Ergodic properties of infinite measure preserving interval maps with indifferent fixed points

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Abstract

We consider piecewise twice differentiable maps T on $[0,1]$ with indifferent fixed points giving rise to infinite invariant measures and study their behaviour on ergodic components. As we do not assume the existence of a Markov partition but only require the first image of the fundamental partition to be finite, we use canonical Markov extensions to first prove pointwise dual-ergodicity which together with an identification of wandering rates leads to distributional limit theorems. We show that T satisfies Rohlin's formula and prove a variant of the Shannon-McMillan-Breiman theorem. Moreover, we give a stronger limit theorem for the transfer operator providing us with a large collection of uniform and Darling-Kac sets. This enables us to apply recent results from fluctuation theory.

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1 Introduction

Interval maps with indifferent (or neutral) fixed points form a rich family of nontrivial infinite measure preserving transformations. They also serve as models for phenomena of intermittency. Most publications on the subject deal with piecewise surjective maps (see e.g. [A3], [T1]-[T3]), whereas the results of [ADU] apply to Markov maps with indifferent fixed points (see also Section 4.8 of [A0]). The present note continues the study of the general non-markovian case begun in [Z1].

To begin with, let us fix some notations. Throughout λ will denote one-dimensional Lebesgue measure, and $\mathcal{B}$ will be the Lebesgue$-\sigma-$field of the space under consideration. For any interval I and any point $x \in cl(I)$, an $I-$*neighbourhood of* x is meant to be a set of the form $(x-\epsilon, x+\epsilon) \cap I$ (and thus need not contain x).

Definition 1 *A* piecewise monotonic system *is a triple (X, T, ξ), where X is the union of some finite family ξ_0 of disjoint bounded open intervals, ξ is a collection of nonempty pairwise disjoint open subintervals (the* cylinders of rank one*) with $\lambda(X \setminus \bigcup \xi) = 0$, and $T : X \to X$ is a map such that $T \mid_Z$ is continuous and strictly monotonic for each $Z \in \xi$.*

Given such a system, we let ξ_n denote the family of *cylinders of rank n*, that is, the nonempty sets of the form $Z = [Z_0, \ldots, Z_{n-1}] := \bigcap_{i=0}^{n-1} T^{-i} Z_i$ with $Z_i \in \xi$. We let $f_Z := (T^n \mid_Z)^{-1}$ be the inverse of the branch $T^n \mid_Z$. $\partial\xi$ will denote the collection of endpoints of members of ξ. The fundamental partition ξ respectively the system (X, T, ξ) are said to be *Markov* if $TZ \cap Z' \neq$ implies $Z' \subseteq TZ$ whenever $Z, Z' \in \xi$. In this case there is an *image partition* $T_*\xi$ (i.e. a coarsest partition into intervals with respect to which each TZ, $Z \in \xi$ is measurable) which is refined by ξ.

Our maps will be assumed to be twice differentiable on each $Z \in \xi$ and satisfy

(A) *Adler's condition*: $T''/(T')^2$ is bounded on $\bigcup \xi$

as well as

(F) *Finite image condition*: $T\xi = \{TZ : Z \in \xi\}$ is finite.

This is equivalent to the existence of a finite image partition $T_*\xi$. Sometimes T will also be

(U) *uniformly expanding*, i.e. $|T'| \geq \tau > 1$ on $\bigcup \xi$.

Definition 2 *If conditions (A), (F), and (U) are satisfied, we will call (X, T, ξ) (respectively T) an* AFU-system *(*AFU-map*).*

We are mainly interested in piecewise monotonic systems (X, T, ξ) satisfying (A) and (F) for which condition (U) may be violated at a finite number of fixed points: Assume that T is possibly *nonuniformly expanding* in that

(N) there is a finite set $\zeta \subseteq \xi$ such that each $Z \in \zeta$ has an *indifferent fixed point* x_Z satisfying *Thaler's assumptions* as one of its endpoints, i.e.

$$\lim_{x \to x_Z,\, x \in Z} Tx = x_Z \quad \text{and} \quad T'x_Z := \lim_{x \to x_Z,\, x \in Z} T'x = 1,$$

and each x_Z, $Z \in \zeta$, is assumed to be a *one-sided regular source*, i.e. T' decreases on $(-\infty, x_Z) \cap Z$, respectively increases on $(x_Z, \infty) \cap Z$. Moreover, T is *uniformly expanding on sets bounded away from $\{x_Z : Z \in \zeta\}$*, in the sense that letting $X_\epsilon := X \setminus \bigcup_{Z \in \zeta} ((x_Z - \epsilon, x_Z + \epsilon) \cap Z)$ we have

$$|T'| \geq \rho(\epsilon) > 1 \quad \text{on } X_\epsilon \quad \text{for each } \epsilon > 0 \ .$$

Definition 3 *If conditions (A), (F), and (N) are satisfied, we call (X, T, ξ) (respectively T) an* AFN-system *(*AFN-map*).*

Remark 1 *Adler's condition (A) ensures that the order of tangency of the graph of T to the diagonal at x_Z is high enough to render the invariant measure infinite. (Recall that maps like $x \mapsto x + x^{1+p}$ mod 1 with $p \in (0,1)$ have finite absolutely continuous invariant measues, cf. Example 3 on p. 312 of [T1].) The condition also excludes other phenomena which to some extent might compensate the effect of an indifferent fixed point, see Example 1 in [Z1].*

Remark 2 *We require the x_Z to be endpoints of cylinder sets Z only for notational convenience. If this condition is not fulfilled in the first place, simply dissect Z at x_Z and replace it by the resulting intervals Z' and Z''. Clearly then $x_{Z'} = x_{Z''}$. Similarly, if the concave-convex condition for a regular source is not satisfied on all of Z but only on some $Z-$neighbourhood of x_Z, we need only regard the latter as a separate cylinder to see that T is AFN anyway.*

Remark 3 *Glueing endpoints together we see that any piecewise monotonic system in a trivial way is equivalent to one for which X is a single interval. Conversely, we can dissect X at a finite number of points from $\partial\xi$ to produce some smaller connected components, thus altering ξ_0. (We will make use of this in Section 4.)*

Our terminology thus is consistent with that of [Z1]. The following basic structure theorem has been established there:

Theorem A (Structure and invariant densities of AFN-maps) *If (X, T, ξ) is an AFN-system, then there is a finite number of pairwise disjoint open sets $X_1, \ldots, X_m$ such that $TX_i = X_i$ mod λ, and $T \mid_{X_i}$ is conservative and ergodic w.r.t. Lebesgue measure. Almost all points of $D := X \setminus \bigcup_i X_i$ are eventually mapped into one of these ergodic components. The tail-σ-field $\mathcal{B}_\infty := \bigcap_{n\geq 1} T^{-n}\mathcal{B}$ is discrete, so that each X_i admits a finite partition $X_i = X_i(1) \cup \ldots \cup X_i(l(i))$ whose members are cyclically permuted by T, and for any $j \in \{1, \ldots, l(i)\}$, $T^{l(i)} \mid_{X_i(j)}$ is exact. The sets $X_i(j)$ are finite unions of open intervals, and hence so are the X_i. Each X_i supports an absolutely continuous invariant measure μ_i (unique up to a constant factor) which has a lower semicontinuous density h_i of the form*

$$h_i(x) = 1_{X_i}(x) \cdot H_i(x) \cdot G(x) \ ,$$

where H_i satisfies $0 < C^{-1} \leq H_i \leq C$ for some constant C, and

$$G(x) := \begin{cases} \frac{x - x_Z}{x - f_Z(x)} & \text{for } x \in Z \in \zeta \\ 1 & \text{for } x \in X \setminus \bigcup \zeta \ . \end{cases}$$

In particular, μ_i is infinite iff X_i contains a $Z-$neighborhood of some x_Z, $Z \in \zeta$.

Example 1 (Long branched piecewise affine maps whose range structure compensates an indifferent fixed point) *It is not possible to replace the finite image condition* **(F)** *by the weaker* long branch condition $\inf_{Z\in\xi_n} \lambda(T^n Z) > 0$ *and still have the conclusions of Theorem A. For simplicity we give a piecewise affine example instead of a smooth one, as the basic idea will become equally clear. Choose $(a_n)_{n\geq 1}$ in $X := (0,1)$ strictly decreasing to 0, and let $a_0 := 1$,*

$A_i := (a_{i+1}, a_i)$, $i \geq 0$. On $A := (0, a_1)$ we define T to be the continuous increasing function mapping each A_i, $i \geq 1$, affinely onto A_{i-1}. Then $T\mid_A$ will be convex with a neutral fixed point at $x_A := 0$ iff the slopes $s_i := \lambda(A_{i-1})/\lambda(A_i)$, $i \geq 1$, decrease to 1. If $\sum_{n\geq 0} a_n = \infty$, tangency to the diagonal is high enough to give an infinite invariant measure if we extend T to $B := (a_1, 1)$ by affinely mapping B onto X. (For example, $a_n \sim c\, n^{-\frac{1}{p}}$, $p \geq 1$, corresponds to $Tx = x + \kappa\, x^{1+p}$ in the smooth setting.)

Still we can construct a map with $T\mid_A$ as above which preserves λ: For a sequence $a_1 = b_1 < b_2 < \cdots < b_j \nearrow 1$ to be determined below, we let $B_j := (b_j, b_{j+1})$ and define $T\mid_{B_j}$, $j \geq 1$, to map B_j affinely onto $C_j := (a_j, 1)$. With $\xi := \{A\} \cup \{B_j : j \geq 1\}$, (X, T, ξ) is a piecewise monotonic system violating **(F)**, *but having long branches, since $T^n Z \supseteq C_1$ for any $Z \in \xi_n$, $n \geq 1$. However, no single branch $T\mid_{B_j}$ covers some neighbourhood of x_A, so that little mass is transported to sets close to x_A, which can compensate the slow escape of mass from there: Writing $t_j := \lambda(C_j)/\lambda(B_j)$ it is clear that T preserves λ iff $1 = s_i^{-1} + \sum_{j\geq i} t_j^{-1}$ for all $i \geq 1$. This will hold iff $t_j^{-1} = s_{j+1}^{-1} - s_j^{-1}$ for all $j \geq 1$. The latter condition implies $b_{j+1} - b_j = \lambda(B_j) = \lambda(C_j) t_j^{-1} = (1 - a_j)(s_{j+1}^{-1} - s_j^{-1})$, which (recalling $b_1 := a_1$) we finally use to define $(b_n)_{n\geq 1}$. It is easy to check then that $b_n \nearrow 1$ as required.*

Due to the simple structure of the sets X_i, and Remark 3, to study the behaviour on ergodic components we need only consider maps of the following type.

Definition 4 *An AFN-map T will be called* basic *if it is conservative ergodic and if also ζ is nonempty.*

The absolutely continuous invariant measure μ of the basic map T thus is infinite and has density $h = H \cdot G$ with $0 < C^{-1} \leq H \leq C$ for some constant C. Due to the fixed points any basic AFN-map in fact is exact (Theorem 2 of [Z1]). The present paper is dedicated to a more detailed study of basic AFN-maps.

Of course, an AFN-map is a *nonsingular* transformation on $(X, \mathcal{B}, \lambda)$, i.e. T is measurable and $\lambda(B) = 0$ implies $\lambda(T^{-1}B) = 0$. Generally, when attributed to such maps, ergodic properties such as *ergodicity*, *conservativity*, and *exactness* will always be understood to hold with respect to Lebesgue measure unless explicitely stated otherwise. To a nonsingular transformation we associate its *Perron-Frobenius (or transfer-) operator (PFO)* $\mathbf{P} : L_1(X, \lambda) \to L_1(X, \lambda)$ defined by the relation

$$\int_B \mathbf{P}u \, d\lambda = \int_{T^{-1}B} u \, d\lambda \quad \text{ for } u \in L_1(X, \lambda) \text{ and } B \in \mathcal{B}.$$

By an obvious approximation procedure, $\mathbf{P}u$ can also be defined for arbitrary measurable functions of constant sign. For an AFN-map T the PFO and its powers have explicit representations

$$\mathbf{P}^n u(x) = \sum_{y : T^n y = x} g_n(y)\, u(y) \ ,$$

where $g = g_1 := 1_{\bigcup \xi} \cdot |T'|^{-1}$ is the *weight function* of $\mathbf{P}$, and $g_{n+1} := g_n \cdot (g \circ T^n)$. A measurable function $u : X \to [0, \infty)$ is the density of a σ−finite invariant measure iff $\mathbf{P}u = u$. If μ is any measure with respect to which T is nonsingular, the

PFO of T w.r.t. μ (defined as above with λ replaced by μ) will be denoted by $\mathbf{P}_\mu$.

A survey of contents. We first collect some background material on induced systems and canonical Markov extensions (C.M.E.s) in Section 2. In particular we show that under suitable conditions the operations of canonically extending a piecewise monotonic map and that of inducing on some subset commute, and give a variant of Hofbauer's results on the structure of the Markov graph for the type of system relevant for us. In Section 3 we prove a lifting theorem for AFU-maps. As we restrict our attention to absolutely continuous measures, our approach to Markov extensions at a first look may appear somewhat pedestrian (compare [Bu], [H1], or [K2]), but it serves in fact as a preparation for the proof of our main limit theorem (Theorem 9). Still, before turning to the latter, we give a series of results which can be derived from what has been established so far: In Section 4 we construct special partitions for our system for which the C.M.E. is particularly nice and apply Aaronson's method to prove pointwise dual ergodicity of the extension, implying that T also shares this property. In Section 5 we identify minimal wandering rates of AFN-maps admitting expansions at the indifferent fixed points, which together with the preceding results shows that the Darling-Kac limit distribution theorem applies to these maps. Sections 6 and 7 discuss entropy. Employing the lifting theorem we show that AFN-maps satisfy Rohlin's formula, and give a criterion for the entropy to be finite. Moreover, we give a variant of the Shannon-McMillan-Breiman theorem extending a result from [T2].

Sections 8 and 9 are devoted to the main result of this paper. Both statement and proof of Theorem 9 generalise that of Thaler's limit theorem ([T3]) to our family of maps: There are positive constants a_n such that for any Riemann-integrable function u, $a_n^{-1}\sum_{k=0}^{n-1}\mathbf{P}^k u \to \lambda(u)\cdot h$ uniformly on sets bounded away from the indifferent fixed points. As a consequence we can apply a beautiful result of fluctuation theory to our situation (Section 11). Section 10 contains a supplementary result showing that any pointwise dual ergodic transformation has lots of sets which are not Darling-Kac, thus proving the assumptions of the limit theorem to be quite natural.

2 Preliminaries on Induced Systems and Canonical Markov Extensions

First-return maps. Let T be a nonsingular transformation of some $\sigma-$finite measure space $(X,\mathcal{B},\lambda)$. Consider a *recurrent set* $Y\in\mathcal{B}$, i.e. one for which $Y\subseteq\bigcup_{n\geq 1}T^{-n}Y \mod \lambda$. (If in fact $X=\bigcup_{n\geq 1}T^{-n}Y \mod \lambda$, then Y is called a *sweep-out set.*) Then the *first return time function* φ given by $\varphi(x):=\min\{n\geq 1: T^n x\in Y\}$ is finite a.e.. We define the *induced or first-return map* $T_Y: Y\to Y$ (mod λ) by $T_Y x := T^{\varphi(x)}x$. The $n-th$ *return time* on Y is then given by $\varphi_n(x):=\sum_{k=0}^{n-1}\varphi(T_Y^k x)$. If μ is a measure on $\mathcal{B}$ with $0<\mu(Y)<\infty$, we let μ_Y denote its normalized restriction to Y: $\mu_Y(A):=\mu(Y)^{-1}\mu(A\cap Y)$. Generally, objects associated with the induced map will notationally be identified by the subscript ${}_Y$. We will need the following standard result (cf. [Sc], [T2])

Lemma 1 (First-return maps and invariant measures) *Let T be a nonsingular transformation of some $\sigma-$finite measure space $(X, \mathcal{B}, \lambda)$, and $Y \in \mathcal{B}$ some recurrent set.*

1. *If T has a $\sigma-$finite invariant measure μ, and $0 < \mu(Y) < \infty$, then μ_Y is invariant for T_Y. If (T, μ) is ergodic, then so is (T_Y, μ_Y).*
2. *If T_Y has a finite invariant measure $\nu \ll \lambda$, then a $\sigma-$finite invariant measure $\mu \ll \lambda$ for T with $\mu_Y = \nu(Y)^{-1}\nu$ is given by*

$$\mu(A) := \sum_{k \geq 1} \nu\left(T^{-k}A \cap \{\varphi \geq k\}\right).$$

If (T_Y, ν) is ergodic, then (T, μ) is conservative ergodic.

