathbb{R}}^d\right)$. Then every finite sharp contraction on X is sharply continuous.*

We are now able to prove a Banach fixed point theorem for finite contractions which is analogous, both in the statement and in the proof, to the classical one.

Theorem 17. *Let $\emptyset \ne K \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$ be a solid set, and let X be a nonempty closed subset of ${}^\rho\mathcal{GF}\left(K, {}^\rho\widetilde{\mathbb{R}}^d\right)$. Let $T : X \longrightarrow X$ be a finite sharp contraction. Then there exists a unique fixed point $\overline{u}$ of T in X. Moreover, for every $u \in X$ the sequence $\left\{T^n(u)\right\}_{n\in\mathbb{N}}$ converges to $\overline{u}$ in the sharp topology.*

Proof. Let $u \in X$. For every $i \in \mathbb{N}$, let α_i be an i-th contraction constant for T on X. We claim that $\{T^n(u)\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the sharp topology. By induction, it is easily checked that

$$\left\|T^{n+1}(u) - T^n(u)\right\|_i \le \alpha_i^n \left\|T(u) - u\right\|_i ,$$

so for every n, $m \in \mathbb{N}$, $n < m$, we have

$$\begin{aligned}\left\|T^m(u) - T^n(u)\right\|_i &\le \left\|T^m(u) - T^{m-1}(u)\right\|_i + \cdots + \left\|T^{n+1}(u) - T^n(u)\right\|_i \le \\ &\le \alpha_i^n \left\|T(u) - u\right\|_i \cdot \sum_{j=0}^{m-1-n} \alpha_i^j = \\ &= \alpha_i^n \left\|T(u) - u\right\|_i \cdot \frac{1-\alpha_i^{m-n}}{1-\alpha_i} = \\ &= \frac{\alpha_i^n - \alpha_i^m}{1-\alpha_i} \cdot \left\|T(u) - u\right\|_i .\end{aligned}$$

The conclusion follows by condition (iii) of Def. 15. Thm. 12 therefore yields that the sequence $\{T^n(u)\}_{n\in\mathbb{N}}$ has a limit $\overline{u} \in X$. As T is sharply continuous, we have that

$$T(\overline{u}) = T\left(\lim_{n\in\mathbb{N}} T^n(u)\right) = \lim_{n\in\mathbb{N}} T^{n+1}(u) = \overline{u},$$

so $\overline{u}$ is a fixed point of T. Finally, let us suppose that v is another fixed point of T. Then

$$\left\|u - v\right\|_0 = \left\|T(u) - T(v)\right\|_0 \le \alpha_0 \left\|u - v\right\|_0$$

and, since $0 \approx \alpha_0 < 1$, this is possible only if $\left\|u - v\right\|_0 \le 0$, and hence $u = v$ by Thm. 8. □

5.2. **A Picard-Lindelöf Theorem for finite iterations.** We first note that, exactly as in the classical case and thanks to the closure of GSF with respect to composition [17, Sec. 3.0.3], the higher order Cauchy problem (1.1) can be reduced to a system of first order equations. Secondly, we introduce the notion of uniformly Lipschitz function with the following

Definition 18. Let $\emptyset \ne K \Subset_{\text{f}} {}^{\rho}\widetilde{\mathbb{R}}$ be a solid set and let $H \subseteq {}^{\rho}\widetilde{\mathbb{R}}^d$ and $F \in {}^{\rho}\mathcal{GC}^{\infty}\left(K \times H, {}^{\rho}\widetilde{\mathbb{R}}^d\right)$. Let $Y \subseteq {}^{\rho}\mathcal{GC}^{\infty}(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$ be such that $y(t) \in H$ for all $y \in Y$ and all $t \in K$. If $y \in Y$, we simply denote by $F(t, y)$ the composition $t \in K \mapsto F(t, y(t)) \in {}^{\rho}\widetilde{\mathbb{R}}^d$. We say that F is *uniformly Lipschitz on Y with constants* $(L_i)_{i\in\mathbb{N}} \in {}^{\rho}\widetilde{\mathbb{R}}^{\mathbb{N}}$ if

$$\forall i \in \mathbb{N}\, \forall x, y \in Y : \ \left\|F(t, x) - F(t, y)\right\|_i \le L_i \cdot \left\|x - y\right\|_i . \tag{5.3}$$

Note that condition (5.3) is formulated in the Fréchet space of GSF ${}^{\rho}\mathcal{GF}(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$, because of the use of the norms $\|-\|_i$. In other words, condition (5.3) involves all the derivatives of $t \in K \mapsto F(t, x(t)) - F(t, y(t))$ of any order i. It is therefore stronger than the usual uniformly Lipschitz condition on F, which involves only values $|F(t, x) - F(t, y)|$ at points x, $y \in {}^{\rho}\widetilde{\mathbb{R}}^d$ and $t \in K$. Moreover, we also note that the bigger is Y the stronger is condition (5.3), and hence the smaller is the class of functions F that satisfy it.

On the other hand, as with ordinary smooth functions, Def. 18 is not restrictive:

Theorem 19. *Let $\emptyset \neq K \Subset_f {}^\rho\widetilde{\mathbb{R}}$ be a solid set. Let $H \Subset_f {}^\rho\widetilde{\mathbb{R}}^d$, and consider the generalized smooth function $F \in {}^\rho\mathcal{GC}^\infty\left(K \times H, {}^\rho\widetilde{\mathbb{R}}^d\right)$, the radii $r_i \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ for all $i \in \mathbb{N}$ and $y_0 \in {}^\rho\widetilde{\mathbb{R}}^d$. Then F is uniformly Lipschitz on the set*

$$\{y \in {}^\rho\mathcal{GC}^\infty(K,H) \mid \|y-y_0\|_i \le r_i \ \forall i \in \mathbb{N}\}. \tag{5.4}$$

Moreover, if we use the symbols $(L_i(K))_{i\in\mathbb{N}}$ to underscore the dependence of these Lipschitz constants by the solid functionally compact set K, then the following property holds: if $\emptyset \neq K' \Subset_f {}^\rho\widetilde{\mathbb{R}}$ is another solid set, then

$$K \subseteq K' \quad \Longrightarrow \quad L_i(K) \le L_i(K') \ \forall i \in \mathbb{N}. \tag{5.5}$$

To prove this result, we first need the classical Faà di Bruno formula for GSF:

Lemma 20. *Let $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}}^d)$, where $U \subseteq {}^\rho\widetilde{\mathbb{R}}^m$ is a solid set. Let $g \in {}^\rho\mathcal{GC}^\infty(V,U)$, $x \in V \subseteq {}^\rho\widetilde{\mathbb{R}}^l$ be a solid set and $i \in \mathbb{N}^l$, with $|i| \ge 1$. For a, $b \in \mathbb{N}^l$, we write $a \prec b$ if $|a| < |b|$ or $|a| = |b|$ and $a_j = b_j$ for all $j = 1,\dots,k$ but $a_{k+1} < b_{k+1}$ for some $k < l$. Note that this relation generalizes the usual strict order on $\mathbb{N}$ if $l = 1$. For $\alpha \in \mathbb{N}^m$ and for $s = 1,\dots,|i|$, set $(k,n) \in p_s(i,\alpha)$ if and only if $(k,n) \in (\mathbb{N}^m)^s \times \left(\mathbb{N}^l\right)^s$, $|k_j| > 0$ for all $j = 1,\dots,s$, $\sum_{j=1}^s k_j = \alpha$, $\sum_{j=1}^s |k_j| n_j = i$ and $0 \prec n_1 \prec \ldots \prec n_s$. Finally, set*

$$p(i,\alpha) := \left\{(0,\overset{|i|-s}{\ldots\ldots},0,k,0,\overset{|i|-s}{\ldots\ldots},0,n) \in (\mathbb{N}^m)^{|i|} \times \left(\mathbb{N}^l\right)^{|i|} \mid (k,n) \in p_s(i,\alpha)\right\}.$$

Then, we have

$$\partial^i(f\circ g)(x) = \sum_{1\le|\alpha|\le|i|} \partial^\alpha f\left(g(x)\right) \cdot \sum_{(k,n)\in p(i,\alpha)} i! \prod_{j=1}^{|i|} \frac{\left[\partial^{n_j} g(x)\right]^{k_j}}{k_j!(n_j!)^{|k_j|}}. \tag{5.6}$$

In this formula, the second factor highlights the polynomial

$$q(i,\alpha) := \left\{n \in \left(\mathbb{N}^l\right)^{|i|} \mid \exists k : \ (k,n) \in p(i,\alpha)\right\}$$

$$B_{i,\alpha} : \left\{\left(z_{n_1},\dots,z_{n_{[i]}}\right)_{n\in q(i,\alpha)} \mid z_{n_j} \in \mathbb{R}^m \ \forall j = 1,\dots,|i|\right\} \longrightarrow \mathbb{R}$$

$$B_{i,\alpha}\left[\left(z_{n_1},\dots,z_{n_{[i]}}\right)_{n\in q(i,\alpha)}\right] := \sum_{(k,n)\in p(i,\alpha)} i! \prod_{j=1}^{|i|} \frac{\left[z_{n_j}\right]^{k_j}}{k_j!(n_j!)^{|k_j|}}$$

which is called (multivariate) Bell polynomial.

Proof. We can proceed as in the classical smooth case (see e.g. [3]), or simply noting that the formula holds true ε−wise, i.e. for all nets (f_ε) and (g_ε) that define f and g. □

We are now ready to prove Thm. 19.È meglio ricontrollare insieme questa dimostrazione perché continuo a trovarci errori stupidi (non concettuali)...

Proof of Thm. 19. If $i = 0$, the mean value theorem gives $|F(t,x(t)) - F(t,y(t))| \le \|F|_{K\times H}\|_1 \cdot \|x-y\|_0$. For $i \ge 1$, we need to apply (5.6) with $F : K \times H \longrightarrow {}^\rho\widetilde{\mathbb{R}}^d$, $m = 1 + d$, $l = 1$, and $g(t) = (t, y(t))$ for all $t \in K$, to evaluate $\frac{d^i}{dt^i}F(t,y(t))$.

Therefore, Bell polynomials become

$$B_{i,\alpha}\left[\left(\delta_{1,n_j},\frac{d^{n_j}y^1}{dt^{n_j}}(t),\dots,\frac{d^{n_j}y^d}{dt^{n_j}}(t)\right)_{\substack{n\in q(i,\alpha)\\ j=1,\dots,i}}\right]=$$

$$=\sum_{(k,n)\in p(i,\alpha)} i!\prod_{j=1}^{i}\frac{\left[\delta_{1,n_j},\frac{d^{n_j}y^1}{dt^{n_j}}(t),\dots,\frac{d^{n_j}y^d}{dt^{n_j}}(t)\right]^{k_j}}{k_j!(n_j!)^{|k_j|}}=$$

$$=\sum_{(k,n)\in p(i,\alpha)} i!\prod_{j=1}^{i}\frac{\left(\delta_{1,n_j}\right)^{k_{j,1}}\cdot\left(\frac{d^{n_j}y^1}{dt^{n_j}}(t)\right)^{k_{j,2}}\cdot\ldots\cdot\left(\frac{d^{n_j}y^d}{dt^{n_j}}(t)\right)^{k_{j,d+1}}}{k_j!(n_j!)^{|k_j|}}.$$

Introducing a simplified notation, we can therefore write

$$\begin{aligned}\frac{d^i}{dt^i}F(t,y(t))&=\sum_{1\le|\alpha|\le i}\partial^\alpha F\left(t,y(t)\right)\cdot B_{i,\alpha}\left[\left(\delta_{1,n_j},\frac{d^{n_j}y^1}{dt^{n_j}}(t),\dots,\frac{d^{n_j}y^d}{dt^{n_j}}(t)\right)_{0\ne n_j\le i}\right]\\&=:\sum_{1\le|\alpha|\le i}\partial^\alpha F\left(t,y(t)\right)\cdot Q_{i,\alpha}\left[y(t)\right].\end{aligned}$$

This yields the following evaluations

$$\begin{aligned}&\left|\frac{d^i}{dt^i}F(t,x(t))-\frac{d^i}{dt^i}F(t,y(t))\right|=\\&\quad\left|\sum_{1\le|\alpha|\le i}\partial^\alpha F\left(t,x(t)\right)\cdot Q_{i,\alpha}\left[x(t)\right]-\sum_{1\le|\alpha|\le i}\partial^\alpha F\left(t,y(t)\right)\cdot Q_{i,\alpha}\left[y(t)\right]\right|\le\\&\quad\left|\sum_{1\le|\alpha|\le i}\partial^\alpha F\left(t,x(t)\right)\cdot Q_{i,\alpha}\left[x(t)\right]-\sum_{1\le|\alpha|\le i}\partial^\alpha F\left(t,x(t)\right)\cdot Q_{i,\alpha}\left[y(t)\right]\right|+\\&\quad+\left|\sum_{1\le|\alpha|\le i}\partial^\alpha F\left(t,x(t)\right)\cdot Q_{i,\alpha}\left[y(t)\right]-\sum_{1\le|\alpha|\le i}\partial^\alpha F\left(t,y(t)\right)\cdot Q_{i,\alpha}\left[y(t)\right]\right|.\end{aligned}\tag{5.7}$$

The first summand in (5.7) gives

$$\begin{aligned}&\left|\partial^\alpha F\left(t,x(t)\right)\cdot Q_{i,\alpha}\left[x(t)\right]-\partial^\alpha F\left(t,x(t)\right)\cdot Q_{i,\alpha}\left[y(t)\right]\right|\le\\&\qquad\|F\|_{i+1}\cdot\left|Q_{i,\alpha}\left[x(t)\right]-Q_{i,\alpha}\left[y(t)\right]\right|\le\|F\|_{i+1}\cdot L_{i,\alpha}\cdot\|x-y\|_i\,,\end{aligned}$$

where $L_{i,\alpha}\in{}^\rho\widetilde{\mathbb{R}}$ satisfies the following Lipschitz-like condition for the polynomial $B_{i,\alpha}$:

$$\begin{aligned}&\Bigg\|B_{i,\alpha}\left[\left(\delta_{1,n_j},\frac{d^{n_j}x^1}{dt^{n_j}}(t),\dots,\frac{d^{n_j}x^d}{dt^{n_j}}(t)\right)_{0\ne n_j\le i}\right]-\\&\qquad-B_{i,\alpha}\left[\left(\delta_{1,n_j},\frac{d^{n_j}y^1}{dt^{n_j}}(t),\dots,\frac{d^{n_j}y^d}{dt^{n_j}}(t)\right)_{0\ne n_j\le i}\right]\Bigg\|_0\le L_{i,\alpha}\cdot\|x-y\|_i\,.\end{aligned}$$

The second summand in (5.7) gives

$$
\begin{aligned}
&\left|\partial^{\alpha}F\left(t,x(t)\right)\cdot Q_{i,\alpha}\left[y(t)\right]-\partial^{\alpha}F\left(t,y(t)\right)\cdot Q_{i,\alpha}\left[y(t)\right]\right|\le\\
&\qquad\le\left|\partial^{\alpha}F\left(t,x(t)\right)-\partial^{\alpha}F\left(t,y(t)\right)\right|\cdot\left|Q_{i,\alpha}\left[y(t)\right]\right|\le\\
&\qquad\qquad\le\left\Vert F\right\Vert_{i+1}\cdot\left\Vert x-y\right\Vert_{i}\cdot\left|Q_{i,\alpha}\left[y(t)\right]\right|.
\end{aligned}
$$

Now

$$
\begin{aligned}
\left|Q_{i,\alpha}\left[y(t)\right]\right| &\le\sum_{(k,n)\in p(i,\alpha)}i!\prod_{j=1}^{i}\frac{\left|\frac{d^{n_j}y^1}{dt^{n_j}}(t)\right|^{k_{j,2}}\cdot\ldots\cdot\left|\frac{d^{n_j}y^d}{dt^{n_j}}(t)\right|^{k_{j,d+1}}}{k_j!(n_j!)^{|k_j|}}\le\\
&\le\sum_{(k,n)\in p(i,\alpha)}i!\prod_{j=1}^{i}\frac{r_{n_j}^{k_{j,2}}\cdot\ldots\cdot r_{n_j}^{k_{j,d+1}}}{k_j!(n_j!)^{|k_j|}}= \qquad (5.8)\\
&=B_{i,\alpha}\left[\left(1,r_{n_j},\ldots,r_{n_j}\right)_{\substack{n\in q(i,\alpha)\\ j=1,\ldots,i}}\right] \qquad (5.9)\\
&=:\bar{B}_{i,\alpha}(r_1,\ldots,r_i).
\end{aligned}
$$

because $y\in\left\{y\in{}^{\rho}\mathcal{GC}^{\infty}(K,H)\mid\left\Vert y-y_0\right\Vert_i\le r_i\ \forall i\in\mathbb{N}\right\}$ and hence $\left|\frac{d^{n_j}y}{dt^{n_j}}(t)\right|\le r_{n_j}$ because $n_j>0$. This shows the stated conclusion with

$$
L_i(K)=\sum_{1\le|\alpha|\le i}\left\{L_{i,\alpha}+\bar{B}_{i,\alpha}(r_1,\ldots,r_i)\right\}\cdot\left\Vert F\right\Vert_{i+1}\ \text{if } i\ge0,
$$

We finally give an estimate of the Lipschitz constants $L_{i,\alpha}$:

$$
\begin{aligned}
&\left|Q_{i,\alpha}\left[x(t)\right]-Q_{i,\alpha}\left[y(t)\right]\right|\le\\
&\le\sum_{(k,n)\in p(i,\alpha)}i!\prod_{j=1}^{i}\frac{1}{k_j!(n_j!)^{|k_j|}}\cdot\left|\prod_{\substack{a=2\\k_{j,a}\ge1}}^{d+1}\left(\frac{d^{n_j}x^{a-1}}{dt^{n_j}}(t)\right)^{k_{j,a}}-\prod_{\substack{a=2\\k_{j,a}\ge1}}^{d+1}\left(\frac{d^{n_j}y^{a-1}}{dt^{n_j}}(t)\right)^{k_{j,a}}\right|.
\end{aligned}
$$

But the mean value theorem in several variables yields

$$
\begin{aligned}
&\left|\prod_{\substack{a=2\\k_{j,a}\ge1}}^{d+1}\left(\frac{d^{n_j}x^{a-1}}{dt^{n_j}}(t)\right)^{k_{j,a}}-\prod_{\substack{a=2\\k_{j,a}\ge1}}^{d+1}\left(\frac{d^{n_j}y^{a-1}}{dt^{n_j}}(t)\right)^{k_{j,a}}\right|\le\\
&\qquad\le\max_{\substack{a=2,\ldots,d+1\\k_{j,a}\ge1}}k_{j,a}r_{n_j}^{k_{j,a}-1}\prod_{b\ne a}k_{j,b}r_{n_j}^{k_{j,b}}\cdot\left\Vert x-y\right\Vert_i
\end{aligned}
$$

so that we can set

$$
\begin{aligned}
L_{i,\alpha}&=:L_{i,\alpha}\left(r_1,\ldots,r_i\right)= \qquad (5.10)\\
&=\sum_{(k,n)\in p(i,\alpha)}i!\prod_{j=1}^{i}\frac{1}{k_j!\left(n_j!\right)^{|k_j|}}\cdot\max_{\substack{a=2,\ldots,d+1\\k_{j,a}\ge1}}k_{j,a}r_{n_j}^{k_{j,a}-1}\prod_{b\ne a}k_{j,b}r_{n_j}^{k_{j,b}}. \qquad (5.11)
\end{aligned}
$$

To prove property (5.5), it suffices to note that the constants $(L_i(K))_{i\in\mathbb{N}}$ depend on K only through the norms $\left\Vert F\right\Vert_i$. □

The key ideas of the proof of Thm. 19 can be summarized by saying that all the Lipschitz constants for the Cauchy problem (1.1) can be effectively computed using the multivariate Faà di Bruno formula, the inequalities $\|y-y_0\|_i \le r_i$ that define the space of functions (5.4), the Lipschitz properties of Bell polynomials and the norms $\|F\|_i$. Let us also notice that in the proof of Thm. 19 we computed possible values of the Lipschitz constants of F, proving the following

Corollary 21. *In the same notations of Thm. 19, values for the Lipschitz constants of F on*

$$\left\{y \in {}^\rho\mathcal{GC}^\infty(K,H) \mid \|y-y_0\|_i \le r_i \ \forall i \in \mathbb{N}\right\}$$

are given for all $i \in \mathbb{N}$ by

$$\begin{aligned} L_i &=: L_i\left(r_1,\dots,r_i\right) = \\ &= \sum_{1\le|\alpha|\le i} \left\{L_{i,\alpha}\left(r_1,\dots,r_i\right) + \bar{B}_{i,\alpha}\left(r_1,\dots,r_i\right)\right\} \cdot \|F\|_{i+1} \end{aligned} \tag{5.12}$$

where

$$L_{i,\alpha}\left(r_1,\dots,r_i\right) = \sum_{(k,n)\in p(i,\alpha)} i! \prod_{j=1}^{i} \frac{1}{k_j!\left(n_j!\right)^{|k_j|}} \cdot \max_{\substack{a=2,\dots,d+1 \\ k_{j,a}\ge 1}} k_{j,a} r_{n_j}^{k_{j,a}-1} \prod_{b\ne a} k_{j,b} r_{n_j}^{k_{j,b}}.$$

$$\bar{B}_{i,\alpha}\left(r_1,\dots,r_i\right) = \sum_{(k,n)\in p(i,\alpha)} i! \prod_{j=1}^{i} \frac{r_{n_j}^{k_{j,2}} \cdot \ldots \cdot r_{n_j}^{k_{j,d+1}}}{k_j!(n_j!)^{|k_j|}}.$$

Note that $L_0 = \|F\|_1$ by (5.12). Finally, for all $i \in \mathbb{N}_{>0}$, we have

$$\lim_{(r_1,\dots,r_i)\to 0^+} L_i(r_1,\dots,r_i) = 0,$$

therefore for all $i \in \mathbb{N}_{>0}$ there exists $R_i \in \mathbb{R}_{>0}$ such that for all $r_1,\dots,r_i \in \mathbb{R}_{>0}$, if $|r_n| < R_i$ for $n = 1,\dots,i$, then

$$L_i(r_1,\dots,r_i) \le \|F\|_i\,.$$

Notice that the values for the Lipschitz constants given in Cor. 21 are not necessarily optimal.

One could ask whether the first factor of (5.12) when $i \ge 1$, which does not depend on the function F, were bounded for $i \to +\infty$ or not. The following result states that, in general, the answer is unfortunately negative.

Corollary 22. *In the notations of Cor. 21, let $S(i,k)$ be the Stirling number of second kind of a set of $i \in \mathbb{N}_{>0}$ elements into $k \in \mathbb{N}_{>0}$ nonempty sets. Let B_i be the complete (univariate) Bell polynomial. If all the radii satisfy $r_j \le 1$ for all $j \in \mathbb{N}$, then*

$$\sum_{1\le|\alpha|\le i} \bar{B}_{i,\alpha}(r_1,\dots,r_i) \le \sum_{k=1}^{i} (1+d)^k \cdot S(i,k) = B_i(1+d,\dots,1+d). \tag{5.13}$$

Moreover

$$\sum_{1\le|\alpha|\le i} \left\{L_{i,\alpha}\left(1,\dots,1\right) + \bar{B}_{i,\alpha}\left(1,\dots,1\right)\right\} \ge B_i(1+d,\dots,1+d).$$

Note that the left-hand side of this inequality depends on the dimension d through the set $p(i,\alpha)$ (see Lem. 20).

Proof. For $\alpha \in \mathbb{N}^{d+1}$, let

$$S(i,\alpha) := \sum_{(k,n)\in p(i,\alpha)} i! \prod_{j=1}^{i} \frac{1}{k_j!(n_j!)^{|k_j|}}$$

be the multivariate Stirling number of second kind. If $r_j \le 1$, then

$$\bar{B}_{i,\alpha}(r_1,\dots,r_i) \le S(i,\alpha). \tag{5.14}$$

To compute $S(i,\alpha)$, in the Faà di Bruno formula (Lem. 20) we recall that $l=1$, $m=1+d$ and we set $g(x) := (e^x, \overset{1+d}{\dots\dots}, e^x)$ for all $x \in (-1,1)$ and $f(y_1,\dots,y_{1+d}) := (e^{y_1-1}\cdot\ldots\cdot e^{y_{1+d}-1}, \overset{d}{\dots\dots}, e^{y_1-1}\cdot\ldots\cdot e^{y_{1+d}-1})$. Therefore, each component of the composition $(f\circ g)_j(x) = e^{(1+d)(e^x-1)}$. By induction on i, like in the classical univariate case, we have

$$\partial^i (f\circ g)_j(0) = B_i(1+d,\dots,1+d) = \sum_{k=1}^{i} (1+d)^k \cdot S(i,k). \tag{5.15}$$

On the other hand, Lem. 20 yields

$$\partial^i (f\circ g)_j(0) = \sum_{1\le|\alpha|\le i} S(i,\alpha).$$

From this equality and (5.15), (5.14), the conclusion (5.13) follows.

