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MEASURE PRESERVING TRANSFORMATIONS SIMILAR TO MARKOV SHIFTS

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Abstract. Similarity, that is, the existence of joint common extensions, defines an interesting equivalence relation for infinite measure preserving transformations T. We provide a sufficient condition, given in terms of return processes to reference sets of finite measure, for T to be similar to a Markov shift. This is then shown to apply to various piecewise smooth dynamical systems.

1. Introduction

The present paper is devoted to a structural property of certain *infinite measure preserving transformations* T. In general, such dynamical systems can have a large variety of interesting ergodic properties, see [A0] for a wealth of information. Here, we focus on situations which exhibit probabilistic properties parallel to classical results for null-recurrent Markov chains. Establishing such limit theorems usually depends on the existence of some reference set of finite measure which has a return process or induced system with good mixing (or related) properties. We refer to [A0], [A2], [A3], [AD1], [ATZ], [T4], [TZ], [Z6], and [Z7] for samples of probabilistic results in this setup. The goal of the present article is to show that many of these systems not only behave like classical Markov chains on the surface, but that there is a deeper structural relation. We are going to prove that many smooth infinite m.p.t.s are in fact *similar* to suitable renewal Markov chains, meaning that both systems are factors of a common extension.

Throughout, all measure spaces $(X, \mathcal{A}, \mu)$ are supposed to be *standard*, i.e. $\mathcal{A}$ is the Borel-σ-algebra of some complete separable metric d_X on X (w.l.o.g. with $\mathrm{diam}(X) \leq 1$), and μ is σ-finite and non-atomic. We abbreviate $\mu(u) := \int_X u \, d\mu$. Let T^* and T be measure preserving transformations (*m.p.t.*s) on $(X^*, \mathcal{A}^*, \mu^*)$ and $(X, \mathcal{A}, \mu)$, respectively. For $c \in (0, \infty)$, a *c-factor map* from the *measure-preserving system* $\mathsf{T}^* := (X^*, \mathcal{A}^*, \mu^*, T^*)$ onto $\mathsf{T} := (X, \mathcal{A}, \mu, T)$ is a measurable map $\pi : X^* \to X$ with

$$\pi \circ T^* = T \circ \pi \quad \text{and} \quad \mu^* \circ \pi^{-1} = c\,\mu.$$

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This situation is denoted by $\pi : \mathsf{T}^* \xrightarrow{c} \mathsf{T}$. If such a π exists, T^* is called a *c-extension* of T, and T a *c-factor* of T^*. We allow $c \neq 1$ here, as there is no natural normalization of infinite measure spaces.

Two m.p. systems T_1 and T_2 are said to be *similar*, denoted $\mathsf{T}_1 \sim \mathsf{T}_2$, if they possess a common extension, that is, if there exist an m.p. system T^* and $c_1, c_2 \in (0,\infty)$ such that $\pi_1 : \mathsf{T}^* \xrightarrow{c_1} \mathsf{T}_1$ and $\pi_2 : \mathsf{T}^* \xrightarrow{c_2} \mathsf{T}_2$ for suitable c_i-factor maps π_i. Otherwise T_1 and T_2 are *strongly disjoint*. While any two probability preserving transformations are similar, this notion provides a highly non-trivial equivalence relation between infinite measure preserving transformations. (The Cartesian product $\mathsf{T}_1 \otimes \mathsf{T}_2$ is an extension, with $c < \infty$, of T_1 iff T_2 preserves a finite measure.) There are invariants enabling the detection of uncountably many strongly disjoint systems within certain classes of well-behaved transformations like, for example, the renewal shifts below.

The first natural class of interest here are the Markov shifts. The basic object of their similarity theory is the *renewal chain* with *return distribution* $f = (f_k)_{k\geq 1}$ (where $f_k \geq 0$, $k \geq 1$, with $\sum_{k\geq 1} f_k = 1$), that is, the irreducible recurrent Markov chain on $\mathbb{N}_0 = \{0, 1, \ldots\}$ with transition matrix $P^{[f]} = P = (p_{j,k})_{j,k\geq 0}$ given through $p_{0,k} := f_{k+1}$, $p_{k+1,k} := 1$, and $p_{j,k} = 0$ otherwise, which has a stationary distribution with weights $r_k := \sum_{j>k} f_j$. The associated *recurrent renewal sequence* $u = (u_n)_{n\geq 0}$ is given by $u_0 := 1$ and $u_n := \sum_{k=1}^{n} f_k u_{n-k}$, $n \geq 1$. Equipping the product space $\Omega := \{\omega = (\omega_i)_{i\geq 0} : \omega_i \in \mathbb{N}_0\}$ with product σ-field $\mathcal{B}_\Omega$ and the Markov measure $\mu^{[f]}$ with $\mu^{[f]}([s_0, \ldots, s_{n-1}]) = r_{s_0}\, p_{s_0,s_1} \cdots p_{s_{n-2},s_{n-1}}$ for cylinder sets $[s_0, \ldots, s_{n-1}] := \{\omega : \omega_0 = s_0, \ldots, \omega_{n-1} = s_{n-1}\}$, and taking T to be the shift on Ω, we obtain the c.e.m.p. system $\mathsf{R}^{[f]} := (\Omega, \mathcal{B}_\Omega, \mu^{[f]}, T)$, the *(one-sided[1]) renewal shift* with return distribution f. Every Markov shift contains lots of renewal shifts as factors (e.g. the return processes to its individual states).

Section 5 of [A0] discusses similarity of Markov shifts and, in particular, of lattice random walks. Similarity of stable random walks on $\mathbb{R}$ and $\mathbb{R}^2$ is studied in [A5]. In [AK] and [ALP], even *isomorphism* (existence of invertible factor maps) of certain random walks is established.

