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WAITING FOR LONG EXCURSIONS AND CLOSE VISITS TO NEUTRAL FIXED POINTS OF NULL-RECURRENT ERGODIC MAPS

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ROLAND ZWEIMÜLLER

Abstract. We determine, for certain ergodic infinite measure preserving transformations T, the asymptotic behaviour of the distribution of the waiting time for an excursion (from some fixed reference set of finite measure) of length larger than l as $l \to \infty$, generalizing a renewal-theoretic result of Lamperti. This abstract distributional limit theorem applies to certain weakly expanding interval maps, where it clarifies the distributional behaviour of hitting-times of shrinking neighbourhoods of neutral fixed points.

1. Introduction

The study of fine probabilistic properties of weakly dependent stochastic processes obtained from ergodic dynamical systems has become a very active field of research. Given a *conservative* (i.e. recurrent) *ergodic measure preserving transformation* (*c.e.m.p.t.*) T on a σ-finite measure space $(X, \mathcal{A}, \mu)$, and an *initial distribution* $\nu \ll \mu$, i.e. a probability measure according to which the initial state $\mathsf{X}_0 \in X$ of the dynamical system is chosen, iteration of T generates the consecutive states of the system, which form a sequence $(\mathsf{X}_n)_{n\geq 0} = (T^n \mathsf{X}_0)_{n\geq 0}$ of random elements of X, defined on the probability space $(X, \mathcal{A}, \nu)$.

One circle of questions which has recently attracted a lot of attention concerns the behaviour of *hitting times* of subsets of X. For $A \in \mathcal{A}$, $\mu(A) > 0$, we let $\varphi_A(x) := \inf\{n \geq 1 : T^n x \in A\}$, $x \in X$, which is finite mod μ. If $A_n \in \mathcal{A}$, $n \geq 1$, are sets of positive measure with $A_n \searrow \varnothing$, we can think of $(A_n)_{n\geq 1}$ as a sequence of *asymptotically rare events* and study, for some fixed ν, the distributions of the φ_{A_n} as $n \to \infty$. It has been shown that for a large variety of probability preserving (piecewise) smooth maps T with uniform or well-controlled weak hyperbolicity, and natural A_n, these *hitting-time distributions* do converge to the expected limit, that is, to an exponential distribution. (And in fact the hitting-time processes often tend to a Poisson process.) Relevant references include [GS], [CC], [AG], and [KL], but this list is far from exhaustive.

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Some prominent families of transformations, parametrized according to the precise degree of weak hyberbolicity, exhibit a dramatic change of stochastic behaviour when we pass from the domain of invariant probabilities (the positively recurrent situation) to the regime of conservative infinite invariant measures (the null-recurrent case) in parameter space. For a prototypical example, consider maps $T : [0,1] \to [0,1]$ which are piecewise $\mathcal{C}^2$ with two full branches and uniformly expanding except for an indifferent fixed point at $x = 0$, e.g.

$$Tx := \begin{cases} x + 2^p x^{p+1} & \text{for } x \in (0, 1/2), \\ 2x - 1 & \text{for } x \in (1/2, 1), \end{cases} \tag{1}$$

where $p > 0$ is the aforementioned parameter determining essential features of the processes $(\mathsf{X}_n)_{n\geq 0}$ generated by T. These T always possess a unique (up to a constant factor) conservative ergodic (even exact) invariant measure $\mu \ll \lambda$, λ denoting Lebesgue measure. For $p < 1$ it is finite, thus leading to an interesting family of weakly hyperbolic probability preserving systems which has been the object of intense study, see, for example, [Yo], [Sa], or [Go]. For $p \geq 1$, however, the measure μ is infinite, and we enter the null-recurrent world of infinite ergodic theory. Here, too, maps like (1) constitute a basic class of well-studied examples, see e.g. [A0]-[A2], [T1]-[T4], or [Z1], [Z2]. While various basic results from standard (finite) ergodic theory cease to hold (most notably the pointwise ergodic theorem with constant normalizations, cf. Section 2.4 of [A0]), some properties of positively recurrent maps survive, in a weak sense, at the threshold parameter $p = 1$ where the measure "has just become infinite". For example, there is a weak law of large numbers for $p = 1$, but not for any $p > 1$, cf. [A1], [ATZ], [T3] and [TZ].