Now, let us consider the particular case where $r_j = 1$ for all $j \in \mathbb{N}$. Then, we have $L_{i,\alpha}(1,\dots,1) \ge S(i,\alpha)$, because

$$\max_{\substack{a=2,\dots,d+1\\ k_{j,a}\ge 1}} k_{j,a} \prod_{b\ne a} k_{j,b} \ge 1,$$

and $\bar{B}_{i,\alpha}(1,\dots,1) = S(i,\alpha)$. Therefore

$$\begin{aligned}\sum_{1\le|\alpha|\le i} L_{i,\alpha}(1,\dots,1) + \bar{B}_{i,\alpha}(1,\dots,1) &\ge \sum_{1\le|\alpha|\le i} S(i,\alpha) = \\ &= \partial^i (f\circ g)_j(0) = B_i(1+d,\dots,1+d).\end{aligned}$$

□

Remark 23. In the majorization (5.13) there are two terms that goes to $+\infty$ as $i \to +\infty$: one of order $(1+d)^i$, and the other of order i^i from the Stirling number $S(i,i)$. One can easily avoid the first one by considering the following equivalent Cauchy problem, where $c \in {}^\rho\widetilde{\mathbb{R}}^*$ is a fixed suitable invertible constant:

$$\begin{cases} z'(t) = F\left(\frac{t}{c}, \frac{z(t)}{c}\right) \\ z(ct_0) = \frac{y_0}{c}. \end{cases} \tag{5.16}$$

In fact, $y \in {}^\rho\mathcal{GC}^\infty([t_0-\alpha, t_0+\alpha], {}^\rho\widetilde{\mathbb{R}}^d)$ is a solution of $y'(t) = F(t, y(t))$, $y(t_0) = y_0$ if and only if $z(t) = cy\left(\frac{t}{c}\right)$ solves (5.16) on the interval $[c(t_0-\alpha), c(t_0+\alpha)]$. Moreover if $F_c(t,z) := F\left(\frac{t}{c}, \frac{z}{c}\right)$ for all $(t,z) \in cK \times cH$, then

$$\|F_c\|_i = \frac{1}{c^i}\|F\|_i \quad \forall i \in \mathbb{N}_{\ge 1}.$$

Therefore, setting $c = 1+d$, we have the desired result.

We can now state and prove our main result of this section.

Theorem 24. *Let $t_0 \in {}^{\rho}\widetilde{\mathbb{R}}$, $y_0 \in {}^{\rho}\widetilde{\mathbb{R}}^d$, α, $r_i \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ for all $i \in \mathbb{N}$. Let $H \subseteq {}^{\rho}\widetilde{\mathbb{R}}^d$ be a sharply closed set such that $\overline{B_{r_0}(y_0)} \subseteq H$ and let $F \in {}^{\rho}\mathcal{GC}^{\infty}([t_0-\alpha, t_0+\alpha] \times H, {}^{\rho}\widetilde{\mathbb{R}}^d)$. Set $M_i := \max_{\substack{t_0-\alpha\le t\le t_0+\alpha \\ |y-y_0|\le r_i, y\in H}} \|F(t,y)\|_i$ and assume that α and $(L_i)_{i\in\mathbb{N}} \in {}^{\rho}\widetilde{\mathbb{R}}^{\mathbb{N}}$ satisfies*

$$\alpha \cdot M_i \le r_i,$$

$$\lim_{n\to+\infty} \alpha^n L_i^n = 0 \quad \forall i \in \mathbb{N}. \tag{5.17}$$

Set

$$Y_\alpha := \{y \in {}^{\rho}\mathcal{GC}^{\infty}\left([t_0-\alpha, t_0+\alpha], H\right) \mid \|y-y_0\|_i \le r_i \ \forall i \in \mathbb{N}\},$$

and assume that F is uniformly Lipschitz on Y_α with constants $(L_i)_{i\in\mathbb{N}}$. Then there exists a unique solution $y \in {}^{\rho}\mathcal{GC}^{\infty}\left([t_0-\alpha, t_0+\alpha], {}^{\rho}\widetilde{\mathbb{R}}^d\right)$ of the Cauchy problem

$$\begin{cases} y'(t) = F(t, y(t)) \\ y(t_0) = y_0 \end{cases} \tag{5.18}$$

We note explicitly that a faithful reformulation of the classical Picard-Lindelöf theorem would involve the set

$$X_\alpha := \{y \in {}^{\rho}\mathcal{GC}^{\infty}([t_0-\alpha, t_0+\alpha], H) \mid \|y-y_0\| \le r_0\}. \tag{5.19}$$

The use of Y_α instead of X_α lies on our Def. 18 of uniformly Lipschitz GSF and hence on Thm. 19. Anyway, since $Y_\alpha \subseteq X_\alpha$, if F is uniformly Lipschitz on X_α, it also satisfies the same property on Y_α.

Lemma 25. *In the assumptions of Thm. 24, Y_α is a sharply closed subset of ${}^{\rho}\mathcal{GF}([t_0-\alpha, t_0+\alpha], {}^{\rho}\widetilde{\mathbb{R}}^d)$.*

Proof. Let y be an adherent point of Y_α, so that for all $i \in \mathbb{N}$ and all $s \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$, there exists a point in $Y_\alpha \cap B_s^i(y)$. Setting $s = \mathrm{d}\rho^n$, we get a sequence $(y_n)_{n\in\mathbb{N}}$ of Y_α such that $\|y_n - y\|_i \to 0$ in the sharp topology. But

$$\|y-y_0\|_i \le \|y-y_n\|_i + \|y_n-y_0\|_i \le \|y-y_n\|_i + r_i. \tag{5.20}$$

By Thm. **??**, $[0,+\infty) = [[0,+\infty)_{\mathbb{R}}]$ is sharply closed, so that taking in (5.20) the limit for $n \to +\infty$ we obtain $\|y-y_0\|_i \le r_i$. To show that $y([t_0-\alpha, t_0+\alpha]) \subseteq H$, we can proceed as in Thm. 12. □

We can now prove Thm. 24:

Proof. We first note that $\overline{B_{r_i}(y_0)}$ is functionally compact by (**??**). The previous lemma and Thm. 12 yield that Y_α is Cauchy complete. For simplicity, set $K_\alpha := [t_0-\alpha, t_0+\alpha]$. The constant function $t \in K_\alpha \mapsto y_0 \in H$ shows that Y_α is not empty. Now let $T : Y_\alpha \to {}^{\rho}\mathcal{GC}^{\infty}(K_\alpha, {}^{\rho}\widetilde{\mathbb{R}}^d)$ be the operator such that, for every $y \in Y_\alpha$

$$T(y)(t) := y_0 + \int_{t_0}^{t} F(s, y(s))\mathrm{d}s \quad \forall t \in K_\alpha.$$

Let us note that Thm. **??** and Thm. **??** imply $T(y) \in {}^{\rho}\mathcal{GC}^{\infty}\left(K_\alpha, {}^{\rho}\widetilde{\mathbb{R}}^d\right)$ for all $y \in {}^{\rho}\mathcal{GC}^{\infty}\left(K_\alpha, {}^{\rho}\widetilde{\mathbb{R}}^d\right)$. Our goal is to show that our assumption on α allow to prove that T is a finite contraction, and hence to find a solution of equation (5.18) as a fixed point of T. Let us observe that, for every $y \in Y_\alpha$, we have

$$\|T(y) - y_0\|_i = \left\| \int_{t_0}^{t} F(s, y(s)) \mathrm{d}s \right\|_i \leq$$
$$\leq \max_{t \in K_\alpha} \int_{t_0}^{t} \|F(s, y(s))\|_i \, \mathrm{d}s \leq \alpha \cdot M_i \leq r_i.$$

Note that the existence of this maximum is guaranteed by Lem. ??. Moreover, for all $t \in K_\alpha$ we have $|T(y)(t) - y_0| \leq \|T(y) - y_0\|_0 \leq r_0$, hence $T(y)(t) \in \overline{B_{r_0}(y_0)} \subseteq H$. Therefore, $T : Y_\alpha \longrightarrow Y_\alpha$. To prove that T is a finite contraction on Y_α, let x, $y \in Y_\alpha$, $i \in \mathbb{N}$, and compute $\|T(y) - T(x)\|_i$:

$$\|T(y) - T(x)\|_i = \left\| \int_{t_0}^{t} [F(s, y(s)) - F(s, x(s))] \, \mathrm{d}s \right\|_i \leq$$
$$\leq \max_{t \in K_\alpha} \int_{t_0}^{t} \|F(s, y) - F(s, x)\|_i \, \mathrm{d}s \leq$$
$$\leq \max_{t \in K_\alpha} \int_{t_0}^{t} L_i \cdot \|x - y\|_i \, \mathrm{d}s \leq \alpha \cdot L_i \cdot \|x - y\|_i \,. \tag{5.21}$$

Therefore $T : Y_\alpha \longrightarrow Y_\alpha$ is a finite contraction because of our assumptions. The existence part of the conclusion follows from Banach theorem 17 (which yields existence and uniqueness in Y_α). To prove uniqueness in ${}^{\rho}\mathcal{GC}^{\infty}\left(K_\alpha, {}^{\rho}\widetilde{\mathbb{R}}^d\right)$, let $z \in {}^{\rho}\mathcal{GC}^{\infty}\left(K_\alpha, {}^{\rho}\widetilde{\mathbb{R}}^d\right)$ be another solution of (5.18). As in (5.21), we have $(1 - \alpha \cdot L_0) \cdot \|y - z\|_0 \leq 0$. But assumption (5.17) implies $\alpha \cdot L_0 \approx 0$, so that $0 < 1 - \alpha \cdot L_0$ and hence $\|y - z\|_0 \leq 0$. Thm. 8 gives hence $y = z$. □

Now, we can connect Thm. 24 and Cor. 21 to obtain clearer conditions for the existence of α.

Corollary 26. *Let $t_0 \in {}^{\rho}\widetilde{\mathbb{R}}$, $y_0 \in {}^{\rho}\widetilde{\mathbb{R}}^d$, β, $r_i \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ for all $i \in \mathbb{N}$. Set $K := [t_0 - \beta, t_0 + \beta]$, $H \supseteq \overline{B_{r_0}(y_0)}$ and let $F \in {}^{\rho}\mathcal{GC}^{\infty}(K \times H, {}^{\rho}\widetilde{\mathbb{R}}^d)$. Set*

$$M_i := \max_{(t,y) \in K \times \left(\overline{B_{r_i}(y_0)} \cap H\right)} |F(t, y)| > 0.$$

Assume that F is uniformly Lipschitz on

$$Y_\alpha := \{y \in {}^{\rho}\mathcal{GC}^{\infty}([t_0 - \alpha, t_0 + \alpha], H) \mid \|y - y_0\|_i \leq r_i \ \forall i \in \mathbb{N}\}$$

with constants $(L_{i\alpha})_{i \in \mathbb{N}}$, and that

$$\exists R \in {}^{\rho}\widetilde{\mathbb{R}}_{>0} \, \forall i \in \mathbb{N} : \ L_{i\alpha} \leq R. \tag{5.22}$$

Take $\alpha \in (0, \beta]$ such that

$$\exists a \in \mathbb{R}_{>0} : \ \alpha \leq \min\left(\frac{\mathrm{d}\rho^a}{R}, \frac{r_i}{M_i}\right). \tag{5.23}$$

Then there exists a unique solution $y \in {}^{\rho}\mathcal{GC}^{\infty}\left([t_0 - \alpha, t_0 + \alpha], {}^{\rho}\widetilde{\mathbb{R}}^d\right)$ of the Cauchy problem (5.18). In particular, if R is infinite or $R \in \mathbb{R}_{>0}$, then $\alpha \approx 0$; if $R \leq \mathrm{d}\rho^b$ for some $b \in \mathbb{R}_{>0}$ and β, $\frac{r_i}{M_i} \in \mathbb{R}$, then any standard real number $0 < \alpha \leq \min\left(\beta, \frac{r_i}{M_i}\right)$ satisfies (5.23).

Proof. Set

$$K_\alpha := [t_0 - \alpha, t_0 + \alpha]$$
$$M_{i\alpha} := \max_{(t,y)\in K_\alpha \times \left(\overline{B_{r_i}(y_0)}\cap H\right)} |F(t,y)|,$$

Since α satisfies (5.23) and $K_\alpha \subseteq K$, we have $\alpha \cdot M_{i\alpha} \le \alpha \cdot M_i \le r_i$, so

$$0 \le \alpha^n \cdot L^n_{i\alpha} \le \alpha^n \cdot R^n \le \mathrm{d}\rho^{an}.$$

Therefore $\lim_{n\to+\infty} \alpha^n \cdot L^n_{i\alpha} = 0$ and Thm. 24 yields existence and uniqueness of the solution in ${}^\rho\mathcal{GC}^\infty(K_\alpha, {}^\rho\widetilde{\mathbb{R}}^d)$.
Finally, if R is infinite or $R \in \mathbb{R}_{>0}$, then $0 < \alpha \le \frac{\mathrm{d}\rho^a}{R} \approx 0$. If $R \le \mathrm{d}\rho^b$ and $0 < \alpha \le \min\left(\beta, \frac{r_i}{M_i}\right)$ is a standard real number, then setting e.g. $a := \frac{b}{2}$, we also get $\alpha \le \mathrm{d}\rho^{-b/2} \le \frac{\mathrm{d}\rho^a}{R}$. □

It is well known that, for classical ODE on the real field $\mathbb{R}$, the semi-amplitude α can be estimated independently on the Lipschitz constant L, see e.g. [50]. For two reasons this result seems not repeatable in a simple way in this generalized framework: first, the classical proof uses terms of the form $e^{\alpha L}$ which find a corresponding in ${}^\rho\widetilde{\mathbb{R}}$ only assuming strong limitations on the product αL; second, to generalize the use of series from $\mathbb{R}$ to ${}^\rho\widetilde{\mathbb{R}}$, e.g. to prove the analogous of Weissinger theorem [50, Thm. 2.4], we need the notion of hyperseries in the ring ${}^\rho\widetilde{\mathbb{R}}$, see [20]. For these reasons, we present only Thm. 24, which is simpler and seems more general (we do not need to assume logarithmic conditions that guarantee the existence of terms like $e^{\alpha L}$), postponing the use of hyperfinite methods to a subsequent paper. An undesired consequence of this is that in each ODE we want to solve, we need to estimate the Lipschitz constants L_i for all $i \in \mathbb{N}$. We finally recall that the necessity to have a countable family of these constants is tied to the same definition of GSF, which involves all the derivatives.

5.3. **Infinite iterations.**

6. A first list of general examples

riprendere alcuni di questi esempi dopo i risultati sugli intervalli massimali di esistenza

In all the following examples, we use the same notations of Cor. 26, Cor. 21 and Thm. 24.

6.1. **Relationship with classical solutions.**

6.2. An ODE with a non-extensible infinitesimal solution interval. The fact that, in general, the semi-amplitude α of the time interval is only infinitesimal could be considered as a deficiency of this approach. On the contrary, this is a general fact of every non-Archimedean theory having at least one positive and invertible infinitesimal h. If fact, the Cauchy problem

$$\begin{cases} y' = -\frac{t}{1+y} \cdot \frac{1}{h} \\ y(0) = 0 \end{cases} \tag{6.1}$$

has solution $y(t) = -1 + \sqrt{1 - \frac{t^2}{h}}$ which is defined and smooth only in the infinitesimal interval $(-\sqrt{h}, \sqrt{h})$. Moreover, we have that $\lim_{t\to\pm\sqrt{h}} y'(t) = +\infty$ (in the

sharp topology) and this clearly shows that the solution cannot be extended (see also Thm. 36).
We recall that a non-Archimedean theory is a necessary consequence of describing generalized functions as set-theoretical functions, so that their free composition can be easily defined. If $h \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$, $h \approx 0$, and $F(t,y) = -\frac{t}{1+y} \cdot \frac{1}{h}$, then it is easy to see that, due to the presence of the infinite term $\frac{1}{h}$, each norm $\left\| F|_{[-a,a]\times\overline{B_r(0)}} \right\|_i$ is infinite, so that the constant R of (5.22) is necessarily infinite as well.
This simple example yields another general remark: On the basis of Cor. 22, a possible simple way to avoid the first factor in (5.12) is to consider sufficiently small radii r_j. For example, we can always trivially take infinitesimal radii $r_j \approx 0$. However, this would necessarily imply that the semi-amplitude $\alpha \approx 0$ because the radii r_j measure how much the solution is far from the initial condition y_0. Even if example (6.1) shows that in general a better result is not possible, in specific cases we can obtain better estimates both of the Lipschitz constants and of the radii r_j, as shown below.

6.3. **How to apply the local existence results.** Mainly in order to understand how to apply the previous Cor. 26 of local existence, let us consider the following ${}^{\rho}\widetilde{\mathbb{R}}$-linear Cauchy problem:

$$\begin{cases} y'(t) = A(t) \cdot y(t); \\ y(t_0) = c, \end{cases} \tag{6.2}$$

where $A \in {}^{\rho}\mathcal{GC}^{\infty}([t_0-\beta, t_0+\beta], {}^{\rho}\widetilde{\mathbb{R}}^{d\times d})$ and $c \in {}^{\rho}\widetilde{\mathbb{R}}^d$. Since GSF are closed with respect to composition, and $x \mapsto x - t_0$ is always a GSF, even when t_0 is an infinite number (e.g., this does not hold using Colombeau generalized functions), without loss of generality, we can assume that $t_0 = 0$. using the notations of Cor. 26, we have $F(t,y) = A(t) \cdot y$ and $K = [-\beta, \beta]$, $H = {}^{\rho}\widetilde{\mathbb{R}}^d$, $M_i = \max_{\substack{-\beta \le t \le \beta \\ |y-c| \le r_i}} |A(t) \cdot y|$.

The basic condition we need to satisfy is (5.23). We therefore start by estimating M_i and by accordingly choosing the radii $r_i \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$, in order to simplify the analysis of (5.23). If $y \in \overline{B_{r_i}(c)}$ and $t \in K$, we have

$$\begin{aligned} |A(t) \cdot y| &\le |A(t) \cdot y - A(t) \cdot c)| + \\ &\quad + |A(t) \cdot c| \le \\ &\le \|A\|_0 \, r_i + |c| \cdot \|A\|_0 = \|A\|_0 \, (r_i + |c|) \end{aligned} \tag{6.3}$$

where the norms $\|-\|_0$ are evaluated on the functionally compact set $K = [-\beta, \beta]$. We assume that $\|A\|_0 > 0$, $M_i > 0$, so that $M_i \le \|A\|_0 \, (r_i + |c|)$ and setting $r_i := \max(|c|, 1) > 0$, from (6.3) we get $\frac{r_i}{M_i} \ge \frac{1}{2\|A\|_0}$, which will be useful to evaluate (5.23).

In the second step, we compute the Lipschitz constants for our particular problem, hoping to obtain better values with respect to the general estimates of Cor. 21. In our case, a direct calculations gives

$$\begin{aligned} \frac{d^i}{dt^i} F(t, y(t)) &= \frac{d^i}{dt^i} \left(A(t) y(t) \right) + b^{(i)}(t) = \\ &= \sum_{k=0}^{i} \binom{i}{k} A^{(i-k)}(t) y^{(k)}(t) + b^{(i)}(t). \end{aligned}$$

Therefore

$$\left|\frac{d^i}{dt^i}F(t,y(t)) - \frac{d^i}{dt^i}F(t,x(t))\right| \le \sum_{k=0}^{i}\binom{i}{k}\|A\|_i\cdot\|y-x\|_i =$$
$$= 2^i\|A\|_i\cdot\|y-x\|_i\,.$$

So, we can set $L_i = 2^i\|A\|_i$ and we can fully analyze the following cases:

(i) At least one norm $\|A\|_i$ is infinite, i.e.,

$$\exists n_i\in\mathbb{N}_{>1}:\ \mathrm{d}\rho^{-n_i+1}\le\|A\|_i<\mathrm{d}\rho^{-n_i} \tag{6.4}$$

but all the exponents n_i are bounded, i.e.

$$\exists n\in\mathbb{N}_0\,\forall i\in\mathbb{N}:\ n_i<n. \tag{6.5}$$

Since $\|A\|_i\le\|A\|_{i+1}$, without loss of generality we can assume that (6.4) holds for all $i\in\mathbb{N}$ sufficiently large. Therefore $L_i=2^i\|A\|_i\le 2^i\mathrm{d}\rho^{-n_i}<\mathrm{d}\rho^{-n}=:R$. We are thus in the first case of Cor. 26, and for any $a\in\mathbb{R}_{>0}$ we can take $\alpha=\min\left(\mathrm{d}\rho^{n+a},\frac{1}{2\|A\|_0}\right)\approx 0$ to satisfy (5.23). In general this infinitesimal solutions cannot be enlarged to a greater symmetric interval, as shown by the trivial ODE $y'=\frac{1}{\mathrm{d}\rho}y$, $y(0)=1$, whose maximal domain is $\{t\in{}^{\rho}\widetilde{\mathbb{R}}\mid\exists N\in\mathbb{N}:\ t\ge N\mathrm{d}\rho\log\mathrm{d}\rho\}\supseteq[-\mathrm{d}\rho^{1+a},\mathrm{d}\rho^{1+a}]$ for all $a\in\mathbb{R}_{>0}$.

(ii) All the norms $\|A\|_i$ are finite i.e.

$$\exists S\in\mathbb{R}_{>0}\,\forall i\in\mathbb{N}:\ \|A\|_i\le S. \tag{6.6}$$

We are hence in the second case of Cor. 26. We can take, e.g., $R:=\mathrm{d}\rho^{-r}$ where $r\in\mathbb{R}_{>0}$ so that $L_i=2^i\|A\|_i\le 2^iS\le R=\mathrm{d}\rho^{-r}$. Condition (5.23) is thus satisfied for all $a\in\mathbb{R}_{>0}$ and taking $\alpha=\min\left(\mathrm{d}\rho^{r+a},\frac{1}{2\|A\|_0}\right)=\mathrm{d}\rho^{r+a}$ because of (6.6). The solution is therefore defined on the union $\bigcup_{b\in\mathbb{R}_{>0}}[-\mathrm{d}\rho^b,+\mathrm{d}\rho^b]$. As shown in Rem. 23, we can easily switch to an equivalent Cauchy problem in order to avoid the term 2^i, which really forces us to take an infinite R. Without loss of generality, we can hence assume that (6.2) is locally Lipschitz with $L_i=\|A(2\cdot -)\|_i$. This shows that the solution of this sub-example is very unsatisfactory because for $\alpha\in\mathbb{R}_{>0}$ sufficiently small, we can get $\alpha L_i<1$. As mentioned at the beginning of Sec. 5, this can be better solved using the notion of hyperfinite contraction, or the results of Sec. 7.3 about maximals sets of existence, see e.g. example 40.

(iii) The norms $\|A\|_i$ are all uniformly infinitesimals, i.e.

$$\exists n\in\mathbb{N}_{>0}\,\forall i\in\mathbb{N}:\ \|A\|_i\le\mathrm{d}\rho^n. \tag{6.7}$$

This is clearly a subcase of the previous one, which is therefore meaningfully different from the present one only if $\|A\|_j=S$ for some $j\in\mathbb{N}$. Taking any $0<h<n$, we have $L_i=2^i\|A\|_i\le 2^i\mathrm{d}\rho^n<\mathrm{d}\rho^{n-h}=:R$. We are thus in the third case of Cor. 26 and hence any $0<\alpha\le\min\left(\beta,\frac{1}{2\|A\|_0}\right)$ satisfies (5.23). Finally note that for any finite β, equation (6.7) yields $\min\left(\beta,\frac{1}{2\|A\|_0}\right)=\beta$ and hence the solution $y\in{}^{\rho}\mathcal{GC}^\infty([-\beta,\beta],{}^{\rho}\widetilde{\mathbb{R}}^d)$.

(iv) For all i sufficiently large, all the norms $\|A\|_i$ are infinite, i.e. (6.4) holds, but the exponents n_i are unbounded, i.e. (6.5) is false. This is the case, for example, of the Cauchy problem: $y'=\delta'\cdot y$, $y(0)=1$. More precisely, using the notations for the embeddings of Schwartz distributions of Thm. ??, the ODE $y'(t) = \iota_{\mathbb{R}}^{b}\left(\delta'\right)(t) \cdot y(t)$ would necessarily have the solution $y(t) = e^{\iota_{\mathbb{R}}^{b}(\delta)(t)} + c$ for some $c \in {}^{\rho}\widetilde{\mathbb{R}}$. If we want that $\iota^{b}|_{\mathcal{C}^{\infty}}$ coincides with the inclusion of sheaf $\mathcal{C}^{\infty}(\Omega) \subseteq {}^{\rho}\mathcal{GC}^{\infty}\Omega, \mathbb{R})$, then we must take $b \geq \mathrm{d}\rho^{-a}$ for some $a \in \mathbb{R}_{>0}$ (see Thm. ??.??). But then $y(0) = e^{\iota_{\mathbb{R}}^{b}(\delta)(0)} + c = e^{b\mu(0)} + c = e^{\mathrm{d}\rho^{-a}} + c \notin {}^{\rho}\widetilde{\mathbb{R}}$, where μ is the fixed Colombeau mollifier. Note that the function $F(t,y) = \iota_{\mathbb{R}}^{b}(\delta')(t) \cdot y$ is uniformly locally Lipschitz with constants $L_i = b(1+b)^i$. Therefore, if we take $b \geq \mathrm{d}\rho^{-a}$, it is easy to prove that

$$\forall b \in \mathbb{R}_{>0}\, \exists i \in \mathbb{N} : \mathrm{d}\rho^{bn} \cdot L_i^n \not\to 0.$$

Therefore, condition (5.17) never holds for any $\alpha \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$.

All this implies that we only have two possibilities: either we negate the sheaf inclusion of smooth functions (and we thus remain with the inclusion of analytic functions, see Thm. ??.??, and only the inclusion up to infinitesimal of smooth functions, see Thm. ??.??), or we take another gauge σ such that $e^{\mathrm{d}\rho^{-a}} \in {}^{\sigma}\widetilde{\mathbb{R}}$, e.g. $\sigma_\varepsilon := e^{-e^{1/\rho_\varepsilon}}$. The second choice would imply that we are indeed able to locally solve $y'(t) = \delta_b'(t) \cdot y(t)$, but where $\delta_b(t) := b\mu(bt)$ is not the σ-embedding (preserving the inclusion of smooth functions) of δ in ${}^{\sigma}\mathcal{GC}^{\infty}(\mathbb{R},\mathbb{R})$, which would instead be e.g. $\iota_{\mathbb{R}}^{\mathrm{d}\sigma^{-1}}(\delta)$. These are unavoidable limitations of this approach.