Induced partitions and induced systems. We shall repeatedly make use of the following notation: If ξ up to some null set is a partition of some space $(X, \mathcal{B}, \lambda)$, we let $\xi(x)$ denote the member of ξ containing x, which is well defined for a.e. $x \in X$. We now let (X, T, ξ) be a piecewise monotonic system and consider some recurrent set $Y \subseteq X$ with return time φ. Moreover, we assume that Y is the union of some finite family $\xi_{Y,0}$ of disjoint open subintervals of X, measurable ξ mod λ. We define the *induced partition of ξ on Y* to be $\xi_Y := \bigcup_{n \geq 1} \{\{\varphi = n\} \cap Z \cap T^{-n}M : Z \in \xi_n,\ M \in \xi_{Y,0}\}$, and for $k \geq 1$ let $\xi_{Y,k} := \bigvee_{i=0}^{k-1} T_Y^{-i}\xi_Y$. We then have

$$\xi_{Y,k}(x) = \xi_{\varphi_k(x)}(x) \cap T^{-\varphi_k(x)}\xi_{Y,0}(T^{\varphi_k(x)}x), \qquad x \in Y,\ k \geq 1.$$

Therefore $T_Y^k \xi_{Y,k}(x) = T^{\varphi_k(x)}\xi_{\varphi_k(x)}(x) \cap \xi_{Y,0}(T^{\varphi_k(x)}x)$ and $\xi_{\varphi_k(x)}(x) \supseteq \xi_{Y,k}(x) \supseteq \xi_{\varphi_k(x)+1}(x)$. T_Y is piecewise monotonic, and ξ_Y is its natural partition into intervals on which it is continuous and monotonic. We shall call (Y, T_Y, ξ_Y) the system which (X, T, ξ) *induces* on Y.

Canonical Markov Extensions. The concept of canonical Markov extensions will constitute the key tool in our analysis. Following [K2],[K3] we use a variant built up of whole image intervals, which is particularly convenient as we wish to study iterated densities. Let (X, T, ξ) be a piecewise monotonic system. For $n \geq 0$ define $\mathcal{M}_n := \{T^k Z : Z \in \xi_k,\ 0 \leq k \leq n\}$ and $\mathcal{M} := \bigcup_{n \geq 0} \mathcal{M}_n$. (All members of $\mathcal{M}$ thus are connected sets.) For $B \in \mathcal{M}$, let $\widehat{B} := \{(x, B) : x \in B\}$, $\widehat{\mathcal{M}_n} := \{\widehat{B} : B \in \mathcal{M}_n\}$ and $\widehat{\mathcal{M}} := \bigcup_{n \geq 0} \widehat{\mathcal{M}_n}$. Finally let $\widehat{X} := \bigcup_{B \in \mathcal{M}} \widehat{B} = \bigcup \widehat{\mathcal{M}}$. The map $\widehat{T}$ given by $\widehat{T}(x, B) := (Tx, T(B \cap \xi(x)))$ is well defined a.e. on $\widehat{X}$, and for $m \geq 1$ we have

$$\widehat{T}^m(x, B) = (T^m x, T^m(B \cap \xi_m(x)))$$

(cf. [K2]). The null set of points for which $\widehat{T}$ is undefined may be ignored from the viewpoint of nonsingular ergodic theory (cf. [A0], Proposition 1.0.5). The natural projection $\pi : \widehat{X} \to X$, $\pi(x, B) := x$ is onto and satisfies $\pi \circ \widehat{T} = T \circ \pi$. Letting $\widehat{\xi} := \widehat{\mathcal{M}} \vee \pi^{-1}\xi$ we obtain a system $(\widehat{X}, \widehat{T}, \widehat{\xi})$ which is Markov by construction and satisfies the definition of a piecewise monotonic system except for the finiteness condition on X. It is called the *canonical Markov extension (C.M.E.)* of (X, T, ξ). Objects associated to the C.M.E. will be written with a hat-accent. Observe that the image partition $\widehat{T}_*\widehat{\xi}$ equals $\widehat{\mathcal{M}}$. Notice also that the construction (and hence $\widehat{T}$) strongly depends on the choice of ξ. $\widehat{X}$ will be regarded as the sum of the spaces $\widehat{B}$, $B \in \mathcal{M}$. One-dimensional Lebesgue measure thereon will again be denoted by λ.

The following notation will be convenient: If M is some object (point, cylinder, or image interval) belonging to $\widehat{\mathcal{M}}_n \setminus \widehat{\mathcal{M}}_{n-1}$, then we say it is on *level* n, and write $\Lambda(M) := n$. Notice that $\Lambda \circ \widehat{T} \leq \Lambda + 1$.

The transfer operators $\mathbf{P}$ and $\widehat{\mathbf{P}}$ respectively associated with T and $\widehat{T}$ are also closely related to each other. For $\widehat{u} : \widehat{X} \to [0, \infty)$ measurable and $(\pi_* \widehat{u})(x) := \sum_{\pi(\widehat{x})=x} \widehat{u}(\widehat{x})$ we have $\pi_*(\widehat{\mathbf{P}}\widehat{u}) = \mathbf{P}(\pi_*\widehat{u})$. ($\pi_*$ is just the PFO of π.) Given $u : X \to [0, \infty)$ we define its *lift to the base of* $(\widehat{X}, \widehat{T}, \widehat{\xi})$ to be the function $\widehat{u}$ vanishing on $\widehat{X} \cap \{\Lambda \geq 1\}$ for which $\pi_*\widehat{u} = u$.

Canonical Markov extensions and First-return maps. We will depend on the observation that under suitable conditions the operations of inducing and canonically extending piecewise monotonic systems essentially commute. (The argument below applies whenever Y is a recurrent set for some nonsingular piecewise invertible system, not necessarily one-dimensional.)

Definition 5 *Let (X, T, ξ) be a piecewise monotonic system and let the recurrent set $Y \subseteq X$ be a finite union of bounded subintervals of X. The partition ξ will be called* adapted to Y *if (up to a set of measure zero) Y is a nonempty union of cylinders from ξ, and if also $\xi_Y \subseteq \xi$.*

Remark 4 *Starting with some system (X, T, ξ') and some $Y \subseteq X$ as above which is measurable ξ', we can refine ξ' on Y by simply replacing it by ξ'_Y thereon, thus obtaining a partition $\xi'' := \xi' \vee \{Y^c, \xi'_Y\}$ which is adapted to Y since $\xi''_Y = \xi'_Y$ as is easily seen.*

Lemma 2 (Canonical Markov extensions and First-return maps) *Let (X, T, ξ) be a piecewise monotonic system, and let $Y \subseteq X$ be a recurrent set. If Y is measurable ξ_0 and ξ is adapted to Y, then the following hold*

1. *Let $(\widehat{Y}, \widehat{T_Y}, \widehat{\xi_Y})$ be the C.M.E. of the induced system (Y, T_Y, ξ_Y), and denote by $(\pi^{-1}Y, \widehat{T}_{\pi^{-1}Y}, \widehat{\xi}_{\pi^{-1}Y})$ the system which $(\widehat{X}, \widehat{T}, \widehat{\xi})$ induces on $\pi^{-1}Y$. Then $\varphi \circ \pi = \widehat{\varphi}$, where φ and $\widehat{\varphi}$ respectively are the first-return times of Y and $\pi^{-1}Y$, and we have $\widehat{Y} \subseteq \pi^{-1}Y$, $\widehat{\xi_Y} = \widehat{\xi}_{\pi^{-1}Y} \cap \widehat{Y}$, and $\widehat{T}_{\pi^{-1}Y}\mid_{\widehat{Y}} = \widehat{T_Y}$. Moreover, $\widehat{Y}$ is forward invariant, $\widehat{T}_{\pi^{-1}Y}\widehat{Y} \subseteq \widehat{Y}$, and $\pi^{-1}Y \setminus \widehat{Y}$ is dissipative: $\pi^{-1}Y = \bigcup_{n\geq 1} (\widehat{T}_{\pi^{-1}Y})^{-n}\widehat{Y}$.*

2. *If, moreover, T_Y and $\widehat{T_Y}$ respectively have finite invariant measures μ_Y and $\widehat{\mu_Y}$ with $\pi_*\widehat{\mu_Y} = \mu_Y$, then by the above $\widehat{\mu_Y}$ trivially extends to an invariant measure for $\widehat{T}_{\pi^{-1}Y}$ by setting $\widehat{\mu_Y}(\pi^{-1}Y \setminus \widehat{Y}) = 0$. The invariant measures μ, $\widehat{\mu}$ of T and $\widehat{T}$ obtained from the respective induced systems T_Y and $\widehat{T}_{\pi^{-1}Y}$ as in the second part of Lemma 1 then satisfy $\pi_*\widehat{\mu} = \mu$.*

Proof. Notice first that due to the extension property and the nature of $\pi^{-1}Y$ we have $\varphi_n \circ \pi = \widehat{\varphi}_n$ for $n \geq 1$, and $\pi^{-1}Y$ is a recurrent set for $\widehat{T}$, so that we can indeed induce thereon. Y being measurable ξ_0 we have $\xi_{Y,0} \subseteq \xi_0$, that is, $\mathcal{M}_{Y,0} \subseteq \mathcal{M}_0$.

Let us make clear the implications of ξ being adapted to Y. Condition $\xi_Y \subseteq \xi$ means that for $x \in Y$ we have $\xi(x) = \xi_Y(x)$ and hence $\xi_k(x) = \xi_Y(x)$ for $k \in \{1, \ldots, \varphi(x)\}$. It follows that $T^k\xi(x) \subseteq Y^c$ for $k < \varphi(x)$, while of course $T^{\varphi(x)}\xi(x) \subseteq Y$. We claim that in fact for each $Z \in \xi_n \cap Y$, $n \geq 1$, $T^n Z$ is either contained in Y or in Y^c. To see this, choose $x \in Z$ so that $Z = \xi_n(x) = \bigcap_{i=0}^{n-1} T^{-i}\xi(T^i x)$, and let $j := \max\{i \in \{1, \ldots, n-1\} : T^i x \in Y\}$. Writing $y := T^j x$ we thus have $\varphi(y) \geq n - j$, and according to the foregoing observation $T^{n-j}\xi_{n-j}(y)$ therefore is contained in one of Y and Y^c. Since on the other hand $Z \subseteq T^{-j}\xi_{n-j}(y)$, we find that $T^n Z \subseteq T^{n-j}\xi_{n-j}(y)$, which proves our claim. As a consequence, the identity $\xi_{Y,k}(x) = \xi_{\varphi_k(x)}(x) \cap T^{-\varphi_k(x)}\xi_{Y,0}(T^{\varphi_k(x)}x)$, $k \geq 1$ in the adapted case thus becomes

$$\xi_{Y,k}(x) = \xi_{\varphi_k(x)}(x) \qquad \text{for } x \in Y.$$

Looking at the construction of $(\widehat{Y}, \widehat{T_Y}, \widehat{\xi_Y})$ we therefore find that $\mathcal{M}_Y = \{T_Y^k W : W \in \xi_{Y,k},\ k \geq 0\}$ is a subclass of $\mathcal{M} \cap Y$, $\mathcal{M} = \{T^k Z : Z \in \xi_k,\ k \geq 0\}$. Thus, $\widehat{Y} = \bigcup \widehat{\mathcal{M}_Y} \subseteq \bigcup (\widehat{\mathcal{M} \cap Y}) = \pi^{-1}Y \subseteq \widehat{X}$. $\widehat{\xi_Y} = \widehat{\xi}_{\pi^{-1}Y} \cap \widehat{Y}$ then is automatically satisfied. Consider now $\widehat{x} = (x, B) \in \pi^{-1}Y$ and notice that for $n \geq 1$,

$$\widehat{T}^n_{\pi^{-1}Y}\widehat{x} = \widehat{T}^{\widehat{\varphi}_n(\widehat{x})}\widehat{x} = \widehat{T}^{\varphi_n(x)}(x, B) = (T^{\varphi_n(x)}x, T^{\varphi_n(x)}(B \cap \xi_{\varphi_n(x)}(x))).$$

B being open, there exists $n_0 \geq 1$ such that $\xi_{\varphi_n(x)}(x) \subseteq B$ whenever $n \geq n_0$. For such n thus

$$\widehat{T}^n_{\pi^{-1}Y}\widehat{x} = (T^{\varphi_n(x)}x, T^{\varphi_n(x)}\xi_{\varphi_n(x)}(x)) = (T_Y^n x, T_Y^n \xi_{Y,n}(x)) \in \widehat{Y},$$

since $T_Y^n \xi_{Y,n}(x) \in \mathcal{M}_Y$. Hence $\pi^{-1}Y = \bigcup_{n \geq 1} (\widehat{T}_{\pi^{-1}Y})^{-n}\widehat{Y}$. If in particular $\widehat{x}$ lies in $\widehat{Y}$ in the first place, then we may take $n = n_0 = 1$ to obtain $\widehat{T}^n_{\pi^{-1}Y}\widehat{x} = \widehat{T_Y}\widehat{x}$.

As for the second part of the Lemma, notice that $\pi^{-1} \circ T^{-k} = \widehat{T}^{-k} \circ \pi^{-1}$ implies that for any $A \in \mathcal{B}$,

$$\begin{aligned}
\widehat{\mu}(\pi^{-1}A) &= \sum_{k \geq 1} \widehat{\mu_Y}\left(\widehat{T}^{-k}\pi^{-1}A \cap \{\widehat{\varphi} \geq k\}\right) = \\
&= \sum_{k \geq 1} \widehat{\mu_Y}\left(\pi^{-1}\left(T^{-k}A \cap \{\varphi \geq k\}\right)\right) = \\
&= \sum_{k \geq 1} \mu_Y\left(T^{-k}A \cap \{\varphi \geq k\}\right) = \mu(A). \qquad \blacksquare
\end{aligned}$$

The transition structure of Canonical Markov Extensions. It will be crucial to know about the special transition structure of C.M.E.'s of finite-image maps. To make this precise we introduce a natural *successor relation* on $\widehat{\xi}$ by writing $C \to D$ iff $D \subseteq \widehat{T}C$. (To avoid excessive use of the hat notation, we will not write $\widehat{C}$ if it is clear that the cylinder belongs to $\widehat{\xi}$. Similarly for members of $\widehat{\mathcal{M}}$). The directed graph $\mathcal{G} := (\widehat{\xi}, \to)$ thus obtained is the *Markov graph* of

$(\widehat{X},\widehat{T},\widehat{\xi})$. C will be called *critical* if $cl(\pi(C))\cap\partial\xi\neq$, that is, if its projection shares an endpoint with the member of ξ it is contained in. The study of Markov graphs associated with extensions of piecewise invertible systems initiated by F.Hofbauer has led to a wealth of deep results. For our variant of Markov extensions the following version of Hofbauer's fundamental structure theorem holds:

Lemma 3 (Structure of the Markov graph) *Consider a piecewise monotonic system (X,T,ξ) with C.M.E. $(\widehat{X},\widehat{T},\widehat{\xi})$. For $C\in\widehat{\xi}$ let $\alpha(C):=\inf\{i\geq 1 : cl(\pi(\widehat{T}^iC))\cap\partial\xi\neq\}$, and assume that C is critical. Then, for $1\leq i<\alpha(C)$, $\widehat{T}^iC$ is a member of both $\widehat{\xi}$ and $\widehat{\mathcal{M}}$, and is in fact the only successor of $\widehat{T}^{i-1}C$ in $\mathcal{G}$. Suppose that $\alpha:=\alpha(C)<\infty$ and let $B:=\widehat{T}^{\alpha}C\in\widehat{\mathcal{M}}$. The family $B\cap\widehat{\xi}$ of successors of $I:=\widehat{T}^{\alpha-1}C$ consists of an (empty, finite, or countably infinite) collection $\{F_j\}_{j\in J}$ of cylinders for which $\pi(F_j)\in\xi$, and at most two cylinders D_k, $k\in K$, where K equals , $\{0\}$, or $\{0,1\}$, for which $\pi(D_k)$ is strictly contained in some member of ξ. Moreover, at most one of the D_k satisfies $\Lambda(\widehat{T}D_k)>\alpha+1$, while trivially $\Lambda(\widehat{T}F_j)\leq 1$ for $j\in J$.*

Proof. The assertions follow by the arguments of Lemmas 12, 13 and Theorem 9 of [H3]: Let $c\in\partial\xi$ be an endpoint of $\pi(C)$ and assume that $\alpha<\infty$. Then at most two of the successors $B\cap Z$, $Z\in\widehat{\xi}$ are proper subsets of cylinders. If there are two, the projection $\pi(D)$ of one of them has $T^{\alpha}c$ as an endpoint and some $d\in\partial\xi$ as the other. But then $T(\pi(D))$ equals $T^{\alpha+1}(C\cap T^{-\alpha}\xi(T^{\alpha}c))= T^{\alpha+1}(\xi_{\alpha+1}(c))\in\mathcal{M}_{\alpha+1}$. ■

If the system is AFN, then $\alpha(C)<\infty$ for $C\in\widehat{\xi}$. Let us call the vertex C a *knot* if it has more than one successor. Although a knot need not be critical, all its successors are. We next observe that the finite image property (F) implies certain important finiteness properties of the C.M.E.:

Lemma 4 (Finiteness properties of the C.M.E.) *Let (X,T,ξ) be a piecewise monotonic system such that $\alpha(C)<\infty$ for each $C\in\widehat{\xi}$, and consider its C.M.E. $(\widehat{X},\widehat{T},\widehat{\xi})$. If (X,T,ξ) satisfies the finite image property (F), then each $\widehat{\mathcal{M}}\cap\{\Lambda=i\}$, $i\geq 1$, is finite. Moreover, there exists $\eta_0\geq 1$ such that any $B\in\widehat{\mathcal{M}}\cap\{\Lambda\geq\eta_0\}$ contains at most one $D\in\widehat{\xi}$ for which $\Lambda(\widehat{T}D)>\Lambda(D)$. In particular, for any $L\geq\eta_0$ we have $\#\widehat{\mathcal{M}}\cap\{\Lambda=L\}\leq\#\widehat{\mathcal{M}}\cap\{\Lambda=\eta_0\}$.*

Proof. Finiteness of the $\widehat{\mathcal{M}}\cap\{\Lambda=i\}$, $i\geq 1$, is immediate. To prove the second statement, notice first that since each $C\in\widehat{\xi}\cap\{\Lambda=0\}$ is critical and any $E\in\widehat{\xi}\cap\{\Lambda>0\}$ is a successor of some $E^-\in\widehat{\xi}$ with $\Lambda(E^-)=\Lambda(E)-1$, any $E\in\widehat{\xi}\cap\{\Lambda>0\}$ is the endpoint E_r of some finite path $\vartheta=(E_0,\ldots,E_r)$, $r\geq 1$, in $\mathcal{G}$ with $\Lambda(E_{i+1})=\Lambda(E_i)+1$ for which E_0 is critical while E_i, $0<i<r$ are not. Such a path will be called an *increasing ladder ending at E*. Notice that in this situation we have $\alpha(E_0)\geq r$. If in addition E is a knot, then $\alpha(E_0)\leq r+1$.