Still, all positive results I am aware of focus on systems which are *a priori* given as Markov chains. The purpose of the present work is to show that various types of piecewise smooth dynamical systems with infinite invariant measure which have been studied in the literature are, in fact, similar to (renewal) Markov chains.

2. Main results

We need to recall a few basic concepts, and to fix notations. A *μ-partition* of the (standard) measure space $(X, \mathcal{A}, \lambda)$ is a measurable partition mod μ. If T is a *nonsingular* map on $(X, \mathcal{A}, \lambda)$, meaning that it is measurable with $\lambda \circ T^{-1} \ll \lambda$, its *transfer operator* $\widehat{T} : L_1(\lambda) \to L_1(\lambda)$ describes the evolution of probability densities under T, that is, $\widehat{T}u := d(\nu \circ T^{-1})/d\lambda$, where ν has density u w.r.t. λ. Equivalently, $\int_X (g \circ T) \cdot u \, d\lambda = \int_X g \cdot \widehat{T}u \, d\lambda$ for all $u \in L_1(\lambda)$ and $g \in L_\infty(\lambda)$, i.e. $\widehat{T}$

[1] For matters of similarity, it is immaterial whether we take one-sided or two-sided shifts, the latter being the natural extensions of the first.

is dual to $g \longmapsto g \circ T$. The operator $\widehat{T}$ naturally extends to $\{u : X \to [0,\infty)$ measurable $\mathcal{A}\}$, and T is *conservative ergodic (c.e.)* iff $\sum_{k\geq 0} \widehat{T}^k u = \infty$ a.e. for all $u \in L_1(\lambda)$ with $\int u \, d\lambda > 0$. In the latter case, we can define, for any $Y \in \mathcal{A}$ with $\lambda(Y) > 0$ the *first return (entrance) time* of Y by $\varphi(x) = \min\{n \geq 1 : T^n x \in Y\}$, $x \in X$, and we let $T_Y x := T^{\varphi(x)} x$, $x \in X$. The *return-time partition* of Y is $\{Y \cap \{\varphi = k\} : k \geq 1\}$, and whenever ξ_Y is a partition refining the latter, we see that the *kth return time* of Y, given by $\varphi_k := \sum_{i=0}^{k-1} \varphi \circ T_Y$, $k \geq 1$, is constant on each $W \in \xi_{Y,k} := \bigvee_{j=0}^{k-1} T_Y^{-1} \xi_Y$ (and we simply write $\varphi_k(W)$ for its value on W).

We are mainly interested in infinite T-invariant measures μ. Given $Y \in \mathcal{A}$ with $0 < \mu(Y) < \infty$, we can regard φ as a random variable on the probability space $(Y, Y \cap \mathcal{A}, \mu_Y)$, $\mu_Y(E) := \mu(Y)^{-1}\mu(Y \cap E)$. Asymptotic properties of its *return distribution*, given by the *(first) return probabilities* $f_k(Y) := \mu_Y(Y \cap \{\varphi_Y = k\})$, $k \geq 1$, determine the stochastic properties of the system. The probability measure μ_Y is invariant under the *first return map*, T_Y restricted to Y. In other words,

$$1_Y = \sum_{k\geq 1} \widehat{T}^k 1_{Y\cap\{\varphi=k\}} \tag{2.1}$$

Note the following interpretation of the functions on the right-hand side, if our system starts with initial density $\mu(Y)^{-1}1_Y$: Normalizing, we obtain the conditional densities $\mu(Y \cap \{\varphi = k\})^{-1} \cdot \widehat{T}^k 1_{Y\cap\{\varphi=k\}} =: H_k$ of our process at time k, given that this is the time of its first return to Y. In other words, having returned at step k, the process starts anew, this time with initial density H_k. The first return to Y therefore constitutes a *proper renewal* if

$$\widehat{T}^k \, 1_{Y\cap\{\varphi=k\}} = \widehat{T_Y} \, 1_{Y\cap\{\varphi=k\}} = \mu(Y \cap \{\varphi = k\}) \cdot 1_Y \quad \text{for } k \geq 1. \tag{2.2}$$

In fact, (2.2) is sufficient for Y to be a *recurrent event for* T, meaning that for arbitrary integers $0 = n_0 < n_1 < \ldots < n_K$,

$$\mu_Y\left(\bigcap_{k=0}^{K} T^{-n_k} Y\right) = \prod_{k=0}^{K} u_{n_k - n_{k-1}}(Y),$$

where $u_n(Y) := \mu_Y(T^{-n}Y)$, $n \geq 0$, cf. [A0], [A1]. In this case, the *return-time process* of Y, $(\varphi(T_Y^k x))_{k\geq 0}$ on $(Y, Y \cap \mathcal{A}, \mu_Y)$, is iid, and

$$\mathsf{T} \sim \mathsf{R}^{[f(Y)]} \tag{2.3}$$

since $\pi : \mathsf{T} \xrightarrow{c} \mathsf{R}^{[f(Y)]}$ with $\pi : X \to \Omega$ given by $\pi(x) := ((1_{Y^c}\,\varphi) \circ T^n(x))_{n\geq 0}$, and $c := 1/\mu(Y)$.

Smooth dynamical systems usually do not come with an easily detected recurrent event. Instead, the best we can hope for is a geometrically nice set Y which allows reasonable control of the $\widehat{T_Y} \, 1_{Y\cap\{\varphi=k\}}$. (For example, [TZ] and [Z6] provide conditions on the size of the collection $\{\mu(Y \cap \{\varphi = k\})^{-1} \cdot \widehat{T_Y} \, 1_{Y\cap\{\varphi=k\}} : k \geq 1\}$ which, in the presence of regularly varying tails of $f(Y) := (f_k(Y))_{k\geq 1}$, are sufficient for various distributional limit theorems.) Below we formalize a condition, central to the present paper, which roughly says that a version of the regeneration property (2.2) holds if we allow parts of the initial density to perform more than just one excursion from Y in order to recover. In fact, it is solely a property of the probability preserving induced system $(Y, Y \cap \mathcal{A}, \mu_Y, T_Y)$ and the return-time partition.