Another instance of a finite-measure result surviving the transition from $p < 1$ to $p = 1$ has been explored in [CGS], [CG] and [CI]: Consider, for T as in (1), the family of intervals $A_\epsilon := [0, \epsilon]$ containing the neutral point $x = 0$, which shrink to zero as $\epsilon \searrow 0$. While $\lambda(A_\epsilon) \to 0$, these sets can, for $p \geq 1$, no longer be regarded as asymptotically rare events in the sense of our dynamical system, since, on the contrary, $\mu(A_\epsilon) = \infty$ and $\mu(A_\epsilon^c) < \infty$ for all ϵ. (See [BZ] for really rare events.) Nevertheless, in the $p = 1$ boundary case, the hitting-time distributions to these sets converge, when suitably normalized, to an exponential law: According to Theorem 5 of [CG] or Theorem 3.3 of [CI], we have, writing $\tau_\epsilon := \varphi_{[0,\epsilon]}$ and $Y := (1/2, 1)$,

$$\frac{1}{\int_Y \tau_\epsilon \, d\mu_Y} \cdot \tau_\epsilon \stackrel{\nu}{\Longrightarrow} \mathcal{E} \quad \text{as } \epsilon \searrow 0, \tag{2}$$

for $\nu = \lambda$ or $\nu = \mu_Y$. Here $\mu_Y(M) := \mu(Y \cap M)/\mu(Y)$ is the conditional measure on Y, the symbol $\stackrel{\nu}{\Longrightarrow}$ indicates distributional convergence w.r.t. the initial distribution ν, and $\mathcal{E}$ denotes an exponentially distributed random variable, i.e. (2) means that for all $t > 0$, $\nu(\{(\int_Y \tau_\epsilon \, d\mu_Y)^{-1} \cdot \tau_\epsilon \leq t\}) \longrightarrow 1 - e^{-t}$ as $\epsilon \searrow 0$. (And it is not hard to see that the normalizing factor is of order $\epsilon \log \epsilon$ as $\epsilon \searrow 0$.) The usual exponential limit law for the hitting-time distributions thus persists at $p = 1$, illustrating once again the amazing robustness of this phenomenon.

To the best of my knowledge, no information for the case $p > 1$ of "seriously infinite" measures is available so far. The abstract distributional limit theorem of the present paper enables us to clarify the asymptotic behaviour of the hitting-time distributions of the sets $[0, \epsilon]$ in this case. We will, in particular, show that for T

as in (1), with $p > 1$,

$$p(2\epsilon)^p \cdot \tau_\epsilon \stackrel{\nu}{\Longrightarrow} \mathcal{J}_{1/p} \quad \text{as } \epsilon \searrow 0, \tag{3}$$

for any probability measure $\nu \ll \lambda$. Here we let $\mathcal{J}_\alpha$, $\alpha \in [0,1)$, denote random variables taking values in $[0,\infty)$, with distributions characterized by the following recursion formulae for their moments (where, by convention, $\mathbb{E}[\mathcal{J}_\alpha^0] := 1$)

$$\mathbb{E}[\mathcal{J}_\alpha^r] = \alpha \sum_{j=0}^{r-1} \binom{r}{j} \frac{\mathbb{E}[\mathcal{J}_\alpha^j]}{r-j-\alpha} \qquad \text{for } r \geq 1. \tag{4}$$

In particular, $\mathcal{J}_0 = 0$, and generally $\mathbb{E}[\mathcal{J}_\alpha] = \alpha/(1-\alpha)$ and $\mathrm{Var}[\mathcal{J}_\alpha] = \alpha/[(2-\alpha)(1-\alpha)^2]$. Regrettably, no explicit expression for the densities of these distributions is available, but partial information, stated in terms of $\mathcal{H}_\alpha := \mathcal{J}_\alpha + 1$, can be found in [La]. From the same paper one can also infer that the Laplace transforms are given by

$$\widehat{\mathcal{J}_\alpha}(s) := \mathbb{E}\left[e^{-s\mathcal{J}_\alpha}\right] = \frac{1}{e^{-s} + s\int_0^1 y^{-\alpha} e^{-sy} dy}, \quad s > 0. \tag{5}$$