Let $\eta_0:=\max\{\alpha(C):C\in\widehat{\xi}\cap\{\Lambda=0\}\}+2$ which is finite by (F), and fix any $B\in\widehat{\mathcal{M}}\cap\{\Lambda\geq\eta_0\}$ containing more than one cylinder from $\widehat{\xi}$ (the other case being trivial). Then $B=\widehat{T}E$ for some knot $E\in\widehat{\xi}$ with $\Lambda(E)=\Lambda(B)-1$.

If E is critical, then we are in the situation of Lemma 3 with $C=E=I$ and $\alpha(C)=1$. Hence there is at most one $D\in\widehat{\xi}\cap B$ for which $\Lambda(\widehat{T}D)>2$, and we are done. If on the other hand E is not critical, we choose an increasing ladder $(E_0, \ldots, E_r)$ ending in E, so that Lemma 3 applies with $C = E_0$, $I = E$, and $\alpha(C) = r+1$. Therfore there is at most one $D \in \widehat{\xi} \cap B$ for which $\Lambda(\widehat{T}D) > r+1$. According to the definition of η_0, however, we must have $\Lambda(E_0) \geq 1$. Hence $\Lambda(B) = \Lambda(E) + 1 = \Lambda(E_0) + r + 1 > r + 1$, which proves our claim. ■

The following central lemma formalizes what is sometimes expressed by saying that the entropy at infinity of the Markov graph is zero. It will enable us to study essential aspects of the dynamics of $\widehat{T}$ by considering sets $\widehat{X} \cap \{\Lambda \leq \eta\}$.

Lemma 5 (Counting paths which remain above their starting level) *Let (X, T, ξ) be a piecewise monotonic system such that $\alpha(C) < \infty$ for each $C \in \widehat{\xi}$, and consider its C.M.E. $(\widehat{X}, \widehat{T}, \widehat{\xi})$. For any $\sigma > 1$ there exist $\eta \geq 1$ and $\kappa > 0$ such that for all $C \in \widehat{\xi}$ with $\Lambda(C) \geq \eta$ we have*

$$\#\Gamma^*(C, l) < \kappa \cdot \sigma^l \quad \text{for } l \geq 1,$$

where $\Gamma^*(C,l) := \{\gamma = (C_0, \ldots, C_{l-1}) : C_0 = C,\ C_i \in \widehat{\xi} \cap \{\Lambda > \Lambda(C)\}$ for $1 \leq i < l$, $C_{i-1} \to C_i$, and $\widehat{T}C_{l-1} \in \widehat{\mathcal{M}} \cap \{\Lambda > \Lambda(C)\}\}$ is the collection of all paths of length $l-1$ in the Markov graph $\mathcal{G}$ which start in C and cannot return to $\widehat{\xi} \cap \{\Lambda \leq \Lambda(C)\}$ even if another edge is adjoined. (If the last condition were omitted, the corresponding family of paths could be infinite.)

When applying this result to finite image systems, we will always assume that η has been chosen larger than η_0 from Lemma 4. In particular then $\#\{C \in \widehat{\xi} \cap \{\Lambda = L\} : \Gamma^*(C, l) \neq \emptyset\} \leq \#\widehat{\mathcal{M}} \cap \{\Lambda = \eta\}$.

Proof. We label the edges of $\mathcal{G}$ as follows: If $I \in \widehat{\xi}$ has the unique successor D we write $\psi(I, D) := 0$. Otherwise I is a knot. For any successor F of I with $\pi(F) \in \xi$ we let $\psi(I, F) := 2$. There are at most two further successors D_k, $k \in K \subseteq \{0, 1\}$ which we assume to be numbered in such a way that $\Lambda(\widehat{T}D_0) \geq \Lambda(\widehat{T}D_1)$ in case there are two of them. We then let $\psi(I, D_k) := k$. The label of an edge (I, D) tells us at what levels the successors of D can be sited. If $\psi(I, D) = 2$, then all of them belong to $\{\Lambda \leq 1\}$.

For fixed $C \in \widehat{\xi}$ with $\Lambda(C) \geq 2$ paths $\gamma = (C_0, \ldots, C_{l-1}) \in \Gamma^*(C, l)$ therefore only contain edges labelled 0 or 1, and can thus be coded unambigously by recording at each step the label of the edge, i.e. the map $\Psi : \Gamma^*(C, l) \to \{0, 1\}^{l-1}$ defined by $\Psi(\gamma)_i := \psi(C_{i-1}, C_i)$ is injective.

We claim that each label 1 edge in γ, except possibly the first one, is preceded by at least $\Lambda(C) - 1$ edges labelled 0, so that $\sum_{i=1}^{l-1} \Psi(\gamma)_i \leq 1 + l(\Lambda(C) - 1)^{-1}$. To see this, we fix such an edge (C_{i-1}, C_i) in γ. Starting from C_i we go back along γ until we first meet a critical vertex C_j, i.e. we let $j := \max\{m \in \{0, \ldots, i-1\} : C_m$ is critical$\}$ which is well defined unless (C_{i-1}, C_i) is the first label 1 edge in γ). By Lemma 3 we then have $\Lambda(\widehat{T}C_i) \leq i - j + 1$, and since $\gamma \in \Gamma^*(C, l)$, it follows that $\Lambda(\widehat{T}C_i) > \Lambda(C)$, so that $i - j \geq \Lambda(C)$, which establishes our claim. Hence, if $\Lambda(C) \geq \eta$,

$$\#\Gamma^*(C, l) \leq \sum_{m \leq (\eta-1)^{-1} l + 1} \binom{l}{m} =: R(\eta, l) \ ,$$

and a straightforward application of Stirling's formula shows that $R(\eta, l) \leq \kappa \cdot \sigma^l$ for $l \geq 1$ provided η is sufficiently large, which gives the criterion for our choice of this constant. ■

3 Lifting absolutely continuous invariant measures of an AFU-system

In order to make use of auxiliary systems like C.M.E.s, we need control of the respective invariant measures. We will in particular depend on the possibility to construct an invariant measure for the C.M.E. of a basic AFN-map which projects down onto the invariant measure of the original system by π_*. Our approach will be to make a detour via some nice (i.e. AFU) induced system for which the lifting theorem to follow applies, and then employ Lemma 2.

Proposition 1 (Lifting a.c.i.m.s of an AFU-system) *Let (X, T, ξ) be an AFU-system, and $\nu \ll \lambda$ an invariant probability measure for T. Then the C.M.E. $(\widehat{X}, \widehat{T}, \widehat{\xi})$ admits an invariant probability measure $\widehat{\nu} \ll \lambda$ with $\pi_* \widehat{\nu} = \nu$, and $\widehat{\nu}$ is ergodic if ν is.*

Nevertheless, $(\widehat{X}, \widehat{T}, \widehat{\xi})$ need not be conservative w.r.t. λ. If for example X is an interval which is not covered by any single branch, then $\widehat{X} \cap \{\Lambda = 0\}$ is dissipative.

The present section is devoted to a proof of this result which is not properly covered by the lifting theorems to be found for example in [K2], since we need to deal with infinite partitions ξ. Equally important, this section serves as a preparation and warm-up for the proof of our main limit theorem (Theorem 9 below), which will follow similar lines. In particular it will also make use of the following application of Lemma 5 to AFU-systems ensuring that $\widehat{T}$ will not let mass initially concentrated on the base of the extension escape to infinity.

Lemma 6 (Controlling the mass escaping to high levels of AFU-extensions) *Let (X, T, ξ) be an AFU-system with C.M.E. $(\widehat{X}, \widehat{T}, \widehat{\xi})$. There exist $\eta \geq 1$, $K \in (0, \infty)$, and $q \in (0, 1)$ such that for any measurable $\widehat{u} : \widehat{X} \to [0, \infty)$ supported on $\{\Lambda = 0\}$ we have:*

$$\sum_{k=0}^{n-1} \widehat{\mathbf{P}}^k \widehat{u}(\widehat{x}) \leq K \cdot q^{\Lambda(\widehat{x})} \cdot \left(\sup_{\{\Lambda \leq \eta\}} \sum_{k=0}^{n-1} \widehat{\mathbf{P}}^k \widehat{u} \right) \qquad \textit{for } \widehat{x} \in \widehat{X} \cap \{\Lambda > \eta\}.$$

Proof. Let $\rho := \sup_X \mid T' \mid^{-1}$, then $\rho \in (0, 1)$, and $\sup_{\widehat{X}} \widehat{g}_l \leq \rho^l$ for $l \geq 1$, where $\widehat{g}_l = \mid (\widehat{T}^l)' \mid^{-1}$ is the weight function of $\widehat{\mathbf{P}}^l$. Choose $\sigma \in (1, \rho^{-1})$ and apply Lemma 5 to obtain η and κ as specified there. Let $q := \sigma\rho \in (0, 1)$. We need some additional notations: For $\widehat{x} \in \widehat{X} \cap \{\Lambda > \eta\}$ and $k \geq 1$ we let $\Gamma(\widehat{x}) := \{(\widehat{y}, l) : \widehat{y} \in \widehat{X} \cap \{\Lambda \leq \eta\}, \widehat{T}\widehat{y}, \ldots, \widehat{T}^l \widehat{y} \in \widehat{X} \cap \{\Lambda > \eta\}, \widehat{T}^l \widehat{y} = \widehat{x}\}$, and $\Gamma(\widehat{x}, k) := \{(\widehat{y}, l) \in \Gamma(\widehat{x}) : l \leq k\}$. Then, since $\widehat{u}$ is supported on $\{\Lambda = 0\}$,

$$\widehat{\mathbf{P}}^k \widehat{u}(\widehat{x}) = \sum_{(\widehat{y}, l) \in \Gamma(\widehat{x}, k)} \widehat{g}_l(\widehat{y}) \cdot \widehat{\mathbf{P}}^{k-l} \widehat{u}(\widehat{y}) \ .$$

Consequently,

$$\sum_{k=0}^{n-1} \widehat{\mathbf{P}}^k \widehat{u}(\widehat{x}) = \sum_{(\widehat{y}, l) \in \Gamma(\widehat{x})} \widehat{g}_l(\widehat{y}) \cdot \sum_{k=0}^{n-1-l} \widehat{\mathbf{P}}^k \widehat{u}(\widehat{y}) \leq \sum_{(\widehat{y}, l) \in \Gamma(\widehat{x})} \widehat{g}_l(\widehat{y}) \cdot \sum_{k=0}^{n-1} \widehat{\mathbf{P}}^k \widehat{u}(\widehat{y}) \ .$$

On the other hand, for any $\widehat{G} : \widehat{X} \to [0, \infty)$ we have

$$\begin{aligned}
\sum_{(\widehat{y},l)\in\Gamma(\widehat{x})} \widehat{g}_l(\widehat{y}) \cdot \widehat{G}(\widehat{y}) &\leq \left(\sup_{\{\Lambda\leq\eta\}} \widehat{G}\right) \sum_{C\in\widehat{\xi},\ \Lambda(C)=\eta} \#\{(\widehat{y},l) \in \Gamma(\widehat{x}) : \widehat{y} \in C\} \cdot \rho^l \\
&\leq \left(\sup_{\{\Lambda\leq\eta\}} \widehat{G}\right) \sum_{C\in\widehat{\xi},\ \Lambda(C)=\eta} \sum_{l\geq\Lambda(\widehat{x})-\eta} \#\Gamma^*(C,l) \cdot \rho^l \\
&\leq \left(\sup_{\{\Lambda\leq\eta\}} \widehat{G}\right) 2(\#(\widehat{\mathcal{M}} \cap \{\Lambda = \eta\}))\, \kappa\, \frac{q^{-\eta}}{1-q} \cdot q^{\Lambda(\widehat{x})}.
\end{aligned}$$

Here we use that each $(\widehat{y}, l) \in \Gamma(\widehat{x})$ with $\widehat{y} \in C$ uniquely determines a path $\gamma(\widehat{y}, l) = (C_0, \ldots, C_{l-1}) \in \Gamma^*(C, l)$ by the requirement that $\widehat{T}^j\widehat{y} \in C_j$ for $j = 1, \ldots, l-1$, that is, $[C_0, \ldots, C_{l-1}] = \widehat{\xi}_l(\widehat{y})$. But of course the map $(\widehat{y}, l) \longmapsto \gamma(\widehat{y}, l)$ is injective on $\Gamma(\widehat{x})$ since $\widehat{T}^l$ is injective on $\widehat{\xi}_l(\widehat{y})$. Therefore for every $l \geq 1$, $\#\{(\widehat{y}, l) \in \Gamma(\widehat{x}) : \widehat{y} \in C\} \leq \#\Gamma^*(C, l)$. Also, it is clear that $(\widehat{y}, l) \in \Gamma(\widehat{x})$ implies $l \geq \Lambda(\widehat{x}) - \eta$ and $\Lambda(\widehat{y}) = \eta$. The final step is immediate from Lemma 5. The assertion of the lemma now follows by taking $\widehat{G} = \sum_{k=0}^{n-1} \widehat{\mathbf{P}}^k\widehat{u}$. ■

The second ingredient is a compactness property of the densities pushed forward on the extension. To formulate it we need

Definition 6 *If (X, T, ξ) is a piecewise monotonic system with C.M.E. $(\widehat{X}, \widehat{T}, \widehat{\xi})$, a function $\widehat{u} : \widehat{X} \to [0, \infty)$ will be called* admissible *if there exists some nonempty family $\mathcal{P}(\widehat{u}) \subseteq \widehat{\mathcal{M}}$ contained in $\{\Lambda \leq m\}$ for m sufficiently large such that $\widehat{u}$ vanishes on $(\bigcup \mathcal{P}(\widehat{u}))^c$, but is strictly positive and differentiable on each $B \in \mathcal{P}(\widehat{u})$. We define the* regularity *of $\widehat{u}$ on B to be $R_B(\widehat{u}) := \sup\{\mid \widehat{u}'(x) \mid / \widehat{u}(x) : x \in B\}$ if $\widehat{u} > 0$ on B, and $R_B(\widehat{u}) := 0$ otherwise. Finally let $R(\widehat{u}) := \sup_{B\in\widehat{\mathcal{M}}} R_B(\widehat{u})$.*

Remark 5 *If B is a bounded interval and $\widehat{u} : B \to (0, \infty)$ satisfies $R_B(\widehat{u}) < \infty$, then $\sup_B \widehat{u} \leq \kappa \inf_B \widehat{u}$, where $\kappa := \exp(\lambda(B)R_B(\widehat{u}))$.*

Lemma 7 (Iterated admissible densities for the C.M.E. of an AFU-map) *Let (X, T, ξ) be an AFU-map and consider an admissible function $\widehat{u}$ on its C.M.E. with $R(\widehat{u}) < \infty$. Let $\widehat{u}_n := \widehat{\mathbf{P}}^n\widehat{u}$, $n \geq 0$, then each $\widehat{u}_n$ is admissible and for each member B of the image partition $\widehat{\mathcal{M}}$ there exists $r_B \in (0, \infty)$ such that $R_B(\widehat{u}_n) \leq r_B$ for all $n \geq 0$, and the same applies to the sequence $(\widehat{U}_n)_{n\geq1}$ with $\widehat{U}_n := \sum_{k=0}^{n-1} \widehat{u}_k$. Moreover, the sequence $(\widehat{u}_n)_{n\geq0}$ is uniformly bounded on B.*

Proof. This gives a nice exercise on transfer operators and Adler's condition. Also, it is essentially contained in Lemma 13 below: The proof of that lemma applies and due to uniform expansiveness we can replace Lemma 18 employed there by a trivial geometric-series estimate. ■

Remark 6 *If $(U_n)_{n\geq1}$ is a sequence of positive functions on the bounded interval B for which $(R(U_n))_{n\geq1}$ is bounded, and $(a_n)_{n\geq1}$ is a sequence in $(0, \infty)$ such that $U_n \leq a_n$ for all n, then the sequence $({a_n}^{-1}U_n)_{n\geq1}$ is uniformly bounded and uniformly Lipschitz on B. Therefore it is relatively compact in $\mathcal{C}(B)$ by the Arzela-Ascoli theorem.*

Proof of Proposition 1. Assume first that the system is ergodic (w.r.t. λ). Let $u := 1_X$ and $\widehat{u} := 1_{\widehat{X}\cap\{\Lambda=0\}}$ be its lift to the base of $\widehat{X}$, then $\widehat{u}$ is admissible for $(\widehat{X}, \widehat{T}, \widehat{\xi})$ and satisfies $R(\widehat{u}) = 0$ as well as $\pi_*\widehat{u} = u$. Write $U_n := \sum_{k=0}^{n-1} \mathbf{P}^k u$, $\widehat{u}_n := \widehat{\mathbf{P}}^n \widehat{u}$, and $\widehat{U}_n := \sum_{k=0}^{n-1} \widehat{u}_k$. Lemma 7 applies to the sequences $(\widehat{u}_n)$ and $(\widehat{U}_n)$. Observe also that for $B \in \widehat{\mathcal{M}}$, $\widehat{U}_n \mid_B$ is positive for $n \geq \Lambda(B)$. Choose η, K and q according to Lemma 6. The lower part $\{\Lambda \leq \eta\}$ of $\widehat{X}$ consists of finitely many image-intervals $B \in \widehat{\mathcal{M}}$. For $n \geq 1$ thus $b_n := \sup_{\{\Lambda\leq\eta\}} \widehat{U}_n$ is finite.

