Unspecified Journal
Volume 00, Number 0, Pages 000–000
S ????-????(XX)0000-0

UNIQUENESS OF THE MAXIMAL FUNCTION IN THE RATIO ERGODIC THEOREM (PREPRINT VERSION)

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Abstract. We show that the maximal operator associated to Hopf's ratio ergodic theorem is injective.

1. Introduction

In a recent paper L. Ephremidze has shown that for a measure preserving transformation (*m.p.t.*) T on a finite measure space $(X, \mathcal{A}, \mu)$ the *ergodic maximal function* $M(f) := \sup_{n\geq 1} n^{-1}\mathbf{S}_n(f)$, where $\mathbf{S}_n(f) := \sum_{k=0}^{n-1} f \circ T^k$, $n \geq 1$, uniquely determines $f \in L_1(\mu)$, i.e. $M(f) = M(g)$ a.e. implies $f = g$ a.e., cf. [E]. (An alternative short proof on this result has been given in [J].)

His article also discusses to what extent this remains true if the measure space is infinite (but σ-finite), proving that the conclusion still holds for nonnegative functions, and showing that it does break down for some others. While this observation certainly is of some interest, one might argue that in infinite measure preserving situations (see [A]), $M(f)$ is not the "correct" object to study (there being no nontrivial limiting behaviour of $n^{-1}\mathbf{S}_n(f)$). Instead, we are going to consider the maximal function corresponding to the proper version of the pointwise ergodic theorem for infinite measure spaces, that is, to Hopf's ratio ergodic theorem (cf. [S], [H]). We briefly recall the statement of the latter (see [KK] and [Z] for short proofs):

Theorem 1.1 (**Hopf's Ratio Ergodic Theorem**). *Let T be a conservative m.p.t. on the $\sigma-$finite measure space $(X, \mathcal{A}, \mu)$. Let $f, p \in L_1(\mu)$ with $p > 0$. Then there exists a measurable function $Q(f, p) : X \to \mathbb{R}$ such that*

$$\frac{\mathbf{S}_n(f)}{\mathbf{S}_n(p)} = \frac{\sum_{k=0}^{n-1} f \circ T^k}{\sum_{k=0}^{n-1} p \circ T^k} \longrightarrow Q(f, p) \qquad a.e.\ on\ X \quad as\ n \to \infty.$$

The limit function $Q(f, p)$ is measurable w.r.t. the $\sigma-$algebra $\mathcal{I} \subseteq \mathcal{A}$ of T-invariant sets and satisfies $\int_I Q(f, p) \cdot p\, d\mu = \int_I f\, d\mu$ for all $I \in \mathcal{I}$. In other words, $Q(f, p) = \mathbb{E}_{\mu_p}[f/p \,\|\mathcal{I}]$, where $d\mu_p := p\, d\mu$.

Following Ephremidze's original approach, we are going to prove:

2000 *Mathematics Subject Classification.* 28D05, 37A30, 37A40 .

This research was supported by an APART fellowship of the Austrian Academy of Sciences.

Date: 17 March 2004.

Theorem 1.2 (**Uniqueness of Hopf's Ergodic Maximal Function**). *Let T be a conservative m.p.t. on the $\sigma-$finite measure space $(X, \mathcal{A}, \mu)$. Fix $p \in L_1(\mu)$ with $p > 0$, and for $f \in L_1(\mu)$ define* Hopf's ergodic maximal function *as*

$$M(f,p) := \sup_{n\geq 1} \frac{\mathbf{S}_n(f)}{\mathbf{S}_n(p)} = \sup_{n\geq 1} \frac{\sum_{k=0}^{n-1} f \circ T^k}{\sum_{k=0}^{n-1} p \circ T^k}.$$

Then $M(f,p)$ uniquely determines f, that is, $M(f,p) = M(g,p)$ a.e. implies $f = g$ a.e. on X.

Notice that even in the case of finite measure this contains a nontrivial generalization of the earlier result.

Remark 1.1. The question of integrability of $M(f,p)$ has been discussed in [D].

