Stable limits for probability preserving maps with indifferent fixed points
(preprint version)

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Abstract

We consider probability preserving maps with indifferent fixed points and prove distributional convergence to stable laws for observables measurable with respect to a dynamical partition.

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1 Introduction

Interval maps with indifferent fixed points constitute one of the most popular classes of dynamical systems at the edge of uniform hyperbolicity. Under reasonable regularity assumptions, such a map T has an absolutely continuous σ-finite invariant measure (acim) μ. The density $h := d\mu/d\lambda$ (λ denoting Lebesgue measure) has singularities at indifferent fixed points determined by the local behaviour of the map at these points. In particular, μ is finite if the degree of tangency of T to the diagonal is moderate, and infinite otherwise. While systems of the latter type are of great interest from the viewpoint of infinite ergodic theory (see e.g. [A0], [T2], [Zw]), we will mainly be concerned with probability preserving maps here like, for example,

$$Tx := \begin{cases} x + 2^p x^{1+p} & x \in [0, \frac{1}{2}) \\ 2x - 1 & x \in [\frac{1}{2}, 1] \end{cases} \qquad \text{with } p \in (0,1). \tag{1}$$

(If $p \geq 1$, the acim μ is infinite.) Among the most interesting features of such systems are their (rather bad) mixing properties, which have only recently been investigated by several authors, cf. [LSV], [Yo], or [Sa].

When specialized to (1), the results on mixing rates obtained in these articles in particular imply a central limit theorem (CLT) analogous to what is well known for uniformly expanding maps (compare [Ke], [R-E], [De], [Ry], [Li] among others), asserting that for observables $f : [0,1] \to \mathbb{R}$ which are, say, Hölder continuous or of bounded variation,

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (f \circ T^k - \mu(f)) \overset{\mu}{\Longrightarrow} \mathcal{N}(0, \sigma^2(f)), \tag{2}$$

provided that $p < 1/2$. Here $\mathcal{N}(0,\sigma^2) = \sigma \cdot \mathcal{N}(0,1)$ denotes (some random variable having) the normal distribution with mean 0 and variance σ^2, and the asymptotic variance $\sigma^2(f) = \lim_{n\to\infty} n^{-1} \int (\sum_{k=0}^{n-1} f \circ T^k - n\mu(f))^2 d\mu$ is positive unless $f = g \circ T - g$ for some measurable function g. However, little information about the situation $p \in [1/2, 1)$ seems to be available, where the mixing rate becomes so slow that we are led to expect that the variance of $\mathbf{S}_n(f) := \sum_{k=0}^{n-1} f \circ T^k$ might grow faster than n, so that (2) cannot hold.

The purpose of the present note is to investigate the asymptotic distributional behaviour of probablility preserving transformations with indifferent fixed points from a class of systems including (1) for all $p \in (0,1)$. We are going to prove that the partial sums $\mathbf{S}_n(f)$ for typical observables f which are measurable w.r.t. some dynamical partition (e.g. occupation times of cylinders) still have a normal limit law, though under a nonstandard normalization, if $p = 1/2$, and that they converge in distribution to (nonnormal) stable laws of order $1/p$ in the $p > 1/2$ case. Remarkably enough, normal limits under the usual normalization, i.e. a standard CLT, will still turn up for a few exceptional (but equally regular) observables even if $p \geq 1/2$.

The occurrence of nonstandard normalizations and nonnormal stable laws is easily understood if we slightly change our point of view and consider the *induced (i.e. first-return) map* $T_Y x := T^{\varphi(x)} x$, $\varphi(x) := \min\{n \geq 1 : T^n x \in Y\}$, on a suitable set Y bounded away from the indifferent fixed points. Important stochastic aspects of the original system are reflected in the behaviour of the successive *return times* $\varphi \circ T_Y^k$, $k \geq 1$, and in contrast to T, T_Y will turn out to have very good mixing properties. However, the neutral fixed points cause the random variable φ on (Y, μ_Y), μ_Y being the normalized restriction of μ to Y, to have a distribution with a *heavy tail.* In particular, the variance of φ is infinite as soon as $p > 1/2$ (and may be if $p = 1/2$), explaining why the standard CLT breaks down for these parameters.

Below we shall follow this idea, which enables us to apply results on distributional limits for heavy-tailed observables under the action of well mixing systems imported form [AD1] and [AD2] (see also [Da], [DJ] and [He] for related results) to φ under T_Y, and discuss how these results carry over to the original map T. The latter step was inspired by (and in fact contains a simplification of) the approach to the CLT used in [ADU].

2 Stable limits for a class of maps with indifferent fixed points. Statements and examples

To keep the exposition transparent, we shall restrict our attention to a class of maps T on $X := [0,1]$ with two full branches and indifferent fixed points at $x = 0$ and possibly also at $x = 1$. Henceforth we assume that

(a) for some $c \in (0,1)$ the restrictions of T to $Z_0 := (0,c)$ and $Z_1 := (c,1)$ are increasing $\mathcal{C}^2$-diffeomorphisms onto $(0,1)$ with inverses v_0 and v_1,

(b) $T \mid_{Z_0}$ extends to a $\mathcal{C}^2$-map on $(0,c]$ and is expanding except for an *indifferent fixed point* at $x = 0$, i.e. for any $\varepsilon > 0$, there exists $\rho(\varepsilon) > 1$ such that $\mid T' \mid \geq \rho(\varepsilon)$ on $[\varepsilon, c]$, while $T(0) = 0$ and $\lim_{x\to 0} T'x = 1$ with T' increasing on $(0, \delta_0)$ for some $\delta_0 > 0$,

(c) there is some decreasing function H_0 on some neighbourhood of zero with

$$\int H_0 \, d\lambda < \infty \text{ and } |v_0''| \leq H_0. \tag{3}$$

Moreover, either

$(\mathbf{d_{(i)}})$ $T \mid_{Z_1}$ extends to a $\mathcal{C}^2$-map on $[c,1]$ and is *uniformly expanding*, i.e. for some $\rho > 1$, $\mid T' \mid \geq \rho$ on Z_1,

or

$(\mathbf{d_{(ii)}})$ $Sx := 1 - T(1-x)$, $x \in (0, 1-c)$, satisfies all the conditions which assumptions b) and c) impose on $T \mid_{Z_0}$.

The family of maps T satisfying **(a)**-**(c)** and $(\mathbf{d}_{(k)})$ will be denoted by $\mathcal{T}_{(k)}$, for $k = $ i or ii. The fundamental partition $\{Z_0, Z_1\}$ will be denoted by ξ. If $T \in \mathcal{T}_{(\text{i})}$, we let $Y(T) := Z_1$ and for $T \in \mathcal{T}_{(\text{ii})}$, set $Y(T) := (y_0, y_1)$, where y_0 is the unique point of period 2 in Z_0 and $y_1 := Ty_0$. Condition **(c)** is to ensure that the map induced outside some neighbourhood of this point satisfies Adler's folklore condition, cf. lemma 1 below, and by arguments like those of [T1], [T2] we then find

Proposition 1 ***(Basic ergodic properties of*** $T \in \mathcal{T}_{(i)} \cup \mathcal{T}_{(ii)}$***)*** *Any* $T \in \mathcal{T}_{(i)} \cup \mathcal{T}_{(ii)}$ *is conservative ergodic and exact w.r.t.* λ *and preserves a* σ*-finite Borel measure* $\mu \ll \lambda$ *with density* h *continuous on* $(0,1)$ *satisfying*

$$h(x) = h_0(x) \frac{x(1-x)}{(x - v_0(x))(v_1(x) - x)}$$

with h_0 *positive and continuous on* $[0,1]$.