We will approach the above question about close visits by slightly shifting our perspective. Instead of chasing small sets, we fix one good reference set Y of finite measure, disjoint from the target sets A, in such a way that hitting a small set A is equivalent to staying away from Y for a long time. This transforms our original question about hitting-times into one about asymptotic distributions of waiting-times for long excursions from Y. In Sections 2 and 3 to follow, we formulate and prove an abstract distributional limit theorem for such waiting times. In Section 4 we use this result to answer the hitting-time question for interval maps.

2. Long excursions from good reference sets

We recall some basic concepts: A function $a : (L,\infty) \to (0,\infty)$ is *regularly varying of index* $\rho \in \mathbb{R}$ at infinity, written $a \in \mathcal{R}_\rho$, if $a(ct)/a(t) \to c^\rho$ as $t \to \infty$ for any $c > 0$, and we shall interpret sequences $(a_n)_{n\geq 0}$ as functions on $\mathbb{R}_+$ via $t \longmapsto a_{[t]}$. *Slow variation* means regular variation of index 0. $\mathcal{R}_\rho(0)$ is the family of functions $r : (0,\delta) \to \mathbb{R}_+$ regularly varying of index ρ at zero (same condition as above, but for $t \searrow 0$). For background information we refer to Chapter 1 of [BGT]. Throughout we use the efficient convention that for $a_n, b_n \geq 0$ and $\vartheta \in [0,\infty)$, $a_n \sim \vartheta \cdot b_n$ as $n \to \infty$ means $\lim_{n\to\infty} a_n/b_n = \vartheta$, even in case $\vartheta = 0$. An analogous convention applies to $f(s) \sim \vartheta \cdot g(s)$ as $s \searrow 0$. We will repeatedly use Karamata's Tauberian theorem (KTT) for Laplace transforms and the Monotone Density theorem, in the versions provided by Proposition 3.2 and Lemma 3.1 of [TZ].

Strong distributional convergence $R_n \stackrel{\mathcal{L}(\mu)}{\Longrightarrow} R$ of a sequence $(R_n)_{n\geq 1}$ of real-valued measurable functions on the σ-finite space $(X,\mathcal{A},\mu)$ means distributional convergence $R_n \stackrel{\nu}{\Longrightarrow} R$ w.r.t. all probability measures $\nu \ll \mu$. Similarly, $R_n \xrightarrow{\mu} R$ means convergence in measure, $R_n \xrightarrow{\nu} R$, for all normalized $\nu \ll \mu$.

Let T be a c.e.m.p.t. on $(X,\mathcal{A},\mu)$. Its *transfer operator* $\widehat{T} : L_1(\mu) \to L_1(\mu)$ describes the evolution of probability densities under T, that is, $\widehat{T}u := d(\nu \circ T^{-1})/d\mu$, where ν has density u w.r.t. μ. Equivalently, $\int_X (g \circ T) \cdot u \, d\mu = \int_X g \cdot \widehat{T}u \, d\mu$ for all

$u \in L_1(\mu)$ and $g \in L_\infty(\mu)$. The operator $\widehat{T}$ naturally extends to $\{u : X \to [0,\infty)$ measurable $\mathcal{A}\}$.