2. Injectivity of a discrete maximal operator

The core of the argument is a discussion of injectivity properties of the discrete maximal operator associated to a class of averaging operations on sequences of real numbers. Let $\Gamma := \mathbb{R}^{\mathbb{N}_0}$ denote the set of realvalued sequences $\alpha = (\alpha_n)_{n\geq 0}$. We consider families of *averaging functions* $\mathbf{A}_{n,m} : \Gamma \to \mathbb{R}$ such that $\mathbf{A}_{n,m}(\alpha)$ only depends on $(\alpha_n, \ldots, \alpha_m)$, $m \geq n \geq 0$, and study their associated *maximal operator*

$$\mathbf{M} : \Gamma \to \overline{\mathbb{R}}^{\mathbb{N}_0}, \quad \mathbf{M}\alpha_n := \sup_{m\geq n} \mathbf{A}_{n,m}(\alpha),\ n \geq 0.$$

Its restriction to $\Gamma^* := \{\alpha \in \Gamma\colon$ for every $n \in \mathbb{N}_0$ there exists $m \geq n$ with $\mathbf{M}\alpha_n = \mathbf{A}_{n,m}(\alpha)\}$, which clearly maps into Γ, will also be denoted by $\mathbf{M}$. The $\mathbf{A}_{n,m}$ are assumed to satisfy the following conditions:

$$(\diamondsuit) \qquad \begin{array}{c} \text{for } 0 \leq n \leq l < m \text{ and } \alpha \in \Gamma, \\ \mathbf{A}_{n,m}(\alpha) \text{ is a nontrivial convex combination of } \mathbf{A}_{n,l}(\alpha) \text{ and } \mathbf{A}_{l+1,m}(\alpha) \end{array}$$

(which automatically extends to partitions of $\{n, \ldots, m\}$ into more than two subintervals), and

$$(\heartsuit) \qquad \begin{array}{c} \text{for } 0 \leq n \leq m \text{ and } \alpha \in \Gamma, \\ \mathbf{A}_{n,m}(\alpha) \text{ and } (\alpha_{n+1}, \ldots, \alpha_m) \text{ uniquely determine } \alpha_n. \end{array}$$

The special case relevant for our ergodic theoretical result is that of inhomogeneous arithmetic averages:

Example 2.1. *For a fixed sequence $\pi = (\pi_k)_{k\geq 0}$ in $(0, \infty)$ define*

$$\mathbf{A}_{n,m}(\alpha) := \frac{\sum_{k=n}^{m} \alpha_k}{\sum_{k=n}^{m} \pi_k}, \quad m \geq n \geq 0.$$

This clearly satisfies our assumptions. The case $\pi_k \equiv 1$ was considered in [E].

We are going to prove the following generalization of proposition 2 of [E], closely following the line of argument given there:

Proposition 2.1 (**Injectivity of the restricted discrete maximal operator**). *The maximal operator $\mathbf{M}$ is injective on Γ^*.*

A *component* of a set $J \subseteq \mathbb{N}_0$ will be understood to be a maximal finite interval $I_{p,q} := \{p, \ldots, q\} \subseteq \mathbb{N}_0$ contained in J. We abreviate $A_{n,m} := \mathbf{A}_{n,m}(\alpha)$ and $\{\mathbf{M}\alpha > \lambda\} := \{n \in \mathbb{N}_0 : \mathbf{M}\alpha_n > \lambda\}$. Whenever an expression like $I_{p,q}$, $A_{p,q}$ etc. appears, we tacitly assume that $p \leq q$.

Lemma 2.1. *Let $m, n, p, q \in \mathbb{N}_0$, $\lambda \in \mathbb{R}$, and $\alpha \in \Gamma^*$.*
a) *If $\mathbf{M}\alpha_n = A_{n,q}$, then $A_{p,q} \geq \mathbf{M}\alpha_n$ for all $p \in I_{n,q}$.*
b) *If $n < m$ and $\mathbf{M}\alpha_n > \mathbf{M}\alpha_m$, then $\mathbf{M}\alpha_n = A_{n,q}$ for some $q \in I_{n,m-1}$.*
c) *If $I_{p,q}$ is a component of $\{\mathbf{M}\alpha > \lambda\}$, then for any $n \in I_{p,q}$, $\mathbf{M}\alpha_n = A_{n,m}$ for some $m = m(n) \in I_{n,q}$.*
d) *If $I_{p,q}$ is a component of $\{\mathbf{M}\alpha > \lambda\}$, then $A_{n,q} > \lambda$ for all $n \in I_{p,q}$.*
e) *If $\mathbf{M}\alpha_{n+1} \leq \mathbf{M}\alpha_n$, then $A_{n,n} = \mathbf{M}\alpha_n$.*
f) *If $I_{n+1,m}$ is a component of $\{\mathbf{M}\alpha > \mathbf{M}\alpha_n\}$, then $A_{n,m} = \mathbf{M}\alpha_n$.*