If μ is finite we always assume that it is normalized. To state the distributional limit theorems for maps of this type, we need to recall a few more concepts. If ν is a probability measure on the measurable space $(X, \mathcal{A})$ and $(R_n)_{n\geq 1}$ is a sequence of measurable real functions on X, distributional convergence of $(R_n)_{n\geq 1}$ w.r.t. ν to some random variable R will be denoted by $R_n \stackrel{\nu}{\Longrightarrow} R$. *Strong distributional convergence* $R_n \stackrel{\mathcal{L}(\mu)}{\Longrightarrow} R$ on the σ-finite measure space $(X, \mathcal{A}, \mu)$ means that $R_n \stackrel{\nu}{\Longrightarrow} R$ for all probability measures $\nu \ll \mu$. If T is a nonsingular ergodic transformation on $(X, \mathcal{A}, \mu)$, (the proof of) proposition 3.6.1 in [A0] shows that if $R_n \circ T - R_n \stackrel{\mu}{\longrightarrow} 0$, then $R_n \stackrel{\mathcal{L}(\mu)}{\Longrightarrow} R$ as soon as $R_n \stackrel{\nu}{\Longrightarrow} R$ for some $\nu \ll \mu$. Specifically, this applies if T is conservative and $R_n = B(n)^{-1}\mathbf{S}_n(f) = B(n)^{-1}\sum_{k=0}^{n-1} f \circ T^k$, $n \in \mathbb{N}$, for some $f \in L_1(\mu)$ and

$B(n) \to \infty$.

Recall (cf. [AD1] or [IL]) that a real valued random variable R (respectively its distribution) is called *stable* if for all $a, b > 0$ there are $c > 0$ and $d \in \mathbb{R}$ such that $aR+bR^* \stackrel{d}{=} cR+d$, where R^* is an independent copy of R and $R \stackrel{d}{=} S$ means equality of distributions. In this case $a^\alpha + b^\alpha = c^\alpha$ for some $\alpha \in (0,2]$, called the *order* of R. Up to translation and scaling, any stable random variable of order $\alpha \in (1,2]$ belongs to the family $(\Xi_{\alpha,\beta})_{\alpha\in(1,2],\beta\in[-1,1]}$ of variables, indexed by the order α and the *skewness* parameter β, and uniquely determined by their characteristic functions

$$\mathbb{E}\left[e^{it\Xi_{\alpha,\beta}}\right] = e^{-\frac{|t|^\alpha}{2}(1-i\beta\mathrm{sgn}(t)\tan(\frac{\alpha\pi}{2}))}. \tag{4}$$

(Notice that for $\alpha = 2$ this is the standard normal distribution $\mathcal{N}(0,1)$.)

We let $\mathcal{R}_\rho$, $\rho \in \mathbb{R}$, denote the collection of functions $a : (L,\infty) \to (0,\infty)$ *regularly varying of index* ρ at infinity, i.e. $a(ct)/a(t) \to c^\rho$ as $t \to \infty$ for any $c > 0$. Regular variation plays a crucial role in many distributional limit theorems, and we refer to chapter 1 of [BGT] for a collection of basic results. Without further mention, we shall represent sequences $(a_n)_{n\geq 0}$ by functions on $\mathbb{R}_+$ defined by $t \longmapsto a_{[t]}$. $\mathcal{R}_\rho(0^+)$ is the family of functions $r : (0,\varepsilon) \to \mathbb{R}_+$ regularly varying of index ρ at zero (same condition as above, but for $t \to 0^+$). Any $a \in \mathcal{R}_\rho(0^+)$, $\rho < 0$, has an *asymptotic inverse*, i.e. a function $a^{-1} \in \mathcal{R}_{1/\rho}$ such that $a^{-1}(a(t)) \sim t$ as $t \to 0^+$ and $a(a^{-1}(t)) \sim t$ as $t \to \infty$.

We are going to restrict our attention to observables $f : X = [0,1] \to \mathbb{R}$ which are measurable w.r.t. one of the dynamical partitions $\xi_K := \bigvee_{k=0}^{K-1} T^{-k}\xi$, $K \in \mathbb{N}$, consisting of the *cylinders of order* K, i.e. the sets $Z_{i_0\ldots i_{K-1}} := \bigcap_{k=0}^{K-1} T^{-k} Z_{i_k}$, and denote the class of such functions by $\mathcal{F}_\xi$. For transformations with a single indifferent fixed point, our limit theorem reads

Theorem 1 ***(Stable distributional limits for*** $T \in \mathcal{T}_{(i)}$***)*** *Let $T \in \mathcal{T}_{(i)}$, $r(x) := Tx - x$, and assume that $r \in \mathcal{R}_{1+p}(0^+)$ for some $p \in (0,1)$. (In particular, the acim μ is finite.) Let $A(t) := t/\,r(t) \in \mathcal{R}_{-p}(0^+)$, $\alpha := \min(1/p, 2)$, and $A^{-1} \in \mathcal{R}_{-\alpha}$ be asymptotically inverse to A.*

a) *If $p \in (0,1/2)$ or if $p = 1/2$ and $\int_1^\infty tA^{-1}(t)\,dt < \infty$, then for every $f \in \mathcal{F}_\xi$ with $f(0^+) \neq \mu(f)$,*

$$\frac{1}{\sqrt{n}}\left(\sum_{k=0}^{n-1} f\circ T^k - n\mu(f)\right) \stackrel{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{N}(0,\sigma^2(f)), \tag{5}$$

with $\sigma^2(f) > 0$.

b) *If $p = 1/2$ and $\int_1^\infty tA^{-1}(t)\,dt = \infty$, then there is some $B_T \in \mathcal{R}_{1/2}$ such that for every $f \in \mathcal{F}_\xi$ with $f(0^+) \neq \mu(f)$,*

$$\frac{1}{B_T(n)}\left(\sum_{k=0}^{n-1} f\circ T^k - n\mu(f)\right) \stackrel{\mathcal{L}(\mu)}{\Longrightarrow} (f(0) - \mu(f))\cdot\mathcal{N}(0,1), \tag{6}$$

where B_T is specified by $[8h(c)/T'(c^+)]\cdot t\int_1^{B_T(t)} sA^{-1}(s)\,ds \sim B_T(t)^2$ as $t \to \infty$.

c) *If $p > 1/2$, then there is some $B_T \in \mathcal{R}_p$ such that for every $f \in \mathcal{F}_\xi$ with $f(0^+) \neq \mu(f)$,*

$$\frac{1}{B_T(n)}\left(\sum_{k=0}^{n-1} f \circ T^k - n\mu(f)\right) \overset{\mathcal{L}(\mu)}{\Longrightarrow} (f(0) - \mu(f)) \cdot \Xi_{\alpha,1}, \tag{7}$$

where $B_T(t) := \alpha[h(c)/(2T'(c^+))]^{1/\alpha} \cdot A(\frac{1}{t})$.

d) *For any $p \in (0,1)$, if $f \in \mathcal{F}_\xi$ with $f(0^+) = \mu(f)$, for which $\sum_{n=0}^{\varphi-1}(f - \mu(f)) \circ T^n$ does not vanish on Z_1, where φ is the first return time of Z_1, then (5) holds with $\sigma(f) > 0$. Finally, if $\sum_{n=0}^{\varphi-1}(f - \mu(f)) \circ T^n$ vanishes on Z_1, then $\lim_{n\to\infty} n^{-1} \int_{[0,1]}((f - \mu(f)) \circ T^k)^2 d\mu = 0 = \sigma(f)$.*

The condition that $f(0^+) \neq \mu(f)$ ensures that visits to neighbourhoods of the indifferent fixed point properly contribute to the ergodic sum of the centered observable $f - \mu(f)$, and leads to the "correct" stable order $\alpha = \min(1/p, 2)$ of T. In the exceptional case $f(0^+) = \mu(f)$, these ergodic sums "simply don't see" the indifferent fixed point.