(i) Can we say that the previous cases exhaust all the possible instances? Since in ${}^{\rho}\widetilde{\mathbb{R}}$ we can have generalized numbers like $\left[\sin\left(\frac{1}{\rho_\varepsilon}\right)\right]$, at a first sight the answer seems to be negative. On the other hand, by recalling the interpretation of Robinson-Colombeau generalized numbers as dynamical numbers in ε (see introductions to Sec. ?? and Sec. 3), the following notion of sample of a point can be of help in the analysis of the remaining instances:

Definition 27.

(i) Let $x,\ x' \in {}^{\rho}\widetilde{\mathbb{R}}^{n}$, then we say that x' is a *sample* of x if there exist representatives $x = [x_\varepsilon]$ and $x' = [x'_\varepsilon]$ such that

$$\forall\varepsilon\, \exists\hat{\varepsilon} \leq \varepsilon :\ x'_\varepsilon = x_{\hat{\varepsilon}}. \tag{6.8}$$

(ii) We say that $X \subseteq {}^{\rho}\widetilde{\mathbb{R}}^{n}$ *contains all the samples of its points* if for all $x \in X$, $x' \in {}^{\rho}\widetilde{\mathbb{R}}^{n}$, if x' is a sample of x, then also $x' \in X$.

Note that, as a consequence of the condition $\bar{\varepsilon} \leq \varepsilon$ in 6.8, we can define $\bar{\varepsilon}(\varepsilon) := \sup\left\{0 < \hat{\varepsilon} \leq \varepsilon \mid x'_\varepsilon = x_{\hat{\varepsilon}}\right\}$ so that we get

$$\begin{aligned}
&\forall\varepsilon \in I :\ 0 < \bar{\varepsilon}(\varepsilon) \leq \varepsilon\\
&\bar{\varepsilon} : I \longrightarrow I\\
&\bar{\varepsilon} \to 0^{+}\\
&x'_\varepsilon = x_{\bar{\varepsilon}(\varepsilon)} \quad \forall\varepsilon \in I,
\end{aligned}$$

therefore, we can roughly state that "being a sample of a point is a property for $\varepsilon \to 0^{+}$".

This notion has interesting properties. For example, it can be easily proved (see [20]) that if X contains all the samples of its points, then the GSF

$f \in {}^{\rho}\mathcal{GC}^{\infty}(X,Y)$ is zero if and only if $f(x)=0$ for all near-standard and for all infinite points $x \in X$. It is important to note that the set $\mathrm{c}(\Omega)$ of compactly supported points in $\Omega \subseteq \mathbb{R}^n$ (see (**??**)) always contains all the samples of its points, therefore every equality between Schwartz distributions can be tested in this way. Analogously, Def. **??** needs to be tested only at all near-standard and all infinite points $x \in X$ (see [20]). Moreover, from Lem. **??** we can also easily derive the following partial replace of the trichotomy law: for all $x \in {}^{\rho}\widetilde{\mathbb{R}}$ there exist x_1, x_2, x_3 samples of x such that

$$x_1 = 0 \quad \text{or} \quad x_2 < 0 \quad \text{or} \quad x_3 > 0,$$

and the inequality $x > 0$ holds if and only if for all x' sample of x we have

$$x' \text{ is near-standard or } x' \text{ is infinite} \quad \Longrightarrow \quad x' > 0.$$

Since from every bounded ε-net we can always extract a convergent subnet, we can say that the previous cases exhaust all the possible instances, in the following weak sense: for all $\bar{\varepsilon}$ there exist a sample A' of $\|A\|_i = [\|A_\varepsilon\|_i] \in {}^{\rho}\widetilde{\mathbb{R}}$ such that $\|A_{\bar{\varepsilon}}\|_i = A'_{\bar{\varepsilon}}$ and

$$A' \text{ is near-standard or } A' \text{ is infinite.}$$

We can therefore roughly state that all the remaining instances of (6.2) are ε-mixing of some of the previously analyzed cases.

6.4. **First order scalar ODEs by quadrature.** Clearly, the calculus of GSF is sufficiently reach to reformulate the usual proof of existence and uniqueness of solutions of first order 1-dimentional affine ODEs. In fact, we have the following

Theorem 28. *Let $z \in {}^{\rho}\mathcal{GC}^{\infty}([t_0-\alpha, t_0+\alpha], {}^{\rho}\widetilde{\mathbb{R}})$ be the solution of the linear Cauchy problem*

$$\begin{cases} z'(t) = -A(t)\cdot z(t); \\ z(t_0) = k, \end{cases} \tag{6.9}$$

where $A \in {}^{\rho}\mathcal{GC}^{\infty}([t_0-\alpha, t_0+\alpha], {}^{\rho}\widetilde{\mathbb{R}})$ and $k \in {}^{\rho}\widetilde{\mathbb{R}}$. Assume that $y \in {}^{\rho}\mathcal{GC}^{\infty}([t_0-\alpha, t_0+\alpha], {}^{\rho}\widetilde{\mathbb{R}})$ solves the problem

$$\begin{cases} y'(t) = A(t)\cdot y(t) + b(t); \\ y(t_0) = c, \end{cases} \tag{6.10}$$

where $b \in {}^{\rho}\mathcal{GC}^{\infty}([t_0-\alpha, t_0+\alpha], {}^{\rho}\widetilde{\mathbb{R}})$ and $c \in {}^{\rho}\widetilde{\mathbb{R}}$. Then it results

$$z(t)\cdot y(t) = kc + \int_{t_0}^{t} z\cdot b \quad \forall t \in [t_0-\alpha, t_0+\alpha]. \tag{6.11}$$

In particular, if for all $t \in [t_0-\alpha, t_0+\alpha]$ there exists $N \in \mathbb{N}$ such that

$$\int_{t_0}^{t} A \geq N \log \mathrm{d}\rho$$

then the GSF

$$z(t) = k \exp\left(-\int_{t_0}^{t} A\right) \quad t \in [t_0-\alpha, t_0+\alpha],$$

satisfies 6.9, and hence, if k is invertible, we have

$$y(t) = \frac{1}{z(t)}\cdot\left[kc + \int_{t_0}^{t} z\cdot b\right] \quad \forall t \in [t_0-\alpha, t_0+\alpha].$$

Proof. By multiplying (6.10) by $z(t)$ and using the Leibniz rule for GSF, we get $(z\cdot y)' = z\cdot b$. Integrating between t_0 and t and using Thm. **??** we obtain (6.11). $\square$

6.5. **Polynomial singular ODEs by local existence results.** Let us now consider an arbitrary first order scalar polynomial ODE

$$\begin{cases} y'(t) = \sum_{i=0}^{n} a_i(t)\cdot y(t)^i \\ y(0) = y_0 \end{cases} \tag{6.12}$$

where $n \in \mathbb{N}_{>0}$, $a_i \in {}^{\rho}\mathcal{GC}^{\infty}([-\beta,\beta], {}^{\rho}\widetilde{\mathbb{R}})$ and $|y_0| \in {}^{\rho}\widetilde{\mathbb{R}}$ is invertible. Without loss of generality, we can always assume $|y_0| = 1$. In fact, if $c \in {}^{\rho}\widetilde{\mathbb{R}}$ is invertible, and we can solve $z'(t) = \sum_{i=0}^{n} a_i(t)z(t)^i$, $z(0) = \frac{y_0}{c}$, then $y := c\cdot z$ solves $y'(t) = \sum_{i=0}^{n} \frac{a_i(t)}{c^{i-1}} y(t)^i$, $y(0) = y_0$. Therefore, it suffices to set $c = |y_0|$ to reduce to the case $|y_0| = 1$. Using the notations of Cor. 26, we set $K := [-\beta, \beta]$, $H := {}^{\rho}\widetilde{\mathbb{R}}^d$, $F(t,y) := \sum_{i=0}^{n} a_i(t)\cdot y^i$ so that $F \in {}^{\rho}\mathcal{GC}^{\infty}(K \times {}^{\rho}\widetilde{\mathbb{R}}, {}^{\rho}\widetilde{\mathbb{R}})$, and $M_i := \max_{\substack{-\beta\le t\le\beta \\ y\in\overline{B_{r_i}(y_0)}}} |F(t,y)|$. In order to properly choose the radii r_i, we start by estimating M_i. For $t \in K$ and $y \in \overline{B_{r_i}(y_0)}$, we have

$$|F(t,y)| \le \sum_{i=0}^{n} |a_i(t)|\,|y|^i.$$

We set $N_0 := \sum_{j=0}^{n} \|a_j\|_0$, so that using $|y_0| = 1$ we get

$$\begin{aligned} |F(t,y)| &\le N_0 \cdot \sum_{i=0}^{n} \left(|y+y_0| + |y_0|\right)^i = \\ &= N_0 \cdot \sum_{i=0}^{n}\sum_{k=0}^{i} \binom{i}{k} |y-y_0|^k \le \\ &\le N_0 \cdot \sum_{i=0}^{n}\sum_{k=0}^{i} \binom{i}{k} r_i^k = N_0 \cdot \sum_{i=0}^{n} (r_i+1)^i. \end{aligned}$$

Therefore, if we choose $r_0 = r_i = 1$ for all $i \in \mathbb{N}$, we obtain

$$|F(t,y)| \le N_0 \frac{(r_0+1)^{n+1}-1}{r_0}$$
$$\frac{r_i}{M_i} \ge \frac{1}{N_0(2^{n+1}-1)}.$$

Now, we estimate Lipschitz constants:

$$\frac{d^i}{dt^i} F(t, y(t)) = \sum_{k=0}^{n}\sum_{j=0}^{i} \binom{i}{j} a_k^{(i-j)}(t) \frac{d^j}{dt^j}\left[y(t)^k\right].$$

We estimate the j-th derivative of $y(t)^k$ by using univariate Faà di Bruno formula. Since $\frac{d^j}{dt^j}\left(z^k\right) = k(k-1)\cdot\ldots\cdot(k-j+1)z^{k-j} = \binom{k}{j} j! z^{k-j}$ for $j \le k$ and 0 otherwise, we get

$$\frac{d^j}{dt^j}\left[y(t)^k\right] = \sum_{m} \frac{j!}{m!} \binom{k}{|m|} |m|! \cdot y(t)^{k-|m|} \cdot \prod_{p=1}^{j} \left(\frac{y^{(p)}(t)}{p!}\right)^{m_p},$$

where the sum is extended to all multi-indices $(m_1,\dots,m_j)\in\mathbb{N}^j$ such that $1\cdot m_1+2\cdot m_2+\ldots+j\cdot m_j=j$ and $|m|\le k$. Simplifying the notations, we can write

$$\frac{d^j}{dt^j}\left[y(t)^k\right]=\sum_m y(t)^{k-|m|}\cdot\prod_{p=1}^{j}A_{jmp}\cdot y^{(p)}(t)^{m_p},$$
$$A_{jmp}:=\left[\frac{j!}{m!}\binom{k}{|m|}|m|!\right]^{1/j}\cdot\frac{1}{(p!)^{m_p}}\in\mathbb{R}_{>0},$$

and hence

$$\left|\frac{d^j}{dt^j}\left[y(t)^k\right]-\frac{d^j}{dt^j}\left[x(t)^k\right]\right|\le$$
$$\sum_m\prod_{p=1}^{j}A_{jmp}\cdot\left|y(t)^{k-|m|}\cdot\prod_{p=1}^{j}y^{(p)}(t)^{m_p}-x(t)^{k-|m|}\cdot\prod_{p=1}^{j}x^{(p)}(t)^{m_p}\right|.$$

But $\left|y^{(p)}(t)^c\right|\le\|y\|_p^c\le\Big(\|y-y_0\|_p+|y_0|\Big)^c\le(r_p+|y_0|)^c=2^c$ for all $c\in\mathbb{R}_{\ge0}$ and $p\in\mathbb{N}$ because the function y belongs to the set defined in (5.4). By the mean value theorem for several variables, we have

$$\left|y(t)^{k-|m|}\cdot\prod_{p=1}^{j}y^{(p)}(t)^{m_p}-x(t)^{k-|m|}\cdot\prod_{p=1}^{j}x^{(p)}(t)^{m_p}\right|\le$$
$$\le\left[(k-|m|)2^{k-|m|-1}2^{|m|}+2^{k-|m|}|m|2^{|m|-1}\right]\cdot\Big(\|y-x\|_0+\|x-y\|_p\Big)\le$$
$$\le k2^k\cdot\|x-y\|_p$$

And finally this yields

$$\left|\frac{d^j}{dt^j}\left[y(t)^k\right]-\frac{d^j}{dt^j}\left[x(t)^k\right]\right|\le$$
$$\le\|y-x\|_j\cdot\sum_m k^j2^{jk}\prod_{p=1}^{j}A_{jmp}=:$$
$$=:\|y-x\|_j\cdot C_{jk}$$

$$\left|\frac{d^i}{dt^i}F(t,y(t))-\frac{d^i}{dt^i}F(t,x(t))\right|\le\sum_{k=0}^{n}\sum_{j=0}^{i}\binom{i}{j}\|a_k\|_{i-j}C_{jk}\|y-x\|_j\le$$
$$\le\|y-x\|_i\sum_{k=0}^{n}\|a_k\|_i2^iD_{ik}\le$$
$$\le\|y-x\|_i2^iN_iD_i,$$

where $D_{ik}:=\sum_{j=0}^{i}C_{jk}\in\mathbb{R}_{>0}$, $D_i:=\sum_{k=0}^{n}D_{ik}\in\mathbb{R}_{>0}$ and $N_i:=\sum_{k=0}^{n}\|a_k\|_i\in{}^{\rho}\widetilde{\mathbb{R}}_{\ge0}$.

Now, we can proceed by analyzing the following main cases:

(i) At least one N_i is infinite, i.e.,

$$\exists n_i\in\mathbb{N}_{>1}:\ \mathrm{d}\rho^{-n_i+1}\le N_i<\mathrm{d}\rho^{-n_i}\tag{6.13}$$

but all the exponents n_i are bounded, i.e.

$$\exists n \in \mathbb{N}_0 \, \forall i \in \mathbb{N} : \; n_i < n.$$

Setting $R := \mathrm{d}\rho^{-n}$, we are in the first case of Cor. 26, and for any $a \in \mathbb{R}_{>0}$ we can take $\alpha = \min\left(\mathrm{d}\rho^{n+a}, \frac{1}{N_0(2^{n+1}-1)}\right) \approx 0$ to satisfy (5.23).

(ii) All the terms N_i are finite i.e.

$$\exists S \in \mathbb{R}_{>0} \, \forall i \in \mathbb{N} : \; N_i \le S. \tag{6.14}$$

We are hence in the second case of Cor. 26. We can take, e.g., $R := \mathrm{d}\rho^{-r}$ where $r \in \mathbb{R}_{>0}$. Condition (5.23) is thus satisfied for all $a \in \mathbb{R}_{>0}$ and taking $\alpha = \min\left(\mathrm{d}\rho^{r+a}, \frac{1}{N_0(2^{n+1}-1)}\right) = \mathrm{d}\rho^{r+a}$ because of (6.14). The solution is therefore defined on the union $\bigcup_{b\in\mathbb{R}_{>0}}[-\mathrm{d}\rho^b, +\mathrm{d}\rho^b]$.

(iii) The terms N_i are all uniformly infinitesimals, i.e.

$$\exists n \in \mathbb{N}_{>0} \, \forall i \in \mathbb{N} : \; N_i \le \mathrm{d}\rho^n. \tag{6.15}$$

Taking any $0 < h < n$, we can set $R := \mathrm{d}\rho^{n-h}$. We are in the third case of Cor. 26 and hence any $0 < \alpha \le \min\left(\beta, \frac{1}{N_0(2^{n+1}-1)}\right)$ satisfies (5.23). For any finite β, equation (6.15) yields $\min\left(\beta, \frac{1}{N_0(2^{n+1}-1)}\right) = \beta$ and hence the solution $y \in {}^\rho\mathcal{GC}^\infty([-\beta,\beta], {}^\rho\widetilde{\mathbb{R}}^d)$.

(iv) For all i sufficiently large, all the terms N_i are infinite, i.e. (6.13) holds, but the exponents n_i are unbounded, i.e. (6.5) is false. As seen above in the linear case, this leads to an ODE which is solvable only changing the gauge ρ, but at the price of problems related to the embedding of smooth functions.

6.6. **Polynomial singular ODEs by quadrature: Bernoulli equation.** A Bernoulli ODE

$$\begin{cases} y'(t) = a_1(t)y(t) + a_n(t)y(t)^n \\ y(0) = y_0 \end{cases} \qquad \forall t \in [-\alpha,\alpha] \tag{6.16}$$

is classically reduced to a linear ODE. The possibility to extend to ${}^\rho\widetilde{\mathbb{R}}$ these trivial steps, depends on the invertibility of the values $y(t) \in {}^\rho\widetilde{\mathbb{R}}$ for t belonging to a sharp neighbourhood of y_0. If we assume that $y_0 \in {}^\rho\widetilde{\mathbb{R}}$ is invertible, it is plausible to expect the following

Theorem 29. *Let $\alpha \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ and $f \in {}^\rho\mathcal{GC}^\infty([t_0-\alpha, t_0+\alpha], {}^\rho\widetilde{\mathbb{R}})$ be a GSF such that $f(t_0)$ is invertible, then there exists $\beta \in (0,\alpha]$ such that $f(t)$ is invertible for all $t \in [t_0-\beta, t_0+\beta]$.*

Proof. By contradiction, let us assume that

$$\forall \beta \in (0,\alpha] \, \exists t \in [t_0-\beta, t_0+\beta] : \; f(t) \text{ is not invertible.}$$

Then setting $\beta = \min(\alpha, \mathrm{d}\rho^n)$, $n \in \mathbb{N}_{>0}$, we obtain the existence of a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \in [t_0 - \mathrm{d}\rho^n, t_0 + \mathrm{d}\rho^n]$ and $f(t_n)$ is not invertible for $n \in \mathbb{N}_{>0}$ sufficiently large. Therefore, (**??**) and continuity of GSF (Thm. **??**.**??**), yield

$$\lim_{n\to+\infty} |f(t_n)| = |f(t_0)| > \mathrm{d}\rho^q$$

for some $q \in \mathbb{R}_{>0}$, because of Lem. **??**. For n sufficiently large, we thus get

$$\frac{1}{2}\mathrm{d}\rho^q < |f(t_0)| - \frac{1}{2}\mathrm{d}\rho^q < |f(t_n)| < |f(t_0)| + \frac{1}{2}\mathrm{d}\rho^q.$$

Once again, Lem. **??** implies then that $f(t_n)$ must be invertible, and this yields a contradiction. $\square$

Therefore, if (6.16) holds and y_0 is invertible, then for some $\beta \in (0,\alpha]$, $y(t)$ is invertible for all $t \in [-\beta,\beta]$. Since $n \in \mathbb{N}_{>1}$, we can hence set $w(t) := y(t)^{1-n}$ for all $t \in [-\beta,\beta]$. The GSF w satisfies the linear Cauchy problem

$$\begin{cases} w'(t) = (1-n)a_1(t)w(t) + (1-n)a_n(t) \\ w(0) = y_0^{1-n} \end{cases} \quad \forall t \in [-\beta,\beta].$$

This can be locally solved using the results of Sec. 6.3. Otherwise, assuming that for all $t \in [-\beta,\beta]$ there exists $N \in \mathbb{N}$ such that

$$\int_{t_0}^t a_1 \ge N \log \mathrm{d}\rho,$$

then

$$\begin{aligned} z(t) &= y_0^{1-n} \exp\left(\int_0^t (n-1)a_1(s)\mathrm{d}s\right) \\ y(t) &= z(t)^{\frac{1}{n-1}} \left[y_0^{2-n} + (1-n)\int_0^t z(s)a_n(s)\mathrm{d}s\right]^{\frac{1}{1-n}} \quad \forall t \in [-\beta,\beta] \end{aligned}$$

solve (6.16).

6.7. **Nonlinear ODEs which are not solvable in any gauge.** Examples of nonlinear ODEs that are not solvable using the present approach and, at the same time, preserving good properties of the embedding of smooth functions, are

$$\begin{cases} y'(t) = \delta\left(y(t)\right) \\ y(0) = 0 \end{cases} \qquad \begin{cases} y'(t) = e^{\delta(0)y(t)} \\ y(0) = c. \end{cases} \tag{6.17}$$

Let us start with the former by showing that Thm. 24 cannot be applied. With the usual notations, we have $F(y) = \delta(y)$ for all $y \in {}^\rho\widetilde{\mathbb{R}}$, $K := [-\alpha,\alpha]$, $H := {}^\rho\widetilde{\mathbb{R}}$. If we use the embedding determined by the infinite generalized number $b \in {}^\rho\widetilde{\mathbb{R}}$ and by the Colombeau mollifier μ, then $\delta(t) = b\mu(bt)$ and hence $\delta^{(i)}(0) = b^{i+1}\mu^{(i)}(0)$. Since μ is an even function (see Lem. **??**), we have

$$\begin{aligned} \delta^{(2i+1)}(0) &= 0 \\ \delta^{(2i)}(0) &= b^{2i+1}\mu^{(2i)}(0). \end{aligned} \tag{6.18}$$

We assume to have chosen μ so that $\mu^{(2i)}(0)$ is not eventually zero. We have

$$\begin{aligned} \frac{d^i}{dt^i}F\left[y(t)\right] &= \sum_m \frac{i!}{m!}\delta^{(|m|)}(y(t))\prod_{j=1}^i \left(\frac{y^{(j)}(t)}{j!}\right)^{m_j} =:, \\ &=: \sum_m \delta^{(|m|)}(y(t))\prod_{j=1}^i y^{(j)}(t)^{m_j} C_{ijm}, \end{aligned} \tag{6.19}$$

where the sum is extended to all multi-indices $(m_1,\ldots,m_i)$ such that $1m_1 + 2m_2 + \ldots + im_i = i$ and $C_{ijm} := \left(\frac{i!}{m!}\right)^{1/i}\frac{1}{(j!)^{m_j}}$. Let us assume that F is uniformly Lipschitz on the space

$$Y = \left\{y \in {}^\rho\mathcal{GC}^\infty(K,H) \mid \|y\|_i \le r_i \ \forall i \in \mathbb{N}\right\}$$

with Lipschitz constant $(L_i)_{i\in\mathbb{N}}$. Assume that β, r_i and α satisfy

$$\begin{aligned} &0<\beta b\le r_i \\ &0<\alpha\le 1, \end{aligned} \tag{6.20}$$

and set $y(t):=\beta bt$, $x(t):=0$ for all $t\in K$. We want to choose β so as to show that condition (5.17) never hold. We have $|y(t)|\le\alpha\beta b\le\beta b\le r_0$ because of (6.20). Therefore, $\|y\|_0\le r_0$. Similarly, we have $\|y\|_i\le r_i$ for all $i\in\mathbb{N}$. This shows that x, $y\in Y$ and that $\|y-x\|_i=\|y-x\|_0=\beta b$. Lipschitz condition and (6.19) for $i\ge 1$ yield

$$\begin{aligned} \beta bL_i &\ge \left|\sum_m \delta^{(|m|)}(0)\prod_{j=1}^i y^{(j)}(0)C_{ijm}\right| = \\ &= \left|\sum_m \delta^{(|m|)}(0)\beta^{m_1}b^{m_1}C_{i1m_1}\right| \ge \\ &\ge \delta^{(i)}(0)\beta^i b^i. \end{aligned}$$

This inequality and (6.18) finally show that

$$L_{2i}\ge b^{4i}\beta^{2i-1}\mu^{(2i)}(0).$$

Now, since α, $\beta>0$, we can consider a, $c\in\mathbb{R}_{>0}$ such that $\alpha\ge \mathrm{d}\rho^a$ and $\beta\ge\mathrm{d}\rho^c$. Moreover, we analyse only the case $b=\mathrm{d}\rho^{-1}$. We thus get

$$\alpha^n\cdot L_i^n\ge \mathrm{d}\rho^{an}\mathrm{d}\rho^{-4ni}\mathrm{d}\rho^{2cni-cn}\mu^{(2i)}(0)^n.$$

Take $i\in\mathbb{N}$ so that $4i>a$, then $4i-a>0>-c$ and hence $an-4ni-cn<0$. If we take $c<\frac{4i-a}{2i-1}$ we have that $an-4ni+2cni-cn<0$ and therefore $\alpha^n\cdot L_i^n\not\to 0$. Finally, for i sufficiently large, the exponent c can be taken close to 2 and hence $\beta b=\mathrm{d}\rho^c\mathrm{d}\rho^{-1}\approx 0$, which gives a lower bound to r_i since we assumed (6.20). We therefore proved that

$$\forall a\in\mathbb{R}_{>0}:\ \alpha\ge\mathrm{d}\rho^a\ \Rightarrow\ \exists c,i:\ \alpha^nL_i^n\ge\mathrm{d}\rho^{an-4ni+2cni-cn}\mu^{(2i)}(0)\not\to 0.$$

This shows that Thm. 24 cannot be applied.
It can be easily proved that the GSF F is locally Lipschitz with constants $L_i=D_i\mathrm{d}\rho^{-2i}$, $D_i\in\mathbb{R}_{>0}$. We can thus note that, once again, changing the gauge by using e.g. $\sigma_\varepsilon:=e^{-\frac{1}{\rho_\varepsilon}}$ and taking $\mathrm{d}\rho^{2i}>\alpha\ge\mathrm{d}\sigma>0$, we have that $\alpha^nL_i^n\to 0$ in ${}^{\sigma}\widetilde{\mathbb{R}}$ and hence the equation $y'(t)=\iota_{\mathbb{R}}^{\mathrm{d}\rho^{-1}}(\delta)(y(t))$ is solvable in the interval $[-\alpha,\alpha]\subseteq{}^{\sigma}\widetilde{\mathbb{R}}$. However, the embedding $\iota_{\mathbb{R}}^{\mathrm{d}\rho^{-1}}$ preserves analytic functions with respect to the equality in ${}^{\sigma}\widetilde{\mathbb{R}}$, but smooth functions only up to infinitesimal of the form $C\cdot\mathrm{d}\rho^n$ for $C\in\mathbb{R}_{>0}$ and for all $n\in\mathbb{N}$ (see Thm. **??**), which does not correspond to equality in ${}^{\sigma}\widetilde{\mathbb{R}}$.