Proof. **a)** The case $p = n$ being trivial, we suppose that $A_{p,q} < \mathbf{M}\alpha_n$ for some $p \in I_{n+1,q}$, then using $A_{n,p-1} \leq \mathbf{M}\alpha_n$, ($\diamondsuit$) implies $A_{n,q} < \mathbf{M}\alpha_n$, which contradicts our assumption.

b) We have $\mathbf{M}\alpha_n = A_{n,q}$ for some $q \geq n$, and part a) shows that $q < m$.

c) Fix any $n \in I_{p,q}$. As $\mathbf{M}\alpha_n > \lambda \geq \mathbf{M}\alpha_{q+1}$, statement b) yields our assertion.

d) Fix $n \in I_{p,q}$. Repeatedly applying c), we obtain $n = n_0 < n_1 < \ldots < n_j = q$ with $A_{n_{i-1},n_i-1} > \lambda$ (take $n_{i+1} := m(n_i) + 1$), and ($\diamondsuit$) implies d).

e) Let $\lambda := \mathbf{M}\alpha_n$, and let $q \geq n$ be an integer satisfying $A_{n,q} = \lambda$. If $q = n$ we are done. Suppose now that $q > n$. The trivial estimate $A_{n+1,q} \leq \mathbf{M}\alpha_{n+1} \leq \lambda$, together with ($\diamondsuit$) shows that $A_{n,n} < \lambda$ would imply $A_{n,q} < \lambda$, contradicting our choice of q.

f) Let λ and q be as in e). Observe first that necessarily $q \geq m$: By statement d), assuming the contrary implies $A_{q+1,m} > \lambda$, and hence (due to $A_{n,q} = \lambda$ and property ($\diamondsuit$)) $A_{n,m} > \lambda$, which is impossible.

If $q = m$, we are done. Suppose now that $q > m$. The trivial inequality $A_{m+1,q} \leq \mathbf{M}\alpha_{m+1} \leq \lambda$, together with ($\diamondsuit$) shows that $A_{n,m} < \lambda$ would imply $A_{n,q} < \lambda$, contradicting our choice of q. Thus, $A_{n,m} \geq \lambda$, and therefore $A_{n,m} = \lambda$. ☐

Lemma 2.2. *Let $\lambda \in \mathbb{R}$, $\alpha, \beta \in \Gamma^*$.*
a) *If $I_{p,q}$ is a component of $\{\mathbf{M}\alpha > \lambda\}$, then $(\mathbf{M}\alpha_p, \ldots, \mathbf{M}\alpha_q)$ determines $(\alpha_p, \ldots, \alpha_q)$.*
b) *If $\mathbf{M}\alpha_n \geq \mathbf{M}\alpha_m$ for some $m > n \geq 0$, then α_n is uniquely determined by $\mathbf{M}\alpha$.*

Proof. **a)** Arrange the values $\{\mathbf{M}\alpha_n : n \in I_{p,q}\}$ in descending order, i.e. $\lambda_1 > \ldots > \lambda_j > \lambda$ where $I_i := \{n \in I_{p,q} : \mathbf{M}\alpha_n = \lambda_i\} \neq \varnothing$ and $\bigcup_{i=1}^{j} I_i = I_{p,q}$. We are going to identify the α_n for $n \in I_i$ by induction on i.

For $i = 1$ and $n \in I_i$, we have $A_{n,n} = \lambda_1$ by lemma 2.1 e), which due to ($\heartsuit$) uniquely determines α_n.

Assume now that the α_n have been found for $n \in I_1 \cup \ldots \cup I_i$. We identify α_n for any fixed $n \in I_{i+1}$: If $\mathbf{M}\alpha_{n+1} \leq \lambda_{i+1}$, then $A_{n,n} = \lambda_1$ by lemma 2.1 e), and we are done. If $\mathbf{M}\alpha_{n+1} > \lambda_{i+1}$, then there exists $m \leq q$ such that $I_{n+1,m}$ is a component of $\{\mathbf{M}\alpha > \lambda_{i+1}\}$, and lemma 2.1 f) ensures that $A_{n,m} = \lambda_{i+1}$. Since α has already been identified on $\{\mathbf{M}\alpha > \lambda_{i+1}\} \supseteq \{n+1, \ldots, m\}$, we see that α_n is uniquely determined, cf. ($\heartsuit$).

b) If $\lambda := \mathbf{M}\alpha_n \geq \mathbf{M}\alpha_{n+1}$, then lemma 2.1 e) shows that $A_{n,n} = \lambda$, which uniquely determines α_n by ($\heartsuit$).