Example 1 *The maps (1) from the introduction satisfy all the assumptions of the theorem with $r(x) = 2^p x^{1+p} \in \mathcal{R}_{1+p}(0^+)$. (Since - as usual - the invariant density h is not known, we cannot expect to determine all the constants explicitely.) Let us first consider the occupation times of the first-order cylinders Z_0, Z_1. Evidently, $\mu(f) \neq f(0) \in \{0,1\}$ for $f = 1_{Z_i}$.*

a) *Assume first that $p \in (0, 1/2)$, then there is some $\sigma(1_{Z_i}) > 0$.*

$$\frac{1}{\sqrt{n}}\left(\sum_{k=0}^{n-1} 1_{Z_i} \circ T^k - n\mu(Z_i)\right) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{N}(0, \sigma(1_{Z_i})).$$

b) *For $p = 1/2$ we find $\int_1^y tA^{-1}(t)\,dt \sim \frac{1}{2}\log y$ as $y \to \infty$, and hence $B_T(t) = \sqrt{h(c)}\sqrt{t\log t}$, so that*

$$\frac{1}{\sqrt{h(c)}}\,\frac{1}{\sqrt{n\log n}}\left(\sum_{k=0}^{n-1} 1_{Z_i} \circ T^k - n\mu(Z_i)\right) \overset{\mathcal{L}(\mu)}{\Longrightarrow} (1_{Z_i}(0) - \mu(Z_i)) \cdot \mathcal{N}(0,1).$$

c) *In the case $p > 1/2$, we obtain*

$$p\left(\frac{8}{h(c)}\right)^p \frac{1}{n^p}\left(\sum_{k=0}^{n-1} 1_{Z_i} \circ T^k - n\mu(Z_i)\right) \overset{\mathcal{L}(\mu)}{\Longrightarrow} (1_{Z_i}(0) - \mu(Z_i)) \cdot \Xi_{\frac{1}{p},1}.$$

To illustrate the case d) of the theorem, we need to consider observables which are not measurable with respect to ξ.

d) *For any $p \in (0,1)$, define $a := \mu(Z_{10})$, $b := \mu(Z_{11})$, and let $f := a1_{Z_{11}} - b1_{Z_{10}}$ which is measurable ξ_2 and centered. Since $f = 0$ on Z_0 we have $\sum_{n=0}^{\varphi-1} f^c \circ T^n = f$ on Z_1, which does not vanish. Hence (5) holds with $\sigma(f) > 0$.*

Example 2 *Let us also illustrate that the case* $p = 1/2$ *and* $\int_1^\infty tA^{-1}(t)\,dt < \infty$ *of part a) of the theorem is not void: The most convenient way to construct suitable examples is by prescribing the inverse branch* v_0*. For instance, let* $v(x) := x + x^{3/2}\log x$, $x > 0$*. It is a matter of straightforward calculation to see that for* $\gamma \in \mathbb{R}$ *large enough (e.g.* $\gamma \geq 10$*),* $v_0(x) := \gamma\, v(x/\gamma)$, $x \in (0,1]$*, can be taken as the inverse branch of some* $T \in \mathcal{T}_{(i)}$ *with* $c := v_0(1)$ *and, say,* T *affine on* $(c,1)$*. For this map* T *we have* $A(x) = x/(Tx - x) \sim \sqrt{\gamma}/(\sqrt{x}\log(\frac{\gamma}{x}))$*, and as* A *is strictly monotone near* 0^+*, we can use this equivalence to check that an asymptotic inverse of* A *is given by* $A^{-1}(y) = \gamma/(2y\log y)^2$*. Consequently,* $\int_1^\infty tA^{-1}(t)\,dt \approx \int_1^\infty \frac{dt}{t(\log t)^2} < \infty$ *as required.*

In the course of the proof it will become clear why we can only obtain the most asymmetric case of $\beta = \pm 1$ laws (notice that $\Xi_{\alpha,-1} = -\Xi_{\alpha,1}$) in the limit if $T \in \mathcal{T}_{(\mathrm{i})}$. Roughly speaking, in the case of a single fixed point, we only have to consider the return time φ which of course is bounded below, but has a heavy tail at ∞. In the case of two indifferent fixed points, any value of the skewness parameter β can occur, since we have to consider "signed return times" which may have a heavy tail at $-\infty$ as well:

Theorem 2 ***(Stable distributional limits for*** $T \in \mathcal{T}_{(ii)}$***)*** *Let* $T \in \mathcal{T}_{(ii)}$ *and define* $r_0(x) := Tx - x$, $r_1(x) := 1 - x - T(1-x)$*. Assume that there are* $c_0, c_1 \geq 0$, $c_0 + c_1 > 0$ *and some* $r \in \mathcal{R}_{1+p}(0^+)$, $p \in (0,1)$*, such that* $r_i(x) \sim (c_i + o(x))r(x)$ *as* $x \to 0^+$*. Let* $A(t) := t/\,r(t) \in \mathcal{R}_{-p}(0^+)$, $\alpha := \min(1/p, 2)$*, and* $A^{-1} \in \mathcal{R}_{-\alpha}$ *be asymptotically inverse to* A*.*

a) *If* $p \in (0, 1/2)$ *or if* $p = 1/2$ *and* $\int_1^\infty tA^{-1}(t)\,dt < \infty$*, then for every* $f \in \mathcal{F}_\xi$ *with* $\{f(0), f(1)\} \neq \{\mu(f)\}$*, (5) holds with* $\sigma(f) > 0$*.*

b) *If* $p = 1/2$ *and* $\int_1^\infty tA^{-1}(t)\,dt = \infty$*, then there is some* $B_T \in \mathcal{R}_{1/2}$ *such that for every* $f \in \mathcal{F}_\xi$ *with* $\{f(0), f(1)\} \neq \{\mu(f)\}$*,*

$$\frac{1}{B_T(n)}\left(\sum_{k=0}^{n-1} f \circ T^k - n\mu(f)\right) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \sqrt{v(f)} \cdot \mathcal{N}(0,1), \tag{8}$$

where B_T *is specified by* $B_T(t)^2 \sim 8h(c) \cdot t \int_1^{B_T(t)} sA^{-1}(s)\,ds$ *as* $t \to \infty$*,* $d_0 := 1/T'(c^+)$, $d_1 := 1/T'(c^-)$*, and*

$$v(f) := \sum_{i \in \{0,1\}} d_i \cdot \left(\frac{|\, f(i) - \mu(f)\,|}{c_i}\right)^\alpha. \tag{9}$$