For $Y \in \mathcal{A}$ with $\mu(Y) > 0$ the *first entrance time* or *hitting time* of Y is $\varphi_Y(x) := \min\{n \geq 1 : T^n x \in Y\}$, $x \in X$, and we define $T_Y x := T^{\varphi_Y(x)}x$, $x \in X$. The restricted measure $\mu\,|_{Y\cap\mathcal{A}}$ is invariant under the *first return map*, that is, T_Y restricted to Y. In other words, $1_Y = \sum_{k\geq 1}\widehat{T}^k 1_{Y\cap\{\varphi=k\}}$ a.e. If $\mu(Y) < \infty$, the *first return time*, i.e. φ_Y restricted to Y, can be regarded as a random variable on the probability space $(Y, Y\cap\mathcal{A}, \mu_Y)$. Assuming that Y is a suitable reference set (to be explained below), the asymptotic behaviour of its *return distribution*, i.e. that of the *(first) return probabilities* $f_k(Y) := \mu_Y(Y \cap \{\varphi_Y = k\})$, is an important characteristic governing the probabilistic properties of the system. For distributional limit theorems regular variation of the *tail probabilities* $q_n(Y) := \sum_{k>n} f_k(Y) = \mu_Y(Y \cap \{\varphi_Y > n\})$, or the *wandering rate* of Y given by $w_N(Y) := \mu(Y)\sum_{n=0}^{N-1} q_n(Y) = \mu(Y^N)$, where $Y^N := \bigcup_{n=0}^{N-1} T^{-n}Y$, $N \geq 1$, is essential. Note that $Y^N := \bigcup_{n=0}^{N-1} Y_n$ (disjoint), where (as in [TZ], [Z3]) we let

$$Y_0 := Y \quad \text{and} \quad Y_n := Y^c \cap \{\varphi_Y = n\},\ n \geq 1. \tag{6}$$

The following theorem is the abstract core of the present paper. It will be established via the renewal-theoretic approach developed in [T3], [TZ], and [Z3]. Condition (8), which formalizes what a good reference set is in this context, is a slightly stronger version of the basic condition used in [Z3]. Via (9) we also impose a variant of the sweeping condition used there (in the Darling-Kac theorem).

For $Y \in \mathcal{A}$ and $l \geq 1$, we let $\mathbf{J}_l(Y)(x)$, $x \in X$, denote the time at which the first excursion from Y of length larger or equal to l starts,

$$\mathbf{J}_l(Y)(x) := \inf\{n \geq 0 : T^n x \in Y^c \cap \{\varphi_Y \geq l\}\}. \tag{7}$$

The asymptotic distributional behaviour of these variables is explained in

Theorem 1 (Waiting for long excursions from sets with compact first returns). *Let T be a c.e.m.p.t. on the σ-finite measure space $(X, \mathcal{A}, \mu)$, and assume that $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, is such that*

$$\mathfrak{H}''_Y = \left\{\frac{1}{f_k(Y)}\widehat{T}^k 1_{Y\cap\{\varphi=k\}} : k \geq 1,\ f_k(Y) > 0\right\} \text{ is precompact in } L_\infty(\mu), \tag{8}$$

and

$$\text{there are } \iota, l_0 \geq 1 \text{ for which} \quad \inf_{l\geq l_0}\inf_Y \sum_{j=0}^{\iota-1}\widehat{T}^j\left(\frac{\widehat{T}^l 1_{Y_l}}{q_l(Y)}\right) > 0. \tag{9}$$

If

$$(w_N(Y)) \in \mathcal{R}_{1-\alpha} \quad \text{for some } \alpha \in [0,1), \tag{10}$$

then

$$\frac{1}{l}\,\mathbf{J}_l(Y) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{J}_\alpha. \tag{11}$$

Remark 1. The first time at which the orbit $(T^n x)_{n\geq 0}$ actually observes a long excursion is $\mathbf{H}_l(Y)(x) := \inf\{n \geq l-1 : T^j x \in Y^c \text{ for } j \in \{n-l+1,\dots,n\}\} = \mathbf{J}_l(Y)(x) + l - 1$. The conclusion (11) is equivalent to $\mathbf{H}_l(Y)/l \stackrel{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{H}_\alpha := \mathcal{J}_\alpha + 1$, which in [La] has been established for processes with an iid sequence of excursion lengths. (That is, in the special case in which the $\varphi \circ T_Y^j$, $j \geq 0$, are independent random variables on $(Y, Y \cap \mathcal{A}, \mu_Y)$.)

