A little addendum to my paper
Invariant measures for general(ized) induced transformations.
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Let T be a measurable map on $(X, \mathcal{A})$, $Y \in \mathcal{A}$, and $\tau : Y \to \mathbb{N}$ measurable such that for all $x \in Y$, $T^\tau x := T^{\tau(x)} x \in Y$. For any measure ν on $(Y, Y \cap \mathcal{A})$,

$$(0.1) \qquad (\tau \times_T \nu)(E) := \sum_{n \geq 0} \nu\left(\{\tau > n\} \cap T^{-n} E\right), \qquad E \in \mathcal{A},$$

defines a measure on $(X, \mathcal{A})$.

It is known that the measure $\tau \times_T \nu$ is T-invariant iff ν is T^τ-invariant. In that case one has $(\tau \times_T \nu) \mid_{Y \cap \mathcal{A}} \ll \nu$. Moreover,

$$(0.2) \qquad (\tau \times_T \nu)(X) = \int_Y \tau \, d\nu$$

(see [T] or Prop 1.1 of [Z] for these statements). In addition, if ν is ergodic for T^τ, then $\tau \times_T \nu$ is ergodic for T (cf. [T]).

The goal of the present short note is to also identify the value $(\tau \times_T \nu)(Y)$. Note that τ can always be represented via the successive return times $\varphi_1 < \varphi_2 < \varphi_3 < \ldots$ to Y, given by $\varphi_m := \sum_{j=0}^{m-1} \varphi \circ T_Y^j$, $m \geq 1$, with T_Y denoting the first return map: There always is some $\rho : Y \to \mathbb{N}$ (necessarily measurable) such that $\tau = \varphi_\rho$, compare Remark 4.2 of [Z]. Now we have the following counterpart to (0.2):

Proposition 1. *Suppose that $\tau = \varphi_\rho$, then*

$$(0.3) \qquad (\tau \times_T \nu)(Y) = \int_Y \rho \, d\nu.$$

Proof. Starting form the def of $\tau \times_T \nu$ decompose Y according to the value r of ρ, and then according to the values $k_1, \ldots, k_r$ of the successive return-times $\varphi_1 < \varphi_2 < \ldots < \varphi_r$. Then use the fact that a set

$$Y \cap \{\rho = r\} \cap \{\varphi_1 = k_1\} \cap \ldots \cap \{\varphi_r = k_r\} \cap T^{-n} Y$$

with $n < k_r$ ($= \tau$ on this set) can only be non-empty if $n \in \{0, k_1, \ldots, k_{r-1}\}$, and in this case equals

$$Y \cap \{\rho = r\} \cap \{\varphi_1 = k_1\} \cap \ldots \cap \{\varphi_r = k_r\}.$$

This gives

$$\begin{aligned}(\tau \times_T \nu)(Y) &= \sum_{n\geq 0} \nu\left(\{\tau > n\} \cap T^{-n}Y\right) \\ &= \sum_{r\geq 1} \sum_{1\leq k_1<\ldots<k_r} \sum_{n=0}^{k_r-1} \nu\left(\{\rho = r\} \cap \{\varphi_1 = k_1\} \cap \ldots \cap \{\varphi_r = k_r\} \cap T^{-n}Y\right) \\ &= \sum_{r\geq 1} \sum_{1\leq k_1<\ldots<k_r} r \cdot \nu\left(\{\rho = r\} \cap \{\varphi_1 = k_1\} \cap \ldots \cap \{\varphi_r = k_r\}\right) \\ &= \sum_{r\geq 1} r \cdot \nu\left(\{\rho = r\}\right),\end{aligned}$$

as required. □

References

[T] M. Thaler: *Transformations on [0,1] with infinite invariant measures.* Isr. J. Math. **46** (1983), 67-96.

[Z] R. Zweimüller: *Invariant measures for general(ized) induced transformations.* Proc. Amer. Math. Soc. **133** (2005), 2283-2295.