By Remark 6, for each $B \in \widehat{\mathcal{M}} \cap \{\Lambda \leq \eta\}$ the sequence $(b_n^{-1}\widehat{U}_n \mid_B)_{n\geq 1}$ is relatively compact in $\mathcal{C}(B)$. If on the other hand $B \in \widehat{\mathcal{M}} \cap \{\Lambda > \eta\}$, then by Lemma 6

$$\widehat{U}_n \mid_B \leq K\, q^{\Lambda(B)} \cdot b_n \qquad \text{for all } n \geq 1, \quad (\diamondsuit)$$

whence $(b_n^{-1}\widehat{U}_n \mid_B)_{n\geq 1}$ again is relatively compact in $\mathcal{C}(B)$. A standard diagonalisation argument therefore yields $n_k \nearrow \infty$ and $\widehat{H} : \widehat{X} \to [0, \infty)$ Lipschitz on members of $\widehat{\mathcal{M}}$ such that

$$\frac{1}{b_{n_k}} \widehat{U}_{n_k} \longrightarrow \widehat{H} \qquad \text{uniformly on each } B \in \widehat{\mathcal{M}}.$$

By our choice of the b_n and finiteness of $\widehat{\mathcal{M}} \cap \{\Lambda \leq \eta\}$, $\widehat{H}$ is not identically zero. Observe that $(\lambda(b_n^{-1}\widehat{U}_n))_{n\geq 1}$ is bounded: By $(\diamondsuit)$ we have

$$\begin{aligned} \lambda\left(\frac{1}{b_n}\widehat{U}_n \cdot 1_{\widehat{X}\cap\{\Lambda>\eta\}}\right) &\leq \lambda(X)(\#\widehat{\mathcal{M}} \cap \{\Lambda = \eta\}) \sum_{l\geq 1} \sup_{\widehat{X}\cap\{\Lambda=\eta+l\}} \left(\frac{1}{b_n}\widehat{U}_n\right) \leq \\ &\leq \lambda(X)(\#\widehat{\mathcal{M}} \cap \{\Lambda = \eta\}) K (1-q)^{-1} \end{aligned}$$

for $n \geq 1$. Also, $(\lambda(b_n^{-1}\widehat{U}_n \cdot 1_{\widehat{X}\cap\{\Lambda\leq\eta\}}))_{n\geq 1}$ is bounded by uniform convergence.

Consequently, by Fatou's lemma, $\lambda(\widehat{H}) < \infty$. Also, since $\lambda(\widehat{U}_n) = n\lambda(u)$ we find that $b_n \to \infty$, which in turn implies that $\widehat{\mathbf{P}}\widehat{H} = \widehat{H}$, since by the last statement in Lemma 7 we have $\widehat{\mathbf{P}}\widehat{H} \leq \widehat{H}$ (see for example p. 726 of [K3]). Thus, $\widehat{h} := \lambda(\widehat{H})^{-1}\widehat{H}$ is the density of some invariant probability measure $\widehat{\nu} \ll \lambda$ of $\widehat{T}$. It remains to prove that $\pi_*\widehat{\nu} = \nu$. The estimate $(\diamondsuit)$ shows that the convergence $c_{n_k}^{-1}\widehat{U}_{n_k} \longrightarrow \widehat{h}$, with $c_n := \lambda(\widehat{H})b_n$, is summably dominated on each fiber $\pi^{-1}(x)$, $x \in X$, hence we have

$$\pi_*(\widehat{h}) = \pi_*\left(\lim_{k\to\infty} \frac{1}{c_{n_k}}\widehat{U}_{n_k}\right) = \lim_{k\to\infty} \pi_*\left(\frac{1}{c_{n_k}}\widehat{U}_{n_k}\right) = \lim_{k\to\infty} \frac{1}{c_{n_k}} U_{n_k}.$$

Moreover it is clear that $\pi_*(\widehat{h})$ is a probability density on X. But as $n^{-1}U_n \to \lambda(X)\, h$ with $h := d\nu/d\lambda$ weakly in L_1 by ergodicity of T (and in fact uniformly, cf. [Ry]), this implies that $\pi_*(\widehat{h}) = h$. For ergodicity of $\widehat{\nu}$ see Lemma 1 of [K2].

To treat the general case we need only consider ergodic measures ν. By Theorem A, the corresponding ergodic component of T contains some cylinder $Z \in \xi_L$ whose indicator now serves as our initial density. Write $u := 1_Z$ and let $\widehat{u}$ be its obvious lift, then it is not hard to see that each $\widehat{u}_n$ is bounded and $\widehat{u}_L$ is admissible for the extension (these are simple special cases of Lemmas 14 and 15 of Section 8), so that the argument used above again applies. ■

4 Canonical Markov extensions of basic AFN-maps and pointwise dual ergodicity

Modified systems. We introduce a few more notations. Let (X, T, ξ) be some basic AFN-system. If $Z \in \zeta$ is a cylinder of T containing an indifferent fixed point, we let $Z(1) := Z \backslash f_Z(Z)$ and $Z(n+1) := f_Z(Z(n))$ for $n \geq 1$, i.e. $Z(n)$ is the set of points in Z leaving this cylinder under the $n-$th iterate of T. Clearly, $Z = \bigcup_{n \geq 1} Z(n)$. If $N \geq 1$, we write $X(N) := cl_X(\bigcup(\xi \backslash \zeta) \cup \bigcup_{Z \in \zeta} \bigcup_{n=1}^{N} Z(n))$, which clearly is a sweep-out set for T.

Fix $N \geq 1$ and let $Y := X(N)$. ζ being finite, Y is the union of a finite number of subintervals of X, and we denote the collection of their interiors by $\xi_{Y,0}$ (cf. Section 2). As we wish to apply Lemma 2, we slightly modify the original system. We first let $\xi'(Y) := \xi \vee \bigvee_{Z \in \zeta} \bigvee_{n=1}^{N} \{Z(n), Z(n)^c\}$ to make Y a union of cylinders. (As this introduces only finitely many new cylinders, $T\xi'(Y)$ is finite.) Next, we remove from X the endpoints of members of $\xi_{Y,0}$, thus obtaining a set $X(Y) \subseteq X$ whose partition into connected components, henceforth denoted by $\xi(Y)_0$, contains $\xi_{Y,0}$. Finally, define $\xi(Y) := \xi'(Y) \vee \{Y^c, \xi'(Y)_Y\}$ as in Remark 4, which gives a partition of $X(Y)$ adapted to Y.

$(X(Y), T, \xi(Y))$ and Y thus satisfy the assumptions of Lemma 2. In studying ergodic properties of T we may restrict our attention to $X(Y)$, since $X \setminus X(Y)$ is finite. However, the C.M.E.s of (X, T, ξ) and $(X(Y), T, \xi(Y))$ are essentially different.

Lemma 8 (Inducing on $Y = X(N)$ and extending modified systems) *If (X, T, ξ) is an AFN-system, and $Y := X(N)$ for some $N \geq 1$, then $(Y, T_Y, \xi(Y)_Y)$ is AFU. Furthermore, although $(X(Y), T, \xi(Y))$ does not satisfy the finite-image property (F), its C.M.E. $(\widehat{X(Y)}, \widehat{T}, \widehat{\xi(Y)})$ still has some important finiteness properties: For any $n \geq 1$, $\widehat{X(Y)} \cap \{\Lambda \leq n\}$ has finite Lebesgue measure, and letting $\widehat{\mathcal{M}}$ denote its image partition, $\widehat{\mathcal{M}} \cap \{\Lambda \leq n\} \cap \pi^{-1} Y$ is finite.*

Proof. Properties (A) and (U) of T_Y do not depend on the partition and hold because Y is bounded away from the indifferent fixed points. (Adler's condition can be verified as in Lemma 10 of [Z1].) Let us prove that $T_Y \xi(Y)_Y$ is finite.

Recall that if $x \in Y \cap \{\varphi = 1\}$, then $\xi'(Y)_Y(x) = \xi'(Y)(x) \cap T^{-1}\xi'(Y)_{Y,0}(Tx) = \xi'(Y)(x) \cap T^{-1}\xi_{Y,0}(Tx)$. Hence $\xi(Y)_Y \cap \{\varphi = 1\} = \xi'(Y)_Y \cap \{\varphi = 1\} \subseteq \{Z \cap T^{-1}M : Z \in \xi'(Y),\, M \in \xi_{Y,0}\}$, so that $T_Y(\xi(Y)_Y \cap \{\varphi = 1\}) \subseteq \{TZ \cap M : Z \in \xi'(Y), M \in \xi_{Y,0}\}$ is finite since $T\xi'(Y)$ and $\xi_{Y,0}$ are.

For $k \geq 2$, $\xi(Y)_Y \cap \{\varphi = k\} = \{W \cap T^{-1}Z(N-1+k) : W \in \xi'(Y),\, Z \in \zeta\}$, hence $T(\xi(Y)_Y \cap \{\varphi = k\}) = \{TW \cap Z(N-1+k) : W \in \xi'(Y), Z \in \zeta\}$ is finite as both ζ and $T\xi'(Y)$ are. Consequently, so is each $T_Y(\xi(Y)_Y \cap \{\varphi = k\})$, $k \geq 2$. We claim that there is some $k_0 \geq 2$ such that

$$T_Y(\xi(Y)_Y \cap \{\varphi = k\}) = T_Y(\xi(Y)_Y \cap \{\varphi = k_0\}) \qquad \text{for } k \geq k_0.$$

By finiteness of $T\xi'(Y)$ there exists $k_0 \geq 2$ such that any image intersecting a set $\bigcup_{k \geq k_0} Z(N+k)$ actually covers it, i.e. $TW \cap Z(N+k)$ equals or $Z(N+k)$ whenever $W \in \xi'(Y), Z \in \zeta$, and $k \geq k_0$, which implies the claim. It follows that $T_Y \xi(Y)_Y = T_Y(\xi(Y)_Y \cap \{\varphi \leq k_0\})$ is finite, proving that $(Y, T_Y, \xi(Y)_Y)$ is AFU.

To see that $(X(Y), T, \xi(Y))$ violates the finite-image property (F), notice that Y contains some $W \in \xi$ such that TW covers a set of the form $V := \bigcup_{n \geq n_0} Z(n)$, where $Z \in \zeta$. On $f_W(V)$ the induced partition $\xi'(Y)_Y$ therefore refines $\beta := \{f_W(Z(n)) : n \geq n_0\}$. But since T is injective on $f_W(V) \subseteq W$, the image of the infinite disjoint collection β is infinite, and hence so is $T\xi(Y)$.

Finally observe that $\mathcal{M}_n = \{T^k Z : Z \in \xi(Y)_k,\ 0 \leq k \leq n\}$ equals $\xi(Y)_0 \cup \mathcal{M}_n^* \cup \{Z(k) : Z \in \zeta,\ k \geq k_0\}$, where $\mathcal{M}_n^* := \{T^k Z : Z \in \xi(Y)_k \cap \{\varphi \leq k_0\}, 0 \leq k \leq n\}$. It is not hard to see that each $\mathcal{M}_n^*$ is finite, which immediately implies the remaining assertions. ■

Remark 7 *In the situation of this Lemma, if (X, T, ξ) is conservative w.r.t. λ (e.g. if it is basic), then for each $Z \in \zeta$ there is some $Z-$neighbourhood of x_Z covered by the image of some member of $\xi \cap Y$, which implies that the inclusion $\widehat{Y} \subseteq \pi^{-1} Y$ from Lemma 2 in this case cannot be strict.*

Corollary 1 (Bounded variation of invariant densities) *Let (X, T, ξ) be an AFN-system. Then any invariant density h has a version which is of bounded variation (and hence has one-sided limits) on each $Y = X(N)$, $N \geq 1$.*

Proof. It is clear that we need only consider the ergodic case. By the above, T_Y is AFU, thus admitting an invariant density h_Y of bounded variation (by [Ry], cf. Section 6 of [Z1]). According to the first part of Lemma 1, we have $h \mid_Y = h_Y$. ■

Lemma 9 (Structure and invariant measure of the extension) *Let (X, T, ξ) be a basic AFN-map, and $Y = X(N)$, $N \geq 1$. Then $(\widehat{X(Y)}, \widehat{T}, \widehat{\xi(Y)})$ is conservative ergodic with respect to some infinite invariant measure $\widehat{\mu} \ll \lambda$ for which $\pi_* \widehat{\mu} = \mu$.*

Proof. Observe first that the induced system $(Y, T_Y, \xi(Y)_Y)$ is AFU by Lemma 8 and ergodic by Lemma 1. It has the invariant probability measure $\mu_Y \ll \lambda$. Proposition 1 shows that $(\widehat{Y}, \widehat{T_Y}, \widehat{\xi(Y)_Y})$ has an invariant probability $\widehat{\mu_Y} \ll \lambda$ with $\pi_* \widehat{\mu_Y} = \mu_Y$, and is ergodic w.r.t. $\widehat{\mu_Y}$. Conservativity, ergodicity, and the existence of a $\sigma-$finite invariant measure $\widehat{\mu}$ for $(\widehat{X(Y)}, \widehat{T}, \widehat{\xi(Y)})$ with $\pi_* \widehat{\mu} = \mu$ now follow from Lemmas 1 and 2. ■

Pointwise dual ergodicity. Recall (cf. [A2] or Section 3.7 in [A0]) that a conservative ergodic measure preserving map T on a $\sigma-$finite space $(X, \mathcal{B}, \mu)$ is called *pointwise dual ergodic* if there are positive constants $a_n(T)$, $n \geq 1$, such that

$$\frac{1}{a_n(T)} \sum_{k=0}^{n-1} \mathbf{P}_\mu^k f \longrightarrow \mu(f) \qquad \text{a.e. as } n \to \infty \text{ for all } f \in L_1(\mu).$$

The sequence $(a_n(T))_{n \geq 1}$ which then is uniquely determined up to asymptotic equivalence is called its *return sequence*. Aaronson's method for proving this property for Markov maps can now be applied to the extension of $(X(Y), T, \xi(Y))$.

Lemma 10 *Let (X, T, ξ) be a basic AFN-map, $N \geq 1$, and $Y := X(N)$. Then $(\widehat{X(Y)}, \widehat{T}, \widehat{\xi(Y)})$ is pointwise dual ergodic with respect to $\widehat{\mu}$.*

Proof. Fix some $Z \in \widehat{\xi(Y)} \cap \widehat{Y}$ with $\widehat{\mu}(Z) > 0$. Then the induced system $(Z, \widehat{T}_Z, \widehat{\xi(Y)}_Z)$ is piecewise onto and AFU. By Proposition 4.3.3, Corollary 4.7.8, and Lemma 3.7.4 of [A0], Z therefore is a Darling-Kac set for $\widehat{T}$, i.e. there are constants a_n such that $a_n^{-1} \sum_{k=0}^{n-1} \widehat{\mathbf{P}}_{\widehat{\mu}}^k 1_Z \longrightarrow \widehat{\mu}(Z)$ uniformly mod $\widehat{\mu}$ on Z (see Section 8). According to Proposition 3.7.5 of [A0], $\widehat{T}$ thus is pointwise dual ergodic. ■

The central result of this section now follows immediately: Since by Proposition 3.7.6 of [A0] any factor of a pointwise dual ergodic system again has this property, we obtain

Theorem 1 *Any basic AFN-map T is pointwise dual-ergodic.*

Remark 8 *For a sequence $(a_n)_{n\geq 1}$ in $(0, \infty)$, its* asymptotic proportionality class *will be the family of all sequences $(c_n)_{n\geq 1}$ asymptotically equivalent to some fixed positive multiple of it, i.e. $c_n \sim c \cdot a_n$ as $n \to \infty$. For a pointwise dual ergodic transformation, the asymptotic proportionality class of its return sequence $(a_n(T))_{n\geq 1}$ is called the* asymptotic type of T. *By Propositions 3.7.1 and 3.3.2 in [A0], this gives an isomorphism invariant for such maps. (See section 3.1 of [A0] for a discussion of notions of isomorphism for infinite measure preserving transformations.)*

Remark 9 *It would also be desirable to directly prove pointwise dual ergodicity of T by finding a set $B \in \mathcal{B}$ satisfying the hypotheses of Lemma 3.7.4 in [A0] for T, since this would give further strong properties (see e.g. [AD], [ADF]).*

5 Wandering rates and asymptotic distributional behaviour

Having established pointwise dual ergodicity, we are going to identify the asymptotic type for a large class of AFN-maps. Besides explicating an isomorphism invariant (cf. Remark 8) this determines the asymptotic distributional behaviour. We begin by identifying minimal wandering rates for T. Aaronson's asymptotic renewal equation together with Karamata's Tauberian theorem then gives the order of growth of $a_n(T)$. Sections 3.6 to 3.8 in [A0] are the basic reference for the material discussed here.