Remark 2. As in [TZ] and [Z3], regular variation of $(w_N(Y))$ is a property of the system $(X, \mathcal{A}, \mu, T)$ rather than a property of a particular set: By Proposition 3.2 and Remark 3.6 of [TZ], (8) implies that Y has *minimal wandering rate*, meaning that $\underline{\lim}_{N\to\infty} w_N(Z)/w_N(Y) \geq 1$ for all $Z \in \mathcal{A}$, $0 < \mu(Z) < \infty$. Such a minimal rate (if it exists) is an important asymptotic characteristic of the system, the *wandering rate of* T, denoted $(w_N(T))$.

Remark 3. For the main application worked out here, Theorem 2 below, a much simpler version of (9) suffices, namely

$$\inf_{k\geq 1, f_k(Y)>0} \inf_Y \left(\frac{1}{f_k(Y)} \widehat{T}^k 1_{Y\cap\{\varphi=k\}} \right). \tag{12}$$

However, we prove Theorem 1 under the more general condition (9), since this paves the way for applications to more complicated situations, cf. Remark 4.

3. Proof of Theorem 1

The argument to follow shows, in particular, that there are variables $\mathcal{J}_\alpha$ with moments given by (4). To begin with, we check that the distributions of the $\mathcal{J}_\alpha$, $\alpha \in [0,1)$, are in fact uniquely determined by these moments. According to a classical result of T. Carleman, it suffices to show that the series $\sum_{r\geq 1} \mathbb{E}[\mathcal{J}_\alpha^r]^{-1/2r}$ diverges. We show that

$$\mathbb{E}[\mathcal{J}_\alpha^r] \leq \left(\frac{r}{1-\alpha}\right)^r \quad \text{for } r \geq 0. \tag{13}$$

If $r = 0$, this is trivial. For the inductive step, fix some $r \geq 1$ and assume that (13) has been shown to hold up to $r-1$. Then use (4) to see that indeed

$$\begin{aligned} \mathbb{E}[\mathcal{J}_\alpha^r] &\leq \frac{\alpha}{1-\alpha} \sum_{j=0}^{r-1} \binom{r}{j} \left(\frac{j}{1-\alpha}\right)^j \\ &\leq \frac{\alpha}{(1-\alpha)^r} \sum_{j=0}^{r-1} \binom{r}{j} (r-1)^j \leq \frac{\alpha}{(1-\alpha)^r} ((r-1)+1)^r, \end{aligned}$$

proving (13) and hence the required divergence statement.

We now use a variant of the renewal-theoretic approach to distributional limit theorems for infinite measure preserving transformations developed in [T3], [TZ], and [Z3], to show that all moments converge. Our starting point is the following dissection identity for $J_l := \mathbf{J}_l(Y)$, $l \geq 1$, on the distinguished reference set Y,

$$J_l = \begin{cases} k + J_l \circ T^k & \text{on } Y \cap \{\varphi = k\},\ 1 \leq k \leq l, \\ 1 & \text{on } Y \cap \{\varphi > l\}. \end{cases} \tag{14}$$

This results in

Lemma 1 (Splitting moments at the first return). *Let T be a c.e.m.p.t. on the σ-finite measure space $(X, \mathcal{A}, \mu)$, let $Y \in \mathcal{A}$ with $\varphi := \varphi_Y$, and $J_l := \mathbf{J}_l(Y)$. Then, for $r \geq 1$, we have*

$$\int_Y \widehat{T}^l 1_{Y_l} \cdot J_l^r \, d\mu = \sum_{j=0}^{r-1} \binom{r}{j} \int_Y \left(\sum_{k=1}^{l} k^{r-j} \, \widehat{T}^k 1_{Y \cap \{\varphi = k\}} \right) \cdot J_l^j \, d\mu + \mu(Y \cap \{\varphi > l\}).$$

Proof. According to (14), we see that

$$\begin{aligned} \int_Y J_l^r \, d\mu &= \sum_{k=1}^{l} \int_{Y \cap \{\varphi = k\}} (k + J_l)^r \circ T^k \, d\mu + \int_{Y \cap \{\varphi > l\}} 1 \, d\mu \\ &= \sum_{k=1}^{l} \int_Y \widehat{T}^k 1_{Y \cap \{\varphi = k\}} \cdot (k + J_l)^r \, d\mu + \mu(Y \cap \{\varphi > l\}) \\ &= \sum_{j=0}^{r} \binom{r}{j} \sum_{k=1}^{l} k^{r-j} \int_Y \widehat{T}^k 1_{Y \cap \{\varphi = k\}} \cdot J_l^j \, d\mu + \mu(Y \cap \{\varphi > l\}). \end{aligned}$$