Analogously, we can deal with the second example in (6.17), where for simplicity we consider the embedding $\iota_{\mathbb{R}}^{\mathrm{d}\rho^{-1}}$. In that case we have $F(t,y):=e^{\delta(0)y}$ which is hence defined on $X:=\left\{y\in{}^{\rho}\widetilde{\mathbb{R}}\mid\exists N\in\mathbb{N}:\ y\le -N\mathrm{d}\rho\log\mathrm{d}\rho\right\}$. Since $X\supseteq[-\mathrm{d}\rho^{1+a},\mathrm{d}\rho^{1+a}]$ for all $a\in\mathbb{R}_{>0}$, we can set $K:=[-\beta,\beta]$ and $H:=B_{\mathrm{d}\rho^{1+a}}(0)$, $r_0:=\mathrm{d}\rho^{1+a}$. Using once again the previous methods, it is not hard to show that F is uniformly Lipschitz with constants $L_i:=\mathrm{d}\rho^{-i}C_i$, where $C_i\in\mathbb{R}_{>0}$. We can hence proceed as in the previous example to show that Thm. 24 cannot be applied, even if the ODE can be solved considering another gauge, as we have seen above for the previous example.

6.8. **Application to physical systems governed by nonlinear ODE with step input.** In the applications, e.g. in the research about viscoelastic materials, one needs to study the response of spring–dashpot systems to a step input, which can be e.g. a step loading or a step deformation or the response of an electrical circuit to a sudden voltage change. The behaviour of these systems is described by an ODE, and the response of nonlinear systems to step inputs can hence be framed in the context of GSF. As shown by [44, 45, 46], in this problem we are interested in the solutions y of Cauchy problems of the type

$$\begin{cases} a(I,y)y + y' = b(I,y)I + c(t,y)I' \\ y(0) = \tilde{y}(0) \end{cases} \tag{6.21}$$

where $I(t) = H(t) \cdot \tilde{I}(t)$, H is the Heaviside function, and a, b, c, $\tilde{y}$, $\tilde{I}$ are fixed smooth functions (see e.g. [44, Thm. 10]). Due to the term

$$\begin{aligned} c(t,y)I' &= c(t,y)\left[\delta(t)\cdot\tilde{I}(t) + H(t)\cdot\tilde{I}'(t)\right] = \\ &= c(t,y)\left[\iota^b_{\mathbb{R}}(\delta)(t)\cdot\tilde{I}(t) + \iota^b_{\mathbb{R}}(H)(t)\cdot\tilde{I}'(t)\right], \end{aligned}$$

and using the methods we have already seen above, it is possible to prove that the Lipschitz constants for this problems are $L_i = C_i b^i$, where $C_i \in \mathbb{R}_{>0}$. Therefore, the only way to solve (6.21) is to change the gauge, e.g. considering $\sigma_\varepsilon := e^{-\frac{1}{\varepsilon}}$. In this case, there exists $\alpha \in {}^\sigma\widetilde{\mathbb{R}}_{>0}$ and a solution $y \in {}^\sigma\mathcal{GC}^\infty([-\alpha,\alpha], {}^\sigma\widetilde{\mathbb{R}})$. Note that using the new gauge we now have $b = [b_\varepsilon] \in {}^\sigma\widetilde{\mathbb{R}}$, which is still an infinite number in the ring ${}^\sigma\widetilde{\mathbb{R}}$. Therefore the GSF $\iota^b_{\mathbb{R}}(\delta) = [b_\varepsilon \mu(b_\varepsilon \cdot -)] = \iota^b_{\mathbb{R}}(H)'$ is still a good model for a unit impulse generalized function. Analogously, one can deal with the second order ODE $y + a(I,t)y' + by'' = 2c(I,y)I' + 2dI''$, which is also considered in [44, 45, 46].

6.9. **A classically non-Lipschitz ODE and the problem of uniqueness.** Let us consider the following classical ODE:

$$\begin{cases} y'(t) = \sqrt{|y(t)|}, \\ y(0) = 0. \end{cases} \tag{6.22}$$

By Peano theorem, we know that Equation 6.22 has a solution; however, this solution is not unique. As the continuous function $\sqrt{|-|} : y \in \mathbb{R} \mapsto \sqrt{|y|}$ can be identified with an element of $\mathcal{D}'(\mathbb{R})$, in the GSF setting equation 6.22 can be interpreted as

$$\begin{cases} y'(t) = \iota^b_{\mathbb{R}}\left(\sqrt{|-|}\right)(y(t)), \\ y(0) = 0. \end{cases} \tag{6.23}$$

However, due to the infinite derivatives of the square root at the origin, Thm. 24 cannot be applied to (6.23). To prove this, take an infinite $b \in {}^\rho\widetilde{\mathbb{R}}$ such that $b \geq \mathrm{d}\rho^{-a}$ for some $a \in \mathbb{R}_{>0}$. Apply Thm. **??.??** to any open set $\Omega \subseteq \mathbb{R}$ such that $0 \notin \Omega$ and

Thm. ??.?? obtaining that

$$\iota_{\mathbb{R}}^{b}\left(\frac{d^{i}}{dy^{i}}\sqrt{|y|}\right)\Bigg|_{\mathrm{c}(\Omega)}=\iota_{\Omega}^{b}\left(\frac{d^{i}}{dy^{i}}\sqrt{|y|}\Big|_{\Omega}\right)=\frac{d^{i}}{dy^{i}}\sqrt{|y|}\Big|_{\Omega}=$$
$$=(-1)^{i+1}2^{-i}y^{\frac{1}{2}-i}\quad\forall i\in\mathbb{N}_{>0}. \tag{6.24}$$

Therefore, $\left|\iota_{\mathbb{R}}^{b}\left(\frac{d^{i}}{dy^{i}}\sqrt{|y|}\right)(\mathrm{d}\rho^{a})\right|=2^{-i}\mathrm{d}\rho^{\frac{a}{2}}\mathrm{d}\rho^{-ia}\ge\mathrm{d}\rho^{-(i+1)a}$ for all $i\ge 2$. Now, assume that $F(y)=\iota_{\mathbb{R}}^{b}\left(\sqrt{|-|}\right)(y)$ were uniformly Lipschitz with constants $(L_{i})_{i\in\mathbb{N}}$, then for all c, $d\in\mathbb{R}_{>0}$ with $c>d$, and $i\ge 2$ we get

$$L_{i}(\mathrm{d}\rho^{c}-\mathrm{d}\rho^{d})\ge\left\|F(\mathrm{d}\rho^{c})-F(\mathrm{d}\rho^{d})\right\|_{i}\ge\mathrm{d}\rho^{-(i+1)c}-\mathrm{d}\rho^{-(i+1)d}\ge\mathrm{d}\rho^{-(i+1)c},$$

where the i-norm is calculated on any closed interval containing $\mathrm{d}\rho^{c}$. Hence, $L_{i}\ge\frac{\mathrm{d}\rho^{-(i+1)c}}{\mathrm{d}\rho^{c}-\mathrm{d}\rho^{d}}\ge\mathrm{d}\rho^{-ic}$. Therefore, condition (5.17) holds only for $\alpha=0$ and hence Thm. 24 cannot be applied. Note that this estimate of the Lipschitz constants is worst than in the examples listed above because of the arbitrariness of the constant $c\in\mathbb{R}_{>0}$. We also note that if $y\in\mathcal{C}^{1}(\mathbb{R},\mathbb{R})$ is a classical solution of (6.22), and we set $z:=\iota_{\mathbb{R}}^{b}(y)\in{}^{\rho}\mathcal{GC}^{\infty}(\mathrm{c}(\mathbb{R}),{}^{\rho}\widetilde{\mathbb{R}})$, then $z'(t)=\left[\iota_{\mathbb{R}}^{b}(y)\right]'(t)=\iota_{\mathbb{R}}^{b}(y')(t)=\iota_{\mathbb{R}}^{b}\left(\sqrt{|y|}\right)(t)\approx\sqrt{\left|\iota_{\mathbb{R}}^{b}(y)(t)\right|}=\sqrt{|z(t)|}$ and $z(0)=\iota_{\mathbb{R}}^{b}(y)(0)\approx y(0)=0$. Therefore the GSF z satisfies (6.23) only up to infinitesimals. To help intuition, in Fig. 6.1 we represent the function $\sqrt{|-|}$ with a grey dash-dot line and its regularization $\iota_{\mathbb{R}}^{b}\left(\sqrt{|-|}\right)$ with a black line.

It is also not hard to prove that no GSF satisfies (6.23). We have to proceed along the following schema:

(i) We prove that $\iota_{\mathbb{R}}^{b}\left(\sqrt{|-|}\right)(0)\ne 0$ by contradiction: using (6.24), we prove that there exists $y\in[0,1]$ such that $\iota_{\mathbb{R}}^{b}\left(\sqrt{|-|}\right)(y)$ is greater than the maximum value of $\iota_{\mathbb{R}}^{b}\left(\sqrt{|-|}\right)$ on the functionally compact set $[0,1]$.

(ii) Therefore the constant map $y(t)=0$ is not a solution of (6.23), not even for $t\ge 0$.

(iii) So, if $y\in{}^{\rho}\mathcal{GC}^{\infty}([-\alpha,\alpha],{}^{\rho}\widetilde{\mathbb{R}})$ were a solution of (6.23), we must have that $y(t_{0})\ne 0$ for some $t_{0}>0$.

(iv) Therefore, there must exist a sample t_{1} of t_{0} (see Def. 27) such that $y(t_{1})>0$ or $y(t_{1})<0$. Both cases can be treated in the same way, so we assume that $y(t_{1})>0$.sbagliata: esiste un sample di $y(t_{0})$ e quindi un'altra soluzione $\bar{y}$, sample di y, e un sample t_{1} (con gli stessi $\bar{\varepsilon}$) tali che $\bar{y}(t_{1})>0$ or $\bar{y}(t_{1})<0$. Si continua considerando $\bar{y}$ invece di y

(v) By the intermediate value theorem (see [20]), the function y assumes all the values between $y(t_{1})>0$ and $y(0)=0$.

(vi) Therefore, proceeding as in (6.24), we can prove that the second derivative y'' is not ρ-moderate at some point $y\in[0,t_{1}]$.

This example clearly shows that in this non-Archimedean theory, we can easily have infinite derivatives corresponding to an infinite number in the ring ${}^{\rho}\widetilde{\mathbb{R}}$, but we cannot have functions with vertical tangent straight lines.

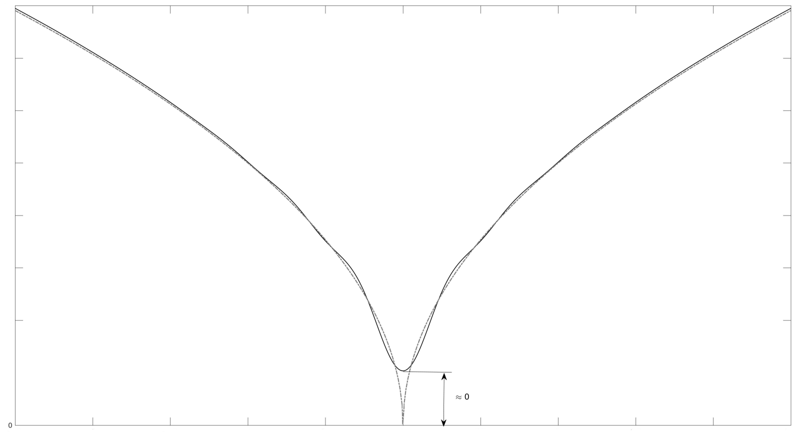

FIGURE 6.1. A representation of the function $\sqrt{|-|}$ (grey dash-dot line) and its regularization $\iota^b_{\mathbb{R}}\left(\sqrt{|-|}\right)$ (black line).

6.10. **Relations with previous studies of local solutions of singular non-linear ODEs.** In [13] E. Erlacher and M. Grosser established a local existence and uniqueness theorem for ODEs in the special Colombeau algebra of generalized functions. While theirs and ours approach have certainly some similarities, they differs in the following aspects:

- Composition of GSF is always defined while, in general, the composition of Colombeau generalized function is not. To avoid this issue, the authors in [13] based the composition of generalized functions, and hence the concept of solution, on the notion of c-boundedness, see [13, 25].
- The domain of existence of the local solution in [13] has a positive standard real radius $\alpha \in \mathbb{R}_{>0}$, while, in general, the domain of existence of the local solution given by Thm. 24 might have an infinitesimal radius $\alpha \approx 0$. This is due to the more restrictive hypotheses considered in [13]. Therefore, in some cases the result in [13] is better since, in general, it gives existence and uniqueness on larger domains; on the other side, Thm. 24 allows the studying of a wider range of problems such as those of sections 6.2, 6.3, 6.5. However, we will see how to obtain similar results in our context in the following Section 7.3 about maximal sets of existence. A large existence interval is a necessary consequence of the fact that Colombeau generalized function must be defined on domain of the type $c(\Omega)$, which always contains finite points and large neighbourhoods. In other words, the definition of Colombeau generalized function does not allow to have functions defined only on infinitesimal intervals (or on domains which contain infinite points).
- One of the more restrictive hypotheses considered in [13] is that F is bounded by some real number on some neighborhood of (t_0, y_0). We do not have to assume such a condition. This boundedness assumption on F does not allow to solve problems such as those of sections 6.3, 6.5, 6.6.

- In [13], the restriction to neighborhoods having a real positive radius forced the authors to assume a logarithmic growth condition on the Lipschitz constant of the function $F(t,y)$ with respect to y in order to prove the uniqueness of solutions. In contrast with their result, Thm. 24 ensures uniqueness under milder hypotheses. This logarithmic assumption does not allow to solve problems such as those of sections 6.8, 6.3, 6.5, 6.6.
- In [13], the gauge is fixed to $\rho_\varepsilon = \varepsilon$. This does not allow to solve problems such as those of sections 6.8, 6.7. In particular, we note that [44, Thm. 10] assume to have a Colombeau generalized function as a solution of the considered ODE (6.21), but unfortunately no existence theorem in Colombeau theory allows us to prove the existence of such a solution.

dobbiamo aggiungere qualcosa su altri articoli di ODE singolari, soprattutto quelli di Lions.

7. Classical results for singular nonlinear ODEs

7.1. Uniqueness results of solutions.

7.2. Continuous dependence on initial data.
Comparison theorem:

Theorem 30. *Let $H \subseteq {}^\rho\widetilde{\mathbb{R}}^d$, $F \in {}^\rho\mathcal{GC}^\infty([t_0-\alpha, t_0+\alpha] \times H, {}^\rho\widetilde{\mathbb{R}}^d)$, u, $v \in {}^\rho\mathcal{GC}^\infty([t_0-\alpha, t_0+\alpha], H)$ be such that*

$$\begin{aligned} &\forall t \in [t_0-\alpha, t_0+\alpha]: \ u'(t) \le F(t, u(t)) \le v'(t) \\ &u(t_0) \le v(t_0). \end{aligned}$$

Then $u(t) \le v(t)$ for all $t \in [t_0-\alpha, t_0+\alpha]$.

Proof. By contradiction, assume that $u(T) >_L v(T)$ for some $T \in [t_0-\alpha, t_0+\alpha]$ and some $L \subseteq_0 I$. □

7.3. Maximal sets of existence.
Properties of the maximal interval of existence of an ODE are some of the most well known results in the classical theory of ODEs. In this section we want to study analogous properties in our generalized framework. As we will show, most of the classical properties have their generalized counterparts.

Crucial for our approach is a sheaf property for GSF proved by Hans Vernaeve in [53], which is based on the concept of *interleave* of a subset of ${}^\rho\widetilde{\mathbb{R}}$:

Definition 31. Let $A \subseteq {}^\rho\widetilde{\mathbb{R}}$. We let

$$interl(A) := \left\{ \sum_{j=1}^m e_{S_j} x_j \mid m \in \mathbb{N}, S_1, \dots, S_m \text{ partition of } (0,1), x_j \in A \right\},$$

where $e_{s_j} = \left[I_{S_j} \right]$ and I_{S_j} is the characteristic function of S_j.

The sheaf property that we will use is the following:

Theorem 32 (H.Vernaeve)**.** *For each $m \in \mathbb{N}$, let $\Omega_m \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ be a union of an increasing sequence $(A_{m,n})_{n\in\mathbb{N}}$ of internal sets with $A_{m,n+1}$ neighborhood of $A_{m,n}$ $\forall n \in \mathbb{N}$. Let $u_m \in {}^\rho\mathcal{GC}^\infty\left(\Omega_m, {}^\rho\widetilde{\mathbb{R}}^h\right)$ $\forall m \in \mathbb{N}$ be such that $u_m|_{\Omega_m \cap \Omega'_m} = u_{m'}|_{\Omega_m \cap \Omega'_m}$ $\forall m, m'$. Let $\Omega = interl\left(\cup_{m\in\mathbb{N}} \Omega_m\right)$. Then there exists a unique $u \in {}^\rho\mathcal{GC}^\infty\left(\Omega, {}^\rho\widetilde{\mathbb{R}}^h\right)$ such that $u|_{\Omega_m} = u_m$ $\forall m \in \mathbb{N}$.*

An useful result about interleaves of finite unions of sets of intervals is the following:

Lemma 33. *Let $n \in \mathbb{N}$, $\alpha_1, \dots, \alpha_n \in {}^\rho\widetilde{\mathbb{R}}_{>0}$. Then:*

(i) *$interl([0,\alpha_1] \cup \dots \cup [0,\alpha_n]) = [0, \alpha_1 \vee \dots \vee \alpha_n]$;*

(ii) *$\exists c_1, \dots, c_n \in [0,1]$, with $\sum_{i=1}^n c_i = 1$ and $c_i^2 = c_i$ for all $i = 1, \dots, n$ such that for every $x \in [0, \alpha_1 \vee \dots \vee \alpha_n]$ we have that $c_i x \in [0, \alpha_i]$ for every $i = 1, \dots, n$;*

(iii) *let $x = \sum_{i=1}^n c_i x_i$ be the decomposition of $x \in [0, \alpha_1 \vee \dots \vee \alpha_n]$, where $c_1, \dots, c_n$ are the constants given in ((ii)); if $x \in [0, \alpha_i]$ then $c_j x \in [0,\alpha_i] \cap [0,\alpha_j]$ for every $i, j \le n$;*

(iv) *for every $i = 1, \dots, n$ let $u_{\alpha_i} \in {}^\rho\mathcal{GC}^\infty\left([0,\alpha_i], {}^\rho\widetilde{\mathbb{R}}\right)$ be such that for every $i, j \le n$*

$$u_{\alpha_i}|_{[0,\alpha_i]\cap[0,\alpha_j]} = u_{\alpha_j}|_{[0,\alpha_i]\cap[0,\alpha_j]}.$$

Then the unique common extension $u \in {}^\rho\mathcal{GC}^\infty\left([0, \alpha_1 \vee \dots \vee \alpha_n], {}^\rho\widetilde{\mathbb{R}}\right)$ of $u_{\alpha_i}, \dots, u_{\alpha_n}$ satisfies the following property: for every $x \in [0, \alpha_1 \vee \dots \vee \alpha_n]$

$$u(x) = \sum_{i=1}^n c_i u_{\alpha_i}(c_i x), \tag{7.1}$$

where $c_1, \dots, c_n$ are the constants given in ((ii)).

Proof. We will prove all the results for $n = 2$, as the general cases can be easily proven by induction on n.

((i)) By definition, all the elements in $interl([0,\alpha_1] \cup [0,\alpha_2])$ can be written as $e_S x_1 + e_{S^c} x_2$ for $S \subseteq (0,1)$, $x_1 \in [0,\alpha_1]$, $x_2 \in [0,\alpha_2]$.

$\subseteq$: Let $e_S x_1 + e_{S^c} x_2 = x$. Let $x_1 = [x_{1,\varepsilon}]$, $x_2 = [x_{2,\varepsilon}]$. Assume that $0 \le x_{1,\varepsilon} \le \alpha_{1,\varepsilon}$ and $0 \le x_{2,\varepsilon} \le \alpha_{2,\varepsilon}$ for every $\varepsilon \in I$. Then $\forall \varepsilon \in I\, x_\varepsilon \le \max(x_{1,\varepsilon}, x_{2,\varepsilon}) \le \max(\alpha_{1,\varepsilon}, \alpha_{2,\varepsilon})$, hence $0 \le x_\varepsilon \le \alpha_{1,\varepsilon} \vee \alpha_{2,\varepsilon}$, and so $x \in [0, \alpha_1 \vee \alpha_2]$.

$\supseteq$: Let $x = [x_\varepsilon] \in [0, \alpha_1 \vee \alpha_2]$, and let us assume that $0 \le x_\varepsilon \le \alpha_{1,\varepsilon} \vee \alpha_{2,\varepsilon}\ \forall \varepsilon \in I$. Let $S = \{\varepsilon \in I \mid \alpha_{1,\varepsilon} \vee \alpha_{2,\varepsilon} = \alpha_{1,\varepsilon}\}$. Let $x_{1,\varepsilon} = I_S(\varepsilon) \cdot x_\varepsilon, x_{2,\varepsilon} = I_{S^x} \cdot x_\varepsilon$. Then, clearly, $x_1 \in [0,\alpha_1], x_2 \in [0,\alpha_2]$ and $x = e_S x_1 + e_{S^c} x_2 \in interl([0,\alpha_1] \cup [0,\alpha_2])$. Notice that by letting $c_1 = e_S$ and $c_2 = e_{S^c}$ we have also proven ((ii)).

((iii)) Just set $x_1 = c_1 x$ and $x_2 = c_2 x$. As $c_1 = e_S$ and $c_2 = e_{S^c}$ with $S = \{\varepsilon \in I \mid \alpha_{1,\varepsilon} \vee \alpha_{2,\varepsilon} = \alpha_{1,\varepsilon}\}$, it follows that $x_1 \in [0,\alpha_1]$ and $x_2 \in [0,\alpha_2]$. Moreover (since $c_1^2 = c_1$ and $c_2^2 = c_2$) we have

$$c_1 x_1 + c_2 x_2 = c_1^2 x + c_2^2 x = (c_1 + c_2)\, x = 1 \cdot x = x.$$

((iv)) The existence of a unique extension of u_α, u_β to $[0, \alpha_1 \vee \alpha_2]$ is a consequence of Theorem 32 together with the fact that $interl([0,\alpha_1] \cup [0,\alpha_2]) = [0, \alpha_1 \vee \alpha_2]$. Therefore to conclude it is sufficient to show that the function u given in equation 7.1 is a GSF that extends $u_{\alpha_1}, u_{\alpha_2}$. And this is trivial: u is a GSF as, for every $n \in \mathbb{N}$, by its definition it is immediate to notice that

$$||u||_{n,[0,\alpha_1\vee\alpha_2]} \le ||u_{\alpha_1}||_{n,[0,\alpha_1]} + ||u_{\alpha_2}||_{n,[0,\alpha_2]},$$

which proves the existence of polynomial bounds for u as both u_{α_1} and u_{α_2} are GSF. To prove that u extends u_{α_1} let $x \in [0,\alpha_1]$. Then by ((iii)) we have that $c_2 x \in [0,\alpha_1]$ and, as $u_{\alpha_2}|_{[0,\alpha_1]} = u_{\alpha_1}|_{[0,\alpha_1]}$, we have

$$u(x) = c_1 u_{\alpha_1}(c_1 x) + c_2 u_{\alpha_2}(c_2 x) = c_1 u_{\alpha_1}(c_1 x) + c_2 u_{\alpha_1}(c_2 x).$$

To conclude, we show that $c_1u_{\alpha_1}(c_1x)+c_2u_{\alpha_1}(c_2x)=u_{\alpha_1}(x)$. We do this ε-wise: let $u_{\alpha_1}=[u_{\alpha_1,\varepsilon}]$, $x=[x_\varepsilon]$. Then

$$c_1u_{\alpha_1}(c_1x)+c_2u_{\alpha_1}(c_2x)=\left[I_S(\varepsilon)u_{\alpha_1,\varepsilon}\left(I_S(\varepsilon)x_\varepsilon\right)+I_{S^c}(\varepsilon)u_{\alpha_1,\varepsilon}\left(I_{S^c}(\varepsilon)x_\varepsilon\right)\right].$$

If $\varepsilon\in S$ then $I_S(\varepsilon)u_{1,\varepsilon}\left(I_S(\varepsilon)x_\varepsilon\right)=u_{\alpha_1,\varepsilon}\left(x_\varepsilon\right)$ whilst $I_{S^c}(\varepsilon)u_{\alpha_1,\varepsilon}\left(I_{S^c}(\varepsilon)x_\varepsilon\right)=0$; if $\varepsilon\in S^c$ then $I_S(\varepsilon)u_{\alpha_1,\varepsilon}\left(I_S(\varepsilon)x_\varepsilon\right)=0$ whilst $I_{S^c}(\varepsilon)u_{\alpha_1,\varepsilon}\left(I_{S^c}(\varepsilon)x_\varepsilon\right)=u_{\alpha_1,\varepsilon}\left(x_\varepsilon\right)$. Therefore $\forall\varepsilon\in I$ we have

$$I_S(\varepsilon)u_{\alpha_1,\varepsilon}\left(I_S(\varepsilon)x_\varepsilon\right)+I_{S^c}(\varepsilon)u_{\alpha_1,\varepsilon}\left(I_{S^c}(\varepsilon)x_\varepsilon\right)=u_{\alpha_1,\varepsilon}\left(x_\varepsilon\right),$$

and so $c_1u_{\alpha_1}(c_1x)+c_2u_{\alpha_1}(c_2x)=\left[u_{\alpha_1,\varepsilon}\left(x_\varepsilon\right)\right]=u_{\alpha_1}(x)$ as desired. The case $x\in[0,\alpha_2]$ can be done similarly. □

From now on, we let

$$\begin{cases}y'(t)=F(t,y(t))\\ y(0)=y_0\end{cases}\tag{7.2}$$

be a generalized ODE satisfying the hypotheses of Corollary 26. To avoid unnecessary complications, we assume that $F(x,y)\in{}^{\rho}\mathcal{GC}^{\infty}\left({}^{\rho}\widetilde{\mathbb{R}}_{>0}\times{}^{\rho}\widetilde{\mathbb{R}}^{n},{}^{\rho}\widetilde{\mathbb{R}}\right)$ and that $\forall t\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}\,\exists L\in{}^{\rho}\widetilde{\mathbb{R}}\,\exists K\Subset_f{}^{\rho}\widetilde{\mathbb{R}}_{\ge0}$ neighborhood of t such that all the Lipschitz constants of $F(x,y)$ on $K\times{}^{\rho}\widetilde{\mathbb{R}}^{n}$ are bounded by L. In this way, by Corollary 26 we deduce that for every $t_0\in{}^{\rho}\widetilde{\mathbb{R}}_{\ge0}$, for every $y_0\in{}^{\rho}\widetilde{\mathbb{R}}^{n}$ the problem

$$\begin{cases}y'(t)=F(t,y(t))\\ y(0)=y_0\end{cases}\tag{7.3}$$

has a unique solution in a neighborhood of t_0. We set

$$T=\left\{\alpha\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}\mid\exists!\,\text{solution of the problem 7.2 in}\,{}^{\rho}\mathcal{GC}^{\infty}\left([0,\alpha],{}^{\rho}\widetilde{\mathbb{R}}\right)\right\}$$

and $M=\bigcup_{\alpha\in T}[0,\alpha]$.