Definition 7 *Let T be a conservative ergodic measure preserving map on $(X, \mathcal{B}, \mu)$, and let $A \in \mathcal{B}^+ := \{A \in \mathcal{B} : 0 < \mu(A) < \infty\}$. The* wandering rate of A *is the sequence given by*

$$L_A(n) := \mu\left(\bigcup_{k=0}^{n-1} T^{-k} A\right) \qquad \text{for } n \geq 1.$$

Definition 8 *For an AFN-map T we let $\mathcal{E}(T) := \{A \in \mathcal{B}^+ \colon A \subseteq X(N)$ for some $N \geq 1\} = \{A \in \mathcal{B}^+ \colon A \subseteq X_\varepsilon$ for some $\varepsilon > 0\}$ denote the class of $\mu-$positive* sets bounded away from the indifferent fixed points.

The following result generalises Theorem 3 in [T2].

Theorem 2 (Universality of wandering rates for $\mathcal{E}(T)$) *If T is a basic AFN-map, then there exists a sequence $(w_n(T))_{n\geq 1}$ in $(0,\infty)$ such that $w_n(T)\nearrow \infty$ and*

$$w_n(T) \sim L_A(n) \quad \text{as } n\to\infty \quad \text{for all } A\in\mathcal{E}(T).$$

The asymptotic equivalence class of $(w_n(T))_{n\geq 1}$ will be called the *wandering rate of* T. The family $\mathcal{E}(T)$ being *hereditary* in the sense that $B\subseteq A\in\mathcal{E}(T)$, $B\in\mathcal{B}^+ \Rightarrow B\in\mathcal{E}(T)$, its members have *minimal wandering rates* (cf. [A0], p. 134).

Proof. The proof of Theorem 3 in [T2] (which applies to piecewise onto maps) can be adapted to work for general AFN-maps. The result also follows from Corollary 3 in Section 8 (which does not depend on this section) and Theorem 3.8.3 of [A0]. ■

Remark 10 *The proportionality class of the wandering rate also gives an isomorphism invariant for AFN-maps (cf. the Proposition on p.80 of [T2]). This can be useful in distinguishing between non-isomorphic maps for which the identification of the asymptotic type $(a_n(T))_{n\geq 1}$ via Theorem 4 below fails due to a lack of regularity.*

Next, we will express the orders of $(w_n(T))$ and $(a_n(T))$ in terms of the local behaviour of T at the indifferent fixed points. We need to observe that the bounded factor H of the invariant density h has one-sided limits at the x_Z, i.e. for $Z\in\zeta$ there exists $H(Z) := \lim_{x\to x_Z, x\in Z} H(x)\in(0,\infty)$. This is immediate from the formula for h obtained in the proof of Theorem 1 in [Z1] together with Thaler's inequality (Lemma 8 there). We will say that T *admits nice expansions* if for all $Z\in\zeta$ there are $a_Z\neq 0$ and $p_Z\geq 1$ such that

$$Tx = x + a_Z\left|x-x_Z\right|^{p_Z+1} + o\left(\left|x-x_Z\right|^{p_Z+1}\right) \qquad \text{as } x\to x_Z \text{ in } Z,$$

in which case we write $p := \max_{Z\in\zeta} p_Z$. Moreover, y_Z will denote the endpoint of $Z\in\zeta$ different from x_Z.

Theorem 3 (Identifying the wandering rate of T) *If T is a basic AFN-map, then*

$$w_n(T) \sim \sum_{Z\in\zeta} H(Z) \sum_{k=0}^{n-1} \left|f_Z^k(y_Z) - x_Z\right| \qquad \text{as } n\to\infty.$$

If T admits nice expansions, then, as $n\to\infty$,

$$w_n(T) \sim \left(\sum_{Z\in\zeta, p_Z=p} \mid a_Z\mid^{-\frac{1}{p}} H(Z)\right)\cdot\begin{cases} \log n & \text{if } p=1\\ p^{-\frac{1}{p}}\frac{p}{p-1}\, n^{1-\frac{1}{p}} & \text{if } p>1.\end{cases}$$

In particular, $(w_n(T))_{n\geq 1}$ then is regularly varying with index $1-\frac{1}{p}$.

Proof. This is established by computing the wandering rate of the particular set $A := X(1)\in\mathcal{E}(T)$, which is convenient since $\bigcup_{k=0}^{n-1} T^{-k}A = X(n)$ for $n\geq 1$. The growth rate of $(\mu(X(n)))_{n\geq 1}$ only depends on the local type of T at the points x_Z, and the estimate is obtained in precisely the same way as in the proof of Theorem 4 in [T2]. ■

Remark 11 *Regular variation of the wandering rate can be characterized using the functions* $u_Z(x) := (x - x_Z)(T'x - 1)(Tx - x)^{-1}$, $x \in Z \in \zeta$. *According to the proof of Theorem 2 in [T4] (in particular Lemma 3 there), if* $\lim_{x \to x_Z, x \in Z} u_Z(x) =: 1 + q_Z$ *exists (where necessarily* $q_Z \in [1, \infty]$*), then the sequence* $(\sum_{k=0}^{n-1} \left| f_Z^k(y_Z) - x_Z \right|)_{n \geq 1}$ *is regularly varying with index* $1 - q_Z^{-1}$. *In case* $q_Z \in (1, \infty)$ *the converse implication holds, too.*

Theorem 4 (Identifying return sequences) *Let* T *be a basic AFN-map for which* $(w_n(T))_{n \geq 1}$ *is regularly varying with index* $1 - \alpha \in [0, 1]$*, then*

$$a_n(T) \sim \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)} \cdot \frac{n}{w_n(T)} \qquad \text{as } n \to \infty,$$

and $(a_n(T))_{n \geq 1}$ *is regularly varying with index* α. *If, in particular,* T *admits nice expansions, and* $\alpha = \frac{1}{p}$*, then, as* $n \to \infty$,

$$a_n(T) \sim \left(\Gamma(1+\alpha)\Gamma(2-\alpha) \sum_{Z \in \zeta, p_Z = p} \mid a_Z \mid^{-\alpha} H(Z) \right)^{-1} \cdot \begin{cases} \frac{n}{\log n} & \text{if } \alpha = 1 \\ (1-\alpha)\alpha^{-\alpha}\, n^{\alpha} & \text{if } \alpha < 1. \end{cases}$$

Proof. This follows from the preceding theorem and Proposition 3.8.7 in [A0] since by Egorov's theorem the hereditary family $\mathcal{E}(T)$ contains uniform sets (in fact, as remarked in the proof of Theorem 2, $\mathcal{E}(T)$ contains uniform sets only, see Section 8). ■

Remark 12 *Theorem 4.8.7 in [A0] shows that in fact every sequence* (b_n) *which is regularly varying with index* $\alpha \in (0, 1)$ *can be realized as the return sequence* $(a_n(T))$ *of some simple AFN-map* T. *See also the remark following that theorem.*

Finally, an application of Corollary 3.7.3 in [A0] shows that such T exhibit nice distributional limiting behaviour:

Theorem 5 (Aaronson-Darling-Kac limit theorem) *Let* T *be a basic AFN-map for which* $(a_n(T))_{n \geq 1}$ *is regularly varying with index* $\alpha \in [0, 1]$. *Then for any* $f \in L_1(\mu)$ *with* $\mu(f) \neq 0$ *we have*

$$\frac{1}{a_n(T)} \sum_{k=0}^{n-1} f \circ T^k \stackrel{d}{\Longrightarrow} \mu(f)\, W^{(\alpha)} \qquad \text{as } n \to \infty,$$

where the distribution of the lefthand sum can be taken with respect to an arbitrary fixed probability measure $P \ll \lambda$*, and* $W^{(\alpha)}$ *is a random variable on* $(0, \infty)$ *having the* normalised Mittag-Leffler distribution of order α*, that is,*

$$\mathbf{E}\left(e^{zW^{(\alpha)}} \right) = \sum_{m \geq 0} \frac{\Gamma(1+\alpha)^m}{\Gamma(1+m\alpha)} z^m \ .$$

Also, Theorem 1 of [ADF] applies in this situation (with additional assumptions if $\alpha \notin (0, 1)$) thus giving a second order ergodic theorem in measure.

6 Entropy and Rohlin's formula

Following [Kr], for any conservative ergodic measure preserving map T of some $\sigma-$finite space $(X, \mathcal{B}, \mu)$, we define the *entropy of* T *(w.r.t.* μ*)* to be

$$h_\mu(T) := \mu(Y)\, h_{\mu_Y}(T_Y),$$

where Y is any set of positive finite measure, and $h_{\mu_Y}(T_Y)$ is the ordinary metric entropy of a probability preserving map. By Abramov's formula for the entropy of induced maps this does not depend on the choice of Y. Basic properties of this extended notion of entropy were established in [Kr]. Combined with the concept of minimal wandering rates (in case these exist), this yields a strong isomorphism invariant called *normalised wandering rate* (cf. Remark 2 on p. 94 of [T2]). There is also the corresponding concept of *normalised asymptotic type*, cf. [A1]. We are going to prove

Theorem 6 (Rohlin's formula) *If* (X, T, ξ) *is a basic AFN-map with invariant measure* $\mu \ll \lambda$*, then*

$$h_\mu(T) = \int_X \log |T'| \; d\mu \; .$$

We shall need part of the following useful observation.

Lemma 11 *Let* T *be a measure preserving map of the probability space* $(X, \mathcal{B}, \nu)$*, and let* $\xi = \{Z_1, Z_2, \ldots\}$ *be a measurable partition of* X *such that for some* $C \geq 1$ *we have* $\nu\left(Z \cap T^{-1}A\right) \leq C \cdot \nu(Z)\, \nu(A)$ *for any* $Z \in \xi$*,* $A \in \mathcal{B}$*. Then* $h_\nu(T) \geq H_\nu(\xi) - \log C$*, where* $H_\nu(\xi) := -\sum_{Z\in\xi} \nu(Z) \log \nu(Z)$*.*

Proof. For $n \geq 1$ we have $\xi_{n+1} = \{Z \cap T^{-1}W : Z \in \xi,\, W \in \xi_n\}$, where of course $\xi_n := \bigvee_{k=0}^{n-1} T^{-k}\xi$. Thus,

$$\begin{aligned}
H_\nu(\xi_{n+1}) &= -\sum_{Z\in\xi, W\in\xi_n} \nu(Z \cap T^{-1}W)\, \log \nu(Z \cap T^{-1}W) \\
&\geq -\sum_{Z\in\xi, W\in\xi_n} \nu(Z \cap T^{-1}W)\, \log(C \cdot \nu(Z)\, \nu(W)) \\
&= -\log C - \sum_{Z\in\xi} \left(\sum_{W\in\xi_n} \nu(Z \cap T^{-1}W) \right) \log \nu(Z) \\
&\quad\; - \sum_{W\in\xi_n} \left(\sum_{Z\in\xi} \nu(Z \cap T^{-1}W) \right) \log \nu(W) \\
&= -\log C + H_\nu(\xi) + H_\nu(\xi_n) \; .
\end{aligned}$$

Hence, by induction, $H_\nu(\xi_n) \geq n\, H_\nu(\xi) - (n-1)\log C$ for $n \geq 1$, and therefore $h_\nu(T, \xi) \geq H_\nu(\xi) - \log C$. Thus, if $H_\nu(\xi) < \infty$, then $h_\nu(T) \geq H_\nu(\xi) - \log C$. If $H_\nu(\xi) = \infty$, apply the same reasoning to the finite partitions $\xi^{(N)} := \{Z_1, \ldots, Z_N, \bigcup_{j>N} Z_j\}$ which obviously satisfy the hypotheses of the lemma. This yields $h_\nu(T) \geq H_\nu(\xi^{(N)}) - \log C$ for any $N \geq 1$, and the assertion follows since $H_\nu(\xi) = \lim_{N\to\infty} H_\nu(\xi^{(N)})$. ■

Remark 13 *If (X, T, ξ) is a piecewise monotonic system satisfying (A) and (F), then there exists some constant D such that $\lambda(Z \cap T^{-1}A) \leq D \cdot \lambda(Z)\,\lambda(A)$ whenever $Z \in \xi$ and $A \in \mathcal{B}$. If also (U) holds, then the density of any invariant probability $\nu \ll \lambda$ is bounded away from zero and infinity (cf. Theorem A), so that the assumptions of the lemma are satisfied. Hence, for any AFU-system (X, T, ξ) with invariant probability $\nu \ll \lambda$, $H_\nu(\xi) = \infty$ implies $h_\nu(T) = \infty$.*

Proposition 2 (Entropy of the extension of an AFU-map) *In the situation of Proposition 1 we have $h_{\widehat{\nu}}(\widehat{T}) = h_\nu(T)$.*

Proof. Since (X, T, ξ, ν) is a measure-preserving factor of $(\widehat{X}, \widehat{T}, \widehat{\xi}, \widehat{\nu})$, we have $h_\nu(T) \leq h_{\widehat{\nu}}(\widehat{T})$. If $H_\nu(\xi) = \infty$, therefore $h_\nu(T) = \infty = h_{\widehat{\nu}}(\widehat{T})$ by the preceding remark. Hence we may assume that $H_\nu(\xi) < \infty$, in which case the corresponding part of the proof of Theorem 3 in [K2] applies. ■

Corollary 2 (Rohlin's formula for AFU-maps) *For any conservative ergodic AFU-map T with invariant probability $\nu \ll \lambda$, we have $h_\nu(T) = \int_X \log |T'|\, d\nu$.*

Proof. See the proof of Proposition 1 in [DKU]. ■

Proof of Theorem 6. Let $Y := X(1)$, then $h_\mu(T) = \mu(Y)\, h_{\mu_Y}(T_Y)$ by definition. On the other hand, $\int_X \log |T'|\, d\mu = \mu(Y) \int_Y \log |T_Y'|\, d\mu_Y$ by the chain rule (cf. p. 32 of [DKU] or the Lemma on p. 90 of [T2]). Hence T satisfies Rohlin's formula for μ iff T_Y does for μ_Y, and the latter is true by Lemma 8 and the corollary above. ■

Rohlin's formula leads to a convenient finiteness criterion: The arguments given in the proof of the second part of Theorem 5 in [T2] easily carry over to our situation to give

Theorem 7 (Finiteness of entropy) *If (X, T, ξ) is a basic AFN-map with invariant measure $\mu \ll \lambda$, then $h_\mu(T)$ is finite iff $H_{\lambda_X}(\xi) < \infty$ (with λ_X denoting normalized Lebesgue measure) and $\lambda(|u_Z|) < \infty$ for all $Z \in \zeta$, where $u_Z(x) := (x - x_Z)(T'x - 1)(Tx - x)^{-1}$, $x \in Z$ is the function which already appeared in Remark 11.*

7 The Shannon-McMillan-Breiman theorem for infinite measure preserving transformations

The pointwise ergodic theorem for information, often referred to as the Shannon-McMillan-Breiman theorem states that for an ergodic probability preserving map T and a generating partition ξ we have

$$-n^{-1} \log \mu(\xi_n(x)) \longrightarrow h_\mu(T) \quad \text{as } n \to \infty \text{ for a.e. } x \in X$$

whenever $H_\mu(\xi) < \infty$. If T preserves an infinite measure μ, Aaronson's ergodic theorem (Theorem 2.4.2 in [A0]) shows that constantly normalised pointwise convergence is impossible even for sums $\mathbf{S}_n(f) := \sum_{k=0}^{n-1} f \circ T^k$, $f \in L_1^+(\mu) := \{f \in L_1(\mu) : f \geq 0,\ \mu(f) > 0\}$, while in the finite situation we have $\mathbf{S}_n(f) \sim$

$\mu(f)\cdot n$. However, as observed in Theorem 5 of [T2] for piecewise surjective AFN-maps, if we substitute this asymptotic equation, thus normalizing $-\log\mu(\xi_n(x))$ by $\mu(f)^{-1}\mathbf{S}_n(f)(x)$, pointwise convergence to the entropy may still take place (see Theorem 8 below). We present a variant of Thaler's approach which is slightly more general and in particular applies to arbitrary AFN-maps.

Remark 14 *Let $(X, \mathcal{B}, \lambda)$ be a $\sigma-$finite measure space and ξ a generating $\lambda-$partition for T, i.e. $\xi_n := \bigvee_{k=0}^{n-1} T^{-k}\xi \nearrow \mathcal{B}$ mod λ. We shall say that a measure $\nu \ll \lambda$ is $\sigma-$*finite mod *λ on $(\xi_n)_n$, if X can be covered mod λ by $\nu-$finite sets from $\bigcup_n \xi_n$, that is, if for $\lambda-$a.e. $x \in X$, $\nu(\xi_n(x)) < \infty$ for $n \geq n_x$.*

If both μ, ν are of this type and $\nu \ll \mu$, then applying the standard increasing martingale theorem on the members of some $\mu-$ and $\nu-$finite cover of X by sets from $\bigcup_n \xi_n$ yields

$$\frac{\nu(\xi_n(x))}{\mu(\xi_n(x))} \longrightarrow \frac{d\nu}{d\mu}(x) \quad \textit{for } \mu - \textit{a.e. } x \in X.$$

Lemma 12 (Localizing the asymptotics of information-type functions) *Let T be a nonsingular conservative transformation on some $\sigma-$finite space $(X, \mathcal{B}, \lambda)$, and let ξ be a generating $\lambda-$partition of X. If there is some sweep-out set Y on which*

$$r(x) := \lim_{n\to\infty} \frac{-\log\lambda(\xi_n(x))}{s_n(x)}$$

exists, where $s_n(x) := \mathbf{S}_n(1_Y)(x)$, then the limit exists a.e. on X and is $T-$invariant, i.e. $r \circ T = r$ mod λ.