Separating the $j = r$ term on the right-hand side and using $1_Y = \sum_{k \geq 1} \widehat{T}^k 1_{Y \cap \{\varphi = k\}}$, we obtain

$$\begin{aligned} \int_Y \left(\sum_{k > l} \widehat{T}^k 1_{Y \cap \{\varphi = k\}} \right) \cdot J_l^r \, d\mu = \sum_{j=0}^{r-1} \binom{r}{j} \int_Y \left(\sum_{k=1}^{l} k^{r-j} \widehat{T}^k 1_{Y \cap \{\varphi = k\}} \right) \cdot J_l^j \, d\mu \\ + \mu(Y \cap \{\varphi > l\}), \end{aligned}$$

which, due to $\widehat{T}^l 1_{Y_l} = \sum_{k > l} \widehat{T}^k 1_{Y \cap \{\varphi = k\}}$, $l \geq 0$, (cf. (2.3) of [TZ]) is what we claimed. □

We can now put the machinery of [TZ] and [Z3] to work.

Proof of Theorem 1. **(i)** Assume w.l.o.g. that $\mu(Y) = 1$, and let $J_l := \mathbf{J}_l(Y)$, $l \geq 1$, and $\varphi := \varphi_Y$. We observe first that the sequence $(J_l/l)_{l \geq 1}$ is *asymptotically T-invariant in measure*, in the sense that

$$\frac{J_l \circ T - J_l}{l} \xrightarrow{\mu} 0 \quad \text{as } l \to \infty. \tag{15}$$

This follows from

$$\{|J_l \circ T - J_l| > 1\} = Y^c \cap \{\varphi = l\} \quad \text{for } l \geq 2, \tag{16}$$

since we have $\nu(\{\varphi = l\}) \to 0$ as $l \to \infty$ for every probability measure $\nu \ll \mu$, as φ is finite a.e. Due to (15), strong distributional convergence (11) is automatic once we prove that

$$\frac{1}{l} J_l \overset{\mu_Y}{\Longrightarrow} \mathcal{J}_\alpha, \tag{17}$$

cf. Proposition 4.1 of [TZ]. Having confirmed that the distributions of the $\mathcal{J}_\alpha$ are determined by their moments, we may verify (17) by showing that for all $r \geq 0$,

$$\int_Y \left(\frac{J_l}{l} \right)^r d\mu \longrightarrow \mathbb{E}[\mathcal{J}_\alpha^r] \quad \text{as } l \to \infty. \tag{$\blacklozenge_r$}$$

The $r = 0$ case is trivial: by our conventions, $\int_Y (J_l/l)^0 \, d\mu = 1$ for all $l \geq 1$.

(ii) By KTT (cf. Proposition 4.2 of [TZ]), $(w_N(Y))_{N\geq 1} \in \mathcal{R}_{1-\alpha}$ means that there is some $\ell \in \mathcal{R}_0$ such that

$$Q_Y(s) := \sum_{l\geq 0} q_l(Y)\, e^{-ls} = \left(\frac{1}{s}\right)^{1-\alpha} \ell\left(\frac{1}{s}\right) \quad \text{for } s > 0.$$

Since $\alpha < 1$, we can also apply the monotone density theorem to see that the non-increasing sequence $(q_l(Y))_{l\geq 0}$ satisfies

$$q_l(Y) \sim \frac{1-\alpha}{\Gamma(2-\alpha)} \cdot l^{-\alpha}\ell(l) \quad \text{as } l \to \infty. \tag{18}$$

Using the differentiation lemma for regularly varying functions (specifically, part b) of Lemma 4.1 of [TZ]), we can also conclude that

$$Q_Y^{(r)}(s) \sim r!\binom{\alpha-1}{r}\left(\frac{1}{s}\right)^{r+1-\alpha} \ell\left(\frac{1}{s}\right) \qquad \begin{array}{l}\text{as } s \searrow 0\\ \text{for all } r \geq 0.\end{array}$$