Theorem 34. *M is the maximal set of existence of a unique solution u of Problem 7.2 in the sense that $\forall\alpha\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$ if $[0,\alpha]\subseteq dom(u)$ then $[0,\alpha]\subseteq M$. Moreover:*

(i) *$M\cap{}^{\rho}\widetilde{\mathbb{R}}_{>0}$ is open;*
(ii) *$\forall\alpha,\beta\in M$ $\alpha\vee\beta\in M$;*
(iii) *$interl(M)=M$;*
(iv) *if $\alpha\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$ is such that $\gamma\in T$ for every $\gamma\in(0,\alpha)$ then $[0,\alpha)\subseteq M$.*

Proof. The maximality of M is trivial.

((i)) As discussed above, this follows from the fact that, under our hypotheses on F, Problem 7.3 has always a unique solution.

((ii)) Let u_α,u_β be the unique solutions of Problem 7.2 respectively on $[0,\alpha]$, $[0,\beta]$. From Theorem 32 we deduce the existence of a unique u defined on $interl([0,\alpha]\cup[0,\beta])=[0,\alpha\vee\beta]$ such that $u|_{[0,\alpha]}=u_\alpha,u|_{[0,\beta]}=u_\beta$. We claim that u is the unique solution of Problem 7.2 on $[0,\alpha\vee\beta]$. Let $\alpha=[\alpha_\varepsilon],\beta=[\beta_\varepsilon]$. By Lemma 33.((iv)), we have that $\forall t\in[0,\alpha\vee\beta]$ $u(t)=c_1u(c_1t)+c_2u(c_2t)$, where $c_1=[I_S],c_2=[I_{S^c}]$ and $S=\{\varepsilon\in I\mid\alpha_\varepsilon\vee\beta_\varepsilon=\alpha_\varepsilon\}$. Hence

$$u'(t)=c_1^2u'(c_1t)+c_2^2u'(c_2t)=$$
$$c_1u'(c_1t)+c_2u'(c_2t)\ (\text{as } c_1^2=c_1,c_2^2=c_2)=$$
$$c_1f(c_1t,u(c_1t))+c_2f(c_2t,u(c_2t))=f(t,u(t)),$$

where the last equality can be proved ε-wise as follows: if $f=[f_\varepsilon]$, $u=[u_\varepsilon]$ and $t=[t_\varepsilon]$ then $\forall\varepsilon\in I$

$$c_1 f\left(c_1 t, u\left(c_1 t\right)\right)+c_2 f\left(c_2 t, u\left(c_2 t\right)\right)=$$
$$\left[I_S(\varepsilon) f_\varepsilon\left(I_S(\varepsilon) t_\varepsilon, u_\varepsilon\left(I_S(\varepsilon) t_\varepsilon\right)\right)+I_{S^c}(\varepsilon) f_\varepsilon\left(I_{S^c}(\varepsilon) t_\varepsilon, u_\varepsilon\left(I_{S^c} t_\varepsilon\right)\right)\right],$$

and we conclude as $\forall\varepsilon\in I$

$$I_S(\varepsilon) f_\varepsilon\left(I_S(\varepsilon) t_\varepsilon, u_\varepsilon\left(I_S(\varepsilon) t_\varepsilon\right)\right)+I_{S^c}(\varepsilon) f_\varepsilon\left(I_{S^c}(\varepsilon) t_\varepsilon, u_\varepsilon\left(I_{S^c} t_\varepsilon\right)\right)=f_\varepsilon\left(t_\varepsilon, u_\varepsilon\left(t_\varepsilon\right)\right),$$

namely

$$\left[I_S(\varepsilon) f_\varepsilon\left(I_S(\varepsilon) t_\varepsilon, u_\varepsilon\left(I_S(\varepsilon) t_\varepsilon\right)\right)+I_{S^c}(\varepsilon) f_\varepsilon\left(I_{S^c}(\varepsilon) t_\varepsilon, u_\varepsilon\left(I_{S^c} t_\varepsilon\right)\right)\right]=$$
$$\left[f_\varepsilon\left(t_\varepsilon, u_\varepsilon\left(t_\varepsilon\right)\right)\right]=f(t,u(t)).$$

To prove that u is the unique solution of Problem 7.2 on $[0,\alpha\vee\beta]$, let v be another such solution. Then necessarily $v|_{[0,\alpha]}=u_\alpha$ and $v|_{[0,\beta]}=u_\beta$, therefore by Theorem 32 $v=u$, as u is the unique common extension of u_α, u_β to $[0,\alpha\vee\beta]$.

((iii)) Let $x=\sum_{i=1}^m e_{S_i}x_i\in interl(M)$. Then $x_1,\dots,x_m\in M$, hence $\exists\alpha_1,\dots,\alpha_m\in T$ such that $x_i\in[0,\alpha_i]$ $\forall i=1,\dots,m$. And so $x\in[0,\alpha_1\vee\dots\vee\alpha_m]\subseteq M$ by Lemma 33.

((iv)) Let $n\in\mathbb{N}$ be such that $\alpha>d\rho^n$. For every $m\in\mathbb{N}$ let $\gamma_m:=(\alpha-d\rho^m)\vee d\rho^n$. Notice that $\bigcup_{m\in\mathbb{N}}[0,\gamma_m]=[0,\alpha)$ and that $interl\left(\bigcup_{m\in\mathbb{N}}[0,\gamma_m]\right)=interl\left([0,\alpha)\right)=[0,\alpha)$. Our thesis follows from Theorem 32 by setting $\Omega_m:=[0,\gamma_m]$ and u_m the unique solution of Problem 7.2 on Ω_m. ☐

In general, as we will show in Example 40, the set M has not the form $[0,\alpha)$ for some $\alpha\in{}^\rho\widetilde{\mathbb{R}}_{>0}\cup\{\infty\}$. However, M can be considered as an "initial interval in ${}^\rho\widetilde{\mathbb{R}}_{\ge0}$" in the following sense: for every $\alpha\in{}^\rho\widetilde{\mathbb{R}}_{>0}$, if $\alpha\in M$ then $[0,\alpha]\subseteq M$.

In the classical theory of ODE it is well known that the boundedness of the maximal interval of existence of the unique solution of an ODE is related with the so-called "blow up" of the solution. In our generalized setting, the situation is similar:

Theorem 35. *Let $\alpha\in{}^\rho\widetilde{\mathbb{R}}_{>0}$ and let us assume that u is the unique solution of Problem 7.2 on $[0,\alpha)$. Then TFAE:*

(i) $\alpha\in T$, *i.e.* $[0,\alpha]\subseteq M$;

(ii) *u can be extended, namely there exists $\alpha'>\alpha$ with $\alpha'\in T$;*

(iii) $\forall n\in\mathbb{N}$ $\frac{\partial^n u}{\partial t^n}$ *is uniformly continuous on* $[0,\alpha)$ *and there exists* $H_n\Subset_f{}^\rho\widetilde{\mathbb{R}}$ *with* $\frac{\partial^n u}{\partial t^n}([0,\alpha))\subseteq H_n$.

Proof. The implication ((i))⇒((ii)) holds as M is open, whilst the converse is immediate. To conclude we will prove that ((i)) and ((iii)) are equivalent.

((i))⇒((iii)) If $\alpha\in T$ then there exists a unique solution $\overline{u}$ of Problem 7.2 defined on $[0,\alpha]$. As $\overline{u}\in{}^\rho\mathcal{GC}^\infty\left([0,\alpha],{}^\rho\widetilde{\mathbb{R}}\right)$, by Theorem 14.((v)) we get that $\overline{u}$ and all its derivatives are uniformly continuous on $[0,\alpha]$. Hence they are also uniformly continuous on $[0,\alpha)$, so we deduce that $\forall n\in\mathbb{N}$ $\frac{\partial^n u}{\partial t^n}$ is uniformly continuous on $[0,\alpha)$. Moreover, as $[0,\alpha]$ is functionally compact, for every $n\in\mathbb{N}$ the set $H_n=\frac{\partial^n\overline{u}}{\partial t^n}([0,\alpha])$ is functionally compact, and $\frac{\partial^n u}{\partial t^n}([0,\alpha))\subseteq\frac{\partial^n u}{\partial t^n}([0,\alpha])\subseteq H_n$.

((iii))⇒((i)) By Theorem 14.((iii)) there exists a unique $\overline{u}\in{}^\rho\mathcal{GC}^\infty\left([0,\alpha],{}^\rho\widetilde{\mathbb{R}}\right)$ such that $\overline{u}|_{[0,\alpha)}=u$. The fact that $\overline{u}$ is the unique solution of Problem 7.2 in

${}^{\rho}\mathcal{GC}^{\infty}\left([0,\alpha],{}^{\rho}\widetilde{\mathbb{R}}\right)$ follows from Lemma **??**: in fact, for every $t\in(0,\alpha)=int([0,\alpha])$ $\overline{u}'(t)=u'(t)=f(t,u(t))=f(t,\overline{u}(t))$, and so the same equality holds on $\overline{(0,\alpha)}=[0,\alpha]$. Moreover, if $v\in{}^{\rho}\mathcal{GC}^{\infty}\left([0,\alpha],{}^{\rho}\widetilde{\mathbb{R}}\right)$ is another solution of Problem 7.2 then $u=v$ on $(0,\alpha)$, and once again from Lemma **??** we deduce that then $u=v$ on $\overline{(0,\alpha)}=[0,\alpha]$. □

Corollary 36. *Let M be the maximal set of existence of the unique solution u of Problem 7.2. Then for every $\alpha\in\overline{M}\setminus M$ there exists $n\in\mathbb{N}$ such that $\lim_{t\to\alpha^{-}}|u^{(n)}(t)|=+\infty$.*

Proof. By Theorem 34,(iv) we have that $[0,\alpha)\subseteq M$. As $\alpha\in\overline{M}\setminus M$, we have that condition (i) of Theorem 35 is not fulfilled, and the thesis hence follows by Theorem 35. □

In the classical case, the implication ((iii))⇒((i)) holds even if we replace ((*iii*)) with the much weaker condition "$u([0,\alpha))$ is bounded". In fact, this condition, together with the fact that u, being a solution of Problem 7.2 on $[0,\alpha)$, satisfies the integral equation

$$u(t)=y_0+\int_0^t f(s,u(s))ds,$$

is sufficient to ensure the uniform continuity of u on $[0,\alpha)$ (and this remains true also for GSF). In our setting, we had to strenghten this condition for the reasons explained in Section 4: in the GSF setting, we always have to take into account all the derivatives of u. In any case, a simple Corollary of Theorem 35 is the following sufficient condition for the non-extendability of solutions:

Corollary 37. *Let $\alpha\in{}^{\rho}\widetilde{\mathbb{R}}$ and let us assume that u is the unique solution of Problem 7.2 on $[0,\alpha)$. If there exists $n\in\mathbb{N}$ such that $\frac{\partial^n u}{\partial t^n}([0,\alpha))$ is unbounded then $\alpha\notin M$.*

We will show some examples of maximal intervals of existence in the next Section.

7.4. **Gronwall Inequalities and sufficient conditions for global solutions.** In the classical theory of ODE there are various versions of Gronwall inequalities, the most widely used being the differential and the integral form. In this section, we want to prove the analogues of these two results for generalized ODE. As we will see, the smoothness of GSF will allow to relax the regularity assumptions needed in the classical case; however, the polynomial asymptotic bounds used to define GSF will force us to assume certain logarithmic bounds similar to those of the previous sections. Notably, our proofs are substantially identical to the classical ones.

Theorem 38 (Gronwall inequality for GSF, differential form)**.** *Let $\alpha\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$. Let $u,a\in{}^{\rho}\mathcal{GC}^{\infty}\left([0,\alpha],{}^{\rho}\widetilde{\mathbb{R}}\right)$. If $u'(t)\le a(t)u(t)$ for every $t\in(0,\alpha)$ and $\|a\|_{[0,\alpha],0}\cdot\alpha<N\cdot\log\left(d\rho^{-1}\right)$ for some finite $N\in\mathbb{N}$ then*

$$\forall t\in[0,\alpha]\;u(t)\le u(0)\cdot e^{\int_0^t a(s)ds}.$$

Proof. Let $v(t)=e^{\int_0^t a(s)ds}$. Our assumption $\|a\|_{[0,\alpha],0}\cdot\alpha<N\cdot\log\left(d\rho^{-1}\right)$ ensures that $v\in{}^{\rho}\mathcal{GC}^{\infty}\left([0.\alpha],{}^{\rho}\widetilde{\mathbb{R}}\right)$, as

$$\forall t\in[0,\alpha]\;0\le e^{\int_0^t a(s)ds}\le e^{\int_0^t\|a\|_{[0,\alpha],0}ds}\le e^{\alpha\cdot\|a\|_{[0,\alpha],0}}<e^{N\cdot\log\left(d\rho^{-1}\right)}=d\rho^{-N}$$

(the estimates for higher order derivatives can be checked similarly). Notice that $\forall t\in[0,\alpha]$ we have

$$v(t)=e^{\int_0^t a(s)ds}\ge e^{-\alpha\cdot\|a\|_{[0,\alpha],0}}\ge$$
$$e^{-N\log\left(d\rho^{-1}\right)}=d\rho^N>0,$$

hence also $\frac{1}{v}\in{}^{\rho}\mathcal{GC}^{\infty}\left([0,\alpha],{}^{\rho}\widetilde{\mathbb{R}}\right)$. Moreover, $v'(t)=a(t)v(t)$ and $v(0)=1$. By the quotient rule (which is satisfied by GSF) we have that

$$\forall t\in[0,\alpha]\,\frac{d}{dt}\frac{u(t)}{v(t)}=\frac{u'v-v'u}{v^2}=\frac{u'-au}{v}\le 0.$$

By applying Corollary **??** to $-\frac{u}{v}$ we deduce that

$$\forall t\in[0,\alpha]\,\frac{u(t)}{v(t)}\le\frac{u(0)}{v(0)}=u(0),$$

namely $u(t)\le u(0)e^{\int_0^t a(s)ds}$. □

It is important to notice that the condition $\|a\|_{[0,\alpha],0}\cdot\alpha<N\cdot\log\left(d\rho^{-1}\right)$ can always be satisfied, provided that we take a sufficiently small α; henceforth the following is a trivial Corollary of our previous result.

Corollary 39. *Let $\alpha\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$. Let $u,a\in{}^{\rho}\mathcal{GC}^{\infty}\left([0,\alpha],{}^{\rho}\widetilde{\mathbb{R}}\right)$. Let $\beta\in(0,\alpha]$ be such that $u'(t)\le a(t)u(t)$ for every $t\in(0,\beta)$. Then there exists $\gamma\in(0,\beta)$ such that*

$$\forall t\in[0,\gamma]\,u(t)\le u(0)\cdot e^{\int_0^t a(s)ds}.$$

Example 40. Let us consider the generalized ODE

$$\begin{cases}y'(t)=2y(t);\\ y(0)=1.\end{cases}\tag{7.4}$$

For every $N\in\mathbb{N}$, e^{2t} is a solution of Problem 7.4 on $\left[0,N\log\left(d\rho^{-1}\right)\right]$. We claim that it is the unique solution: let $u(t)$ be another solution, and let $v(t):=e^{2t}-u(t)$. Notice that $v(0)=0$. For every $t\in\left[0,N\log\left(d\rho^{-1}\right)\right]$ we have

$$v'(t)\le 2v(t),$$

therefore, as $a(t)\equiv 2$ and $2\cdot N\log\left(d\rho^{-1}\right)<(2N+1)\log\left(d\rho^{-1}\right)$, we can apply Theorem 38 to get that, for every $t\in\left[0,N\log\left(d\rho^{-1}\right)\right]$

$$v(t)\le v(0)e^{2t}=0,$$

namely $v(t)=e^{2t}$. Combining this fact with Theorem 34 we also get that the maximal set of existence of the unique solution of Problem 7.4 is

$$M=\bigcup_{n\in\mathbb{N}}\left[0,N\left(\log d\rho^{-1}\right)\right],$$

as if $t\ge 0$ is such that $t\notin M$ then e^{2t} is not a GSF (since it is not polynomially bounded by $d\rho^{-1}$). Notice that this provides an example of a maximal set of existence of a solution of a generalized ODE which is not of the form $[0,\alpha)$ for any $\alpha\in{}^{\rho}\widetilde{\mathbb{R}}_{\ge 0}$.

We now prove to the integral form of Gronwall inequality which, once again, is completely analogue (in the statement and the proof) to the classical one:

Theorem 41 (Gronwall inequality for GSF, integral form)**.** *Let* $\alpha \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$*. Let* $u, a, b \in {}^{\rho}\mathcal{GC}^{\infty}\left([0.\alpha], {}^{\rho}\widetilde{\mathbb{R}}\right)$ *and assume that* $\|a\|_{[0,\alpha],0} \cdot \alpha < N \cdot \log\left(d\rho^{-1}\right)$ *for some finite* $N \in \mathbb{N}$*. Assume that* $a(t) \geq 0 \, \forall t \in [0, \alpha]$ *and that* $u(t) \leq b(t) + \int_0^t a(s)u(s)ds$*. Then:*

(i) *for every* $t \in [0, \alpha]$ *we have*

$$u(t) \leq b(t) + \int_0^t a(s)b(s)e^{\int_s^t a(r)dr}ds;$$

(ii) *if* $b(t)$ *is non-decreasing, for every* $t \in [0, \alpha]$ *we have*

$$u(t) \leq b(t)e^{\int_0^t a(s)ds};$$

(iii) *in particular, if* $b(t) \equiv b \in {}^{\rho}\widetilde{\mathbb{R}}$ *then for every* $t \in [0, \alpha]$ *we have*

$$u(t) \leq b \cdot e^{\int_0^t a(s)ds}.$$

Proof. As in Theorem 38, the assumption $\|a\|_{[0,\alpha],0} \cdot \alpha < N \cdot \log\left(d\rho^{-1}\right)$ is sufficient to prove that for every $s \in [0, \alpha]$ the function $e^{\int_s^t a(r)dr} \in {}^{\rho}\mathcal{GC}^{\infty}\left([0, \alpha], {}^{\rho}\widetilde{\mathbb{R}}\right)$.

Notice that ((iii)) is a particular case of ((ii)) which is a particular case of ((i)): in fact, as $b(t)$ is non-decreasing then

$$b(t) + \int_0^t a(s)b(s)e^{\int_s^t a(r)dr}ds \leq b(t) \cdot \left(1 + \int_0^t a(s)e^{\int_s^t a(r)dr}ds\right) =$$

$$b(t)\cdot\left(1 + \int_0^t a(s)e^{\int_0^t a(r)dr}e^{-\int_0^s a(r)dr}ds\right) = b(t)\cdot\left(1 + e^{\int_0^t a(r)dr} \cdot \left(-e^{\int_0^s a(r)dr}|_{s=0}^{s=t}\right)\right) =$$

$$b(t)e^{\int_0^t a(r)dr}.$$

We are left to prove ((i)). Let

$$v(t) = e^{-\int_0^t a(s)ds} \cdot \int_0^t a(s)u(s)ds \in {}^{\rho}\mathcal{GC}^{\infty}\left([0, \alpha], {}^{\rho}\widetilde{\mathbb{R}}\right). \tag{7.5}$$

As GSF satisfy the fundamental theorem of calculus, as well as product and chain rules for derivation, we get that

$$v'(t) = \left(u(t) - \int_0^t u(s)a(s)ds\right) a(t)e^{-\int_0^t a(s)ds} \leq$$

$$b(t)a(t)e^{-\int_0^t a(s)ds},$$

since, by assumption, for every $t \in [0, \alpha]$ $a(t) \geq 0$ and $u(t) \leq b(t) + \int_0^t u(s)a(s)ds$. By integrating in t, since $v(0) = 0$ from Theorem **??** we get

$$v(t) \leq \int_0^t b(s)a(s)e^{-\int_0^s a(r)dr}ds$$

hence by equation (7.5) we obtain

$$e^{-\int_0^t a(s)ds} \cdot \int_0^t a(s)u(s)ds \leq \int_0^t b(s)a(s)e^{-\int_0^s a(r)dr}ds,$$

namely

$$\int_0^t a(s)u(s)ds \leq e^{\int_0^t a(s)ds} \int_0^t b(s)a(s)e^{-\int_0^s a(r)dr}ds \leq$$

$$\int_0^t b(s)a(s)e^{\left(\int_0^t a(r)dr-\int_0^s a(r)dr\right)}ds=\int_0^t b(s)a(s)e^{\int_s^t a(r)dr}ds,$$

and so $u(t)\le b(t)+\int_0^t a(s)u(s)ds\le b(t)+\int_0^t a(s)b(s)e^{\int_s^t a(r)dr}ds.$ □

Example 42. Let us consider the generalized ODE

$$\begin{cases}u'(t)=(1+\delta)\,u(t);\\ u(0)=1.\end{cases}\tag{7.6}$$

By the embedding properties we know that δ has its max in 0 and that $(1+\delta)(t)\ge 0$ for every $t\in{}^{\rho}\widetilde{\mathbb{R}}$. Let $\alpha\in{}^{\rho}\widetilde{\mathbb{R}}_{\ge0}$ be such that $(1+\delta(0))\cdot\alpha<N\log\left(d\rho^{-1}\right)$ for some $N\in\mathbb{N}$ and let $u(t)$ be a solution of Problem 7.6 in $[0,\alpha]$. Then $u(t)$ satisfies the integral equation

$$u(t)=1+\int_0^t(1+\delta(s))u(s)ds\le1+\int_0^t(1+\delta(0))\,u(s)ds.$$

Therefore we can apply Theorem 41 to deduce that

$$u(t)\le(1+\delta(0))\,e^{(1+\delta(0))t}$$

for every $t\in[0,\alpha]$.

Classically, one of the consequences of Gronwall's Lemma is that if $f(t,x)$ satisfies the inequality

$$|f(t,x)|\le a(t)|x|+b(t)\tag{7.7}$$

with a,b positive functions then the ODE

$$u'=f(t,u)$$

admits a unique global solution for every initial condition (see e.g. [50, Theorem 2.17]). As Example 40 shows, this result is false in the GSF setting due to the fact exponential bounds of the form $e^{g(t)}$ are not sufficient to prove boundedness in the GSF setting. Moreover, the Fréchet structure of GSF spaces forces to extend the control given by condition 7.7 to all derivatives. Nevertheless, an analogue of the classical global existence theorem given by the estimate 7.7 is the following:

Theorem 43. *Let $\alpha\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$, let $f\in{}^{\rho}\mathcal{GC}^{\infty}\left([0,\alpha]\times{}^{\rho}\widetilde{\mathbb{R}},{}^{\rho}\widetilde{\mathbb{R}}\right)$, let $u_0\in{}^{\rho}\widetilde{\mathbb{R}}$ and let M be the maximal set of existence of the unique solution u of the initial value problem*

$$\begin{cases}u'=f(t,u);\\ u(0)=u_0.\end{cases}\tag{7.8}$$

Moreover, let us assume that $\forall n\in\mathbb{N}$ there exists positive functions $A_n(t),B_n(t)\in{}^{\rho}\mathcal{GC}^{\infty}\left([0,\alpha],{}^{\rho}\widetilde{\mathbb{R}}\right)$ such that:

(i) $\|A_n\|_{[0,\alpha],0}\cdot\alpha<N\cdot\log\left(d\rho^{-1}\right)$ *for some $N\in\mathbb{N}$;*

(ii) *for every $t\in M$* $\left|\frac{\partial^n}{\partial t^n}u(t)\right|\le A_n(t)\,|u(t)|+B_n(t)$.