Definition 1. Let $(Y, \mathcal{B}, \nu, S)$ be a probability preserving system, and η a non-trivial ν-partition of Y. A *regenerative partition of* 1_Y *subordinate to* η, is a collection $\mathfrak{W} = \{w_W : k \geq 1, W \in \eta_k\}$, where $\eta_k := \bigvee_{l=0}^{k-1} S^{-l}\eta$, and each[2] w_W is a non-negative measurable function on W, such that

$$\sum_{k \geq 1} \sum_{W \in \eta_k} w_W = 1_Y, \tag{2.4}$$

and

$$\widehat{S}^k(w_W) = \nu(w_W) \cdot 1_Y \quad \text{for all } k \geq 1 \text{ and } W \in \eta_k. \tag{2.5}$$

The sequence $\mathfrak{d}(\mathfrak{W})$ with $\mathfrak{d}_k(\mathfrak{W}) := \sum_{W \in \eta_k} \nu(w_W)$, $k \geq 1$, is the *delay distribution* of $\mathfrak{W}$, and we shall say that $\mathfrak{W}$ has *integrable delay* if $\nu(\mathfrak{W}) := \sum_{k \geq 1} k\, \mathfrak{d}_k(\mathfrak{W}) < \infty$.

The first main result of the present paper states that the mere existence of an induced system admitting such a $\mathfrak{W}$ implies similarity to a certain renewal shift:

Theorem 1 (Regenerative extensions from regenerative partitions of unity). *Let* $\mathsf{T} = (X, \mathcal{A}, \mu, T)$ *be a c.e.m.p. system, and* $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$. *If the induced system* $\mathsf{S} = (Y, Y \cap \mathcal{A}, \mu_Y, T_Y)$ *admits a regenerative partition of* 1_Y $\mathfrak{W} = \{w_W : k \geq 1, W \in \eta_k\}$, *subordinate to a partition* η *refining the return-time partition of* Y, *and if* $\mathfrak{W}$ *has integrable delay, then*

$$\mathsf{T} \sim \mathsf{R}^{[\mathfrak{f}(\mathfrak{W}, \varphi)]} \tag{2.6}$$

where the regeneration distribution $\mathfrak{f}(\mathfrak{W}, \varphi)$ of $\mathfrak{W}$ under T *is given by*

$$\mathfrak{f}_j(\mathfrak{W}, \varphi) := \sum_{k \geq 1, W \in \eta_k : \varphi_k(W) = j} \mu_Y(w_W), \quad j \geq 1.$$

Specifically, there exists an extension $\mathsf{T}^* = (X^*, \mathcal{A}^*, \mu^*, T^*)$ *of* T, *which possesses a recurrent event* Y^* *with return distribution* $\mathfrak{f}(\mathfrak{W}, \varphi)$.

We give an explicit direct construction of a suitable extension in Section 3. This sufficient condition is readily verified for various classes of piecewise invertible infinite measure preserving transformations, since we will show that regenerative partitions of unity exist whenever the induced system belongs to some folklore family of transformations.

Consider a *nonsingular piecewise invertible system* $(Y, \mathcal{B}, \lambda, S, \eta)$, meaning that η is a λ-partition of Y into open sets (w.r.t. a metric d_Y, w.l.o.g. with $\mathrm{diam}(Y) \leq 1$) such that the restriction of S to any of its *cylinders* $W \in \eta$ is a nonsingular homeomorphism onto SW. Let $\eta_n := \bigvee_{k=0}^{n-1} T^{-k}\eta$ denote the family of *cylinders of rank* n. The n-th iterate $(Y, \mathcal{B}, \lambda, S^n, \eta_n)$ again is a system of the same type. All *inverse branches*, denoted[3] $v_W := (S^n \mid_W)^{-1} : S^nW \to W$, $W \in \eta_n$,

[2] In writing w_W we tacitly assume that $W = \bigcap_{l=0}^{k-1} W_l$ (with $W_l \in \eta$) knows its rank k, i.e. we identify the set W with the formal expression $[W_0, \ldots, W_{k-1}]$. While we may have $W \cap T^{-k}Z = W \pmod{\nu}$ for some $W \in \eta_k$ and $Z \in \eta$, this (rather common) abuse of notation will not cause any confusion. (Actually, in the presence of (2.5), no ambiguities are possible: We have $\nu(W \setminus W \cap T^{-k}Z) = \nu(W \cap T^{-k}Z^c) = \int_{Z^c} \widehat{S}^k(1_W)\, d\nu > 0$ since η is non-trivial and (2.5) ensures $\widehat{S}^k(1_W) > 0$ a.e. on Y.)

[3] Same convention as in the previous footnote.

have Radon-Nikodym derivatives $v'_W := d(\lambda \circ v_W)/d\lambda = \widehat{S}^n 1_W$. The system $(Y, \mathcal{B}, \lambda, S, \eta)$ is called *uniformly expanding* if there is some $\rho = \rho(S) \in (0,1)$ such that $d_Y(v_W(x), v_W(y)) \leq \rho \cdot d_Y(x,y)$ whenever $x, y \in W \in \eta$. To ensure good ergodic properties, we will need some distortion control. As in [Z3], a real function u will be called *admissible on* $B \subseteq Y$ if it is Lipschitz with $\inf u > 0$ or, equivalently, if $u > 0$ and there is some $r \in (0, \infty)$ for which $u(x)/u(y) \leq 1 + r \cdot d_Y(x,y)$ for $x, y \in B$. In this case, the inf of all such r is the *regularity* $R_B(u)$ *of* u on B. A natural version of *Adler's condition*, suitable for this setup, is that there should be some $A = A(S) \in [0, \infty)$ for which

$$\sup_{W \in \eta} R_{SW}(v'_W) \leq A.$$

The system is *Markov* if $SW \cap W' \neq \varnothing$ for $W, W' \in \eta$ implies $W' \subseteq W$, and *piecewise onto* if $SW = X$ for all $W \in \eta$. We call it a *Rényi-system* if it has a uniformly expanding iterate S^N, is piecewise onto, and satisfies Adler's condition. Any Rényi system has a unique invariant probability measure $\nu \ll \lambda$ (hence is ergodic), and is also Rényi with respect to the latter. In this situation, there is always a regenerative partition of unity:

Theorem 2 (**Regenerative partitions of unity for Rényi-systems**). *Let* $(Y, \mathcal{B}, \nu, S, \eta)$ *be a probability preserving Rényi-system. Then there exists a regenerative partition* $\mathfrak{W} = \{w_W : k \geq 1, W \in \eta_k\}$ *of* 1_Y*, subordinate to* η*, with exponentially decaying delay distribution. In particular,* $\mathfrak{W}$ *has integrable delay.*

Combining these results we therefore obtain:

Theorem 3 (**Induced Rényi systems mean similarity to Markov shifts**). *A c.e.m.p. system* $\mathsf{T} = (X, \mathcal{A}, \mu, T)$ *inducing a Rényi-system* $\mathsf{S} = (Y, Y \cap \mathcal{A}, \mu_Y, T_Y, \eta)$ *with* η *refining the return-time partition of* $Y \subseteq X$ *is similar to a Markov shift.*

This applies to large collections of (piecewise) smooth systems:

Example 1 (**Random walks driven by expanding Markov maps**). *Suppose that* $(Y, \mathcal{B}, \lambda, S, \eta)$ *is a probability preserving uniformly expanding piecewise invertible Markov system with* $\#S\eta < \infty$ *satisfying Adler's condition (hence Gibbs-Markov in the sense of* [A0], [AD2]*). Any* η*-measurable map* $\phi : X \longrightarrow \mathbb{Z}$ *defines a* $\mathbb{Z}$-extension $T = S_\phi$ *of* S*, that is the m.p.t. on the* σ*-finite infinite measure space* $(Y \times \mathbb{Z}, \mathcal{B} \otimes \mathfrak{P}(\mathbb{Z}), \nu \otimes \iota)$*,* ι *denoting counting measure on* $\mathbb{Z}$*, given by*

$$T(y, g) := (Sy, g + \phi(y)).$$

We henceforth assume (see [AD2] *for sufficient conditions) that* T *is conservative and ergodic. A natural candidate for a good reference set of finite measure is the* $g = 0$ *section* $Y_0 := Y \times \{0\}$*. The first return map* T_{Y_0} *is easily seen to be a map of the same type as* S*. Therefore, we can induce once more, on any cylinder of* T_{Y_0}*, to finally obtain a Rényi-system. Consequently,* T *is similar to a renewal shift.*

Example 2 (**Interval maps with indifferent fixed points**). *An important family of infinite measure preserving dynamical systems is given by piecewise* $\mathcal{C}^2$ *interval maps with* indifferent (neutral) fixed points*. In* [Z1], [Z2] *we introduced and studied a large class of such maps, generalizing earlier work from* [A0]*,* [A3]*,* [ADU]*, and* [T1]-[T3]*. For the precise description of these c.e.m.p. piecewise invertible systems* $\mathsf{T} = (X, \mathcal{A}, \mu, T, \xi)$ *on an interval, called* basic AFN-maps*, we refer to* [Z1]*,* [Z2]*. Much of their analysis relies on the fact that there are good sets* $Y = Y(T)$

(constructed there) with uniformly expanding induced map T_Y satisfying Adler's condition. In the special case of a Markov system, it is then clear that we can induce once again, to get another set $Y' \subseteq Y$ on which T induces a Rényi-system. In the general non-markovian case, we appeal to Lemma 9 of [Z2] *which provides us with a markovian* 1*-extension $\widehat{\mathsf{T}}$ of T possessing sets $\widehat{Y}$ on which it induces a Rényi-system. Hence, any basic AFN-map is similar to a renewal shift.*

Example 3 (**S-unimodal Misiurewicz maps with flat tops**). *This is another type of interval maps with dynamics dominated by some neutral orbit. Extending results of* [BM], [Z4] *studied S-unimodal Misiurewicz maps with degenerate critical points, which are c.e.m.p. piecewise invertible systems $\mathsf{T} = (X, \mathcal{A}, \mu, T, \xi)$ on an interval, with μ infinite if the map is sufficiently flat at the critical point. Any map of this type admits an interval Y on which it induces a Rényi-system (see the proof of Theorem 1 of* [Z4]*), and hence is similar to a renewal shift.*

3. Constructing a regenerative extension

Here is a proof of our first core result.

Proof of Theorem 1. Given a c.e.m.p. system $\mathsf{T} = (X, \mathcal{A}, \mu, T)$ with a set $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, and a regenerative partition $\mathfrak{W} = \{w_W : k \geq 1, W \in \eta_k\}$ of 1_Y with integrable delay, we give an explicit construction of a $\mu_Y(\mathfrak{W})$-extension $\mathsf{T}^* = (X^*, \mathcal{A}^*, \mu^*, T^*)$ of T with a suitable recurrent event Y^*.