Otherwise, if $\lambda < \mathbf{M}\alpha_{n+1}$, then there is some $q \leq m$ for which $I_{n+1,q}$ is a component of $\{\mathbf{M}\alpha > \lambda\}$. According to statement a), $(\alpha_{n+1}, \ldots, \alpha_q)$ is uniquely determined, and by lemma 2.1 f), $A_{n,m} = \lambda$. Consequently, cf. ($\heartsuit$), α_n is uniquely determined as well. ☐

The injectivity result now follows easily:

Proof of proposition 2.1. Due to lemma 2.2 b), it is enough to show that each $\alpha \in \Gamma^*$ has the following property:

$$\text{for each } n \geq 0 \text{ there is some } m > n \text{ s.t. } \mathbf{M}\alpha_n \geq \mathbf{M}\alpha_m.$$

Fix α and n. We have $\lambda := \mathbf{M}\alpha_n = A_{n,p}$ for some $p \geq n$. Due to ($\diamondsuit$), existence of some $q > p$ with $A_{p+1,q} > \lambda$ would imply $A_{n,q} > \lambda$, which is impossible. Hence, $A_{p+1,q} \leq \lambda$ for all $q > p$, so that $\mathbf{M}\alpha_m \leq \lambda$ where $m := p+1$. □

3. Proof of the theorem

In proving our result for Hopf's ergodic maximal function, we will stick to arguments specific to the ergodic theory of point transformations (rather than operators). If T is a conservative m.p.t. on the σ-finite measure space $(X, \mathcal{A}, \mu)$, and $Y \in \mathcal{A}$ with $0 < \mu(Y) < \infty$, we let $\varphi_Y(x) := \min\{n \geq 1 : T^n x \in Y\}$, $x \in Y$, denote the *first return time* of Y, which is finite a.e. on Y, and consider the *first return* (or *induced*) *map* $T_Y : Y \to Y$ given by $T_Y x := T^{\varphi_Y(x)} x$. According to basic classical results, T_Y is an m.p.t. of the finite measure space $(Y, \mathcal{A} \cap Y, \mu\mid_{\mathcal{A}\cap Y})$, and the invariant measures μ and $\mu\mid_{\mathcal{A}\cap Y}$ are related via

$$\int_{I(Y)} F \, d\mu = \int_Y F_Y \, d\mu \quad \text{for } F \in L_1(\mu), \tag{3.1}$$

where $F_Y := \sum_{j=0}^{\varphi_Y - 1} F \circ T^j$, and $I(Y) := \bigcup_{n \geq 0} T^{-n} Y \in \mathcal{I}$.

The one auxiliary result from ergodic theory we need for the proof of our theorem has long been known in the ergodic finite measure preserving case (see e.g. [P], p.84). It is not hard to extend it to conservative infinite measure preserving situations, thus obtaining the following generalization of proposition 1 in [E].

Proposition 3.1 (Zero chance of strictly constant signs). *Let T be a conservative m.p.t. on the σ-finite measure space $(X, \mathcal{A}, \mu)$. Let $F \in L_1(\mu)$ with $\int_I F \, d\mu = 0$ for $I \in \mathcal{I}$. Then*

$$\mu\left(\{\mathbf{S}_n(F) < 0 \text{ for all } n \geq 1\}\right) = 0.$$

Proof. **a)** Assume first that μ is finite. For the reader's convenience we briefly recall the beautiful argument given in [P]. Let $Y := \{\mathbf{S}_n(F) \leq 0 \text{ for all } n \geq 1\}$ and suppose that $\mu(Y) > 0$ (otherwise there is nothing to prove). Then it is easy to see that

$$\sup_{n \geq 1} \mathbf{S}_n(F) = F_Y \quad \text{a.e. on } Y.$$

Recalling (3.1) we obtain $\int_Y \sup_{n\geq 1} \mathbf{S}_n(F) \, d\mu = \int_Y F_Y \, d\mu = \int_{I(Y)} F \, d\mu = 0$, and as $\sup_{n\geq 1} \mathbf{S}_n(F) \leq 0$ on Y, we conclude that $\sup_{n\geq 1} \mathbf{S}_n(F) = 0$ a.e. on Y. Since for a.e. $x \in Y$ this supremum is attained, we have $\mu\left(\{\mathbf{S}_n(F) < 0 \text{ for all } n \geq 1\}\right) = 0$.