Then $[0,\alpha)\subseteq M$.

Proof. If $\beta\in M$ for every $\beta<\alpha$ then the thesis follows from Theorem 34,(iv). Hence let us assume, by contrast, that there exists $\beta<\alpha$ with $\beta\in\overline{M}\setminus M$. As u solves the initial value problem 7.8, by condition ((ii)) with $n=0$ we deduce that

$$|u(t)|=\left|u_0+\int_0^t f(s,u(s))ds\right|\le|u_0|+\int_0^t|f(s,u(s))|\,ds\le$$

$$|u_0| + \int_0^t B_0(s)ds + \int_0^t A_0(s)\,|u(s)|\,ds,$$

so by our assumptions ((i)) we can apply Theorem 41 with $a(t) \equiv A_0(t)$ and $b(t) \equiv |u_0| + \int_{t_0}^t B_0(s)ds$ to get

$$|u(t)| \le b(t)e^{\int_0^t A_0(r)dr}.$$

In particular, $\lim_{t\to\beta^-} b(t)e^{\int_0^t A_0(r)dr} \in {}^\rho\widetilde{\mathbb{R}}$ by condition ((i)). Therefore by condtition ((ii)) it follows that $\lim_{t\to\beta^-}\left|\frac{\partial^n}{\partial t^n}u(t)\right| \in {}^\rho\widetilde{\mathbb{R}}$ for every $n \in \mathbb{N}$, and this contradicts that $\beta \in \overline{M} \setminus M$ by Corollary 36. $\square$

Remark 44. Notice that Condition (ii) in Theorem 43 could be substitued with the weaker condition $\left|\frac{\partial^{n+1}}{\partial t^{n+1}}u(t)\right| \le A_n(t)\left|\frac{\partial^n}{\partial t^n}u(t)\right| + B_n(t)$, which is sometimes easier to prove.

We will study the structure of the space of solutions of linear generalized ODEs in Section **??**. However, let us show an interesting particular consequence of Theorem 43 for GSF solutions of classical linear smooth problems:

Corollary 45. *Let $a(t), b(t) \in \mathcal{C}^\infty(\mathbb{R})$. Then for every $u_0 \in {}^\rho\widetilde{\mathbb{R}}$ the initial value problem*

$$\begin{cases} u' = \iota^b_{\mathbb{R}}(a)u + \iota^b_{\mathbb{R}}(b); \\ u(0) = u_0, \end{cases} \tag{7.9}$$

has a unique solution in ${}^\rho\mathcal{GC}^\infty\left({}^\rho\widetilde{\mathbb{R}}_{fin} \cap {}^\rho\widetilde{\mathbb{R}}_{>0}, {}^\rho\widetilde{\mathbb{R}}\right)$.

Proof. Let $\gamma = \log\left(\log\left(d\rho^{-1}\right)\right)$ and let $\alpha \in \mathbb{R}_{>0}$.

Claim 46. For every $n \in \mathbb{N}$ there exist $p_n(x), q_n(x) \in \mathbb{R}(x)$ such that $\left|\frac{\partial^{n+1}}{\partial t^{n+1}}u(t)\right| \le p_n(\gamma)\,|u(t)| + q_n(\gamma)$.

Let us prove the claim by induction on n: for $n = 0$, since $\iota^b_{\mathbb{R}}(a)(t) \le \gamma, \iota^b_{\mathbb{R}}(b)(t) \le \gamma$ for every $t \in [0, \alpha)$, we have

$$|u'(t)| = |\iota^b_{\mathbb{R}}(a)(t)u(t) + \iota^b_{\mathbb{R}}(b)(t)| \le \gamma|u(t)| + \gamma.$$

Now assume the claim proven for every $j \le n$, and let us prove it for $n + 1$:

$$\left|\frac{\partial^{n+1}}{\partial t^{n+1}}u(t)\right| = \left|\frac{\partial^n}{\partial t^n}\left(\iota^b_{\mathbb{R}}(a)(t)u(t) + \iota^b_{\mathbb{R}}(b)(t)\right)\right| =$$

$$\left|\sum_{i=0}^n \binom{n}{i}\frac{\partial^i}{\partial t^i}\left(\iota^b_{\mathbb{R}}(a)(t)\right)\frac{\partial^{n-i}}{\partial t^{n-i}}u(t) + \frac{\partial^n}{\partial t^n}\iota^b_{\mathbb{R}}(b)(t)\right| =$$

$$\left|\sum_{i=0}^n \binom{n}{i}\iota^b_{\mathbb{R}}\left(\frac{\partial^i}{\partial t^i}a\right)(t)\frac{\partial^{n-i}}{\partial t^{n-i}}u(t) + \iota^b_{\mathbb{R}}\left(\frac{\partial^n}{\partial t^n}b\right)(t)\right| \le$$

$$\sum_{i=0}^n \binom{n}{i}\left|\iota^b_{\mathbb{R}}\left(\frac{\partial^i}{\partial t^i}a\right)(t)\right|\left|\frac{\partial^{n-i}}{\partial t^{n-i}}u(t)\right| + \left|\iota^b_{\mathbb{R}}\left(\frac{\partial^n}{\partial t^n}b\right)(t)\right|$$

For every $t \in {}^\rho\widetilde{\mathbb{R}}_{\le\alpha}$ we have that $\left|\iota^b_{\mathbb{R}}\left(\frac{\partial^i}{\partial t^i}a\right)(t)\right| < \gamma$ and $\left|\iota^b_{\mathbb{R}}\left(\frac{\partial^n}{\partial t^n}b\right)(t)\right| < \gamma$, hence

$$\left|\frac{\partial^{n+1}}{\partial t^{n+1}}u(t)\right| \le \sum_{i=0}^n \binom{n}{i}\gamma\left|\frac{\partial^{n-i}}{\partial t^{n-i}}u(t)\right| + \gamma;$$

to conclude, we use the inductive hypothesis to get

$$\left|\frac{\partial^{n+1}}{\partial t^n}u(t)\right| \leq \sum_{i=0}^{n}\binom{n}{i}\gamma\left(p_{n-1}(\gamma)\left|u(t)\right| + q_n(\gamma)\right) + \gamma,$$

which is an expression of the desired form.

Having proven the claim, we now let $A_n = p_n$ and $B_n = q_n$. Conditions ((*i*)), ((ii)) of Theorem 43 are immediately proven, and this shows that $[0,\alpha) \in M$ for every $\alpha \in \mathbb{R}_{>0}$. Therefore ${}^{\rho}\widetilde{\mathbb{R}}_{fin} \cap {}^{\rho}\widetilde{\mathbb{R}}_{>0} \subseteq M$ and our thesis is proven. □

Remark 47. More in general, assume that we are given the initial value problem

$$\begin{cases} u' = A(t)u + B(t); \\ u(0) = u_0, \end{cases} \tag{7.10}$$

and assume that $A(t), B(t) \in {}^{\rho}\mathcal{GC}^{\infty}\left({}^{\rho}\widetilde{\mathbb{R}}, {}^{\rho}\widetilde{\mathbb{R}}\right)$ are "logarithmic in all derivatives in a Fermat neighborhood of 0" (namely $\forall n \in \mathbb{N}\, \exists r \in \mathbb{R}_{>0}, N \in \mathbb{N}$ such that $||A||_{n,[-r.r]} < N \cdot \log\left(d\rho^{-1}\right), ||B||_{n,[-r.r]} < N \cdot \log\left(d\rho^{-1}\right)$). Then with a very similar proof to that of Corollary 45 it is possible to show that the maximal set of existence of the unique solution of the initial value problem 7.10 contains a Fermat neighborhood of 0.

Of course, the converse of Remark 47 does not hold true, as the following example shows:Questo esempio va ricontrollato nei dettagli

Example 48. Consider the initial value problem

$$\begin{cases} u' = \iota_{\mathbb{R}}^{b}(\delta) \cdot u; \\ u(0) = 1. \end{cases} \tag{7.11}$$

This problem has a global solution, namely $u(t) = e^{\iota_{\mathbb{R}}^{b}(H)(t)}$. Observe that, in the notations of Remark 47, we have $A(t) = \delta(t)$, $B(t) = 0$, and $A(t)$ is not logarithmically bounded on any Fermat neighborhood of 0, as $\delta(0) > N \cdot \log\left(d\rho^{-1}\right)$ for every $N \in \mathbb{N}$. However, notice that whilst the initial value problem 7.11 has a unique solution, the seemingly similar problem

$$\begin{cases} u' = \iota_{\mathbb{R}}^{b}(\delta') \cdot u; \\ u(0) = 1, \end{cases}$$

has no solutions, as its solution would have the form $e^{\iota_{\mathbb{R}}^{b}(\delta)(t)}$, which is not a GSF in any sharp neighborhood of 0.

7.5. **The flow of a generalized ODE.** Flow for weak solutions/Mardsen/Spornberger thesis

Theorem 49. *Let $\emptyset \neq K \Subset_f {}^{\rho}\widetilde{\mathbb{R}}$ be a solid set and let $F \in {}^{\rho}\mathcal{GC}^{\infty}\left(K \times {}^{\rho}\widetilde{\mathbb{R}}^d, {}^{\rho}\widetilde{\mathbb{R}}^d\right)$. Let us assume CHECK. Then there exists a unique GSF $\Phi(t,x) : \widetilde{M} \times {}^{\rho}\widetilde{\mathbb{R}}^d \to {}^{\rho}\widetilde{\mathbb{R}}$ such that*

(i) $\frac{\partial}{\partial t}\Phi(t,x) = f\left(t, \Phi(t,x)\right);$
(ii) $\Phi(0,\cdot) = id;$
(iii) $\Phi(t+s,\cdot) = \Phi\left(t, \Phi(s,\cdot)\right).$

Proof. First of all, let us notice that in such a function Φ (which, in analogy with the classical case, we will call "flux of the ODE") exists, then for every $x \in {}^{\rho}\widetilde{\mathbb{R}}^{d}$ we have that $u_x(t) := \Phi(t,x)$ is the unique solution of the initial value problem

$$\begin{cases} u' = F(t,u); \\ u(0) = x, \end{cases} \tag{7.12}$$

hence the uniqueness of Φ is an immediate consequence of Theorem 24.

Regarding the existence, for every $(t,x) \in \widetilde{M} \times {}^{\rho}\widetilde{\mathbb{R}}^{d}$ we let $\Phi(t,x)$ be the unique solution of the initial value problem 7.12. It remains to prove that $(t,x) \to \Phi(t,x)$ is a GSF. Va fatta ϵ wise, copiare Sporneberg sistemando solo la cosa sulle derivate. $\square$

8. Characterization of distributions among GSF

8.1. **Embedded distributions.** In this section we want to characterize those GSF that are obtained as embedding of distributions with respect to an abstract embedding ι_{Ω}^{b}

$$\iota_{\Omega}^{b} : \mathcal{D}' \to {}^{\rho}\mathcal{GC}^{\infty}(\Omega^{\bullet}, {}^{\rho}\widetilde{\mathbb{R}})$$

defined by using an infinite $b \in {}^{\rho}\widetilde{\mathbb{R}}$ and a Colombeau mollifier μ, where

$$\Omega^{\bullet} = \{x \in \mathrm{c}(\Omega) \mid \exists x^{\circ} \in \Omega\}$$

(see Section **??** for details).

Remark 50. By Theorem 18 of [20], we have that if $f \in {}^{\rho}\mathcal{GC}^{\infty}(\mathrm{c}(\Omega), {}^{\rho}\widetilde{\mathbb{R}})$ is such that $f|_{\Omega^{\bullet}} = 0$ then $f = 0$, and by Theorem 21 of [20] we have that ι_{Ω}^{b} is injective. Thus also

$$T \in \mathcal{D}'(\Omega) \longmapsto \iota_{\Omega}^{b}(T)|_{\Omega^{\bullet}} \in {}^{\rho}\mathcal{GC}^{\infty}(\Omega^{\bullet}, {}^{\rho}\widetilde{\mathbb{R}})$$

is injective. We identify a distribution $T \in \mathcal{D}'(\Omega)$ with a function $g \in \mathcal{C}^{k}(\Omega)$ if $T(f) = \int_{\Omega} f(x)g(x)dx$ for every $f \in \mathcal{D}(\Omega)$.

Definition 51. Let $V \subseteq \Omega^{\bullet}$, $f \in {}^{\rho}\mathcal{GC}^{\infty}(V, {}^{\rho}\widetilde{\mathbb{R}})$ and $F \subseteq \mathcal{D}'(\Omega)$. Then we say that:

(i) f is an embedded F-function (relatively to ι_{Ω}^{b}) if $V = \Omega^{\bullet}$ and there exists $T \in F$ such that $f = \iota_{\Omega}^{b}(T)$; if the distribution T corresponds to $g \in \mathcal{C}^{k}(\Omega)$ we will say that f is an embedded $\mathcal{C}^{k}(\Omega)$ function;

(ii) f is a restricted embedded F-function (relatively to ι) if $\exists T \in F$ such that $f = \iota_{\Omega}^{b}(T)|_{V}$.

Remark 52. Let us notice that from the definition it easily follows that f is a restricted embedded F-function iff $\exists \hat{f} \in {}^{\rho}\mathcal{GC}^{\infty}(\Omega^{\bullet}, {}^{\rho}\widetilde{\mathbb{R}})$ such that $\hat{f}$ is an embedded F-function with $\hat{f}|_{V} = f$.

In this section we want to characterize which GSF are embedded distributions. The main technical result involving embedded GSF that we will use is the following localization Lemma.

Lemma 53. *Let $f \in \mathcal{GC}^{\infty}\left(\Omega^{\bullet}, {}^{\rho}\widetilde{\mathbb{R}}\right)$. Assume that $F \subseteq \mathcal{D}'$ be a subsheaf such that*

$$\forall W \in \tau_{\mathbb{R}^n}\ \iota_W|_{F(W)} : F(W) \rightarrowtail {}^{\rho}\mathcal{GC}^{\infty}(\Omega^{\bullet}, {}^{\rho}\widetilde{\mathbb{R}})$$

and

$$\left(\iota_W|_{F(W)}\right) : F \to {}^{\rho}\mathcal{GC}^{\infty}((-)^{\bullet}, {}^{\rho}\widetilde{\mathbb{R}})$$

is a natural transformation. Then the following facts are equivalent:

(i) *f is an embedded $F(\Omega)$-function;*

(ii) *$\forall x \in \Omega\, \exists W \in \tau_{\mathbb{R}^n}$ such that $x \in W \subseteq \Omega$ and $f|_{W^\bullet}$ is an embedded $F(W)$-function;*

(iii) *$\forall x \in \Omega\, \exists W \in \tau_{\mathbb{R}^n}$ such that $x \in W \subseteq \Omega$, W is relatively compact in Ω and $f|_{W^\bullet}$ is an embedded $F(W)$-function*

Proof. (i)⇒(ii) It sufficies to set $W := \Omega$, so $f|_{^\rho\widetilde{\Omega'}}$ is an embedded distribution.

(ii) ⇒(i) For every $x \in \Omega$ let W_x be an open subset of Ω with $x \in \Omega$, and let $g_x \in F(W_x)$ be such that

$$f|_{^\rho\widetilde{W_x}} = \iota^b_{W_x}(g_x).$$

$\mathcal{C} = \{W_x \mid x \in \Omega\}$ is an open covering of Ω and $\{g_x \mid x \in \Omega\}$ is a coherent family of distributions with respect to $\mathcal{C}$, since for every $x, y \in \Omega$ we have

$$\iota^b_{W_x\cap W_y}\left(g_x|_{W_x\cap W_y}\right) = \iota^b_{W_x}(g_x)|_{^\rho(\widetilde{W_x\cap W_y})} = f|_{^\rho(\widetilde{W_x\cap W_y})} =$$

$$\iota^b_{W_y}(g_y)|_{^\rho(\widetilde{W_x\cap W_y})} = \iota^b_{W_x\cap W_y}\left(g_y|_{W_x\cap W_y}\right),$$

and as $\iota^b_{\Omega_x\cap\Omega_y}$ is injective by hypothesis, we get that $g_x|_{\Omega_x\cap\Omega_y} = g_y|_{\Omega_x\cap\Omega_y}$.

By the sheaf properties of F we get that there exists $g \in F$ such that for every $x \in \Omega$ $g|_{W_x^\bullet} = g_x = f|_{W_x^\bullet}$. In particular, for every $y \in \Omega^\bullet$ we have that $y \in W^\bullet_{y^\circ}$, hence we deduce that $\forall y \in \Omega^\bullet\, g(y) = f(y)$, and we conclude by using Remark 50.

(iii)⇒(ii) is trivial.

(ii)⇒(iii) Let $x \in W = \bigcup\{A \subseteq W \mid \overline{A}\,\text{is relatively compact in}\, W\}$. Hence there exists a relatively compact set A such that $x \in A \subseteq W \subseteq \Omega$. In particular, $\Omega^\bullet \subseteq W^\bullet$. Let $T \in F(W)$ be such that $f|_{W^\bullet} = \iota^b_W(T)$. Then

$$f|_{A^\bullet} = \left(f|_{W^\bullet}\right)|_{A^\bullet} = \iota^b_W(T)|_{A^\bullet} = \iota^b_A\left(T|_A\right),$$

and we conclude as $T|_A \in F$ by hypothesis. □

Our characterization relies on the notion of "standard part" of a GSF.

Definition 54. Let $f \in {}^\rho\mathcal{GC}^\infty(\Omega^\bullet, {}^\rho\widetilde{\mathbb{R}})$. Then we say that:

(i) f is near standard if and only if $\forall x \in \Omega\, f(x) \in {}^\rho\widetilde{\mathbb{R}}^\bullet$;

(ii) f is strongly near standard if and only if $\forall x \in \Omega^\bullet\, f(x) \in {}^\rho\widetilde{\mathbb{R}}^\bullet$;

(iii) if f is near standard, then we will denote by $f^\circ : \Omega \to \mathbb{R}$ the function such that $\forall x \in \Omega\ f^\circ(x) = (f(x))^\circ \in \mathbb{R}$;

(iv) f is standard if and only if there exists $g \in \mathcal{C}^0(\Omega, \mathbb{R})$ such that $f = \iota^b_\Omega(g)$.

Remark 55. If f is near standard then $\forall x \in \Omega\, f^\circ(x) = (f(x))^\circ \sim f(x)$ exactly as if $x \in \Omega^\bullet$ then $x^\circ \sim x$. This justifies the names in Definition 54.

The following result is a modification of Theorem controllare nella versione definitiva del vostro lavoro in [20].

Lemma 56. *Let $g \in \mathcal{C}^0(\Omega, \mathbb{R})$. Then:*

(i) *$\iota^b_\Omega(g)$ is strongly near standard;*

(ii) *$\forall x \in \Omega^\bullet\ \left(\iota^b_\Omega(g)(x)\right)^\circ = g(x^\circ)$. In particular $\left(\iota^b_\Omega(g)\right)^\circ = g$.*

Therefore, every standard GSF is strongly near standard.

Proof. (i) Let $x = [x_\varepsilon] \in \Omega^\bullet$ with $y = x^\circ \in \Omega$. Let $r, s \in \mathbb{R}_{>0}$ be such that $\forall^\circ\varepsilon\, d(x_\varepsilon, \Omega^c) \geq r$, $|x_\varepsilon| \leq s$. Then $\chi_\varepsilon(x_\varepsilon) = 1$, and so

$$\iota_\Omega^b(g)(x) = \left[\left(g * \mu_\varepsilon^b\right)(x_\varepsilon)\right].$$

As $x^\circ \in \Omega$, there exists a relatively compact set $K \subset \Omega$ such that for ε sufficiently small $x_\varepsilon \in K$. But $\left(g * \mu_\varepsilon^b\right) \to g$ as $\varepsilon \to 0^+$ on compact subsets of Ω, therefore

$$\left|\left(g * \mu_\varepsilon^b\right)(x_\varepsilon) - g(x^\circ)\right| \leq \left|\left(g * \mu_\varepsilon^b\right)(x_\varepsilon) - g(x_\varepsilon)\right| + |g(x_\varepsilon) - g(x^\circ)|$$
$$\left\|\left(g * \mu_\varepsilon^b\right) - g\right\|_K^\infty + |g(x_\varepsilon) - g(x^\circ)| \to 0.$$

Then $f(x) = \left[\left((\chi_\varepsilon \cdot g) * \mu_\varepsilon^b\right)(x_\varepsilon)\right]$. Then

$$(f(x))^\circ = \lim_{\varepsilon\to 0^+} f_\varepsilon(x_\varepsilon) = \lim_{\varepsilon\to 0^+} \left((\chi_\varepsilon \cdot g) * \mu_\varepsilon^b\right)(x_\varepsilon) = g(y)$$

as g is continuous. Hence f strongly admits a standard part, and $f^\circ = g$, so $f = \iota_\Omega^b(g) = \iota_\Omega^b(f^\circ)$.

(ii) is an immediate consequence of (i). □

Remark 57. Notice that Lemma 56 is easily generalizable to all ι that satisfy the following property: $\forall g \in \mathcal{C}^0(\Omega)\, \exists\, (g_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I$:

- (g_ε) defines $\iota_\Omega(g)$;
- $(g_\varepsilon) \to g$ as $\varepsilon \to 0^+$ uniformly on compact subsets of Ω.

Together with Lemma 53, the following result will provide a characterization of all embedded distributions.

Theorem 58. *Let $f \in {}^\rho\mathcal{GC}^\infty(\Omega^\bullet, {}^\rho\widetilde{\mathbb{R}})$, $F(\Omega) \subseteq \mathcal{C}^0(\Omega)$, and let ι_Ω^b satisfy the properties of Remark 57. The following properties are equivalent:*

(i) *f is an embedded $F(\Omega)$-function;*
(ii) *f is near standard, $f^\circ \in F(\Omega)$, $f = \iota_\Omega^b(f^\circ)$;*
(iii) *f is strongly near standard, $f^\circ \in F(\Omega)$, $f = \iota_\Omega^b(f^\circ)$.*

Proof. (i)⇒(ii) Let $g \in F(\Omega)$ be such that $f = \iota_\Omega^b(g)$. As $F(\Omega) \subseteq \mathcal{C}^0(\Omega)$, by applying Lemma 56 we get that $\iota_\Omega^b(g) = f$ is near standard. But then $f^\circ = \left(\iota_\Omega^b(g)\right)^\circ = g \in F(\Omega)$, and so $f = \iota_\Omega^b(f^\circ)$ as desired.

(ii)⇒(iii) As $f^\circ \in F(\Omega) \subseteq \mathcal{C}^0(\Omega)$, by Lemma 56 we get that $\iota_\Omega^b(f^\circ) = f$ is strongly near standard.

(iii)⇒(i) This is trivial. □

As an immediate corollary we obtain the following characterization of embedded $\mathcal{C}^\alpha(\Omega)$ functions.

Corollary 59. *Let $\alpha \in \mathbb{N}^n$ and let $f \in {}^\rho\mathcal{GC}^\infty(\Omega^\bullet, {}^\rho\widetilde{\mathbb{R}})$. The following properties are equivalent:*

(i) *there exists $g \in \mathcal{C}^\alpha(\Omega)$ such that $f = \iota_\Omega^b(g)$;*
(ii) *f is near standard, $f^\circ \in \mathcal{C}^\alpha(\Omega)$, $f = \iota_\Omega^b(f^\circ)$;*
(iii) *f strongly admits a standard part $f^\circ \in \mathcal{C}^\alpha(\Omega)$ and $f = \iota_\Omega^b(f^\circ)$.*

Both requests in Condition (iii) of Theorem 58 are needed to prove that f is an embedded continuous function.

Example 60. Let f be the GSF such that for every $x \in \Omega^\bullet$ $f(x) = b^{-1}$. Then f strongly admits 0 as its standard part; however, $i_\Omega^b(0) = 0 \neq b^{-1}$.

Definition 61. Let $F \subseteq \mathcal{D}'(\Omega)$ be a subpresheaf, and let $\alpha \in \mathbb{N}^n$. We set

$$F^{-\alpha}(\Omega) = \{D^\alpha T \mid T \in F\}\,.$$

Theorem 62. *Let $\alpha \in \mathbb{N}^n$ and let $f \in {}^\rho\mathcal{GC}^\infty(\Omega^\bullet, {}^\rho\widetilde{\mathbb{R}})$. Let $F \subseteq \mathcal{D}'$ be a subpresheaf. The following properties are equivalent:*

(i) *f is an embedded $F^{-\alpha}(\Omega)$-function;*

(ii) *there exists $g \in {}^\rho\mathcal{GC}^\infty(\Omega^\bullet, {}^\rho\widetilde{\mathbb{R}})$ which is an embedded $F(\Omega)$-function such that $\partial^\alpha g = f$.*

Proof. (i)⇒(ii) Let $T \in F^{-\alpha}(\Omega)$ be such that $f = \iota^b_\Omega(T)$. Let $g \in F$ be such that $T = D^\alpha g$. Then $\partial^\alpha\left(\iota^b_\Omega(g)\right) = \iota^b_\Omega(D^\alpha(g)) = i^b_\Omega(T) = f$.