Letting $F_Y(s) := \sum_{k\geq 1} f_k(Y)\, e^{-ks}$, which satisfies $1 - F_Y(s) = (1-e^{-s})Q_Y(s)$, $s > 0$, we furthermore obtain

$$\begin{aligned} -F_Y^{(m)}(s) &= (-1)^{m+1} \sum_{k\geq 1} k^m f_k(Y)\, e^{-ks} \\ &\sim m!\binom{\alpha}{m}\left(\frac{1}{s}\right)^{m-\alpha} \ell\left(\frac{1}{s}\right) \qquad \begin{array}{l}\text{as } s \searrow 0\\ \text{for all } m \geq 1.\end{array} \end{aligned}$$

Hence, appealing to KTT once again, we get

$$\sum_{k=1}^{l} k^m f_k(Y) \sim \frac{\alpha(1-\alpha)}{(m-\alpha)\,\Gamma(2-\alpha)} \cdot l^{m-\alpha}\ell(l) \qquad \begin{array}{l}\text{as } l \to \infty\\ \text{for all } m \geq 1.\end{array} \tag{19}$$

(iii) Next we establish, by induction on r, the statement that for all $r \geq 0$,

$$\int_Y J_l^r \, d\mu = O(l^r) \quad \text{as } l \to \infty. \tag{$\Diamond_r$}$$

For $r = 0$ this is trivial. For the inductive step we assume that $(\Diamond_j)$ holds for $0 \leq j < r$, where $r \geq 1$ is fixed. Consider the terms on the right-hand side of Lemma 1: For each j we have

$$\int_Y \left(\sum_{k=1}^{l} k^{r-j}\, \widehat{T}^k 1_{Y\cap\{\varphi=k\}}\right) \cdot J_l^j \, d\mu = \left(\sum_{k=1}^{l} k^{r-j} f_k(Y)\right) \cdot \int_Y J_l^j \cdot u_{r-j,l}\, d\mu, \tag{20}$$

where, for $l \geq l_0 := \min\{k \geq 1 : f_k(Y) > 0\}$,

$$u_{m,l} := \frac{\sum_{k=1}^{l} k^m\, \widehat{T}^k 1_{Y\cap\{\varphi=k\}}}{\sum_{k=1}^{l} k^m f_k(Y)} \in \overline{\text{co}}(\mathfrak{H}_Y''), \tag{21}$$

the closed convex hull of $\mathfrak{H}''_Y$ in $L_\infty(\mu)$, which is compact and, in particular, bounded. Combining this with (18), (19) and $(\Diamond_j)$, we see for the complete right-hand side in Lemma 1 that

$$\sum_{j=0}^{r-1} \binom{r}{j} \int_Y \left(\sum_{k=1}^{l} k^{r-j}\, \widehat{T}^k 1_{Y\cap\{\varphi=k\}} \right) \cdot J_l^j \, d\mu + \mu(Y \cap \{\varphi > l\}) = O\left(l^{r-\alpha}\ell(l)\right) \tag{22}$$

as $l \to \infty$. On the other hand we have

$$J_l \circ T^j = J_l - j \le J_l \quad \text{ on } Y \cap T^{-j}Y \text{ for } l \ge j,$$

and hence

$$\int_Y \widehat{T}^{l+j} 1_{Y_l} \cdot J_l^r \, d\mu = \int_{Y\cap T^{-j}Y} \widehat{T}^l 1_{Y_l} \cdot (J_l^r \circ T^j)\, d\mu \le \int_Y \widehat{T}^l 1_{Y_l} \cdot J_l^r \, d\mu \quad \text{ for } l \ge j.$$

Letting $v_l := q_l(Y)^{-1}\widehat{T}^l 1_{Y_l} \in \overline{\mathrm{co}}(\mathfrak{H}''_Y)$, we therefore see, due to (9) and (18), that

$$\begin{aligned} \int_Y J_l^r \, d\mu &= O\left(\int_Y J_l^r \cdot \sum_{j=0}^{\iota-1} \widehat{T}^j v_l \, d\mu \right) \\ &= O\left(\int_Y J_l^r \cdot v_l \, d\mu \right) = O\left(\frac{\int_Y \widehat{T}^l 1_{Y_l} \cdot J_l^r \, d\mu}{l^{-\alpha}\ell(l)} \right) \quad \text{ as } l \to \infty. \end{aligned}$$

Using Lemma 1 we can combine this with (22) to obtain $(\Diamond_r)$.