Proof. We first show that for a.e. $x \in X$ the sequence $(q_n(x))_{n\geq 1}$, where $q_n(x) := \log\lambda(\xi_{n+k}(x)) - \log\lambda(\xi_n(T^k x))$, is bounded for each $k \geq 1$. We prove that for fixed k and $Z \in \xi_k$ it is in fact convergent for a.e. $x \in Z$.

If $x \in Z$, then $\lambda(\xi_{n+k}(x)) = \lambda(Z \cap T^{-k}\xi_n(T^k x)) = m(\xi_n(T^k x))$, where $m(A) := T_*^k(\lambda\mid_Z)(A) = \lambda(Z \cap T^{-k}A)$. Since T is nonsingular, we have $m \ll \lambda$. The assumption that $r(x)$ should exist on some sweep-out set implies λ is $\sigma-$finite mod λ on $(\xi_n)_n$, and so is m. By Remark 14 and nonsingularity again,

$$\frac{m(\xi_n(T^k x))}{\lambda(\xi_n(T^k x))} \longrightarrow \frac{dm}{d\lambda}(T^k x) \quad \text{for } \lambda - \text{a.e. } x \in Z.$$

We claim that $(dm/d\lambda)(T^k x)$ is positive and finite a.e. on Z. In fact, $\lambda(Z \cap T^{-k}\{dm/d\lambda = 0\}) = m(\{dm/d\lambda = 0\}) = 0$, and analogously for $\{dm/d\lambda = \infty\}$. Hence for $\lambda-$a.e. $x \in Z$,

$$\frac{\lambda(\xi_{n+k}(x))}{\lambda(\xi_n(T^k x))} = \frac{m(\xi_n(T^k x))}{\lambda(\xi_n(T^k x))} \longrightarrow \frac{dm}{d\lambda}(T^k x) \in (0, \infty),$$

so that $q_n(x) = \log(\lambda(\xi_{n+k}(x))/\lambda(\xi_n(T^k x)))$ converges as $n \to \infty$.

As for the assertion of the Lemma, for $\lambda-$a.e. $x \in X$ there is some $k \geq 1$ such that $T^k x \in Y$. By conservativity also $s_n(x) \to \infty$ a.e. Existence of the limit $r(x)$ and its $T-$invariance now follow from the identity

$$\frac{-\log(\lambda(\xi_{n+k}(x)))}{s_{n+k}(x)} = \frac{s_n(T^k x)}{s_{n+k}(x)} \left(\frac{-\log(\lambda(\xi_n(T^k x)))}{s_n(T^k x)} - \frac{1}{s_n(T^k x)} \log \frac{\lambda(\xi_{n+k}(x))}{\lambda(\xi_n(T^k x))} \right). \blacksquare$$

Theorem 8 (Shannon-McMillan-Breiman for AFN-maps) *Let (X,T,ξ) be a basic AFN-system with invariant measure $\mu \ll \lambda$. If $h_\mu(T) < \infty$, then for any $f \in L_1^+(\mu)$ we have*

$$\frac{-\log \mu(\xi_n(x))}{\mathbf{S}_n(f)(x)} \longrightarrow \frac{h_\mu(T)}{\mu(f)} \qquad \textit{as } n \to \infty \textit{ for } \mu-\textit{a.e. } x \in X.$$

Moreover, for any measure $\nu \ll \mu$ which is $\sigma-$finite mod λ on the sequence $(\xi_n)_{n\geq 1}$, and $\nu-$a.e. $x \in X$ we have $-\log \nu(\xi_n(x))/\mathbf{S}_n(f)(x) \longrightarrow \mu(f)^{-1}h_\mu(T)$.

Proof. Notice first that $h_\mu(T) < \infty$ means that $h_{\mu_Y}(T_Y) < \infty$, where $Y := X(1)$ (cf. Section 4), and by Lemma 8 and Remark 13 this implies $H_{\mu_Y}(\xi(Y)_Y) < \infty$. The standard version of the Shannon-McMillan-Breiman theorem thus applies to T_Y and $\xi(Y)_Y$ proving that

$$\frac{-\log \mu_Y(\xi(Y)_{Y,n}(x))}{n} \longrightarrow h_{\mu_Y}(T_Y) = \mu(Y)^{-1}h_\mu(T) \quad \text{ for } \mu_A - \text{a.e. } x \in Y.$$

(Clearly $\xi(Y)_Y$ is a generator for T_Y, and $\xi'(Y)$ is for T.) The following observation is crucial: Writing $s_n(x) := \mathbf{S}_n(1_Y)(x)$ and recalling the construction of $\xi'(Y)$ and $\xi(Y)$ in Section 4, and the fact that $\xi'(Y)_Y = \xi(Y)_Y$, we find that

$$\xi(Y)_{Y,s_n(x)}(x) \subseteq \xi'(Y)_n(x) \subseteq \xi(Y)_{Y,s_n(x)-1}(x)$$

whenever these cylinders are well defined. Consequently, for $\mu-$a.e. $x \in Y$,

$$\frac{-\log \mu_Y(\xi(Y)_{Y,s_n(x)-1}(x))}{s_n(x)} \leq \frac{-\log \mu_Y(\xi'(Y)_n(x))}{s_n(x)} \leq \frac{-\log \mu_Y(\xi(Y)_{Y,s_n(x)}(x))}{s_n(x)}$$

for $n \geq 1$, which together with the result for T_Y yields

$$\frac{-\log \mu_Y(\xi'(Y)_n(x))}{s_n(x)} \longrightarrow \frac{h_\mu(T)}{\mu(Y)} \qquad \text{as } n \to \infty \text{ for } \mu - \text{a.e. } x \in Y.$$

Here we use that due to conservativity $s_n(x) \to \infty$ and hence also $s_{n-1}(x) \sim s_n(x)$ a.e. For the same reason, we may replace μ_Y above by its constant multiple μ. Furthermore we can pass from $\xi'(Y)$ to ξ, since $\xi'(Y)$ refines ξ and is coarser than ξ_2, so that $\xi'(Y)_{n-1}(x) \supseteq \xi_n(x) \supseteq \xi'(Y)_n(x)$. Hence

$$\frac{-\log \mu(\xi_n(x))}{\mathbf{S}_n(1_Y)(x)} \longrightarrow \frac{h_\mu(T)}{\mu(1_Y)} \qquad \text{as } n \to \infty \text{ for } \mu - \text{a.e. } x \in Y,$$

and Lemma 12 then shows that convergence in fact takes place on almost all of X. Hopf's ratio ergodic theorem finally ensures that we may replace 1_Y by any $f \in L_1^+(\mu)$.

The generalization to measures ν is immediate from Remark 14. ■

This pointwise convergence result immediately gives a distributional limit theorem for the size of cylinders: In the situation of Theorem 5, if $h_\mu(T) < \infty$ and $\nu \ll \lambda$ is as in Theorem 8, then

$$\frac{-\log \nu(\xi_n(.))}{a_n(T)} \overset{d}{\Longrightarrow} h_\mu(T) \cdot W^{(\alpha)},$$

where the distribution of the lefthand variable may be taken with respect to any fixed probability measure $P \ll \lambda$.

Remark 15 *To justify the title of the section we emphasize that the argument given is quite general: The conclusions of Theorem 8 hold whenever T is a c.e.m.p.t. on a $\sigma-$finite space $(X, \mathcal{B}, \mu)$ and ξ is a generating partition mod μ such that there exists some $Y \in \mathcal{B}^+$ measurable ξ for which $H_{\mu_Y}(\xi_Y) < \infty$. (Here we should actually employ a slightly modified definition of the induced partition ξ_Y, e.g. as $\bigcup_{n\geq 1}\{\{\varphi = n\} \cap Z : Z \in \xi_n\}$, since the one we have been using makes sense in the piecewise monotonic situation only.)*

8 A stronger limit theorem for the transfer operator: Uniform convergence and Darling-Kac sets

The following generalisation of Thaler's limit theorem ([T3]) to our situation is the main result of the present paper. It considerably sharpens Theorem 1 and has beautiful probabilistic consequences (cf. Section 11 below).

Theorem 9 (Uniform convergence for the PFO of AFN-maps) *Let (X, T, ξ) be a basic AFN-system. Then there is a sequence $(a_n(T))_{n\geq 1}$ of positive real numbers such that for any Riemann-integrable function u on X, we have*

$$\frac{1}{a_n(T)} \sum_{k=0}^{n-1} \mathbf{P}^k u \longrightarrow \lambda(u) \cdot h \qquad \text{as } n \to \infty,$$

uniformly on members of $\mathcal{E}(T)$, where h is a version of the invariant density $d\mu/d\lambda$.

For a pointwise dual ergodic map T of some $\sigma-$finite space $(X, \mathcal{B}, \mu)$ with return sequence $(a_n)_{n\geq 1}$, $\mathcal{U}(T)$ shall denote the family of *uniform sets*, i.e. those $B \in \mathcal{B}^+$ on which $a_n^{-1} \sum_{k=0}^{n-1} \mathbf{P}_\mu^k f$ converges uniformly (mod μ) for some $f \in L_1^+(\mu)$. $B \in \mathcal{B}^+$ is called a *Darling-Kac set* if $a_n^{-1} \sum_{k=0}^{n-1} \mathbf{P}_\mu^k 1_B$ converges uniformly (mod μ) on B. The collection of these sets is $\mathcal{DK}(T)$. As the invariant density h of a basic AFN-map is bounded away from zero and infinity on each member of $\mathcal{E}(T)$, we find:

Corollary 3 (Uniform sets and Darling-Kac sets) *Let (X, T, ξ) be a basic AFN-map, then $\mathcal{E}(T) \subseteq \mathcal{U}(T)$, and any $B \in \mathcal{E}(T)$ satisfying $\lambda(\partial B) = 0$ is a Darling-Kac set for T.*

We turn to the proof of Theorem 9, deferring the proofs of the following lemmas to the next section. Although the proof could be shortened a bit by making use of our previous results on pointwise dual ergodicity (Theorem 1 and in particular Lemma 10), we prefer to give an independent approach, since the arguments hidden in the results referred to there are even less elementary.

Below we shall work with C.M.E.s $(\widehat{X(Y)}, \widehat{T}, \widehat{\xi(Y)})$, where $Y = X(N)$ for some $N \geq 1$, and $X(Y)$, $\xi(Y)$ are as defined in Section 4. Recall that objects associated with the C.M.E. $(\widehat{Y}, \widehat{T_Y}, \widehat{\xi_Y}) = (\widehat{Y}, \widehat{T_Y}, \widehat{\xi(Y)_Y})$ of the induced system (Y, T_Y, ξ_Y) are identified by subscripts: $\widehat{\mathcal{M}_Y}$ is the family of image-intervals of T_Y which $\widehat{Y}$ is built up from (see Lemma 2), while $\Lambda_{\widehat{Y}}$ and $\widehat{\mathbf{P}}_{\widehat{Y}}$ respectively are the level function and the transfer operator of this tower.

Lemma 13 (Iterated admissible densities for the C.M.E. of a basic AFN-map) *Let* (X, T, ξ) *be a basic AFN-map, and* $Y = X(N)$ *for some* $N \geq 1$. *Consider an admissible function* $\widehat{u}$ *on the C.M.E. of* $(X(Y), T, \xi(Y))$ *satisfying* $R(\widehat{u}) < \infty$. *Let* $\widehat{u}_n := \widehat{\mathbf{P}}^n \widehat{u}$, $n \geq 0$, *then each* $\widehat{u}_n$ *is admissible, and for each member* B *of* $\widehat{\mathcal{M}_Y} \subseteq \widehat{\mathcal{M}}$ *there is some* $r_B \in (0, \infty)$ *such that* $R_B(\widehat{u}_n) < r_B$ *for* $n \geq 0$. *The same is true for* $(\widehat{U}_n)_{n \geq 1}$, *where* $\widehat{U}_n := \sum_{k=0}^{n-1} \widehat{u}_k$. *Moreover, the sequence* $(\widehat{u}_n)_{n \geq 0}$ *is uniformly bounded on* B.

For $L \geq 1$ we let $\mathcal{F}_L$ denote the collection of functions of the form $u = 1_A$, where $A = \bigcup \mathcal{A}$, $\mathcal{A} \subseteq \xi_L$, and $A \in \mathcal{E}(T)$. Furthermore we define $\mathcal{F} := \bigcup_{L \geq 1} \mathcal{F}_L$.

Lemma 14 (Eventually admissible functions) *If* $u \in \mathcal{F}_L$, $Y = X(N)$ *for some* $N \geq 1$, *and* $\widehat{u}$ *is the lift of* u *to the base of* $(\widehat{X(Y)}, \widehat{T}, \widehat{\xi(Y)})$, *then* $\widehat{\mathbf{P}}^L \widehat{u}$ *is admissible for this extension, and* $R(\widehat{\mathbf{P}}^L \widehat{u}) < \infty$.

Lemma 15 *If* $u : X \to [0, \infty)$ *is bounded and measurable, then each* $\mathbf{P}^n u$, $n \geq 1$, *is bounded on members of* $\mathcal{E}(T)$.

The following result is the key tool for our proof. It extends Lemma 6 to the lift of $Y = X(N)$ to the appropriate tower above T by dynamically embedding the extension of (Y, T_Y, ξ_Y) therein as described in Lemma 2.

Lemma 16 (Controlling the mass escaping to high levels of the embedded AFU-extension) *Let* (X, T, ξ) *be a basic AFN-map, and* $Y = X(N)$ *for some* $N \geq 1$. *For the C.M.E. of* $(X(Y), T, \xi(Y))$ *there exist* $\eta \geq 1$, $K \in (0, \infty)$, *and* $q \in (0, 1)$ *such that for any measurable* $\widehat{u} : \widehat{X(Y)} \to [0, \infty)$ *supported on* $\widehat{Y} \cap \{\Lambda = 0\}$ *we have*

$$\sum_{k=0}^{n-1} \widehat{\mathbf{P}}^k \widehat{u}(\widehat{x}) \leq K \cdot q^{\Lambda_{\widehat{Y}}(\widehat{x})} \cdot \left(\sup_{\widehat{Y} \cap \{\Lambda_{\widehat{Y}} \leq \eta\}} \sum_{k=0}^{n-1} \widehat{\mathbf{P}}^k \widehat{u} \right) \qquad \text{for } \widehat{x} \in \widehat{Y} \cap \{\Lambda_{\widehat{Y}} > \eta\}.$$

Lemma 17 (Uniform convergence of projected iterated densities) *In the situation of the preceding lemma let* $u : X \to [0, \infty)$ *be a bounded measurable function supported on* Y, *and let* $\widehat{u}$ *be its lift to the base of* $\widehat{X(Y)}$. *If for some sequence* $n_j \nearrow \infty$, *positive constants* b_{n_j}, *and some bounded measurable* $\widehat{H} : \widehat{Y} \longrightarrow [0, \infty)$ *we have* $b_{n_j}^{-1} \sum_{k=0}^{n_j - 1} \widehat{\mathbf{P}}^k \widehat{u} \to \widehat{H}$ *uniformly on members of* $\widehat{\mathcal{M}_Y}$, *then*

$$\frac{1}{b_{n_j}} \sum_{k=0}^{n_j - 1} \mathbf{P}^k u \longrightarrow H \qquad \text{uniformly on } Y$$

with $H := \pi_* \widehat{H}$. *If the functions* $\sum_{k=0}^{n_j - 1} \widehat{\mathbf{P}}^k \widehat{u}$ *are continuous on members of* $\widehat{\mathcal{M}_Y}$, *then* H *is continuous in each point of* $X \setminus \bigcup_{B \in \widehat{\mathcal{M}_Y}} \partial(\pi B)$.