(i) We can represent $\mathfrak{W}$ as a sequence $(w_j)_{j\geq 1}$, and let $\tau_j := k$ if $w_j = w_W$ for some $W \in \eta_k$. We will construct, by specifying X^*, a subspace $(X^*, \mathcal{A}^*, \nu^*)$ of the (σ-finite) product

$$(X, \mathcal{A}, \mu) \otimes (I, \mathcal{B}, \lambda) \otimes (\mathbb{N}, \mathfrak{P}(\mathbb{N}), \iota) \otimes (\mathbb{N}_0, \mathfrak{P}(\mathbb{N}_0), \iota) \tag{3.1}$$

where $I := (0, 1)$ with Borel-σ-field $\mathcal{B}$ and Lebesgue measure λ, and ι is counting measure on the power set $\mathfrak{P}(\mathbb{N}_0)$. Points in the product space will be written as $x^* = (x, z, j, l)$, and we denote the canonical projections onto the first and the last coordinate by π and Λ, respectively. We obtain X^* by defining its *level sets* $X^*_l := X^* \cap \{\Lambda = l\}$, $l \geq 0$, each of which is the union $X^*_l = \bigcup \mathcal{M}_l$ of some countable family $\mathcal{M}_l$ of pairwise disjoint measurable sets. The $\mathcal{M}_l$ will be constructed inductively: As soon as $\mathcal{M}_l$ is given, we define T^* a.e. on X^*_l, which will be a nonsingular map into $X^* \cap \{\Lambda \leq l + 1\}$. We then choose $\mathcal{M}_l$ to be a suitable measurable partition of $T^* X^*_l \cap \{\Lambda = l\}$.

The construction below involves the sets $W_j \in \mathcal{A}$, $j \geq 1$, and their images. To avoid measurability problems, we replace the $T^l W_j$ by (fixed versions of) the sets

$$W_{j,l} := \{\widehat{T}^l\, 1_{W_{j,0}} > 0\}, \quad j \geq 1 \text{ and } l \geq 0. \tag{3.2}$$

(The same convention has tacitly been used in [Z5].) Note that (mod μ)

$$W_{j,l} \cap Y = \begin{cases} W_{j,l} & \text{if } l = \varphi_i(W_j) \text{ for some } i \in \{0, 1, \ldots, \tau_j - 1\}, \\ \varnothing & \text{otherwise,} \end{cases} \tag{3.3}$$

for $0 \leq l < \varphi^*(j)$ with $\varphi^*(j) := \varphi_{\tau_j}(W_j)$, $j \geq 1$, since $\mathfrak{W}$ is subordinate to some partition (mod μ_Y) ξ_Y of Y refining the return-time partition.

(ii) Defining $\mathcal{M}_0$, and T^* on X_0^*, requires more preparation: For $x \in Y$ we define a sequence of (possibly void) intervals $V_{x,j} := (\sum_{i=0}^{j-1} w_i(x), \sum_{i=0}^{j} w_i(x)) \subseteq I$, $j \geq 1$, where $w_0(x) := 0$. By (2.4) the $V_{x,j}$ form a partition (mod λ) η_x of I. The sets $W'_j := \{(x,z) : x \in X \text{ and } z \in V_{x,j}\}$, $j \geq 1$, then form a measurable partition (mod $\mu \otimes \lambda$) of $Y \times I$ with $\mu \otimes \lambda(W'_j) = \mu(w_j)$, and we define (a.e. on $Y \times I$) the integer $J(x,z)$ through $(x,z) \in W'_{J(x,z)}$. For $x \in Y$ we let $F_x : I \to I$ denote a Rényi-map with fundamental partition η_x mapping each nonempty $V_{x,j}$ affinely onto I, so that

$$\widehat{F_x}\, 1_{V_{x,j}} = \lambda(V_{x,j}) \cdot 1_I = w_j(x) \cdot 1_I \qquad \text{for } x \in Y \text{ and } j \geq 1.$$

Then the skew-product map $G_j : W'_j \to X \times I$ given by $G_j(x,z) := (Tx, F_x(z))$ is nonsingular with

$$\widehat{G_j}\, 1_{W'_j}(x,z) = \widehat{T}\, w_j(x).$$

Now let $\mathcal{M}_0 := \{W^*_{j,0} : j \geq 1\}$ where $W^*_{j,0} := W'_j \times (j,0)$, and define T^* on $X_0^* := \bigcup \mathcal{M}_0$ by

$$T^*(x,z,j,0) := \begin{cases} (Tx, F_x(z), j, 1) & \text{if } 1 < \varphi^*(j), \\ (Tx, F_x(z), J(Tx, F_x(z)), 0) & \text{if } 1 = \varphi^*(j), \end{cases} \tag{3.4}$$

By the above, this map is nonsingular, and we have

$$\widehat{T^*}\, 1_{W^*_{j,0}} = \left\{ \begin{array}{ll} 1_{W^*_{j,1}} & \text{if } 1 < \varphi^*(j) \\ 1_{X_0^*} & \text{if } 1 = \varphi^*(j) \end{array} \right\} \cdot \left(\left(\widehat{T}\, w_j \right) \circ \pi \right), \tag{3.5}$$

where $W^*_{j,1} := W_{j,1} \times I \times (j,1)$ for $j \geq 1$ with $1 < \varphi^*(j)$ (recall our convention about measurable images), and $W^*_{j,1} := \varnothing$ otherwise. Accordingly, we let $\mathcal{M}_{l+1} := \{W^*_{j,1} : j \geq 1\}$.

(iii) Given $\mathcal{M}_l = \{W^*_{j,l} : j \geq 1\}$ for some $l \geq 1$, we define T^* on $X_l^* := \bigcup \mathcal{M}_l$ by

$$T^*(x,z,j,l) := \begin{cases} (Tx, z, j, l+1) & \text{if } l+1 < \varphi^*(j), \\ (Tx, z, J(Tx,z), 0) & \text{if } l+1 = \varphi^*(j). \end{cases} \tag{3.6}$$

This is clearly nonsingular, and we see that for any $g \in L_1(\mu)$,

$$\widehat{T^*}\left(1_{W^*_{j,l}} \cdot (g \circ \pi)\right) = \left\{ \begin{array}{ll} 1_{W^*_{j,l+1}} & \text{if } l+1 < \varphi^*(j) \\ 1_{X_0^*} & \text{if } l+1 = \varphi^*(j) \end{array} \right\} \cdot \left(\left(\widehat{T}\, g \right) \circ \pi \right). \tag{3.7}$$

where $W^*_{j,l+1} := W_{j,l+1} \times I \times (j,1)$ for $j \geq 1$ with $l+1 < \varphi^*(j)$, and $W^*_{j,l+1} := \varnothing$ otherwise. To conclude the inductive step, we let $\mathcal{M}_{l+1} := \{W^*_{j,l+1} : j \geq 1\}$.