(ii)⇒(i) Let $g \in {}^\rho\mathcal{GC}^\infty(\Omega^\bullet, {}^\rho\widetilde{\mathbb{R}})$ be such that $f = \partial^\alpha g$, and let $g = \iota^b_\Omega(T)$ for $T \in F$. Then $f = \partial^\alpha(g) = \left(\iota^b_\Omega D^\alpha(T)\right) = \partial^\alpha(\iota^b_\Omega(g))$. □

The characterization of embedded distributions can be deduced from Theorem **??** and Lemma 53 by applying the local characterization theorem of distributions as weak derivatives of continuous functions (see e.g. [49], Chapter 6):

Theorem 63. *Let $f \in {}^\rho\mathcal{GC}^\infty(\Omega^\bullet, {}^\rho\widetilde{\mathbb{R}})$. The following properties are equivalent:*

(i) *there exists $T \in \mathcal{D}'(\Omega)$ such that $f = \iota^b_\Omega(T)$;*

(ii) *for every $x \in \Omega$ there exists $K \Subset \Omega$, a multi-index α and a continuous function $g \in \mathcal{C}(K)$ such that $x \in \overset{\circ}{K}$ and $f_{|_{\overset{\circ}{K}}} = \partial^\alpha\left(\iota^b_{\overset{\circ}{K}}(g)\right)$.*

Proof. (i)⇒(ii) Let $x \in \Omega$ and let $K \Subset \Omega$ be such that $x \in \overset{\circ}{K}$. By the sheaf properties of i^b_Ω (see Theorem **??**) we have that $f_{|_{\overset{\circ}{K}}} = \iota^b_{\overset{\circ}{K}}\left(T_{|_K}\right)$. $T_{|_K}$ is a compactly supported distribution, hence there exists $g \in \mathcal{C}(\Omega)$ and a multi-index α such that $T_{|_K} = \partial^\alpha(g)$. Hence $f_{|_{\overset{\circ}{K}}} = \iota^b_{\overset{\circ}{K}}\left(T_{|_K}\right) = \iota^b_{\overset{\circ}{K}}\left(\partial^\alpha(g)\right) = \partial^\alpha\left(\iota^b_{\overset{\circ}{K}}(g)\right)$.

(ii)⇒(ii) For every $x \in \Omega$ let $K_x \Subset \Omega$, α, $g_x \in \mathcal{C}(K)$ be such that $x \in \overset{\circ}{K}_x$ and $f_{|_{\overset{\circ}{K}_x}} = \partial^\alpha\left(\iota^b_{\overset{\circ}{K}_x}(g_x)\right)$. In particular, this shows that for every $x \in \Omega$ there exists $T_x \in \mathcal{D}'\left(\overset{\circ}{K}_x\right)$ such that $f_{|_{\overset{\circ}{K}}} = \iota^b_{\overset{\circ}{K}}(T_x)$. The family $\{\Omega_x \mid x \in \Omega\}$ gives an open covering of Ω; moreover, for every $x, y \in \Omega$ we have that

$$(T_x)_{|_{\overset{\circ}{K}_x \cap \overset{\circ}{K}_y}} = (T_x)_{|_{\overset{\circ}{K}_x \cap \overset{\circ}{K}_y}}\,.$$

By the sheaf properties of $\mathcal{D}'(\Omega)$ hence there is $T \in \mathcal{D}'(\Omega)$ such that for every $x \in \Omega$ $T_{|_{\overset{\circ}{K}_x}} = T_x$, and hence by Lemma 53 we deduce that $f = \iota^b_\Omega(T)$. □

Example 64. Let $\left(e^{\left(\frac{1}{\varepsilon}\right)}\right) = O(\rho_\varepsilon)$ and let $f(x) = e^{\left[\frac{1}{\varepsilon}\right]t}$. Let $b \in {}^\rho\widetilde{\mathbb{R}}$ be positive infinite and let Ω be an open neighborhood of 0. Is $f(x)$ an embedded distribution in ${}^\rho\mathcal{GC}^\infty(\Omega^\bullet, {}^\rho\widetilde{\mathbb{R}})$ w.r.t ι^b_Ω? The answer is no: no primitive of $f(x)$ strongly admits a standard part on any Fermat neighborhood of zero, hence we conclude by applying Theorem 63.

Condition (ii) in Theorem 63 can be summarized by saying that f, to be an embedded distributions, needs to have locally standard $\mathcal{C}^k$-primitives. As in the classical setting, two different primitives of a GSF differ by a polynomial (which, in the GSF case, has coefficients in ${}^{\rho}\widetilde{\mathbb{R}}$). Hence, for applications, it is important to know two things: when a generalized polynomial admits a standard part, and when it is an embedded distribution.

Lemma 65. *Let $P(x) \in {}^{\rho}\widetilde{\mathbb{R}}[x]$. Then $P(x)$ admits a standard part if and only if all its coefficients are near standard.*

Proof. Let $P(x) = \sum_{i=1}^{n} c_i x^i$. If c_i is near standard for every $i \leq n$ then $P^{\circ}(x) = \sum_{i=1}^{n} c_i^{\circ} x^i$, so $P(x)$ admits a standard part.

Conversely: let us assume that $P(x)$ admits a standard part and, by contrast, let us assume that at least one of the coefficients c_i is not near standard. Let $c_i = [c_{i,\varepsilon}]$ for every $i \leq n$. There are two possible cases:

Case 1: all the nets $(c_{i,\varepsilon})$ are bounded. Let c_j be a coefficient that does not admit a standard part; as c_j does not admit a standard part, there are two sequences $(\delta_m)_{m\in\mathbb{N}}$, $(\delta'_m)_{m\in\mathbb{N}}$ such that $\lim_{m\to+\infty} \delta_m = \lim_{m\to+\infty} \delta'_m = 0$, and $\lim_{m\to+\infty} c_{j,\delta_m}$ and $\lim_{m\to+\infty} c_{j,\delta'_m}$ exist finite with $\lim_{m\to+\infty} c_{j,\delta_m} \neq \lim_{m\to+\infty} c_{j,\delta'_m}$. By using a classical diagonal argument, as all the nets $(c_{i,\varepsilon})$ are bounded we can extract subsequences $(\varepsilon_m) \subset (\delta_m)$, $(\varepsilon'_m) \subset (\delta'_m)$ such that for every $i \leq n$ the limits $\lim_{m\to+\infty} c_{i,\varepsilon_m}$ and $\lim_{m\to+\infty} c_{i,\varepsilon'_m}$ exist finite. As $P(x)$ admits a standard part, for every $x \in \Omega$ we have that

$$P^{\circ}(x) = \lim_{m\to+\infty} \left(\sum_{i=1}^{n} c_{i,\varepsilon_m} x^i \right) = \lim_{m\to+\infty} \left(\sum_{i=1}^{n} c_{i,\varepsilon'_m} x^i \right).$$

Since the limits $\lim_{m\to+\infty} c_{i,\varepsilon_m}$ and $\lim_{m\to+\infty} c_{i,\varepsilon'_m}$ exist finite, we also have that

$$\lim_{m\to+\infty} \left(\sum_{i=1}^{n} c_{i,\varepsilon_m} x^i \right) = \sum_{i=1}^{n} \left(\lim_{m\to+\infty} c_{i,\varepsilon_m} \right) x^i$$

and

$$\lim_{m\to+\infty} \left(\sum_{i=1}^{n} c_{i,\varepsilon'_m} x^i \right) = \sum_{i=1}^{n} \left(\lim_{m\to+\infty} c_{i,\varepsilon'_m} \right) x^i.$$

In particular, we have that

$$\forall x \in \Omega \sum_{i=1}^{n} \left(\lim_{m\to+\infty} c_{i,\varepsilon_m} \right) x^i = \sum_{i=1}^{n} \left(\lim_{m\to+\infty} c_{i,\varepsilon'_m} \right) x^i,$$

and this is possible if and only if $\forall i \leq n \lim_{m\to+\infty} c_{i,\varepsilon_m} = \lim_{m\to+\infty} c_{i,\varepsilon'_m}$, which is false for $i = j$. Hence we have an absurd.

Case 2: There exists an index $j \leq n$ and a subsequence (ε_m) s.t. $\lim_{m\to+\infty} \varepsilon_m = 0$ and $\lim_{m\to+\infty} c_{j,\varepsilon_m} = \pm\infty$. By the usual diagonal arguments, it is possible to extract a subsequence $(\varepsilon'_m) \subseteq (\varepsilon_m)$ such that $\lim_{m\to+\infty} c_{i,\varepsilon'_m}$ exists (finite or infinite) for every $i \leq n$. We set $I = \left\{ i = 0, \ldots, n \mid \lim_{m\to+\infty} c_{i,\varepsilon'_m} \in \{+\infty, -\infty\} \right\}$. For every $i \in I$ we let

$$M_i = \left\{ m \in \mathbb{N} \mid \left| c_{i,\varepsilon'_m} \right| \geq \left| c_{j,\varepsilon'_m} \right| \ \forall j \in \{0, \ldots, n\} \right\}.$$

As $\mathbb{N} = \bigcup_{i=1}^{n} M_i$, there exists an index $\bar{i}$ such that $M_{\bar{i}}$ is infinite. For every $m \in \mathbb{N}$ let $f(m)$ be the m-th element of M_i and let $\delta_m = \varepsilon'_{f(m)}$. By construction, there are sequecences $(r_{0,\delta_m}), \ldots, (r_{n,\delta_m})$ such that

- $\forall i = 0, \ldots, n$, $\forall m \in \mathbb{N}$ $|r_{i,\delta_m}| \leq 1$;

- $\forall i = 0, \dots, n$, $\forall m \in \mathbb{N}$ $c_{i,\delta_m} = r_{i,\delta_m} \cdot c_{\bar{i},\delta_m}$.

For every $x \in \Omega$ we have that $c_{n,\delta_m} x^n + \dots + c_{0,\delta_m} = c_{\bar{i},\delta_m} \cdot (r_{n,\delta_m} x^n + \dots + r_{0,\delta_m})$. As $P(x)$ has a standard part and $\lim_{m\to+\infty} c_{\bar{i},\delta_m} = \pm\infty$, it must be

$$\lim_{m\to+\infty} (r_{n,\delta_m} x^n + \dots + r_{0,\delta_m}) = 0.$$

By extracting a subsequence $(\delta'_m) \subseteq (\delta_m)$ we can suppose that for every $i = 0, \dots, n$ $\lim_{m\to+\infty} r_{i,\delta_m}$ exists finite. Then for every $x \in \Omega$

$$0 = \lim_{m\to+\infty} \left(r_{n,\delta'_m} x^n + \dots + r_{0,\delta'_m}\right) = \sum_{i=0}^{n} \left(\lim_{m\to+\infty} r_{i,\delta'_m}\right) x^i,$$

and this is possible if and only if $\lim_{m\to+\infty} r_{i,\delta'_m} = 0$ for every $i \le n$, which is absurd as $\lim_{m\to+\infty} r_{\bar{i},\delta'_m} = 1$. □

Theorem 66. *Let $P(x) = \sum_{i=0}^{n} c_i x^i \in {}^{\rho}\widetilde{\mathbb{R}}(x)$. Then $P(x)$ is an embedded distribution if and only if $c_i \in \mathbb{R}$ for every $i = 0, \dots, n$.*

Proof. If $P(x) \in \mathbb{R}(x)$ then $P(x) \in \mathcal{C}^\infty(\Omega)$, and so $\iota^b_\Omega(P(x)) = P(x)$, which shows that $P(x)$ is an embedded distribution.

Conversely, let us assume that $P(x) \in {}^{\rho}\widetilde{\mathbb{R}}(x)$ is an embedded distribution. Let $x \in \Omega$. By Theorem 63 locally $P(x)$ is the derivative of an embedded continuous function, namely there exists $k \in \mathbb{N}$ such that locally a k-th primitive of $P(x)$ is an embedded continuous function. The k-th primitives of $P(x)$ have the form

$$P^{-k}(x) = \sum_{i=0}^{n} c_i x^{i+k} + Q(x),$$

where $Q(x) \in {}^{\rho}\widetilde{\mathbb{R}}(x)$ is a generalized polynomial with degree smaller than k. Let $Q(x) = \sum_{j=0}^{k-1} q_j x^j$. As $P^{-k}(x)$ is an embedded continuous function, it must have a standard part, so by Lemma 65 we deduce that $\forall i = 0, \dots, n$ c_i is near standard and $\forall j = 0, \dots, k-1$ q_j is near standard. Hence

$$\left(P^{-k}\right)^\circ(x) = \sum_{i=0}^{n} (c_i)^\circ x^i + \sum_{j=0}^{k-1} (q_j)^\circ x^j \in \mathcal{C}^\infty(\Omega),$$

so, as by Theorem 58 it must be $P^{-k}(x) = i^b_\Omega\left(\left(P^{-k}\right)^\circ(x)\right)$, we deduce that $P^{-k}(x) = \sum_{i=0}^{n} (c_i)^\circ x^i + \sum_{j=0}^{k-1} (q_j)^\circ x^j$. Therefore $\forall i = 0, \dots, n$ $c_i = c_i^\circ \in \mathbb{R}$. □

Example 67. Let $\Omega \subseteq \mathbb{R}$ be open, and let $r \in \widetilde{\mathbb{R}}$. Let $f \in {}^{\rho}\mathcal{GC}^\infty(\Omega^\bullet, {}^{\rho}\widetilde{\mathbb{R}})$ be such that $f(x) = r$ for every $x \in \Omega^\bullet$. Is $f(x)$ an embedded distribution? Let $x \in \Omega$, and let us assume that f is an embedded distribution. By Theorem **??** we find a Fermat open neighborhood Ω' of x, a natural number k and a continuous function $g \in \mathcal{C}^0(\Omega)$ such that

$$F_k = i^b_{\Omega'}(g)|_{{}^{\rho}\widetilde{\Omega'}},$$

where F_k is a k-th primitive of f. As f is constant, this k-th primitive has the form $F_k(x) = rx^k + P(x)$, where $P(x)$ is a polynomial whose degree is less than k. From Theorem **??** we get that F_k has to have a standard part F_k° and $F_k = i^b_\Omega(F_k^\circ)$. As F_k has to be equal to the embedding of its standard part, which is a polynomial, we also deduce from Theorem 66 that it must be $r = r^\circ$, and this happens if and only if $r \in \mathbb{R}$. Therefore we have that f is an embedded distribution if and only if $r \in \mathbb{R}$, in which case f is actually an embedded $\mathcal{C}^\infty$-function.

Example 68. Let $\Omega \subseteq \mathbb{R}$ be open, let $h \in \mathcal{D}(\Omega)$, $h \neq 0$ and let $J \in {}^\rho\widetilde{\mathbb{R}}$ be infinite. Let $f(x) = J \cdot h(x)$ for every $x \in \Omega^\bullet$ (we identify $h(x)$ with its embedding). Let us assume that f is an embedded distribution. Let $x \in \Omega$. As usual, by applying Theorem **??** we find a Fermat neighborhood Ω' of x, a natural number k and a continuous function g such that

$$F_k = i^b_{\Omega'}(g)$$

on Ω', where F_k is a k-th primitive of f. The k-th primitives of f have the following expression:

$$F_k = J \cdot (H_k(x) + P(x)),$$

where $H_k(x)$ is (the embedding of) a (classical) k-th primitive of h, and $P(x) \in {}^\rho\widetilde{\mathbb{R}}[x]$ is a polynomial whose degree is less than k. From Theorem **??** we get that F_k must have a standard part. As J is infinite, this entails that $H_k(x) + P(x)$ must have the function constantly equal to 0 as its standard part. In particular, as $(H_k)^\circ = H_k$, we deduce that P has to have a standard part, and so $H_k = P^\circ$. As $\deg P < k$ we deduce that $\deg P^\circ < k$, namely H_k is a polynomial whose degree is less than k. But then $h = \partial^k H_k = 0$, and this is absurd. Hence we have that $f(x)$ is not an embedded distribution on Ω.

Example 69. Let Ω, h and J be as in Example 68. Let $f(x) = J \cdot h(J \cdot x)$ (notice that $f(x)$ has some similarities with $\iota^b_\Omega(\delta)$). Let us assume that $f(x)$ is an embedded distribution, and let $x \in \Omega$. By applying Theorem **??** we find a Fermat neighborhood Ω' of x, a natural number k and a continuous function g such that

$$F_k = i^b_{\Omega'}(g)$$

on Ω', where F_k is a k-th primitive of f. The k-th primitives of f have the following expression:

$$F_k = J^{1-k} H_k(J \cdot x) + P(x),$$

where $H_k(x)$ is (the embedding of) a (classical) k-th primitive of h, and $P(x) \in {}^\rho\widetilde{\mathbb{R}}[x]$ is a polynomial whose degree is less than k. From Theorem **??** we get that F_k must have a standard part. But, as $H_k(x) \in \mathcal{D}(\Omega)$, we have that

$$F_k^\circ(x) = \begin{cases} P^\circ(x), & \text{if}\, x \neq 0; \\ J^{1-k} H_k(0) + P^\circ(0), & \text{if}\, x = 0. \end{cases}$$

In particular, this means that P° exists, and so $P^\circ(0) \in \mathbb{R}$. Therefore, as J is infinite, if $k = 0$ and $H_k(0) \neq 0$ we have an absurd. If $k = 0$ and $H_k(0) = 0$ we have that F_k° is a polynomial with degree less than k and, since $F_k = i_{\Omega'}(F_k^\circ)$, we deduce that F_k is a polynomial with degree less than k and so $f = D^k F_k = 0$, and this entails $h \equiv 0$, which is absurd. So we can assume that $k > 0$. It must be $J^{1-k} H_k(0) + P(0) \in \mathbb{R}^\bullet$. If $k > 1$ then $\left(J^{1-k} H_k(0)\right)^\circ = 0$, hence $F_k^\circ = P^\circ$ and we can argue as before. And this concludes the proof as, if F_1 is an embedded $\mathcal{C}^0$-function then also F_2 is (for some $P(x)$), and we showed that this is impossible.

8.2. **Association.** In Colombeau theory it has been also considered a weaker correspondence between distributions and generalized functions, called association (see e.g. REFERENCES). An analogous notion can be introduced also for GSF:

Definition 70. Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $f, g \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\Omega}, {}^\rho\widetilde{\mathbb{R}})$. We say that f, g are associated (notation: $f \sim g$) if for every $h \in \mathcal{D}(\Omega)$ we have that

$$\int_{\widetilde{\Omega}} (f-g)h dx \approx 0.$$

A straightforward consequence of Definition 70 is that, for every GSF f, g, if $f \sim g$ then $Df \sim Dg$.

In Section 8.1 we showed that not all the functions that have a continuous standard part are embedded distributions. However, they are always associated to the embedding of their standard part:

Theorem 71. *Let $f \in {}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}})$. If f strongly admits a continuous standard part f° then $f \sim \iota_\Omega^b(f^\circ)$.*

To prove Theorem 71 we will use the following Lemma

Lemma 72. *Let $f, g \in {}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}})$. If $f(x) \approx g(x)$ for every near standard point $x \in \mathrm{c}(\Omega)$ then $f(x) \approx g(x)$ for every $x \in \mathrm{c}(\Omega)$.*

Proof. By contrast, let $x = [x_\varepsilon] \in \mathrm{c}(\Omega)$ be such that $f(x) \not\approx g(x)$. Hence there exists a sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ and a real number $r > 0$ such that $\lim_{m\to+\infty} \varepsilon_m = 0$ and $\lim_{m\to+\infty} |f_{\varepsilon_m}(x_{\varepsilon_m}) - g_{\varepsilon_m}(x_{\varepsilon_m})| > r$. As $x \in \mathrm{c}(\Omega)$, we can extract a subsequence $(\delta_m) \subset (\varepsilon_m)$ such $\lim_{m\to+\infty} \delta_m$ exists finite. For every $\varepsilon \in (0, 1]$ let $m(\varepsilon) := \min\{m \in \mathbb{N} \mid \delta_m < \varepsilon\}$. Let $y = \left[y_{m(\varepsilon)}\right]$. Then y is near standard but $|f(y) - g(y)| > r$, which is absurd. $\square$

We can now prove Theorem 71.

Proof. f strongly admits a standard part f°, hence for every near standard point $x \in \mathrm{c}(\Omega)$ we have that $f(x) \approx f^\circ(x^\circ)$. By hypothesis, f is continuous, therefore for every near standard point $x \in \mathrm{c}(\Omega)$ $f^\circ(x^\circ) \approx \iota_\Omega^b(f^\circ)(x)$. Therefore for every near standard point $x \in \mathrm{c}(\Omega)$ we have that $f(x) \approx \iota_\Omega^b(f^\circ)(x)$, and hence by Lemma 72 we get that $f(x) \approx \iota_\Omega^b(f^\circ)(x)$ for every $x \in \mathrm{c}(\Omega)$. Now let $\varphi \in \mathcal{D}(\Omega)$. As φ has a compact support, we have that

$$\int_{{}^\rho\widetilde{\Omega}} \left(f - \iota_\Omega^b(f^\circ)\right) \cdot \varphi dx = \int_{{}^\rho\widetilde{supp(\varphi)}} \left(f - \iota_\Omega^b(f^\circ)\right) \cdot \varphi dx.$$

Let $M = \max|\varphi|$. Then

$$\left|\int_{{}^\rho\widetilde{supp(\varphi)}} \left(f - \iota_\Omega^b(f^\circ)\right) \cdot \varphi dx\right| \le \int_{{}^\rho\widetilde{supp(\varphi)}} \left|f - \iota_\Omega^b(f^\circ)\right| \cdot |\varphi| \, dx \le$$

$$M \cdot \max\left|f - \iota_\Omega^b(f^\circ)\right| \cdot \mu(supp(\varphi))$$

which is infinitesimal as $M \in \mathbb{R}$, $\mu(\mathrm{stsupp}(\varphi)) \in \mathbb{R}$ and $\max\left|f - \iota_\Omega^b(f^\circ)\right| \approx 0$ as ${}^\rho\widetilde{\mathrm{stsupp}}(\varphi) \subset \mathrm{c}(\Omega)$. $\square$

As a consequence we get a criterion to test if a given GSF is associated with a distribution.

Theorem 73. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $f \in {}^\rho\mathcal{GC}^\infty(\Omega^\bullet, {}^\rho\widetilde{\mathbb{R}})$ be a GSF that strongly admits a continuous standard part g on Ω. Then $f \sim g$.*

Proof. f strongly admits a standard part, therefore

$$\forall x \in \Omega^\bullet f(x) \approx g(x^\circ).$$

As g is continuous, by Lemma 72 we have that $g(x^\circ) \approx \iota^b_\Omega(g)(x)$ for every $x \in \Omega^\bullet$. Hence we have that

$$\forall x \in \Omega^\bullet\ f(x) \approx \iota^b_\Omega(g)(x).$$

From Lemma 72 we get that

$$\forall x \in \mathrm{c}(\Omega)\ f(x) \approx \iota^b_\Omega(g)(x). \tag{8.1}$$

Now let $\varphi \in \mathcal{D}(\Omega)$. As $supp(\varphi) = K$ is a compact subset of Ω, we get that

$$\int_{{}^\rho\widetilde{\Omega}} \left(f(x) - \iota^b_\Omega(g)(x)\right) \cdot \varphi(x) dx = \int_{{}^\rho\widetilde{K}} \left(f(x) - \iota^b_\Omega(g)(x)\right) \cdot \varphi(x) dx.$$

Now let $M = \max_{x \in K} |\varphi(x)|$. We have that

$$\left|\int_{{}^\rho\widetilde{K}} \left(f(x) - \iota^b_\Omega(g)(x)\right) \cdot \varphi(x) dx\right| \le \int_{{}^\rho\widetilde{K}} \left|f(x) - \iota^b_\Omega(g)(x)\right| \cdot |\varphi(x)|\, dx \le$$

$$M \cdot \int_{{}^\rho\widetilde{K}} \left|f(x) - \iota^b_\Omega(g)(x)\right| dx.$$

But K is compact (hence it has a finite measure) and, as an immediate consequence of equation 8.1, we have that $\forall x \in {}^\rho\widetilde{K}\ f(x) \approx \iota^b_\Omega(g)(x)$, so the integral

$$\int_{{}^\rho\widetilde{K}} \left|f(x) - \iota^b_\Omega(g)(x)\right| dx$$

is infinitesimal. As M is finite, we deduce that $\left|\int_{{}^\rho\widetilde{K}} \left(f(x) - \iota^b_\Omega(g)(x)\right) \cdot \varphi(x) dx\right|$ is infinitesimal, which entails that $f \sim \iota^b_\Omega(g)$ as claimed. □

Example 74. Let $c \neq b$ be two infinite positive numbers and let $h \in \mathcal{C}(\Omega)$ be such that $\iota^b_\Omega(h) \neq \iota^c_\Omega(h)$. The standard part of $\iota^c_\Omega(h)$ is h, so we deduce from Theorem 71 that $\iota^c_\Omega(h) \sim \iota^b_\Omega(h)$, even if $\iota^c_\Omega(h)$ is not associated with a distribution with respect to ι^b_Ω: in fact, let T be such a distribution. Then we would have $\iota^b_\Omega(h) \sim \iota^c_\Omega(h) = \iota^b_\Omega(T)$, and so $\iota^b_\Omega(h) = \iota^b_\Omega(T)$, hence $h = T$, which is absurd as we assumed that $\iota^b_\Omega(h) \neq \iota^c_\Omega(h)$.

We want to prove that Theorem 73 gives a sufficient condition for a GSF to be associated with a distribution. To prove this result we will need the following technical Lemma:

Lemma 75. *Let $n \in \mathbb{N}$, let $\Omega \subseteq \mathbb{R}^n$ be open and let $f \in {}^\rho\mathcal{GC}^\infty(\Omega^\bullet, {}^\rho\widetilde{\mathbb{R}})$. Let $\{\Omega_i \mid i \in I\}$ be an open covering of Ω such that for every $i \in I$ $f|_{{}^\rho\widetilde{\Omega}_i}$ is associated with a distribution $T_i \in \mathcal{D}'(\Omega_i)$. Then f is associated with a distribution $T \in \mathcal{D}'(\Omega)$.*

Proof. The hypotheses of the Lemma ensures that $\{T_i \mid i \in I\}$ is a coherent family of distributions with respect to the open covering $\{\Omega_i \mid i \in I\}$ (this can be proved as in Lemma 53). Hence there exists $T \in \mathcal{D}'(\Omega)$ that is a gluing of the family $\{T_i \mid i \in I\}$. We claim that $f \sim T$. To prove the claim, let $\varphi \in \mathcal{D}(\Omega)$. Let $K = supp(\varphi)$; as K is compact, there is a finite subfamily $\{\Omega_1, \dots, \Omega_n\} \subseteq \{\Omega_i \mid i \in I\}$ that covers K.