Proof of Theorem 9. We first show the asserted convergence for functions $u \in \mathcal{F}$. By the Chacon-Ornstein theorem, $(a_n)_{n \geq 1}$ then does not depend on u. Fix $u \in \mathcal{F}_L$ supported on $X(M)$, $M \geq 1$. The main step is to prove the following:

CLAIM: There are positive constants a_n such that for any sequence $m_l \nearrow \infty$ of integers there is some subsequence $(n_j)_{j\geq 1}$ of $(m_l)_{l\geq 1}$ for which

$$\frac{1}{a_{n_j}} \sum_{k=0}^{n_j-1} \mathbf{P}^k u \longrightarrow h \qquad \text{uniformly on members of } \mathcal{E}(T) \text{ as } j \to \infty \ .$$

Proof of the claim. We abreviate $\sum_{k=L}^{n-1} \mathbf{P}^k u$ to U_n. To begin with, let $Y := X(M)$, let $\widehat{u}$ be the lift of u to the base of $(\widehat{X(Y)}, \widehat{T}, \widehat{\xi(Y)})$, and write $\widehat{U}_n := \sum_{k=L}^{n-1} \widehat{\mathbf{P}}^k \widehat{u}$, $n > L$. Choose η, K, and q as in Lemma 16. By Lemma 2 and Remark 7, $\widehat{Y} = \pi^{-1} Y$. As $\widehat{Y} \cap \{\Lambda_{\widehat{Y}} \leq \eta\}$ consists of finitely many image-intervals $B \in \widehat{\mathcal{M}_Y}$, Lemmas 13, 14 and Remark 5 imply that $b_n := \sup_{\widehat{Y}\cap\{\Lambda_{\widehat{Y}} \leq \eta\}} \widehat{U}_n$ is finite for each $n > L$. By Remark 6 and Lemma 13 for each $B \in \widehat{\mathcal{M}_Y} \cap \{\Lambda_{\widehat{Y}} \leq \eta\}$ the sequence $(b_n^{-1} \widehat{U}_n \mid_B)_{n\geq L}$ is relatively compact in $\mathcal{C}(B)$. If on the other hand $B \in \widehat{\mathcal{M}_Y} \cap \{\Lambda_{\widehat{Y}} > \eta\}$, then by Lemma 16

$$\widehat{U}_n \mid_B \leq K \, q^{\Lambda_{\widehat{Y}}(B)} \cdot b_n \qquad \text{for all } n \geq L, \quad ()$$

whence $(b_n^{-1} \widehat{U}_n \mid_B)_{n\geq L}$ again is relatively compact in $\mathcal{C}(B)$. A standard diagonalisation argument therefore yields $r_j \nearrow \infty$ (which can be chosen to be contained in any given subsequence $m_l \nearrow \infty$) and a function $\widehat{H} : \widehat{Y} \to [0, \infty)$ Lipschitz on members of $\widehat{\mathcal{M}_Y}$ such that $(b_{r_j})^{-1} \widehat{U}_{r_j} \longrightarrow \widehat{H}$ uniformly on each $B \in \widehat{\mathcal{M}_Y}$. Lemma 17 ensures that $(b_{r_j})^{-1} \sum_{k=L}^{r_j-1} \mathbf{P}^k u \longrightarrow H := \pi_* \widehat{H}$ uniformly on $Y = X(M)$. By our choice of the b_n and finiteness of $\widehat{\mathcal{M}_Y} \cap \{\Lambda_{\widehat{Y}} \leq \eta\}$, $\widehat{H}$ is bounded and not identically zero, and because of () the same holds for H (on $X(M)$).

The argument just given also shows that whenever $N \geq M$, $n_j^{(N)} \nearrow \infty$, $b_j^{(N)} \in (0, \infty)$, and $H^{(N)} : X(N) \to [0, \infty)$ are such that $(b_j^{(N)})^{-1} U_{n_j^{(N)}} \longrightarrow H^{(N)}$ uniformly on $X(N)$, then there are some subsequence $n_j^{(N+1)} = n_{i_j}^{(N)} \nearrow \infty$ of $(n_j^{(N)})_{j\geq 1}$, positive constants $b_j^{(N+1)}$, and a bounded measurable function $H^{(N+1)} : X(N+1) \to [0, \infty)$ not identically zero, such that $(b_j^{(N+1)})^{-1} U_{n_j^{(N+1)}} \longrightarrow H^{(N+1)}$ uniformly on $X(N+1)$. In this case, however, there must be some $c \in (0, \infty)$ for which $H^{(N+1)} = c \cdot H^{(N)}$ on $X(N)$ and $b_j^{(N+1)} \sim c \cdot b_{i_j}^{(N)}$ as $j \to \infty$. Hence, we may take $b_j^{(N+1)} := b_{i_j}^{(N)}$ and $H^{(N+1)}$ with $H^{(N+1)} \mid_{X(N)} = H^{(N)}$ in the first place.

Given any sequence $m_l \nearrow \infty$ of integers, we thus obtain $H : X \to [0, \infty)$ not identically zero, and a collection $\{(n_j^{(N)})_{j\geq 1}\}_{N\geq M}$ of sequences, each contained in $(m_l)_{l\geq 1}$, for which each $(n_j^{(N+1)})_{j\geq 1}$ is a subsequence of $(n_j^{(N)})_{j\geq 1}$, such that for any $N \geq M$, we have $(b_{n_j^{(N)}})^{-1} U_{n_j^{(N)}} \longrightarrow H$ uniformly on $X(N)$ as $j \to \infty$, where the b_n are those obtained in the first step (i.e. for $N = M$). The diagonal sequence $n_j := n_j^{(M+j)}$ then satisfies

$$\frac{1}{b_{n_j}} \sum_{k=L}^{n_j-1} \mathbf{P}^k u \longrightarrow H \qquad \text{uniformly on members of } \mathcal{E}(T) \text{ as } j \to \infty \ ,$$

which by the last statement in Lemma 13 implies $\mathbf{P}H \leq H$ (see p. 726 of [K3]), and conservativity then gives $\mathbf{P}H = H$. Therefore $H = c \cdot h$ a.e. for some $c = c(H) \in (0, \infty)$.

To ensure that all subsequences converge to the same multiple of h, we now fix $x_0 \in X(M) \setminus \bigcup_{B \in \widehat{\mathcal{M}_{X(M)}}} \partial(\pi B)$. Then each limit function H is continuous in x_0 by Lemma 17, so that $H(x_0)$ determines the factor $c(H)$ (notice that $H(x_0) > 0$ since $\inf h > 0$). Hence $H(x_0)^{-1} H =: H_1$ is a fixed multiple of h, continuous in x_0, which does not depend on the subsequence and is characterized by $H_1(x_0) = 1$. Moreover there is some n_0 for which $U_{n_0}(x_0) > 0$, whence $U_n(x_0) > 0$ for $n \geq n_0$, and for such n we define $a_n := U_n(x_0) \in (0, \infty)$. For any sequence $(n_j)_{j \geq 1}$ as above, $a_{n_j} \sim H(x_0) b_{n_j}$, and thus

$$\frac{1}{a_{n_j}} \sum_{k=L}^{n_j - 1} \mathbf{P}^k u \longrightarrow H_1 \qquad \text{uniformly on members of } \mathcal{E}(T) \text{ as } j \to \infty.$$

T being conservative ergodic, we have $a_n \nearrow \infty$. In view of Lemma 15, we may therefore extend our range of summation to include $k = 0, \ldots, L-1$, which completes the proof of the claim.

To finish the first part of the proof observe that by a straightforward subsequence-in-subsequence argument uniform convergence to H_1 in fact takes place along the full sequence of positive integers.

The proof of the theorem is completed by an approximation argument. It is sufficient to show that for any Riemann-integrable function $u : X \to [0, \infty)$ and every $\varepsilon > 0$ there are functions $\underline{u}$ and $\overline{u}$ satisfying the conclusion of the theorem for which $\underline{u} \leq u \leq \overline{u}$ and $\lambda(\overline{u} - \underline{u}) < \varepsilon$. Since ξ generates mod λ, on members of $\mathcal{E}(T)$ this can easily be done by finite linear combinations of functions in $\mathcal{F}$, but any such function vanishes near the indifferent fixed points.

Since T^2 is basic and has the finite-image property (F), for each $Z \in \zeta$, there is some $W_Z \in \xi_2 \cap \mathcal{E}(T)$ such that $T^2 W_Z$ covers some $Z-$neighbourhood V_Z of x_Z. Hence, $w_Z := \mathbf{P}^2 1_{W_Z} = 1_{T^2 W_Z} \left| f'_{W_Z} \right|$ is positive and continuous on V_Z. Also, since $1_{W_Z} \in \mathcal{F}$, it is clear that w_Z satisfies the conclusion of the theorem, and it is easily seen that u can be approximated as required by finite linear combinations of the w_Z and functions from $\mathcal{F}$. ■

9 Proof of the lemmas

For the proof of Lemma 13 we need

Lemma 18 *In the situation of Lemma 13, if $B \in \widehat{\mathcal{M}_Y}$, then there is some $c_B \in (0, \infty)$ such that*

$$\widehat{F}_Z := \sum_{s=0}^{n} \left| \widehat{f}'_{[Z_{s+1}, \ldots, Z_{n-1}]} \right| \leq c_B \quad \text{on } B$$

whenever $n \geq 1$ and $Z = [Z_0, \ldots, Z_{n-1}] \in \widehat{\xi}_n(B) := \{Z \in \widehat{\xi}_n : \widehat{T}^n Z \supseteq B\}$.

Proof. The proof of Lemma 1 in [T3] shows that if (X, T, ξ) is an AFN-map, there is some $K \in (0, \infty)$ such that for $X_\varepsilon := X \setminus \bigcup_{Z \in \zeta} Z \cap (x_Z - \varepsilon, x_Z + \varepsilon)$ we have $F_Z := \sum_{s=0}^{n} \mid f'_{[Z_{s+1},\ldots,Z_{n-1}]} \mid \leq K \cdot \widetilde{G}(\varepsilon)$ on $T^n Z \cap X_\varepsilon$ for all $n \geq 1$ and $Z = [Z_0, \ldots, Z_{n-1}] \in \xi_n$, where $\widetilde{G}(\varepsilon) := \max_{Z \in \zeta} G(x_Z \pm \varepsilon)$. This immediately implies our lemma. ■

Proof of Lemma 13. Fix $B \in \widehat{\mathcal{M}_Y}$ and $n \geq 1$. It is clear that $\widehat{u}_n$ is supported on some set $\{\Lambda \leq m\}$ which by Lemma 8 has finite Lebesgue measure. By the Markov property, $\widehat{\xi}_n(B) := \{Z \in \widehat{\xi}_n : \widehat{T}^n Z \supseteq B\}$ equals $\{Z \in \widehat{\xi}_n : \widehat{T}^n Z \cap B \neq\}$, whence $\widehat{u}_n \mid_B = \sum_{Z \in \widehat{\xi}_n(B)} (\widehat{u} \circ \widehat{f}_Z) \cdot \mid \widehat{f}'_Z \mid$. Since each summand either vanishes on B or is strictly positive thereon, the same is true for $\widehat{u}_n$. Also, all summands are differentiable on B, and writing $\sigma_Z := sign(\widehat{f}'_Z)$, we formally obtain

$$\widehat{u}'_n = \underbrace{\sum_{Z \in \widehat{\xi}_n(B)} (\widehat{u}' \circ \widehat{f}_Z) \cdot (\widehat{f}'_Z)^2 \cdot \sigma_Z}_{=:\sum_n^{(1)}} + \underbrace{\sum_{Z \in \widehat{\xi}_n(B)} (\widehat{u} \circ \widehat{f}_Z) \cdot (\widehat{f}''_Z) \cdot \sigma_Z}_{=:\sum_n^{(2)}} \quad \text{on } B. \qquad (\triangle)$$

By Adler's condition (A), $A := \sup_{Z \in \widehat{\xi}} \sup_Z \left| \widehat{f}''_Z / \widehat{f}'_Z \right|$ is finite. Logarithmic differentiation and Lemma 18 for $Z = [Z_0, \ldots, Z_{n-1}] \in \widehat{\xi}_n(B)$ thus give

$$\left| \frac{\widehat{f}''_Z}{\widehat{f}'_Z} \right| \leq \left| \sum_{s=0}^{n-1} \frac{\widehat{f}''_{Z_s} \circ \widehat{f}_{[Z_{s+1},\ldots,Z_{n-1}]}}{\widehat{f}'_{Z_s} \circ \widehat{f}_{[Z_{s+1},\ldots,Z_{n-1}]}} \cdot \widehat{f}'_{[Z_{s+1},\ldots,Z_{n-1}]} \right| \leq A \cdot c_B \quad \text{on } B. \qquad ()$$

Therefore $(\widehat{u} \circ \widehat{f}_Z) \cdot \mid \widehat{f}''_Z \mid \leq A c_B \cdot (\widehat{u} \circ \widehat{f}_Z) \cdot \mid \widehat{f}'_Z \mid$ and $\mid \widehat{u}' \circ \widehat{f}_Z \mid \cdot \mid \widehat{f}'_Z \mid^2 \leq R(\widehat{u}) \cdot (\widehat{u} \circ \widehat{f}_Z) \cdot \mid \widehat{f}'_Z \mid$, whence $(\triangle)$ yields $\mid \widehat{u}'_n \mid \leq (R(\widehat{u}) + A c_B) \cdot \widehat{u}_n$ on B, and we can take $r_B := R(\widehat{u}) + A c_B$.

To actually justify $(\triangle)$ we prove that both $\sum_n^{(1)}$ and $\sum_n^{(2)}$ are uniformly convergent on B by first noticing that both are majorized by some constant multiple of $\sum_n := \sum_{\{Z \in \widehat{\xi}_n(B) : \widehat{u} > 0 \text{ on } Z\}} \mid \widehat{f}'_Z \mid$. Now (in view of ()) $\sup_B \mid \widehat{f}'_Z \mid \leq \inf_B \mid \widehat{f}'_Z \mid \cdot \exp(A c_B \lambda(B)) \leq \lambda(Z) \cdot \lambda(B)^{-1} \exp(A c_B \lambda(B))$, and since $\widehat{u}$ is supported on a set of finite Lebesgue measure, we obtain a common upper bound for the $\sum_n$, $n \geq 1$.

The last assertion follows from Remark 5 since $\lambda(\widehat{u}_1) = \lambda(\widehat{u}_n) \geq \int_B \widehat{u}_n d\lambda \geq \lambda(B) \inf_B \widehat{u}_n \geq \lambda(B) \kappa_B^{-1} \sup_B \widehat{u}_n$, where $\kappa_B := \sup_{n \geq 1} \exp(\lambda(B) R_B(\widehat{u}_n)) < \infty$. ■

Proof of Lemma 14. Since on $B \in \widehat{\mathcal{M}}$, $\widehat{\mathbf{P}}^L \widehat{u} = \widehat{u}_L = \sum_{Z \in \widehat{\mathcal{A}} \cap \widehat{\xi}_L(B)} \mid \widehat{f}'_Z \mid$, where $\widehat{\mathcal{A}} \subseteq \widehat{\xi}_L \cap \{\Lambda = 0\}$ is the obvious lift of $\mathcal{A}$ to $\widehat{X}$, the assertion is implicit in the preceding proof. ■

Proof of Lemma 15. For any fixed $N \geq 1$, let $Y := X(N)$ and $\widehat{u}$ be the lift of u to the base of $(\widehat{X(Y)}, \widehat{T}, \widehat{\xi(Y)})$. Then for some $c \in (0, \infty)$, $\widehat{u}$ is dominated by the admissible function $\widehat{w} := c \cdot 1_{\widehat{X} \cap \{\Lambda = 0\}}$. By Lemma 13, each $\widehat{\mathbf{P}}^n \widehat{w}$ is bounded on members of $\widehat{\mathcal{M}} \cap \pi^{-1} Y$, and supported on $\{\Lambda \leq n\}$. Hence $\mathbf{P}^n u = \pi_*(\widehat{\mathbf{P}}^n \widehat{u}) \leq \pi_*(\widehat{\mathbf{P}}^n \widehat{w})$ is bounded on Y by Lemma 8. ■

Proof of Lemma 16. In view of Remark 7, $\widehat{Y} = \pi^{-1}Y$. Consider the particular AFU-system $(\widehat{Y}, \widehat{T_Y}, \widehat{\xi_Y})$ and recall the proof of Lemma 6. The estimate there, when applied with $\widehat{G} = \sum_{k=0}^{n-1} \widehat{\mathbf{P}}^k \widehat{u}$ (and not $\sum_{k=0}^{n-1} \widehat{\mathbf{P}}^k_{\widehat{Y}} \widehat{u}$ as was the case then), shows that for suitable η, K, and q we have for $\widehat{x} \in \widehat{Y} \cap \{\Lambda_{\widehat{Y}} > \eta\}$ and $\widehat{u}$ of the specified type

$$\sum_{(\widehat{y},j)\in\Gamma_{\widehat{Y}}(\widehat{x})} \widehat{g}_{\widehat{Y},j}(\widehat{y}) \cdot \sum_{k=0}^{n-1} \widehat{\mathbf{P}}^k \widehat{u}(\widehat{y}) \leq K \cdot q^{\Lambda_{\widehat{Y}}(\widehat{x})} \cdot \left(\sup_{\widehat{Y}\cap\{\Lambda_{\widehat{Y}}\leq\eta\}} \sum_{k=0}^{n-1} \widehat{\mathbf{P}}^k \widehat{u} \right),$$

where $\Gamma_{\widehat{Y}}(\widehat{x}) := \{(\widehat{y}, l) : \widehat{y} \in \widehat{Y} \cap \{\Lambda_{\widehat{Y}} \leq \eta\},\ \widehat{T}_{\widehat{Y}}\widehat{y}, \ldots, \widehat{T}^l_{\widehat{Y}}\widehat{y} \in \widehat{Y} \cap \{\Lambda_{\widehat{Y}} > \eta\},\ \widehat{T}^l_{\widehat{Y}}\widehat{y} = \widehat{x}\}$ and $\Gamma_{\widehat{Y}}(\widehat{x}, k) := \{(\widehat{y}, l) \in \Gamma_{\widehat{Y}}(\widehat{x}) : l \leq k\}$ correspond to $\Gamma(\widehat{x})$, $\Gamma(\widehat{x}, k)$ in that proof.