b) If μ is infinite, we show that for any $Y \in \mathcal{A}$ with $0 < \mu(Y) < \infty$,

$$\mu\left(Y \cap \{\mathbf{S}_n(F) < 0 \text{ for all } n \geq 1\}\right) = 0.$$

Fix such a set Y, and let $\mathbf{S}_m^Y(F_Y) := \sum_{k=0}^{m-1} F_Y \circ T_Y^k$, $m \geq 1$. Since $1_J F_Y = (1_{I(J)} F)_Y$ for T_Y-invariant sets J, we can apply the finite-measure version of the proposition to the induced system and F_Y to obtain

$$\mu\left(Y \cap \left\{\mathbf{S}_m^Y(F_Y) < 0 \text{ for all } m \geq 1\right\}\right) = 0.$$

Since $(\mathbf{S}_m^Y(F_Y)(x))_{m\geq 1}$ is a subsequence of $(\mathbf{S}_n(F)(x))_{n\geq 1}$, the result follows. □

All the tools required for proving our main result are now available.

Proof of theorem 1.2. **a)** For $x \in X$, we let $\pi_x \in \Gamma$ be given by $(\pi_x)_k := p \circ T^k(x)$, $k \geq 0$, and define $\mathbf{A}_{x,n,m} : \Gamma \to \mathbb{R}$ by $\mathbf{A}_{x,n,m}(\alpha) := \sum_{k=n}^{m} \alpha_k / \sum_{k=n}^{m} (\pi_x)_k$ for $m \geq n \geq 0$, as in example 2.1. Then, for any $n \geq 0$,

$$(\mathbf{M}_x \alpha_x)_n = \sup_{m \geq n} \mathbf{A}_{x,n,m}(\alpha_x) = M(f \circ T^n, p \circ T^n)(x),$$

where $\alpha_x \in \Gamma$ is given by $(\alpha_x)_n := f \circ T^n(x)$.

b) Observe first that since $M(f \circ T^k, p \circ T^k) = M(f, p) \circ T^k$ for $k \geq 0$, the assumption $M(f, p) = M(g, p)$ a.e. of the theorem immediately implies

$$M(f \circ T^k, p \circ T^k) = M(g \circ T^k, p \circ T^k) \text{ for all } k \geq 0 \quad \text{a.e. on } X,$$

meaning that $\mathbf{M}_x \alpha_x = \mathbf{M}_x \beta_x$ for a.e. $x \in X$, where $(\beta_x)_n := g \circ T^n(x)$.

c) Proposition 2.1 ensures that the sequence α_x (and hence, in particular, $(\alpha_x)_0 = f(x)$) is uniquely determined by $\mathbf{M}_x \alpha_x$ provided that $\alpha_x \in \Gamma_x^* = \{\alpha \in \Gamma\colon$ for every $n \in \mathbb{N}_0$ there exists $m \geq n$ with $(\mathbf{M}_x \alpha)_n = \mathbf{A}_{x,n,m}(\alpha)\}$. We claim that this holds for a.e. $x \in X$: By Hopf's ergodic theorem, $\mathbf{S}_n(f \circ T^k)/\mathbf{S}_n(p \circ T^k) \to Q(f \circ T^k, p \circ T^k) = Q(f, p)$ a.e. as $n \to \infty$, and hence

$$M(f \circ T^k, p \circ T^k) \geq Q(f, p) \text{ for all } k \geq 0 \quad \text{a.e. on } X.$$

Applying proposition 3.1 to $F := (f \circ T^k) - Q(f, p)(p \circ T^k)$, we see that for all $k \geq 0$,

$$\mu\left(\left\{\frac{\mathbf{S}_n(f \circ T^k)}{\mathbf{S}_n(p \circ T^k)} < Q(f, p) \text{ for all } n \geq 1\right\}\right) = 0.$$

Consequently, for a.e. $x \in X$, and any $k \geq 0$, there is some $j = j(x, k)$ such that $\mathbf{S}_j(f \circ T^k)(x)/\mathbf{S}_j(p \circ T^k)(x) \geq Q(f, p)(x)$, and hence some index $m = m(x, k)$ for which $\sup_n \mathbf{S}_n(f \circ T^k)(x)/\mathbf{S}_n(p \circ T^k)(x)$ is attained. Therfore, $\alpha_x \in \Gamma_x^*$ as required. □

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