(iv) Steps (ii) and (iii) provide us with a nonsingular map T^* on $(X^*, \mathcal{A}^*, \nu^*)$. The canonical projection π, henceforth restricted to X^*, obviously satisfies $T \circ \pi = \pi \circ T^*$. The prospective recurrent event for T^* is the set $Y^* := X_0^* = \bigcup_{j \geq 1} W^*_{j,0}$. Our new system has a simple tower structure w.r.t. the sets $W^*_{j,l}$ partitioning X^*, as

$$W^*_{j,0} \xrightarrow{T^*} W^*_{j,1} \xrightarrow{T^*} W^*_{j,2} \xrightarrow{T^*} \ldots \xrightarrow{T^*} W^*_{j,\varphi^*(j)-1} \xrightarrow{T^*} Y^*. \tag{3.8}$$

The first $\varphi^*(j) - 1$ steps in this chain are onto (mod ν^*) by construction. For the last step, surjectivity (mod ν^*) will follow from (3.12) below. Note that the first-return time $\varphi^*(x^*) := \min\{n \geq 1 : (T^*)^n x^* \in Y^*\}$ for Y^* equals $\varphi^*(j)$ on $W^*_{j,0}$ (and, in particular, is finite) a.e..

(v) We can now construct a σ-finite invariant measure $\mu^* \ll \nu^*$ for T^* by explicitly specifying its density $h^* = d\mu^*/d\nu^*$ on each $W^*_{j,l}$. We start from ν^* restricted to $Y^* \cap \mathcal{A}^*$, that is, $h^* := 1$ on Y^*. In particular,

$$\mu^*(W^*_{j,0}) = \mu(w_j) \quad \text{for } j \geq 1, \quad \text{and } \mu^*(Y^*) = \mu(Y). \tag{3.9}$$

Then we just push this measure forward, i.e. given h^* on $W^*_{j,l}$ with $l + 1 < \varphi^*(j)$, we let

$$h^* := \widehat{T^*}\,(1_{W^*_{j,l}}\, h^*) \quad \text{on } W^*_{j,l+1}. \tag{3.10}$$

This defines a measurable function $0 \leq h^* \leq 1$ a.e. on X^*, and hence a σ-finite measure μ^* on $\mathcal{A}^*$, for which

$$\mu^*(W^*_{j,l}) = \mu^*(W^*_{j,0}) \quad \text{for } j \geq 1 \text{ and } 0 < l < \varphi^*(j). \tag{3.11}$$

Recalling (3.8) and the fact that the $W^*_{j,l}$ form a partition of X^*, we see that T^*-invariance of μ^* is immediate if we check that $\mu^* \mid_{Y^* \cap \mathcal{A}^*} = \nu^* \mid_{Y^* \cap \mathcal{A}^*}$ is invariant under the induced map $T^*_{Y^*}$, i.e. that $\widehat{T^*_{Y^*}}\, 1_{Y^*} = 1_{Y^*}$. The latter follows if we prove that

$$\widehat{T^*_{Y^*}}\, 1_{W^*_{j,0}} = \mu^*(W^*_{j,0}) \cdot 1_{Y^*} \quad \text{for all } j \geq 1. \tag{3.12}$$

Since φ^* is constant on each $W^*_{j,0}$, this also shows that Y^* is a recurrent event. (And, of course, that T^* is conservative.) Now combine the previously obtained bits of information about $\widehat{T^*}$, given in (3.5) and (3.7), together with assumption (2.5), and the fact that $\widehat{T}^{\,\varphi^*(j)}\, h = \widehat{T}^{\,\varphi_{\tau_j}(W_j)}\, h = \widehat{T_Y}^{\,\tau_j}\, h$ for h supported on W_j, to see that in case $1 = \varphi^*(j)$,

$$\begin{aligned}
\widehat{T^*_{Y^*}}\, 1_{W^*_{j,0}} &= \widehat{T^*}\, 1_{W^*_{j,0}} \\
&= 1_{Y^*}\left(\left(\widehat{T}\, w_j\right) \circ \pi\right) \\
&= 1_{Y^*}\left(\left(\widehat{T_Y}\, w_j\right) \circ \pi\right) \\
&= \mu(w_j) \cdot 1_{Y^*},
\end{aligned}$$

while in case $1 < \varphi^*(j)$,

$$\begin{aligned}
\widehat{T^*_{Y^*}}\, 1_{W^*_{j,0}} &= \widehat{T^*}^{\varphi^*(j)-1} \left(\widehat{T^*}\, 1_{W^*_{j,0}}\right) \\
&= \widehat{T^*}^{\varphi^*(j)-1} \left(1_{W^*_{j,1}} \left(\left(\widehat{T}\, w_j\right) \circ \pi\right)\right) \\
&= \widehat{T^*}^{\varphi^*(j)-2} \left(1_{W^*_{j,2}} \left(\left(\widehat{T}^{\,2} w_j\right) \circ \pi\right)\right) \\
&\vdots \\
&= 1_{Y^*} \left(\left(\widehat{T}^{\,\varphi^*(j)} w_j\right) \circ \pi\right) \\
&= 1_{Y^*} \left(\left(\widehat{T_Y}^{\tau_j}\, w_j\right) \circ \pi\right) \\
&= \mu(w_j) \cdot 1_{Y^*},
\end{aligned}$$

proving (3.12). Hence μ^* is invariant, conservative, and Y^* is a recurrent event.