c) *If* $p > 1/2$*, then there is* $B_T \in \mathcal{R}_p$, $B_T(t) := \alpha(2h(c)\Gamma(1-\alpha)\cos(\frac{\pi\alpha}{2}))^{\frac{1}{\alpha}} \cdot A(\frac{1}{t})$*, such that for every* $f \in \mathcal{F}_\xi$ *with* $\{f(0), f(1)\} \neq \{\mu(f)\}$*,*

$$\frac{1}{B_T(n)}\left(\sum_{k=0}^{n-1} f \circ T^k - n\mu(f)\right) \overset{\mathcal{L}(\mu)}{\Longrightarrow} v(f)^{\frac{1}{\alpha}} \cdot \Xi_{\alpha,\beta(f)}, \tag{10}$$

where $v(f)$ *is as in (9), and*

$$\beta(f) := \frac{1}{v(f)} \sum_{i \in \{0,1\}} \operatorname{sgn}(f(i) - \mu(f))\, d_i \cdot \left(\frac{|\, f(i) - \mu(f)\,|}{c_i}\right)^\alpha.$$

d) *For any $p \in (0,1)$, if $f \in \mathcal{F}_\xi$ with $f(0) = f(1) = \mu(f)$, for which $\sum_{n=0}^{\varphi-1}(f - \mu(f)) \circ T^n$ does not vanish on Y, where φ is the first return time of the set $Y = Y(T)$ introduced above, (5) holds with $\sigma(f) > 0$. Finally, if $\sum_{n=0}^{\varphi-1}(f - \mu(f)) \circ T^n$ vanishes on Y, then $\lim_{n\to\infty} n^{-1} \int_{[0,1]}((f - \mu(f)) \circ T^k)^2 d\mu = 0 = \sigma(f)$.*

Various extensions of these results follow by the same method. For example, we could consider $T \in \mathcal{T}_{(\mathrm{ii})}$ with fixed points of different order. In this case T behaves like maps from $\mathcal{T}_{(\mathrm{i})}$, the weaker fixed point (i.e. the one with smaller p) won't effect the limit law (unless $f - \mu(f)$ vanishes at the stronger fixed point). We leave the details to the interested reader.

3 Preliminary remarks about interval maps and observables

As announced before, an important technical point is the following observation concerning the distortion properties of iterates of indifferent branches satisfying condition **(c)**. Recall that the *regularity* of a positive differentiable function v on an interval J is given by $R_J(v) := \sup_J \mid v' \mid /v$, cf. [Zw]. It is straightforward that a piecewise $\mathcal{C}^2$-map T on the interval satisfies the classical *Adler folklore condition* $\sup_X \mid T'' \mid /(T')^2 < \infty$ iff its inverse branches v have uniformly bounded regularity.

Lemma 1 ***(Inducing Adler's condition)*** *Let $v \in \mathcal{C}^1([0,\varepsilon_0]) \cap \mathcal{C}^2((0,\varepsilon_0])$ be a concave function satisfying $0 < v(x) < x$ for $x \in (0,\varepsilon_0]$, $v'(0) = 1$, and $v' > 0$. Assume that there is some decreasing function H on $(0,\varepsilon_0]$ with $\int H\, d\lambda < \infty$ such that $|v''| \leq H$. Then the sequence $(v^n)_{n\geq 1}$ has uniformly bounded regularity on compact subsets of $(0,\varepsilon_0]$, i.e. $\sup_{n\geq 1} R_{[\varepsilon,\varepsilon_0]}(v^n) < \infty$ for any $\varepsilon \in (0,\varepsilon_0)$.*

First proof. As in Example 3 of [T1], see also [Got]. ■

Second proof. (Inspired by lemma 5 of [Yo]) Let $a_n := v^n(\varepsilon_0)$, $n \geq 0$. Since $(v^n)'$ decreases, both $(a_0 - a_1)(v^n)'(t)$, $t \in [a_1, a_0]$, and $a_n - a_{n+1} = \int_{a_1}^{a_0} (v^n)'(s)\, ds$ belong to the interval $[(a_0 - a_1)(v^n)'(a_0), (a_0 - a_1)(v^n)'(a_1)]$, and as $(v^n)'(a_0) \sim v'(a)(v^n)'(a_1)$, we see that there is some constant $\kappa > 1$ such that

$$0 < \kappa^{-1} \leq \frac{(v^n)'}{a_n - a_{n+1}} \leq \kappa < \infty \qquad \text{on } [a_1, a_0] \text{ for } n \geq 1.$$

In particular, we find that for any $a_1 \leq x < y \leq a_0$ and $n \geq 1$,

$$v^n(y) - v^n(x) = \int_x^y (v^n)'(t)\, dt \leq \kappa(a_n - a_{n+1})(y - x)\ .$$

Observe then that for suitable $\xi_k \in [a_{k+1}, a_k]$,

$$\begin{aligned} \log \frac{(v^n)'(x)}{(v^n)'(y)} &\leq \sum_{k=0}^{n-1} \left|\log v'(v^k(x)) - \log v'(v^k(y))\right| \\ &= \sum_{k=0}^{n-1} \frac{|v''(\xi_k)|}{v'(\xi_k)} \left|v^k(x) - v^k(y)\right| \end{aligned}$$

$$\le \frac{\kappa}{\inf v'} \, |x-y| \sum_{k=0}^{n-1} |v''(\xi_k)| \, (a_k - a_{k+1})$$
$$\le \frac{\kappa}{\inf v'} \, |x-y| \sum_{k=0}^{n-1} H(a_{k+1})(a_k - a_{k+1}),$$

and $\sum_{k=0}^{n-1} H(a_{k+1})(a_k - a_{k+1}) \le (\inf v')^{-1} \int_0^{a_0} H \, d\lambda < \infty$. Therefore, $\log \frac{(v^n)'(x)}{(v^n)'(y)} \le K \, |x-y|$ on $[a_1, a_0]$, implying the lemma. ■

A *piecewise monotonic map* is a triple (Y, S, η), where Y is an interval, η is a collection of nonempty pairwise disjoint open subintervals (the *cylinders of rank one*) with $\lambda(Y \setminus \bigcup_{Z \in \eta} Z) = 0$, and $S : Y \to Y$ is a map such that $S \mid_Z$ is continuous and strictly monotonic for each $Z \in \eta$. Let us say that a piecewise monotonic map (Y, S, η) is a *Rényi map*, if it satisfies Adler's condition and is piecewise onto, i.e. $SZ = Y$ for all $Z \in \eta$. In this case each iterate (Y, S^k, η_k), $k \in \mathbb{N}$, where $\eta_k := \bigvee_{j=0}^{k-1} S^{-j}\eta$, is Rényi, too, and any Rényi map is Gibbs-Markov in the sense of [AD1], [AD2]. Now lemma 1 is easily seen to imply the following observation, from which proposition 1 readily follows by standard arguments, the details of which we omit.

Lemma 2 ***(Induced Rényi maps)*** *Let $T \in \mathcal{T}_{(i)} \cup \mathcal{T}_{(ii)}$ and $Y := Y(T)$. Then the induced system (Y, T_Y, ξ_Y) is a Rényi map.*

(The natural partition for the induced map T_Y is given by $\xi_Y := \xi \vee \{\{\varphi = k\} : k \ge 1\}$.) To conclude this section, we recall an important observation concerning high iterates of the map near an indifferent fixed point, which will be important later on.