For every $i = 1, \dots, n$ let $\psi_i \in \mathcal{C}^\infty(\Omega_i)$ be such that $\sum_{i=1}^n \psi_i \equiv 1$ on K. Moreover, let $\varphi_i := \varphi \cdot \psi_i \in \mathcal{D}(\Omega_i)$. Then

$$\left|\int_{^\rho\widetilde{\Omega}} (f-T)(x)\varphi(x)dx\right| = \left|\int_{^\rho\widetilde{K}} (f-T)(x)\varphi(x)dx\right| \le$$

$$\sum_{i=1}^n \left|\int_{^\rho\widetilde{\Omega_i}} (f-T)(x)\varphi_i(x)dx\right| = \sum_{i=1}^n \left|\int_{^\rho\widetilde{\Omega_i}} (f-T_i)(x)\varphi_i(x)dx\right|.$$

As $f|_{^\rho\widetilde{\Omega_i}} \sim T_i$ for every $i = 1, \dots, n$, we have that $\sum_{i=1}^n \left|\int_{^\rho\widetilde{\Omega_i}} (f-T_i)(x)\varphi_i(x)dx\right|$ is a finite sum of infinitesimals. Hence it is an infinitesimal, and so

$$\left|\int_{^\rho\widetilde{\Omega}} (f-T)(x)\varphi(x)dx\right| \approx 0$$

for every $\varphi \in \mathcal{D}'(\Omega)$, namely $f \sim T$. □

Theorem 76. *Let $n \in \mathbb{N}$, let $\Omega \subseteq \mathbb{R}^n$ be open and let $f \in {}^\rho\mathcal{GC}^\infty(\Omega^\bullet, {}^\rho\widetilde{\mathbb{R}})$. Let us suppose that for every point $p \in \Omega$ there exists a bounded open subset Ω'_p of Ω that contains p, with $\overline{\Omega'_p} \subseteq \Omega$, a GSF $F_p \in {}^\rho\mathcal{GC}^\infty(\Omega^\bullet, {}^\rho\widetilde{\mathbb{R}})$ that is associated with a distribution g_p and a multi-index $\alpha_p \in \mathbb{N}_0^n$ such that $f|_{\widetilde{\Omega'}} = D^{\alpha_p}(F_p)|_{\widetilde{\Omega'}}$. Then f is associated with a distribution.*

Proof. For every $p \in \Omega$, let Ω_p, F_p, g_p and α_p be given as in the statement of the Theorem. From Theorem 73 we deduce that $F_p \sim g_p$ for every $p \in \Omega$, and hence $f_{\widetilde{\Omega_p}} \sim \partial^{\alpha_p} g_p$ for every $p \in \Omega$. As $\partial^{\alpha_p} g_p \in \mathcal{D}'(\Omega_p)$ for every $p \in \Omega$, and as $\{\Omega_p \mid p \in \Omega\}$ is an open covering of Ω, we deduce our thesis as a consequence of Lemma 75. □

Let us notice that, in the hypotheses of Theorem 76, Lemma 75 gives a procedure to find the distribution associated with f by means of a gluing procedure involving the continuous standard parts g_p. Moreover, by mixing Theorem 76 with Theorem 73 we directly deduce the following result:

Theorem 77. *Let $n \in \mathbb{N}$, let $\Omega \subseteq \mathbb{R}^n$ be open and let $f \in \mathcal{GC}^\infty\left(\widetilde{\Omega}, \widetilde{\mathbb{R}}\right)$. Let us suppose that for every point $p \in \Omega$ there exists a bounded open subset Ω'_p of Ω that contains p, with $\overline{\Omega'_p} \subseteq \Omega$, a GSF $F_p \in \mathcal{GC}^\infty\left(\widetilde{\Omega}, \widetilde{\mathbb{R}}\right)$ that strongly admits a continuous standard part and a multi-index $\alpha_p \in \mathbb{N}_0^n$ such that $f|_{\widetilde{\Omega'}} = D^{\alpha_p}(F_p)|_{\widetilde{\Omega'}}$. Then f is associated with a distribution.*

Notice that the converse of this result does not hold: there are GSF associated to continuous functions that does not admit a continuous standard part

We conclude this section by proving for GSF a very well known fact in Colombeau theory regarding the product of embedded smooth functions with embedded distributions.

Theorem 78. *Let $T \in \mathcal{D}'(\Omega)$, $f \in \mathcal{C}^\infty(\Omega)$. Then in general $\iota^\rho_\Omega(f \cdot T) \neq \iota^\rho_\Omega(f) \cdot \iota^\rho_\Omega(T)$; however, $\iota^\rho_\Omega(f \cdot T) \approx \iota^\rho_\Omega(f) \cdot \iota^\rho_\Omega(T)$.*

Proof. To prove the first claim just let $f(x) = x$, $T = \delta$. As $x \cdot \delta = 0$ in $\mathcal{D}'(\Omega)$, we have that $\iota^\rho_\Omega(x \cdot \delta) = 0$. However, $\iota^\rho_\Omega(x) = x$, hence $\iota^\rho_\Omega(x) \cdot \iota^\rho_\Omega(\delta) \neq 0$ on any sharp open neighborhood of 0.

To prove the second claim, let $\psi \in \mathcal{D}(\Omega)$. Then

$$\int f(x)\iota_\Omega^\rho(T)(x)\psi(x)dx = \int \iota_\Omega^\rho(T)(x)\left(f\cdot\psi\right)(x)dx \sim$$
$$\langle T, f\cdot\psi\rangle = \langle f\cdot T,\psi\rangle \sim \int \iota_\Omega^\rho\left(f\cdot T\right)\cdot\psi dx.$$

□

9. The classical theory of linear ODEs for GSF

In this section we want to give a few simple examples involving linear (homogeneous and non homogeneous) ODEs. As we already discussed, in Colombeau Theory (see e.g. [13, 22]) existence and uniqueness results for generalized ODEs need some additional hypotheses involving, usually, certain "$log\varepsilon$" growth conditions for the coefficients. On the contrary, the following analogue of the classical theory holds also for GSF (where we have to talk about "modules" instead of "vector spaces" as ${}^\rho\widetilde{\mathbb{R}}$ is not, in general, a field).

Theorem 79. *Let $a_0,\dots,a_N \in {}^\rho\mathcal{GC}^\infty(\Omega^\bullet,{}^\rho\widetilde{\mathbb{R}})$, $a_N = 1$. Then:*

(i) the space S of local solutions of the linear homogeneous ODE

$$\sum_{i=0}^{N} a_i y^{(i)} = 0 \tag{9.1}$$

is an ${}^\rho\widetilde{\mathbb{R}}$-module of dimension N;

(ii) let $f \in {}^\rho\mathcal{GC}^\infty(\Omega^\bullet,{}^\rho\widetilde{\mathbb{R}})$ and let $y_p \in {}^\rho\mathcal{GC}^\infty(\Omega^\bullet,{}^\rho\widetilde{\mathbb{R}})$ be a solution of the equation

$$\sum_{i=0}^{N} a_i y^{(i)} = f. \tag{9.2}$$

Then the set of all solutions of equation 9.2 is $y_p + S$.

Proof. (i) It is immediate to prove that the set of local solutions S of equation 9.1 is an ${}^\rho\widetilde{\mathbb{R}}$-module. To prove that its dimension is N, as in the classical case we first rewrite equation 9.1 as a system $z' = F(x,z)$ (this can be done as $a_N = 1$). For every $j = 1,\dots,N$ consider the Cauchy problem

$$\begin{cases} z' = F(x,z), \\ z_i = 1, \\ z_j = 0, & \forall j \neq i. \end{cases}$$

Let $\widetilde{z_i}$ be its local solution, whose existence is ensured by Theorem 24. We claim that $\{\widetilde{z_1},\dots,\widetilde{z_N}\}$ is a basis for S. Obviously, $\widetilde{z_1},\dots,\widetilde{z_N}$ are linearly indipendent. Now assume that $y \in S$. For every $i = 1,\dots,N$ let $y_i = y^{(i)}(0)$. Then y and $z = \sum_{i=1}^{M} y_i\widetilde{z_i}$ both solve the Cauchy problem

$$\begin{cases} z' = F(x,z), \\ z_i = y_i, \end{cases}$$

hence by Theorem 24 we have that $y = z$.

(ii) is a trivial consequence of the linearity of the problem. □

The following result is an immediate consequence of Theorem 79.

Corollary 80. *Let $a_0,\dots,a_N \in \mathcal{C}^\infty(\Omega)$, with $a_N = 1$. Let $f_1,\dots,f_N \in \mathcal{C}^\infty(\Omega)$ be a basis of the vector space of solutions of the classical homogeneous equation $\sum_{i=0}^N a_i y^{(i)} = 0$. Then $\iota_\Omega^\rho(f_1),\dots,\iota_\Omega^\rho(f_N)$ is a basis of the ${}^\rho\widetilde{\mathbb{R}}$-module of solutions of the problem $\sum_{i=0}^N \iota_\Omega^\rho(a_i)\, y^{(i)} = 0$ in ${}^\rho\mathcal{GC}^\infty(\Omega^\bullet, {}^\rho\widetilde{\mathbb{R}})$.*

Proof. The only nontrivial point to prove is that $\iota_\Omega^\rho(f_1),\dots,\iota_\Omega^\rho(f_N)$ are linearly indipendent on ${}^\rho\widetilde{\mathbb{R}}$. Assume, by contrast, that there are constants $c_1,\dots,c_N \in {}^\rho\widetilde{\mathbb{R}}$ such that $\sum_{i=0}^N c_i\iota_\Omega^\rho(f_i) = 0$. Following the same ideas of the proof of Lemma 65, without loss of generality we can assume that $c_1,\dots,c_N$ are near standard, and that there exists $j \le N$ such that $c_j = 1$. Then

$$0 = st(0) = st\left(\sum_{i=0}^N c_i\iota_\Omega^\rho(f_i)\right) = \sum_{i=0}^N st\left(c_i\iota_\Omega^\rho(f_i)\right) =$$

$$\sum_{i=0}^N st(c_i)\,\iota_\Omega^\rho(f_i) = f_j(x) + \sum_{i=0, i\ne j}^N st(c_i)\, f_i(x),$$

and this is absurd as $f_1,\dots,f_N$ are linearly indipendent. □

For the non homogeneous case we have the following result, whose proof is trivial due to the linearity of the problem and the fact that ι_Ω^ρ is a differential embedding.

Theorem 81. *Let $T \in \mathcal{D}'(\Omega)$ and let $a_0,\dots,a_N \in \mathbb{R}$. Let $y \in \mathcal{D}'(\Omega)$ be such that $\sum_{i=0}^N a_i y^{(i)} = T$. Then $\iota_\Omega^\rho(T)$ solves the ODE $\sum_{i=0}^N a_i y^{(i)} = \iota_\Omega^\rho(T)$ in ${}^\rho\mathcal{GC}^\infty(\Omega^\bullet, {}^\rho\widetilde{\mathbb{R}})$.*

Example 82. Let us consider the equation

$$\left(\frac{d}{dx} + a\right) y = \delta, \tag{9.3}$$

where $a \ne 0 \in \mathbb{R}$. In [28] it is shown that the weak solutions of equation 9.3 are $H(x)e^{-ax} + Ce^{-ax}$. Hence, by Theorem 81 we have that the GSF solutions of equation 9.3 are $\iota_\Omega^\rho(H(x)e^{-ax}) + \widetilde{C}\,(e^{-ax})$, where $\widetilde{C} \in {}^\rho\widetilde{\mathbb{R}}$.

Let us notice that Theorem 81 does not hold true if we let $a_0,\dots a_N \in \mathcal{C}^\infty(\Omega)$, as the following examples show. In, particular, this means that in GSF theory we are able to detect certain infinitesimal differences between weak and GSF solutions.

Example 83. Let $m \in \mathbb{N}$, $m \ge 1$ and consider the equation

$$x^m y' = 0. \tag{9.4}$$

It is known (see e.g. [28]) that for every $c_1,\dots,c_m \in \mathbb{R}$

$$y = c_1 + c_2 H + \dots + c_{m+1}\delta^{(m-1)}$$

is a weak solution of equation 9.4. However $i_\Omega^\rho(y)$ is not a solution of equation 9.4 in ${}^\rho\mathcal{GC}^\infty(\Omega^\bullet, {}^\rho\widetilde{\mathbb{R}})$ if some of the coefficients $c_1,\dots,c_{m+1}$ is nonzero. In fact,

$$x^m \cdot \left(\frac{d\left(\iota_\Omega^\rho(c_1 + c_2 H + \dots + c_{m+1}\delta^{(m-1)}\right)}{dx}\right) =$$

$$x^m \cdot \iota_\Omega^\rho\left(\frac{d\left(c_1 + c_2 H + \dots + c_{m+1}\delta^{(m-1)}\right)}{dx}\right) \ne 0,$$

as $\iota_\Omega^\rho\left(\frac{d\left(c_1+c_2H+\cdots+c_{m+1}\delta^{(m-1)}\right)}{dx}\right)=0$ if and only if $\frac{d\left(c_1+c_2H+\cdots+c_{m+1}\delta^{(m-1)}\right)}{dx}=0$, which is not the case.

Nevertheless, as a consequence of Theorem 78, we have that

$$x^m\cdot\iota_\Omega^\rho\left(\frac{d\left(c_1+c_2H+\cdots+c_{m+1}\delta^{(m-1)}\right)}{dx}\right)\approx 0.$$

Finally, it is immediate to prove that the unique local solutions of equation 9.4 in ${}^\rho\mathcal{GC}^\infty(\Omega^\bullet,{}^\rho\widetilde{\mathbb{R}})$ are the constant GSF. In fact, let $y\in\mathcal{GC}^\infty(\Omega^\bullet,{}^\rho\widetilde{\mathbb{R}})$ be a solution. Let O be a sharp open neighborhood of 0, and assume that $x^m\cdot y'=0$ on O. Hence $y'=0$ on every invertible internal point in O and, as this is a dense subset of O, this means that $y'=0$ on O, hence y is constant on O.

Example 84. Consider the equation

$$xy'=1. \tag{9.5}$$

In [28] it is shown that the weak solutions of equation 9.5 are $y=c_1+c_2H+log|x|$. However,

$$x\cdot\frac{d\left(\iota_\Omega^\rho\left(c_1+c_2H+log|x|\right)\right)}{dx}=x\cdot\iota_\Omega^\rho\left(c_2H+P_{fin}\left(\frac{1}{|x|}\right)\right)\neq 1$$

(but $x\cdot\iota_\Omega^\rho\left(c_2H+P_{fin}\left(\frac{1}{|x|}\right)\right)\approx 1$ by Theorem 78). Finally, let us observe that equation 9.5 has no GSF solution in any neighborhood of 0: in fact, if y solves this equation then $y(x)=\frac{1}{x}$ on every invertible point x, and this is absurd as $\frac{1}{x}$ is not a GSF.

10. Conclusions and future developments

(i) GSF as a framework very similar to smooth functions. Differences
(ii) The theory resembles the classical one
(iii) Considerations about solutions on infinitesimal intervals. Hyperfinite contractions
(iv) application of the theory using only real numbers and using the ε-definitions
(v) replication of these results for another set of indices by using our intrinsic proofs. Axiomatic approach
(vi) a general transfer theorem using the notion of sample of points and restricting only to properties of near-standard or infinite points
(vii) Similar approach using NSA
(viii) morphisms of gauges and ε-wise solutions

References

[1] Ambrosio, L., Transport equation and Cauchy problem for BV vector fields, Invent. math. 158, 227–260, 2004.

[2] Cain, G.L., Nashed, M.Z., Fixed points and stability for a sum of two operators in locally convex spaces. Pacific Journal of Mathematics, Vol. 39, No. 3, 1971.

[3] Constantine, G.M., Savits, T.H., A multivariate Faà di Bruno formula with applications. Transaction of the American Mathematical Society, Vol. 348, No. 2, 1996.

[4] Colombeau, J.F., *New generalized functions and multiplication of distributions.* North-Holland, Amsterdam, 1984.

[5] Colombeau, J.F., *Elementary introduction to new generalized functions.* North-Holland, Amsterdam, 1985.

[6] Colombeau, J.F., *Multiplication of distributions - A tool in mathematics, numerical engineering and theoretical Physics*. Springer-Verlag, Berlin Heidelberg, 1992.

[7] Cutland, N., Kessler, C., Kopp, E., Ross, D., On Cauchy's notion of infinitesimal. *British J. Philos. Sci.*, 39(3):375–378, 1988.

[8] Delcroix, A., Scarpalezos, D., Topology on Asymptotic Algebras of Generalized Functions and Applications. Monatshefte für Mathematik 129, pp. 1-14, 2000.

[9] Delcroix, A., Dévoué, V., Marti, J.-A., Generalized solutions of singular differential problems. Relationship with classical solutions. J. Math. Anal. Appl. 353, pp. 386-402, 2009.

[10] Dirac, P.A.M., The physical interpretation of the quantum dynamics, *Proc. R. Soc. Lond. A,* ***113****,* 1926–27, 621–641.

[11] Egorov, Yu. V., A contribution to the theory of generalized functions, *Russ. Math. Surveys*, 45(5), 3–40, 1990.

[12] Engquist, B., Tornberg, A.K., Tsai, R., Discretization of Dirac delta functions in level set methods. Journal of Computational Physics, 207:28–51, 2005.

[13] Erlacher, E., Grosser, M., Ordinary Differential Equations in Algebras of Generalized Functions, in Pseudo-Differential Operators, Generalized Functions and Asymptotics, S. Molahajloo, S. Pilipović, J. Toft, M. W. Wong eds, Operator Theory: Advances and Applications Volume 231, 2013, 253–270.

[14] Garetto, C., Topological structures in Colombeau algebras: Topological $\widetilde{\mathbb{C}}$-modules and duality theory. *Acta Appl. Math.* 88, no. 1, 81–123 (2005).

[15] Garetto, C., Topological structures in Colombeau algebras: investigation of the duals of $\mathcal{G}_c(\Omega), \mathcal{G}(\Omega)$ and $\mathcal{G}_{\mathcal{S}}(\mathbb{R}^n)$, *Monatsh. Math.* 146 (2005), no. 3, 203–226.

[16] Garetto, C., Closed graph and open mapping theorems for topological $\widetilde{\mathbb{C}}$-modules and applications*, Math. Nachr.* 282 (2009), no. 8, 1159–1188.

[17] Giordano, P., Kunzinger, M., Inverse Function Theorems for Generalized Smooth Functions. Invited paper for the Special issue ISAAC - Dedicated to Prof. Stevan Pilipovic for his 65 birthday. Eds. M. Oberguggenberger, J. Toft, J. Vindas and P. Wahlberg, Springer series Operator Theory: Advances and Applications, Birkhäuser Basel, 2016.

[18] Giordano, P., Kunzinger, M., A convenient notion of compact sets for generalized functions. Accepted in Proceedings of the Edinburgh Mathematical Society, 2016. See arXiv 1411.7292v1.

[19] Giordano, P., Kunzinger, M., New topologies on Colombeau generalized numbers and the Fermat-Reyes theorem, *Journal of Mathematical Analysis and Applications* 399 (2013) 229–238. http://dx.doi.org/10.1016/j.jmaa.2012.10.005

[20] Giordano, P., Kunzinger, M., Vernaeve, H., A Grothendieck topos of generalized functions I: Basic theory. See `http://www.mat.univie.ac.at/~giordap7/ToposI.pdf`

[21] Giordano, P., Kunzinger, M., Vernaeve, H., Strongly internal sets and generalized smooth functions. Journal of Mathematical Analysis and Applications, volume 422, issue 1, 2015, pp. 56–71.

[22] Giordano, P., Luperi Baglini, L., Asymptotic gauges: generalization of Colombeau type algebras. Math. Nachr. Volume 289, Issue 2-3, pages 247–274, 2016.

[23] Giordano, P., Luperi Baglini, L., A Grothendieck topos of generalized functions III: normal PDE. See `http://www.mat.univie.ac.at/~giordap7/ToposIII.pdf`

[24] Giordano, P., Nigsch, E., Unifying order structures for Colombeau algebras. Math. Nachr. 1–17 (2015) / DOI 10.1002/mana.201400277. See arXiv 1408.1242.

[25] Grosser, M., Kunzinger, M., Oberguggenberger, M., Steinbauer, R., *Geometric theory of generalized functions*, Kluwer, Dordrecht, 2001.

[26] Hairer, M., A theory of regularity structures. Invent. math. 198, 269–504, 2014.

[27] Hosseini, B. , Nigam, N., Stockie, J.M., On regularizations of the Dirac delta distribution, Journal of Computational Physics, Volume 305, 2016, Pages 423–447.

[28] Kanwal, R.P., *Generalized functions. Theory and technique.* 2nd edition, Birkhäuser, 1998.

[29] Katz, M.G., Tall, D., A Cauchy-Dirac delta function. *Foundations of Science*, 2012. See http://dx.doi.org/10.1007/s10699-012-9289-4 and http://arxiv.org/abs/1206.0119.

[30] Lampret, V., Estimating powers with base close to unity and large exponents, Divulgaciones Matemáticas Vol. 13 No. 1(2005), pp. 21–34.

[31] Laugwitz, D., Definite values of infinite sums: aspects of the foundations of infinitesimal analysis around 1820. Arch. Hist. Exact Sci. **39** (1989), no. 3, 195–245.

[32] Laugwitz, D., Early delta functions and the use of infinitesimals in research. Revue d'histoire des sciences, tome 45, 1, 1992. pp. 115-128.

[33] Lecke, A., Luperi Baglini, L., Giordano, P., The classical theory of calculus of variations for generalized functions, 2016. See arXiv 1608.03080.

[34] Ligeza, J., Generalized periodic solutions of ordinary differential equations in the Colombeau algebra. Annales Mathematicae Silesianae 11. Katowice, pp. 67 - 87, 1997.

[35] Ligeza, J., Remarks on generalized solutions of ordinary differential equations in the Colombeau algebra. Mathematica Bohemica 123, No. 3, pp. 301-316, 1998.

[36] Luperi Baglini L., Giordano P., The category of Colombeau algebras. Monatshefte für Mathematik 182, pp. 649-674, 2017.

[37] Marti, J.-A., Dévoué, V., Delcroix, A., Allaud, E., Vernaeve, H., On the Well-posedness of Some generalized Characteristic Cauchy Problems, Math. Model. Nat. Phenom. 11, No. 2, pp. 89-99, 2016.

[38] Nedeljkov, M., Oberguggenberger, M., Ordinary differential equations with delta function terms, Publications de l'Institut Mathématique, Nouvelle série, 91(105) (2012), 125–135.

[39] Oberguggenberger, M., *Multiplication of Distributions and Applications to Partial Differential Equations*, volume **259** of *Pitman Research Notes in Mathematics*. Longman, Harlow, 1992.

[40] Oberguggenberger, M., Vernaeve, H., Internal sets and internal functions in Colombeau theory, *J. Math. Anal. Appl.* 341 (2008) 649–659.

[41] Persson, J., Generalized nonlinear ordinary differential equations linear in measures, Ann. Mat. Pura Appl. (4) 132 (1982), 177–187.

[42] Persson, J., Linear distribution differential equations, Comment. Math. Univ. St. Paul. 33(1) (1984), 119–126.

[43] Persson, J., Fundamental theorems for linear measure differential equations, Math. Scand. 62 (1988), 19–43.

[44] Průša, V., Rajagopal, K.R., On the response of physical systems governed by nonlinear ordinary differential equations to step input, International Journal of Non-Linear Mechanics 81 (2016) 207–221.

[45] Průša, V., Řehoř, M., Tůma, K., Colombeau algebra as a mathematical tool for investigating step load and step deformation of systems of nonlinear springs and dashpots, Z. Angew. Math. Phys. (2017) 68:24.

[46] Řehoř, M., Průša, V., Tůma, K., On the response of nonlinear viscoelastic materials in creep and stress relaxation experiments in the lubricated squeeze flow setting, Physics of Fluids 28, 103102 (2016).

[47] Robinson, A., Function theory on some nonarchimedean fields, Amer. Math. Monthly 80 (6) (1973) 87–109; Part II: Papers in the Foundations of Mathematics.

[48] Schmieden, C., Laugwitz, D., Eine Erweiterung der Infinitesimalrechnung, *Math. Zeitschr.*, 69, l–39, 1958.

[49] Strichartz, R. S., A guide to Distribution Theory and Fourier Transforms, World Scientific, Singapore, 1994.

[50] Teschl, G., *Ordinary Differential Equations and Dynamical Systems*, Graduate Studies in Mathematics, American Mathematical Society, 2012.

[51] Tornberg, A.K., Engquist, B., Numerical approximations of singular source terms in differential equations, Journal of Computational Physics 200 (2004) 462–488.

[52] van der Pol, B., Bremmer, H., *Operational calculus* (3rd ed.), New York: Chelsea Publishing Co., 1987.

[53] Vernaeve, H., Generalized Analytic Functions on Generalized Domains, 2008. See arXiv:0811.1521.

Faculty of Mathematics, University of Vienna, Austria, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria

Email address: paolo.giordano@univie.ac.at, Lorenzo.Luperi@unimi.it