(vi) It remains to validate that π is a $\mu_Y(\mathfrak{W})$-factor map, i.e. that

$$\mu^* \circ \pi^{-1} = \mu_Y(\mathfrak{W}) \cdot \mu. \tag{3.13}$$

The projection π is nonsingular as a map of the space (3.1) containing $(X^*, \mathcal{A}^*, \nu^*)$ onto $(X, \mathcal{A}, \mu)$. Therefore $\mu^* \circ \pi^{-1} \ll \mu$, and due to $T \circ \pi = \pi \circ T^*$, this is a T-invariant measure. T being conservative ergodic, we therefore have $\mu^* \circ \pi^{-1} = c\, \mu$ for some $c \in [0, \infty]$, and it suffices to show that

$$\mu^*(\pi^{-1} Y) = \mu_Y(\mathfrak{W}) \cdot \mu(Y). \tag{3.14}$$

By (3.3), $W^*_{j,l} \cap \pi^{-1} Y = \varnothing$ unless $l = \varphi_i(W_j)$ for some $i \in \{0, 1, \ldots, \tau_j - 1\}$, in which case $W^*_{j,l} \subseteq \pi^{-1} Y$. In view of (3.9) and (3.11) we therefore see that indeed

$$\mu^*(\pi^{-1} Y) = \sum_{j \geq 1} \sum_{i=0}^{\tau_j - 1} \mu^*(W^*_{j, \varphi_i(W_j)}) = \sum_{j \geq 1} \tau_j\, \mu(w_j) = \mu_Y(\mathfrak{W}) \cdot \mu(Y).$$

Using (3.13), and recalling what has been said about φ^* before, the return probabilities of Y^* turn out to be

$$\begin{aligned}
f^*_k(Y^*) &= \mu^*_{Y^*}(Y^* \cap \{\varphi^* = k\}) \\
&= \frac{1}{\mu^*(Y^*)} \cdot \mu^* \left(\bigcup_{j \geq 1 : \varphi^*(j) = k} W^*_{j,0} \right) \\
&= \frac{1}{\mu(Y)} \cdot \sum_{j \geq 1 : \varphi^*(j) = k} \mu\left(w_j\right) = \mathfrak{f}_k(\mathfrak{W})
\end{aligned}$$

for $k \geq 1$, as required. This completes the proof of the theorem. □

4. Regenerative partitions of unity for Rényi-systems

The existence of regenerative partitions of unity for probability preserving Rényi-maps $(Y, \mathcal{B}, \nu, S, \eta)$, subordinate to η, will be derived from a variant of the coupling argument which in [Z3] has been used to give an easy proof of exponential uniform convergence $\widehat{S}^n\, u \to 1_Y$ for admissible probability densities u.

Proof of Theorem 2. **(i)** First, some preparations. It suffices to prove the theorem for some fixed iterate $(Y, \mathcal{B}, \nu, S^M, \eta_M)$ of our system: Uniform expansion and Adler's condition imply (by standard calculations) that there is some $A_\infty(S) < \infty$ such that $\sup_{n\geq 1} \sup_{W\in\eta_n} R(v'_W) \leq A_\infty(S)$, where R is regularity. Therefore, $\rho(S^{jN})(1 + A(S^{jN})) \leq \rho(S^N)^j(1 + A_\infty(S)) \to 0$ as $j \to \infty$, and we may (replacing S by S^M for $M = jN$ sufficiently large if necessary) assume w.l.o.g. that $\rho_0 := \rho(1 + A) < 1$.

The powers $\widehat{S}^m$, $m \geq 1$, of the transfer operator $\widehat{S}$ are more explicitly given by

$$\widehat{S}^m\, w = \sum_{W\in\eta_m} \widehat{S}^m\, (1_W\, w) = \sum_{W\in\eta_m} (w \circ v_W) \cdot v'_W,$$

where all v'_W are admissible and hence positive. A *partition* of a measurable function $u \geq 0$ on Y will mean a representation as a countable sum $u = \sum_i u_i$ with $u_i \geq 0$ and measurable. Let $w \geq 0$ be a measurable function on $W \in \eta_m$. Then any partition $\widehat{S}^m\, w = \sum_i u_i$, can be *pulled back* to W, i.e. there is a partition $w = \sum_i w_i$ such that $\widehat{S}^m\, w_i = u_i$. Simply take $w_i := (u_i/v'_W) \circ S^m$ on W.

The same calculation as in the proof of Lemma 2 of [Z3] shows that for any admissible u on Y and any $Z \in \eta$, $\widehat{S}\,(1_Z u)$ is admissible with regularity satisfying a Doeblin-Fortet type inequality,

$$R\left(\widehat{S}\,(1_Z u)\right) \leq \rho_0 R(u) + A \leq \max(R(u), A_0), \tag{4.1}$$

where $A_0 := A/(1 - \rho_0)$. For $p \in (0, 1)$ and $\widetilde{u}$ admissible with $p(1 + R\,(\widetilde{u})) < 1$, Lemma 3 of [Z3] ensures that $w := \widetilde{u} - p\,\nu(\widetilde{u}) \cdot 1_Y > 0$ is admissible with

$$R\,(w) \leq R\,(\widetilde{u}) + \frac{p(1 + R\,(\widetilde{u}))^2}{1 - p(1 + R\,(\widetilde{u}))}. \tag{4.2}$$

To obtain w from $\widetilde{u}$ means to remove the proportion p of its mass by subtracting the appropriate multiple of 1_Y, hence $p\,\nu(\widetilde{u}) \cdot 1_Y$, interpreted as a part of $\widetilde{u}$, is said to be *coupled* with 1_Y. Obviously, $\nu(w) = (1 - p)\nu(\widetilde{u})$.