Remark 1 ***(Iterating inverse branches)*** *Assume that $I = (0, C]$ for some $C > 0$, and that $v : I \longrightarrow I$ is strictly increasing, continuous, and concave in some neighbourhood of 0, satisfying $v(x) < x$. Moreover, suppose that $r(x) := x - v(x)$ is in $\mathcal{R}_{1+p}(0^+)$ for some $p > 0$. Then,*

$$\text{for any } y \in (0, C], \quad v^n(y) \sim \frac{A^{-1}(n)}{p^{1/p}} \in \mathcal{R}_{-1/p} \text{ as } n \to \infty, \tag{11}$$

where A^{-1} is asymptotically inverse to $A(t) := t / r(t) \in \mathcal{R}_{-p}$, cf. lemma 4.8.6 of [A0]. In addition, since these differences eventually decrease, the monotone density theorem, theorem 1.7.2. of [BGT], implies that

$$\text{for any } y \in (0, C], \quad v^n(y) - v^{n+1}(y) \sim \frac{A^{-1}(n)}{p^{1+1/p} n} \in \mathcal{R}_{-(1+1/p)}. \tag{12}$$

In particular, if we let $C_n := v^n(C)$, $n \ge 0$, and $I(k) := (C_k, C_{k-1}]$, $k \ge 1$, the above determines the asymptotics of $\lambda(I(k))$ as $k \to \infty$.

4 Distributional limits from Rényi mixing auxiliary systems

The present section contains the probabilistic core of our argument. In a manner inspired by section 7 of [ADU], we show how to obtain distributional limit theorems for a system from a corresponding result for some Rényi mixing auxiliary system (like an induced map or a jump transformation). A measure preserving transformation S on a probability space $(Y, \mathcal{A}, P)$ will be called *Rényi mixing* with respect to the P-partition η if there is some *Rényi constant* $C_{\mathcal{R}} \in [1, \infty)$ such that

$$C_{\mathcal{R}}^{-1} \leq \frac{P(A \cap B)}{P(A)P(B)} \leq C_{\mathcal{R}}$$

whenever $k \geq 1$, $A \in \sigma(\eta \vee \ldots \vee S^{-(k-1)}\eta)$, $P(A) > 0$, and $B \in \sigma(S^{-k}\eta \vee S^{-(k+1)}\eta \vee \ldots)$, $P(B) > 0$. (As is well known, if (Y, S, η) is a Rényi map in the sense of the preceding section, then this is satisfied for its acim P.) If in this case $g : Y \to \mathbb{R}$ is measurable η, then $X_n := g \circ S^n$, $n \geq 0$, defines a stationary sequence of random variables on $(Y, \mathcal{A}, P)$ satisfying

$$C_{\mathcal{R}}^{-1} \leq \frac{P(A \cap B)}{P(A)P(B)} \leq C_{\mathcal{R}}$$

whenever $k \geq 1$, $A \in \sigma(X_0, \ldots, X_{k-1})$, $B \in \sigma(X_k, \ldots)$, and $P(A), P(B) > 0$.

The first part of the following lemma generalizes Etemadi's maximal inequality (cf. [Et]) for partial sums of iid sequences to ergodic sums of observables measurable with respect to some Rényi mixing partition. The conclusion of the second part is analogous to that of Lemma 7.4 of [ADU].

Lemma 3 ***(Maximal Inequalities for Rényi mixing sequences)*** *Let (Y, S, η) be a Rényi map, $g : Y \to \mathbb{R}$ measurable η_K for some $K \in \mathbb{N}$, $X_n := g \circ S^n$, $n \geq 0$, and $S_n := X_0 + \ldots + X_{n-1}$, $n \geq 1$.*

a) *For all $\kappa \in \mathbb{R}_+$ and $n \in \mathbb{N}$ we have*

$$P\left(\max_{1 \leq k \leq n} |S_k| \geq \kappa\right) \leq 4KC_{\mathcal{R}}^2 \max_{1 \leq k \leq n/K+1} P\left(|S_k| \geq \frac{\kappa}{4K}\right). \tag{13}$$

b) *If, moreover, $B(n)^{-1}S_n \overset{d}{\Longrightarrow} G$ for some real random variable G and normalizing function B which is regularly varying of index $\beta > 0$, then for any $\varepsilon, \kappa > 0$ there are $\delta > 0$ and $n_0 \geq 1$ such that*

$$P\left(\max_{1 \leq k \leq \delta n} |S_k| \geq \kappa\, B(n)\right) < \varepsilon \qquad \text{for } n \geq n_0. \tag{14}$$

Proof. **a)** Assume first that $K = 1$ and notice that for $k \in \{1, \ldots, n\}$, S_k is measurable $\eta \vee \ldots \vee S^{-(k-1)}\eta$ and $S_n - S_k$ is measurable $S^{-k}\eta \vee \ldots \vee S^{-(n-1)}\eta$. Therefore, partitioning by the pairwise disjoint sets $B_k := \{|S_k| \geq \kappa$ and $|S_j| < \kappa$ for $j < k\}$, $k \in \{1, \ldots, n\}$, we find

$$\begin{aligned}
P\left(\max_{1 \leq k \leq n} |S_k| \geq \kappa\right) &\leq P\left(|S_n| \geq \frac{\kappa}{2}\right) + \sum_{k=1}^{n-1} P\left(B_k \cap \left\{|S_n| < \frac{\kappa}{2}\right\}\right) \\
&\leq P\left(|S_n| \geq \frac{\kappa}{2}\right) + \sum_{k=1}^{n-1} P\left(B_k \cap \left\{|S_n - S_k| > \frac{\kappa}{2}\right\}\right) \\
&\leq P\left(|S_n| \geq \frac{\kappa}{2}\right) + C_{\mathcal{R}} \sum_{k=1}^{n-1} P(B_k)\, P\left(|S_n - S_k| > \frac{\kappa}{2}\right)
\end{aligned}$$

$$\leq \quad 2C_{\mathcal{R}} \max_{0\leq k<n} P\left(|S_n - S_k| \geq \frac{\kappa}{2}\right)$$
$$\leq \quad 2C_{\mathcal{R}} P\left(\max_{0\leq k<n} |S_n - S_k| \geq \frac{\kappa}{2}\right).$$

Now replace κ by $\frac{\kappa}{2}$ and apply the same argument to the partial sums in reverse order, $\widetilde{S}_j := \sum_{k=n-j}^{n-1} g \circ T^k = S_n - S_{n-j}$, $j \in \{0, \ldots, n\}$, to obtain

$$P\left(\max_{1\leq j\leq n} \left|\widetilde{S}_j\right| \geq \frac{\kappa}{2}\right) \quad \leq \quad 2C_{\mathcal{R}} \max_{0\leq j<n} P\left(\left|\widetilde{S}_n - \widetilde{S}_j\right| \geq \frac{\kappa}{2}\right)$$
$$= \quad 2C_{\mathcal{R}} \max_{1\leq k\leq n} P\left(|S_k| \geq \frac{\kappa}{4}\right).$$

Combining the two estimates proves (13) in the case $K = 1$. From this the general case follows easily by considering the subsequences $(X_{jK+r})_{j\geq 0}$ with $r \in \{0, 1, \ldots, K-1\}$.

b) For the second assertion we may assume w.l.o.g. that B is nondecreasing. According to (13) we then have

$$P\left(\max_{1\leq k\leq \delta n} |S_k| \geq \kappa\, B(n)\right) \leq 4KC_{\mathcal{R}}^2 \max_{1\leq k\leq \delta n/K+1} P\left(\frac{|S_k|}{B(k)} \geq \frac{\kappa}{4K}\frac{B(n)}{B(\delta n)}\right),$$

which by the uniform convergence theorem for regularly varying functions (cf. theorem 1.5.2 of [BGT]) and tightness of the convergent sequence of distributions implies (14). ■

Remark 2 *Instead of* ***a)*** *above we could just as well use an Ottaviani-Skorohod type maximal inequality like*

$$P\left(\max_{1\leq k\leq n} |S_k| \geq a + b\right) \leq KC_{\mathcal{R}} \frac{P(|S_n| \geq a)}{\min_{1\leq k\leq n} P(|S_k| \geq b)} \qquad \text{for } a, b > 0,\ n \in \mathbb{N}.$$

The maximal inequality **b)** enables us to prove the following.