For $\widehat{x} \in \widehat{Y} \cap \{\Lambda_{\widehat{Y}} > \eta\}$ and $k \geq 1$ we let $\Upsilon(\widehat{x}) := \{(\widehat{y}, l) : \widehat{y} \in \widehat{Y} \cap \{\Lambda_{\widehat{Y}} \leq \eta\},\ \widehat{T}\widehat{y}, \ldots, \widehat{T}^l\widehat{y} \notin \widehat{Y} \cap \{\Lambda_{\widehat{Y}} \leq \eta\},\ \widehat{T}^l\widehat{y} = \widehat{x}\}$, $\Upsilon(\widehat{x}, k) := \{(\widehat{y}, l) \in \Upsilon(\widehat{x}) : l \leq k\}$, and $\widehat{\varphi}_j$ be the j-th return time to $\widehat{Y}$. Then, as $\widehat{u}$ is supported on $\widehat{Y} \cap \{\Lambda = 0\} = \widehat{Y} \cap \{\Lambda_{\widehat{Y}} = 0\}$, we have

$$\begin{aligned}
\widehat{\mathbf{P}}^k \widehat{u}(\widehat{x}) &= \sum_{(\widehat{y},l)\in\Upsilon(\widehat{x},k)} \widehat{g}_l(\widehat{y}) \cdot \widehat{\mathbf{P}}^{k-l} \widehat{u}(\widehat{y}) \\
&= \sum_{(\widehat{y},l)\in\Upsilon(\widehat{x},k)} \widehat{g}_{\widehat{Y},\sum_{i=1}^{l} 1_{\widehat{Y}}(\widehat{T}^i\widehat{y})}(\widehat{y}) \cdot \widehat{\mathbf{P}}^{k-l} \widehat{u}(\widehat{y}) \\
&= \sum_{\substack{(\widehat{y},j)\in\Gamma_{\widehat{Y}}(\widehat{x},k) \\ \widehat{\varphi}_j(\widehat{y})\leq k}} \widehat{g}_{\widehat{Y},j}(\widehat{y}) \cdot \widehat{\mathbf{P}}^{k-\widehat{\varphi}_j(\widehat{y})} \widehat{u}(\widehat{y}) \ ,
\end{aligned}$$

since the map $\Upsilon(\widehat{x}, k) \to \{(\widehat{y}, j) \in \Gamma_{\widehat{Y}}(\widehat{x}, k) : \widehat{\varphi}_j(\widehat{y}) \leq k\}$ given by $(\widehat{y}, l) \mapsto (\widehat{y}, \sum_{i=1}^{l} 1_{\widehat{Y}}(\widehat{T}^i\widehat{y}))$ is a bijection. We therefore obtain

$$\sum_{k=0}^{n-1} \widehat{\mathbf{P}}^k \widehat{u}(\widehat{x}) = \sum_{k=0}^{n-1} \sum_{\substack{(\widehat{y},j)\in\Gamma_{\widehat{Y}}(\widehat{x},k) \\ \widehat{\varphi}_j(\widehat{y})\leq k}} \widehat{g}_{\widehat{Y},j}(\widehat{y}) \cdot \widehat{\mathbf{P}}^{k-\widehat{\varphi}_j(\widehat{y})} \widehat{u}(\widehat{y}) \leq \sum_{(\widehat{y},j)\in\Gamma_{\widehat{Y}}(\widehat{x})} \widehat{g}_{\widehat{Y},j}(\widehat{y}) \cdot \sum_{k=0}^{n-1} \widehat{\mathbf{P}}^k \widehat{u}(\widehat{y}).$$

Combining these observations we indeed find that if $\widehat{x} \in \widehat{Y} \cap \{\Lambda_{\widehat{Y}} > \eta\}$, then

$$\sum_{k=0}^{n-1} \widehat{\mathbf{P}}^k \widehat{u}(\widehat{x}) \leq K \cdot q^{\Lambda_{\widehat{Y}}(\widehat{x})} \cdot \left(\sup_{\widehat{Y}\cap\{\Lambda_{\widehat{Y}}\leq\eta\}} \sum_{k=0}^{n-1} \widehat{\mathbf{P}}^k \widehat{u} \right) \quad \text{for } n \geq 1. \ \blacksquare$$

Proof of Lemma 17. Notice first that $\widehat{u}$ satisfies the assumptions of Lemma 16, and for any $m \geq 1$ and $x \in X$,

$$\left| \frac{1}{b_n} \sum_{k=0}^{n-1} \mathbf{P}^k u(x) - H(x) \right| = \left| \sum_{\widehat{x}\in\pi^{-1}(x)} \left(\frac{1}{b_n} \sum_{k=0}^{n-1} \widehat{\mathbf{P}}^k \widehat{u}(\widehat{x}) - \widehat{H}(\widehat{x}) \right) \right| \leq$$

$$\leq \underbrace{\left| \sum_{\substack{\widehat{x}\in\pi^{-1}(x) \\ \Lambda_{\widehat{Y}}(\widehat{x})\leq m}} \left(\frac{1}{b_n} \sum_{k=0}^{n-1} \widehat{\mathbf{P}}^k \widehat{u}(\widehat{x}) - \widehat{H}(\widehat{x}) \right) \right|}_{=:D_n(x,m)} + \underbrace{\sum_{\substack{\widehat{x}\in\pi^{-1}(x) \\ \Lambda_{\widehat{Y}}(\widehat{x})>m}} \left(\frac{1}{b_n} \sum_{k=0}^{n-1} \widehat{\mathbf{P}}^k \widehat{u}(\widehat{x}) + \widehat{H}(\widehat{x}) \right)}_{=:S_n(x,m)} .$$

Let us again write $\widehat{U}_n := \sum_{k=0}^{n-1} \widehat{\mathbf{P}}^k \widehat{u}$. As $\widehat{\mathcal{M}_Y} \cap \{\Lambda_{\widehat{Y}} \leq \eta\}$ is finite, $(b_{n_j})^{-1}\widehat{U}_{n_j} \to \widehat{H}$ uniformly on $\widehat{Y} \cap \{\Lambda_{\widehat{Y}} \leq \eta\}$, so that $K_\infty := \sup_{j\geq 1} \sup_{\widehat{Y}\cap\{\Lambda_{\widehat{Y}}\leq\eta\}} (b_{n_j})^{-1}\widehat{U}_{n_j}$ is finite. According to Lemma 16, if $\widehat{x} \in \widehat{Y} \cap \{\Lambda_{\widehat{Y}} > \eta\}$, then

$$\frac{1}{b_{n_j}} \widehat{U}_{n_j}(\widehat{x}) \leq K\,K_\infty \cdot q^{\Lambda_{\widehat{Y}}(\widehat{x})} \qquad \text{for } j \geq 1,$$

and the same estimate clearly applies to $\widehat{H}(\widehat{x})$. Therefore, if $m > \eta$, then for $x \in Y$ and $j \geq 1$, since $\#(\widehat{\mathcal{M}_Y} \cap \{\Lambda_{\widehat{Y}} = r\}) \leq \#(\widehat{\mathcal{M}_Y} \cap \{\Lambda_{\widehat{Y}} = \eta\})$ for $r \geq \eta$ by Lemma 5,

$$S_{n_j}(x,m) \leq \sum_{r>m} (\#\{\widehat{x} \in \pi^{-1}(x) : \Lambda_{\widehat{Y}}(\widehat{x}) = r\})\, 2\,K\,K_\infty \cdot q^r \leq const \cdot q^m.$$

Let $\varepsilon > 0$ be given. Choose $m > \eta$ so large that $q^m < const^{-1}\varepsilon/2$, then $S_{n_j}(x,m) < \varepsilon/2$ for $x \in Y$ and $j \geq 1$. But as $\widehat{\mathcal{M}_Y} \cap \{\Lambda_{\widehat{Y}} \leq m\}$ is finite, $(b_{n_j})^{-1}\widehat{U}_{n_j} \to \widehat{H}$ uniformly on $\widehat{Y} \cap \{\Lambda_{\widehat{Y}} \leq m\}$, and there is some j_0 such that $D_{n_j}(x,m) < \varepsilon/2$ for $x \in Y$ and $j \geq j_0$. Hence $|\, (b_{n_j})^{-1} \sum_{k=0}^{n_j-1} \mathbf{P}^k u(x) - H(x) \,| < \varepsilon$ for $x \in Y$ and $j \geq j_0$ as claimed.

To prove the second assertion observe that for $x \in X \setminus \bigcup_{B\in\widehat{\mathcal{M}_Y}\cap\{\Lambda_{\widehat{Y}}\leq m\}} \partial(\pi B)$ there is some neighbourhood $V \subseteq Y$ such that each of the finitely many members of $\widehat{\mathcal{M}_Y} \cap \{\Lambda_{\widehat{Y}} \leq m\} \cap \pi^{-1}V$ has $\pi-$image covering V. The uniform limit of $\pi_*(1_{\{\Lambda_{\widehat{Y}}\leq m\}\cap\pi^{-1}V} \cdot (b_{n_j})^{-1}\widehat{U}_{n_j})$ thus is continuous on V. Therefore, the oscillation of H in x, $osc_H(x) := \lim_{\delta\to 0} \sup\{H(s) - H(t) : s,t \in (x-\delta, x+\delta)\}$ cannot exceed

$$\sum_{r>m} \sum_{B\in\widehat{\mathcal{M}_Y}\cap\{\Lambda_{\widehat{Y}}=r\}} \sup_{j\geq 1} \sup_B (b_{n_j})^{-1}\widehat{U}_{n_j} \leq \sum_{r>m} \#(\widehat{\mathcal{M}_Y} \cap \{\Lambda_{\widehat{Y}} = r\})\, K\,K_\infty\, q^r \leq const \cdot q^m$$

provided $m \geq \eta$. Hence $osc_H(x) = 0$ for $x \in X \setminus \bigcup_{B\in\widehat{\mathcal{M}_Y}} \partial(\pi B)$. ∎

10 A general remark on Darling-Kac sets

Let $(X, \mathcal{B}, \mu)$ be a nonatomic $\sigma-$finite measure space, T a c.e.m.p.t. thereon which is pointwise dual ergodic, and write $\mathcal{B}^+ := \{B \in \mathcal{B} : 0 < \mu(B) < \infty\}$. There are several interesting subclasses of $\mathcal{B}^+$ associated with T. We have already met the families $\mathcal{DK}(T) \subseteq \mathcal{U}(T)$ and mentioned minimal wandering rates. The collection of sets having minimal wandering rates will be denoted $\mathcal{W}(T)$. According to Theorem 3.8.3 of [A0] we have $\mathcal{U}(T) \subseteq \mathcal{W}(T)$, and Proposition 3.8.2 there shows that $\mathcal{W}(T)$ is strictly smaller than $\mathcal{B}^+$. While $\mathcal{U}$ and $\mathcal{W}$ are hereditary, the following observation shows that in any case there are a lot of sets which are not Darling-Kac: Any set can in a rather strong sense be approximated arbitrarily close from the inside as well as from the outside (unless it equals the entire space) by sets which are not Darling-Kac.

Theorem 10 (Density of Non-DK sets) *Let $(X, \mathcal{B}, \mu)$ be a nonatomic $\sigma-$finite measure space, T a c.e.m.p.t. thereon which is pointwise dual ergodic. Suppose* *that $A, M \in \mathcal{B}$, $A \subseteq M$, are given, and that $0 < \mu(A) < \alpha < \mu(M)$. Then there exists some set $E \in \mathcal{B}$ satisfying $A \subseteq E \subseteq M$ and $\mu(E) = \alpha$, which is not Darling-Kac.*

Notice that this has nothing to do with μ being infinite.

Lemma 19 *Let $(X, \mathcal{B}, \mu)$ be a nonatomic $\sigma-$finite measure space, T a c.e.m.p.t. thereon which is pointwise dual ergodic. Let $B \in \mathcal{B}^+$ be given. Then there exists some $C \in \mathcal{B}^+$ such that*

$$\frac{1}{a_n}\sum_{k=0}^{n-1}\mathbf{P}_\mu^k 1_C \longrightarrow \mu(C)$$

pointwise but not uniformly mod μ on B. Moreover, given any set $D \in \mathcal{B}$ of nonzero measure, and $\gamma \in (0, \mu(D))$, C may be chosen to be contained in D, and satisfy $\mu(C) = \gamma$.

Proof. Let $0 < \beta < \mu(X)$ be given and choose some sequence $(\beta_n)_{n\geq 1}$ of positive numbers satisfying $\sum_{n\geq 1} n\beta_n < \beta$. Since μ is nonatomic, for any $M \in \mathcal{B}^+$ and $\delta \in (0, \mu(M))$ there is some $M_1 \subseteq M$ of measure δ. We can therefore find a nonincreasing sequence $(B_n)_{n\geq 1}$ of subsets of B with $0 < \mu(B_n) \leq \beta_n$. Consider $Z := \bigcap_{n\geq 1}\bigcap_{k=0}^{n-1} T^{-k}B_n^c$, the set of points which do not enter any B_n before the nth iteration. By our construction of $(B_n)_{n\geq 1}$ we have $\mu(Z^c) < \beta$.

Let C be any subset of Z with positive measure. Then for any $n \geq 1$, $C \subseteq \bigcap_{k=0}^{n-1} T^{-k}B_n^c$ which implies that $\sum_{k=0}^{n-1}\mathbf{P}_\mu^k 1_C$ vanishes on B_n. Therefore

$$\left|\frac{1}{a_n}\sum_{k=0}^{n-1}\mathbf{P}_\mu^k 1_C - \mu(C)\right| = \mu(C) > 0 \qquad \text{on } B_n,$$

which is a set of positive measure in B. Thus $a_n^{-1}\sum_{k=0}^{n-1}\mathbf{P}_\mu^k 1_C$ cannot converge uniformly mod μ on B.

If now $D \in \mathcal{B}$, $\mu(D) > 0$, and $\gamma \in (0, \mu(D))$ are given, choose $\beta < \mu(D) - \gamma$ in the first place. Then $\mu(Z \cap D) > \gamma$, and we can indeed find some $C \subseteq Z \cap D$ with $\mu(C) = \gamma$. ■

Proof of Theorem 10. If f, f_n, g, g_n ($n \geq 1$) are realvalued functions on some set B, and $f_n \to f$, $g_n \to g$ pointwise on B, where convergence is uniform only for one of these sequences, then $f_n + g_n \longrightarrow f + g$ nonuniformly on B.

Employing Egorov's theorem choose some $B \subseteq A$ of positive measure such that

$$\frac{1}{a_n}\sum_{k=0}^{n-1}\mathbf{P}_\mu^k 1_A \longrightarrow \mu(A) \qquad \begin{array}{c}\text{uniformly on } B\\ \text{as } n \to \infty.\end{array}$$

Apply the Lemma to obtain a set $C \subseteq D := M\backslash A$ of measure $\gamma := \alpha - \mu(A)$ such that

$$\frac{1}{a_n}\sum_{k=0}^{n-1}\mathbf{P}_\mu^k 1_C \longrightarrow \mu(C) \qquad \text{nonuniformly mod } \mu \text{ on } B.$$

Then $E := A \cup C$ satisfies $A \subseteq E \subseteq M$, $\mu(E) = \alpha$, and

$$\frac{1}{a_n}\sum_{k=0}^{n-1}\mathbf{P}_\mu^k 1_E = \frac{1}{a_n}\sum_{k=0}^{n-1}\mathbf{P}_\mu^k 1_A + \frac{1}{a_n}\sum_{k=0}^{n-1}\mathbf{P}_\mu^k 1_C \longrightarrow \mu(E) \qquad \begin{array}{c}\text{nonuniformly mod } \mu \\ \text{on } B \subseteq E,\ B \in \mathcal{B}^+\end{array}$$

by the introductory remark. Hence E cannot be Darling-Kac. ∎

The moral of this result is that for the conclusion of Theorem 9 to hold, the functions considered there must indeed be taken from some rather small subclass of $L_1(\lambda)$, and the restriction to R-integrable ones is not just due to an inferior method of proof.

11 Fluctuation theory

We finally describe a beautiful application of Theorem 9. Let T be a basic AFN-map, and let $A \subseteq X$ be any set of positive Lebesgue measure. We consider successive visits to A and inspect the process at time n. Define $Z_n(x) := \max\{k \leq n : T^k x \in A\}$, where we follow the convention to put $\max = 0$. Furthermore let $Y_n(x) := \min\{k > n : T^k x \in A\}$, and $V_n(x) := Y_n(x) - Z_n(x)$. To state the result we let $Z^{(0)} := 0$, $Z^{(1)} := 1$, and for $\alpha \in (0,1)$ let $Z^{(\alpha)}$ denote a random variable on $(0,1)$ with density

$$z^{(\alpha)}(x) := \frac{\sin \pi\alpha}{\pi}\frac{1}{x^{1-\alpha}(1-x)^{\alpha}}, \quad x \in (0,1),$$

while we set $V^{(0)} := \infty$, $V^{(1)} := 0$, and for $\alpha \in (0,1)$ let $V^{(\alpha)}$ be a random variable on $(0,\infty)$ with density

$$v^{(\alpha)}(x) := \frac{\sin \pi\alpha}{\pi}\frac{1-(\max\{1-x,0\})^{\alpha}}{x^{1+\alpha}}, \quad x > 0.$$

Since by Corollary 3 we have $\mathcal{E}(T) \subseteq \mathcal{U}(T)$ for any basic AFN-map T, Theorem 1 of [T4] applies to generalise Theorem 2 there as follows.

Theorem 11 (Thaler-Dynkin-Lamperti Arc-Sine law for AFN-maps) *Let T be a basic AFN-map for which $(a_n(T))_{n\geq 1}$ is regularly varying with index $\alpha \in [0,1]$. Then for any $A \in \mathcal{E}(T)$ the sequences (Z_n), (Y_n), and (Z_n) defined above satisfy*

$$\frac{1}{n}Z_n \overset{d}{\Longrightarrow} Z^{(\alpha)}, \quad \frac{1}{n}Y_n \overset{d}{\Longrightarrow} (Z^{(\alpha)})^{-1}, \quad \text{and} \quad \frac{1}{n}V_n \overset{d}{\Longrightarrow} V^{(\alpha)}$$

as $n \to \infty$, where the distributions of the respective lefthand variables can be taken with respect to any fixed probability measure $P \ll \lambda$.

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