Now let $r_0 := 2A_0$, and henceforth fix $p_0 \in (0, 1)$ so small that

$$p_0(1 + r_0)^2 < 1 \quad \text{and} \quad 2r_0 \cdot \frac{p_0(1 + r_0)^2}{1 - p_0(1 + r_0)^2} \leq (1 - \rho_0)(r_0 - A_0).$$

A calculation based on (4.1) and (4.2) shows that

$$\begin{aligned} &\text{for } p \in (0, p_0) \text{ and } Z \in \eta, \text{ if } u \text{ is admissible with } R(u) \leq r_0, \\ &\quad \text{then so is } w := \widehat{S}\,(1_Z u) - p\,\nu(1_Z u) \cdot 1_Y, \text{ with } R(w) \leq r_0, \end{aligned} \tag{4.3}$$

compare [Z3]. This will enable us to carry on coupling a definite proportion of mass along individual cylinders of S.

(ii) We can now describe the construction of $\mathfrak{W} = \{w_W : k \geq 1, W \in \eta_k\}$, by induction on k. Put $\eta_0 := \{Y\}$, so that $\eta_{k+1} = \{W \cap S^{-k} Z : W \in \eta_k,\, Z \in \eta\}$ for $k \geq 0$, and let $u_Y := \widetilde{w}_Y := 1_Y$ (with $R(u_Y) = 0 \leq r_0$), as well as $w_Y := 0$.

Suppose that, for some $K \geq 0$, all $k \in \{0, \dots, K\}$ and $W \in \eta_k$, we have constructed non-negative measurable functions w_W, and $\widetilde{w}_W$ on W such that

$$\sum_{k=1}^{K} \sum_{W \in \eta_k} w_W + \sum_{W \in \eta_K} \widetilde{w}_W = 1_Y \tag{4.4}$$

with

$$w_W, \widetilde{w}_W \leq 1_W \quad \text{for } 1 \leq k \leq K \text{ and } W \in \eta_k, \tag{4.5}$$

satisfying

$$\widehat{S}^k\, w_W = \nu(w_W) \cdot 1_Y \quad \text{for } 1 \leq k \leq K \text{ and } W \in \eta_k, \tag{4.6}$$

and

$$\nu\left(\sum_{W \in \eta_K} \widetilde{w}_W\right) \leq (1 - p_0)^K, \tag{4.7}$$

where all

$$u_W := \widehat{S}^K\, \widetilde{w}_W,\ W \in \eta_K, \text{ are admissible with } R(u_W) \leq r_0. \tag{4.8}$$

Below we show how to obtain, for all $V \in \eta_{K+1}$, functions w_V and $\widetilde{w}_V$ such that (4.4) to (4.8) are still satisfied with K replaced by $K+1$. It is then clear that this procedure yields a regenerative partition $\mathfrak{W} = \{w_W : k \geq 1, W \in \eta_k\}$ of 1_Y subordinate to η, with $\mathfrak{d}_k(\mathfrak{W})$ exponentially small, as required.

(iii) Pulling back the partition

$$\widehat{S}^K\, \widetilde{w}_W = \textstyle\sum_{Z \in \eta} 1_Z\, u_W,$$

to $W \in \eta_K$, we obtain a partition

$$\widetilde{w}_W = \textstyle\sum_{Z \in \eta} \overline{w}_{W,Z}, \tag{4.9}$$

with

$$\widehat{S}^K\, \overline{w}_{W,Z} = 1_Z\, u_W, \quad Z \in \eta, \tag{4.10}$$

which entails $\overline{w}_{W,Z} \leq S^{-K}Z$, so that in view of $\overline{w}_{W,Z} \leq \widetilde{w}_W \leq 1_W$, we have

$$0 \leq \overline{w}_{W,Z} \leq 1_{W \cap S^{-K}Z}. \tag{4.11}$$

For $Z \in \eta$, we define functions $u_{W \cap S^{-K}Z}$ via

$$\widehat{S}\ (1_Z\, u_W) = u_{W \cap S^{-K}Z} + p_0\, \nu(1_Z\, u_W) \cdot 1_Y, \tag{4.12}$$

which, according to (4.3), gives admissible functions with $R(u_{W \cap S^{-K}Z}) \leq r_0$ and $\nu(u_{W \cap S^{-K}Z}) = (1 - p_0)\, \nu(1_Z\, u_W)$. Due to (4.10) we can interpret (4.12) as a partition of $\widehat{S}^{K+1}\, \overline{w}_{W,Z}$, and pulling the latter back to $W \cap S^{-K}Z$ (recall (4.11)), we obtain a partition

$$\overline{w}_{W,Z} = \widetilde{w}_{W \cap S^{-K}Z} + w_{W \cap S^{-K}Z},$$

(hence (4.5) for $K+1$) such that

$$\begin{aligned} \widehat{S}^{K+1}\, \widetilde{w}_{W \cap S^{-K}Z} &= u_{W \cap S^{-K}Z}, \quad \text{and} \\ \widehat{S}^{K+1}\, w_{W \cap S^{-K}Z} &= p_0\, \nu(1_Z\, u_W) \cdot 1_Y, \end{aligned} \tag{4.13}$$

ensuring (4.6) and (4.8) for $K+1$. Therefore (cf. (4.9)) we end up with

$$\widetilde{w}_W = \sum_{Z\in\eta} \widetilde{w}_{W\cap S^{-K}Z} + \sum_{Z\in\eta} w_{W\cap S^{-K}Z}, \tag{4.14}$$

showing that (4.4) carries over to $K+1$. The same is true for (4.7), since

$$\begin{aligned}\nu\left(\sum_{V\in\eta_{K+1}} \widetilde{w}_V\right) &= \sum_{W\in\eta_K, Z\in\eta} (1-p_0)\,\nu(1_Z\, u_W) \\ &= (1-p_0)\sum_{W\in\eta_K} \nu\,(u_W) = (1-p_0)\,\nu\left(\sum_{W\in\eta_K} \widetilde{w}_W\right).\end{aligned}$$

This completes the inductive step and hence the proof of the theorem. □

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