Lemma 4 ***(Distributional convergence of random Rényi sums)*** *Let* $(X_n)_{n\geq 1}$ *be as in lemma 3 and such that* $B(n)^{-1}S_n \overset{d}{\Longrightarrow} G$ *for some real random variable* G *and normalizing function* B *which is regularly varying of index* $\beta > 0$*, where* $S_n := X_0 + \ldots + X_{n-1}$*,* $n \geq 1$*. Assume that* N_n*,* $n \geq 1$*, are random variables for which* $D(n)^{-1}N_n \overset{P}{\longrightarrow} 1$ *with normalizing constants* $D(n) \to \infty$*. Then*

$$\frac{1}{B(D(n))} S_{N_n} \overset{d}{\Longrightarrow} G\ .$$

Proof. Fix $\varepsilon > 0$ and any continuity point t of $P(G \leq t)$. For arbitrary $\delta, \delta' > 0$ we find that

$$P\left(\frac{S_{N_n}}{B(D(n))} \leq t\right) \quad \leq \quad P\left(\left|\frac{N_n}{D(n)} - 1\right| > \delta\right) + P\left(\frac{S_{D(n)}}{B(D(n))} \leq t + \delta'\right) \tag{15}$$
$$+P\left(\left|\frac{N_n}{D(n)} - 1\right| \leq \delta \text{ and } \frac{1}{B(D(n))}\left|S_{D(n)} - S_{N_n}\right| > \delta'\right).$$

Choose $\delta' > 0$ so small that $P(G \le t + \delta') \le P(G \le t) + \varepsilon$ and in such a way that $t + \delta'$ is a point of continuity of $P(G \le t)$. Then

$$P\left(\frac{S_{D(n)}}{B(D(n))} \le t + \delta'\right) \le P(G \le t) + 2\varepsilon \qquad \text{for } n \ge n_0.$$

Observe that on $\left\{\left|D(n)^{-1}N_n - 1\right| \le \delta\right\}$ we have $\left|S_{D(n)} - S_{N_n}\right| = |X_{L_n} + \ldots + X_{U_n}| \le |X_{l_n} + \ldots + X_{U_n}| + |X_{l_n} + \ldots + X_{L_n-1}|$, where $l_n := (1-\delta)D(n) \le L_n(\omega) \le U_n(\omega) \le (1+\delta)D(n) = l_n + 2\delta D(n)$. Therefore, by stationarity of (X_n),

$$P\left(\left|\frac{N_n}{D(n)} - 1\right| \le \delta \text{ and } \frac{\left|S_{D(n)} - S_{N_n}\right|}{B(D(n))} > \delta'\right) \le P\left(\max_{1\le k\le 2\delta D(n)} |S_k| > \frac{\delta' B(D(n))}{2}\right).$$

According to Lemma 3 we can choose $\delta > 0$ so small that

$$P\left(\max_{1\le k\le 2\delta D(n)} |S_k| > \frac{\delta' B(D(n))}{2}\right) < \varepsilon \qquad \text{for } n \ge n_1,$$

and since $D(n)^{-1}N_n \xrightarrow{P} a$, we have $P\left(\left|D(n)^{-1}N_n - 1\right| > \delta\right) < \varepsilon$ for $n \ge n_2$. Consequently, (15) shows that

$$P\left(\frac{S_{N_n}}{B(D(n))} \le t\right) \le P(G \le t) + 4\varepsilon \qquad \text{for } n \ge \max(n_0, n_1, n_2).$$

The lower estimate is proved analogously. Hence $B(D(n))^{-1}S_{N_n} \stackrel{d}{\Longrightarrow} G$. ■

We are now ready to turn to the main objective of the present section, the question of how to pass from distributional limit theorems for some auxiliary system to corresponding results for the original one. Assume then that $(X, \mathcal{A}, \mu, T)$ is an ergodic probability preserving system, and that $(Y, \mathcal{A}\cap Y, \mu_\tau, T_\tau)$ is an *associated auxiliary system* where $Y \in \mathcal{A}$, $\mu(Y) > 0$, $\tau : X \to \mathbb{N}$ is some integrable function such that $T_\tau x := T^{\tau(x)}x \in Y$ for a.e. $x \in X$ and $\tau \circ T = \tau - 1$ on $\{\tau > 1\}$, and $T_\tau : Y \to Y$ is ergodic and preserves the probability measure μ_τ canonically related to μ by

$$\mu_\tau(\tau)\,\mu(E) = \sum_{n\ge 0} \mu_\tau\left(\{\tau > n\} \cap T^{-n}E\right) = \int_Y \sum_{n\ge 0} 1_{\{\tau>n\}}(1_E \circ T^n)\,d\mu_\tau \quad (16)$$

for $E \in \mathcal{A}$. In particular, $\mu_\tau(E) \le \mu_\tau(\tau)\mu(E)$ for $E \in \mathcal{A}\cap Y$. When studying Birkhoff sums $\mathbf{S}_n(f) = \sum_{k=0}^{n-1} f \circ T^k$ for T by means of T_τ, we have to pass to the *induced version* of the measurable function $f : X \to \mathbb{R}$ defined as $g = f_\tau := \sum_{n=0}^{\tau-1} f \circ T^n = \sum_{n\ge 0} 1_{\{\tau>n\}}(f \circ T^n)$. Evidently, $f \mapsto f_\tau$ is linear, and the identity (16) immediately shows that $\mu_\tau(f_\tau) = \mu_\tau(\tau)\,\mu(f)$ for quasi-integrable f, so that $\mu_\tau(f_\tau) = 0$ iff $\mu(f) = 0$.

We can now relate the asymptotic distributional behaviour of T_τ to that of T.

Proposition 2 ***(Distributional convergence via auxiliary systems)*** *Suppose that $(X, \mathcal{A}, \mu, T)$ is an ergodic probability preserving system and $(Y, \mathcal{A}\cap Y, \mu_\tau, T_\tau)$ an associated auxiliary system as above, Rényi mixing with respect* *to some μ_τ-partition η, such that τ is measurable η. Assume that $f : X \to \mathbb{R}$ is a measurable function for which there are a real random variable G and a normalizing function $B \in \mathcal{R}_\beta$, $\beta > 0$, such that*

$$\frac{1}{B(n)}\mathbf{S}_n^\tau(f_\tau) := \frac{1}{B(n)}\sum_{k=0}^{n-1} f_\tau \circ T_\tau^k \overset{\mu_\tau}{\Longrightarrow} G.$$

If, in addition, f_τ is measurable η_K for some $K \in \mathbb{N}$, then

$$\frac{1}{B(n/\mu_\tau(\tau))}\mathbf{S}_n(f) = \frac{1}{B(n/\mu_\tau(\tau))}\sum_{k=0}^{n-1} f \circ T^k \overset{\mathcal{L}(\mu)}{\Longrightarrow} G.$$

Proof. According to the remarks on strong distributional convergence (recall proposition 3.6.1 of [A0]), we need only check that $B(n/\mu_\tau(\tau))^{-1}\mathbf{S}_n(f) \overset{\mu_\tau}{\Longrightarrow} G$. (Using this principle one can also simplify the proof of theorem 7.2 of [ADU] a bit.) Writing $\tau_n := \mathbf{S}_n^\tau(\tau) = \sum_{k=0}^{n-1} \tau \circ T_\tau^k$, $n \geq 0$, we have $n^{-1}\tau_n \longrightarrow \mu_\tau(\tau)$ a.e. on Y. Let $N_n(x) := \min\{j \geq 1 : \tau_j(x) \geq n\}$ and $M_n(x) := \tau_{N_n(x)}(x) = \min\{\tau_i : i \geq 1 \text{ and } \tau_i \geq n\}$, $n \geq 1$. Then $M_n \geq n$, and $\mathbf{S}_n(f) = \mathbf{S}_{M_n}(f) - \mathbf{S}_{M_n - n}(f) \circ T^n = \mathbf{S}_{N_n}^\tau(f_\tau) - \mathbf{S}_{M_n-n}(f) \circ T^n$ for $n \geq 1$. By Lemma 4 we have

$$\frac{1}{B(n/\mu_\tau(\tau))}\mathbf{S}_{N_n}^\tau(f_\tau) \overset{\mu_\tau}{\Longrightarrow} G$$

since $n^{-1}N_n \longrightarrow 1/\mu_\tau(\tau)$ a.e. on Y (and hence on X).Therefore it remains to show that

$$\frac{1}{B(n/\mu_\tau(\tau))}\mathbf{S}_{M_n-n}(f) \circ T^n \overset{\mu_\tau}{\longrightarrow} 0.$$

To this end fix any $\varepsilon > 0$ and observe that $M_n(x) - n = k \geq 1$ implies $\tau(T^n x) = k$. For arbitrary $\delta > 0$ and $k_0 \geq 1$ we therefore find

$$\begin{aligned} \mu_\tau\left(\frac{|\mathbf{S}_{M_n-n}(f) \circ T^n|}{B(n/\mu_\tau(\tau))} > \delta\right) &\leq \mu_\tau\left(T^{-n}\{\tau > k_0\}\right) + \sum_{k=1}^{k_0} \mu_\tau\left(\frac{|\mathbf{S}_k(f) \circ T^n|}{B(n/\mu_\tau(\tau))} > \delta\right) \\ &\leq \mu_\tau(\tau)\left[\mu(\{\tau > k_0\}) + \sum_{k=1}^{k_0} \mu\left(\frac{|\mathbf{S}_k(f)|}{B(n/\mu_\tau(\tau))} > \delta\right)\right]. \end{aligned}$$

As $\tau < \infty$ a.e., we have $\mu_\tau(\tau)\,\mu(\{\tau > k_0\}) < \varepsilon$ for k_0 large enough, and since $B(x) \to \infty$, the second term tends to 0 as $n \to \infty$ for any fixed k_0. ■

5 Application to the interval maps T. Proof of theorems 1 and 2

We are now ready to prove our main results.

Proof of theorem 1. Fix $T \in \mathcal{T}_{(\mathrm{i})}$ satisfying the assumptions of the theorem and consider the specific auxiliary system (Y, T_Y, ξ_Y) from lemma 2 above, $\tau := \varphi$ being the first-return time of $Y = Y(T)$. Remark 1 enables us to understand the tail behaviour of φ: Since $Y \cap \{\varphi > m\} = v_1((0, c_m))$ with $c_m := v_0^m(c)$, $m \geq 0$, we see (using the continuity of the invariant density h and its normalized restriction $h_Y := \mu(Y)^{-1} h \mid_Y$) that

$$\mu_Y(\{\varphi > m\}) \sim \frac{h_Y(c)}{T'(c^+)} \cdot c_m \sim \frac{h_Y(c)}{T'(c^+)\, p^{\frac{1}{p}}} \cdot A^{-1}(m) \in \mathcal{R}_{-\frac{1}{p}} \qquad \text{as } m \to \infty, \tag{17}$$

and

$$\mu_Y(\{\varphi = m\}) \sim \frac{h_Y(c)}{T'(c^+)\, p^{1+\frac{1}{p}}} \cdot \frac{A^{-1}(m)}{m} \in \mathcal{R}_{-(1+\frac{1}{p})} \qquad \text{as } m \to \infty. \tag{18}$$

This enables us to derive the following criterion for the variance of φ to be finite:

$$\int_Y \varphi^2 d\mu_Y < \infty \quad \text{ iff } \quad \int_1^\infty t A^{-1}(t)\, dt < \infty. \tag{19}$$

(In particular, $p < 1/2$ implies finite variance.) To see this, consider the truncated variance $V_\varphi(y) := \int_{Y \cap \{\varphi \leq y\}} \varphi^2 d\mu_Y$, $y > 0$, which will also be important for part b). Now (18) implies that $V_\varphi(y) \approx \int_1^y t A^{-1}(t)\, dt$, proving (19), and also that

$$V_\varphi(y) \sim \frac{h_Y(c)}{T'(c^+)\, p^{1+\frac{1}{p}}} \cdot \int_1^y t A^{-1}(t)\, dt \qquad \text{as } y \to \infty \tag{20}$$

in case $\int_Y \varphi^2 d\mu_Y = \infty$.

Take an observable f which is measurable w.r.t. some ξ_K, $K \geq 1$. As it is most convenient to work with centered random variables, we let $f^c := f - \mu(f)$. Since for any $k \in \mathbb{N}$, $\xi_{Y,k} = \bigvee_{j=0}^{k-1} T_Y^{-j} \xi_Y$ refines the restriction of ξ_k to Y, it is easily seen that the induced observable $f_Y^c := f_\varphi^c = \sum_{k=0}^{\varphi - 1} f^c \circ T^k$ is measurable $\xi_{Y,K}$. Notice also that $f^c = f^c(0)$ on $T(Y \cap \{\varphi > K\})$, so that for any $k \in \mathbb{N}$, $f^c \circ T^k = f^c(0)$ on $Y \cap \{\varphi \geq K + k\}$. Consequently, $f_Y^c = f^c + (\varphi - K)^+ f^c(0) + \sum_{k=\varphi-K+1}^{\varphi-1} f^c \circ T^k$, proving that

$$\begin{gathered} f_Y^c = f^c(0) \cdot \varphi + \psi \\ \text{with } \psi : Y \to \mathbb{R} \text{ measurable } \xi_{Y,K} \text{ and bounded.} \end{gathered} \tag{21}$$

Moreover, letting $\beta := \mathrm{sign}(f^c(0)) \in \{\pm 1\}$ if $f^c(0) \neq 0$, (17) shows that

$$\mu_Y(\{\beta f_Y^c > t\}) \sim \left(\frac{f^c(0)}{p}\right)^{\frac{1}{p}} \frac{h_Y(c)}{T'(c^+)} \cdot A^{-1}(t) \in \mathcal{R}_{-\frac{1}{p}} \qquad \text{as } t \to \infty, \tag{22}$$

in this case, while βf_Y^c is bounded from below anyway.

a) If $\int \varphi^2 d\mu_Y$ is finite, then so is $\int (f_Y^c)^2 d\mu_Y$, and we are in the domain of the standard CLT for Gibbs-Markov maps, cf. [GH], [AD1]. Hence

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f_Y^c \circ T_Y^k \overset{\mathcal{L}(\mu_Y)}{\Longrightarrow} \mathcal{N}(0, \sigma_Y^2(f_Y^c)),$$

provided that the limit $\sigma_Y^2(f_Y^c) := \lim_{n\to\infty} n^{-1} \int_Y (f_Y^c \circ T_Y^k)^2 d\mu_Y = \int (f_Y^c)^2 d\mu_Y + 2\sum_{k\geq 1} \int f_Y^c \cdot (f_Y^c \circ T_Y^k)\, d\mu_Y$ (which always exists) is positive. But $\sigma_Y^2(f_Y^c) = 0$ iff $f_Y^c = g \circ T_Y - g$ for some measurable g. According to theorem 3.1 of [AD1], g has to be constant in this case, and hence $f_Y^c = 0$ on Y. However, the latter is impossible if $f^c(0) \neq 0$. Proposition 2 now implies (5).

b) If $p = 1/2$ and $\int \varphi^2 d\mu_Y = \infty$, (17) shows that the distribution of φ still is in the ("non-normal") domain of attraction of the normal distribution, cf. theorem 2.6.2 of [IL]. By (21) the same is true for f_Y^c provided that $f^c(0) \neq 0$, and we can apply the last corollary of [AD2] which ensures that there is some normalizing function $B_T \in \mathcal{R}_{1/2}$, characterized by $xV_\varphi(B_T(x)) \sim B_T(t)^2$ (which by (20) is the same as the condition given in the theorem), such that

$$\frac{1}{f^c(0) \cdot B_T(n)} \sum_{k=0}^{n-1} f_Y^c \circ T_Y^k \overset{\mathcal{L}(\mu_Y)}{\Longrightarrow} \mathcal{N}(0,1)$$

for all $f \in \mathcal{F}_\xi$ with $f^c(0) \neq 0$. Proposition 2 now yields (6).

c) In case $p > 1/2$, theorem 6.1 of [AD1] applies to f_Y^c under T_Y and provides us with a normalizing function $B \in \mathcal{R}_p$ such that

$$\frac{1}{B(n)} \sum_{k=0}^{n-1} f_Y^c \circ T_Y^k \overset{\mathcal{L}(\mu_Y)}{\Longrightarrow} \Xi_{\alpha,\beta} \qquad \text{as } n \to \infty.$$

where $t\ell_Y(B(t)) = B(t)^\alpha$ with $\ell_Y(t) := \left(\frac{f^c(0)}{p}\right)^{\frac{1}{p}} \frac{h_Y(c)}{2T'(c^+)} \cdot t^\alpha A^{-1}(t)$. Finally, proposition 2 immediately gives

$$\frac{1}{B(\mu(Y)\,n)} \sum_{k=0}^{n-1} f^c \circ T^k = \frac{1}{B(\mu(Y)\,n)} \left(\sum_{k=0}^{n-1} f \circ T^k - n\mu(f) \right) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \Xi_{\alpha,\beta},$$

completing the proof of the theorem in this case.

d) Let us finally consider the case $f^c(0) = f(0) - \mu(f) = 0$. Here f_Y^c itself is bounded and not only measurable $\xi_{Y,K}$, but w.r.t. some coarser finite partition of Y into intervals. By the standard CLT for uniformly expanding maps, cf. [Ry], we get

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f_Y^c \circ T_Y^k \overset{\mathcal{L}(\mu_Y)}{\Longrightarrow} \mathcal{N}(0, \sigma_Y^2(f_Y^c)),$$

provided that $\sigma_Y^2(f_Y^c) > 0$. Again, $\sigma_Y^2(f_Y^c) = 0$ iff $f_Y^c = 0$ on Y. Applying proposition 2 once again, we end up with (5). On the other hand, if $f_Y^c = 0$ on Y, then proposition 2 applies with $G = 0$ and any B, which implies the last statement. ■

Proof of theorem 2. Fix $T \in \mathcal{T}_{(\text{ii})}$ satisfying the assumptions of the theorem and consider the specific auxiliary system (Y, T_Y, ξ_Y) from lemma 2 above, $\tau := \varphi$ being the first-return time of $Y = Y(T)$. We proceed as in the proof of theorem 1, so it is enough to indicate how to obtain the tail behaviour and the parameters.

Since $A_i(t) := t/r_i(t) \sim A(t)/c_i$ as $t \to 0^+$, we have $A_i^{-1}(t) \sim A^{-1}(c_i t)$ as $t \to \infty$. Therefore, letting $d_0 := 1/T'(c^+)$ and $d_1 := 1/T'(c^-)$, we find (via remark 1) that

$$\mu_Y \left(T^{-1} Z_i \cap \{\varphi = k\}\right) \sim \frac{d_i h_Y(c)}{c_i^{1/p} p^{1+1/p}} \cdot \frac{A^{-1}(k)}{k} \quad \text{as } k \to \infty. \tag{23}$$

Now $f_Y^c = f^c(0) 1_{Y \cap T^{-1} Z_0} \cdot \varphi + f^c(1) 1_{Y \cap T^{-1} Z_1} \cdot \varphi + \psi$ with $\psi : Y \to \mathbb{R}$ measurable $\xi_{Y,K}$ and bounded. Consequently, the truncated variance of f_Y^c, $V(y) := \int_{Y \cap \{|f_Y^c| \le y\}} (f_Y^c)^2 d\mu_Y$, $y > 0$, satisfies

$$V(y) \approx \frac{h_Y(c)}{p^{1/p}} \left(\sum_{i \in \{0,1\}} d_i \left(\frac{f^c(i)}{c_i} \right)^2 \right) \int_1^y t A^{-1}(t)\, dt,$$

which is actually an asymptotic equivalence if $\int_1^\infty t A^{-1}(t)\, dt = \infty$, i.e. if the variance of f_Y^c is infinite. Moreover,

$$\begin{aligned} \mu_Y \left(\{f_Y^c > t\}\right) &\sim \sum_{i \in \{0,1\}} 1_{(0,\infty)}(f^c(i))\, \mu_Y \left(T^{-1} Z_i \cap \{\varphi > t/f^c(i)\}\right) \\ &\sim \frac{h_Y(c)}{p^{1/p}} \sum_{i \in \{0,1\}} 1_{(0,\infty)}(f^c(i))\, d_i A_i^{-1}(t/f^c(i)) \\ &\sim \frac{h_Y(c)}{p^{1/p}} \left(\sum_{i \in \{0,1\}} 1_{(0,\infty)}(f^c(i))\, d_i \left(\frac{f^c(i)}{c_i} \right)^{\frac{1}{p}} \right) \cdot A^{-1}(t) \end{aligned}$$

as $t \to \infty$. Inverting the sign, we see that

$$\mu_Y \left(\{f_Y^c < -t\}\right) \sim \frac{h_Y(c)}{p^{1/p}} \left(\sum_{i \in \{0,1\}} 1_{(-\infty,0)}(f^c(i))\, d_i \left(\frac{-f^c(i)}{c_i} \right)^{\frac{1}{p}} \right) \cdot A^{-1}(t).$$

From these observations all the assertions of theorem 2 follow by arguments analogous to those used in the preceding proof. ■

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