(iv) We need some information on the behaviour of the J_l outside Y. Generally,

$$J_l \le n + J_l \circ T^n \quad \text{ on } X \quad \text{ for } n \ge 0,\, l \ge 1. \tag{23}$$

Recalling the notation $Y^M = \bigcup_{n=0}^{M-1} Y_n$, we claim that for every $r \ge 0$ and $M \ge 1$,

$$\left\{ 1_{Y^M} \cdot \left(\frac{J_l}{l} \right)^r : l \ge 1 \right\} \quad \text{ is uniformly integrable.} \tag{24}$$

In case $M = 1$ this is immediate from the $(\Diamond_{r+1})$, $r \ge 0$. Now fix M and r, and let $R_l := (J_l/l)^r$. For $l \ge M > n$ we see, using $\widehat{T}^n 1_{Y_n} \le 1$, and (23) plus its consequence

$$\begin{aligned} Y_n \cap \{R_l > K\} &\subseteq Y_n \cap T^{-n}\{(J_l/l + 1)^r > K\} \\ &\subseteq Y_n \cap T^{-n}\{R_l > 2^{-r}K - 1\} \quad \text{ for } K > 0, \end{aligned}$$

that

$$\begin{aligned} \int_{Y_n\cap\{R_l>K\}} R_l \, d\mu &\le \int_{Y_n\cap\{R_l>K\}} \frac{(n + J_l \circ T^n)^r}{l^r} \, d\mu \\ &\le \int_{Y\cap\{R_l>2^{-r}K-1\}} \frac{(M + J_l)^r}{l^r} \, d\mu \le 2^r \int_{Y\cap\{R_l>2^{-r}K-1\}} (1 + R_l)\, d\mu. \end{aligned}$$

For fixed n, the rightmost integral tends to 0 as $K \to \infty$, uniformly in l, since $\{1_Y R_l : l \ge 1\}$ is uniformly integrable. Taking the union over $n \in \{0, \dots, M-1\}$, we obtain (24).

Similarly, it is not hard to check that for every $r \geq 1$, and any bounded probability density u supported on Y^M for some $M = M(u)$, we have

$$\left\| \left(\left(\frac{J_l}{l} \right)^r \circ T - \left(\frac{J_l}{l} \right)^r \right) \cdot u \right\|_1 \longrightarrow 0 \quad \text{as } l \to \infty. \tag{25}$$

(Note that, by (16), $Y^M \subseteq \{|J_l \circ T - J_l| \leq 1\}$ for $l > M$, then use the mean-value theorem.)

(v) We are now ready for the inductive step in the proof of $(\blacklozenge_r)$. The crude information given by the $(\diamondsuit_r)$, i.e. boundedness of all moment sequences $(\int_Y (J_l/l)^r \, d\mu)_{l \geq 1}$, $r \geq 0$, enables us to refine the previous argument. We claim that for all $r > j \geq 0$,

$$\int_Y J_l^j \cdot u_{r-j,l} \, d\mu \sim \int_Y J_l^j \, d\mu \quad \text{as } l \to \infty, \tag{26}$$

and that for all $r \geq 0$,

$$\int_Y J_l^r \cdot v_l \, d\mu \sim \int_Y J_l^r \, d\mu \quad \text{as } l \to \infty.$$

To see this, we can appeal to parts a) and c) of Proposition 3.2 of [Z3], with $R_l := (J_l/l)^j$ and $R_l := (J_l/l)^r$, respectively: Although condition (3.10) there is not satisfied in the present situation, we may replace it by (24) above, since the role of condition (3.10) in Proposition 3.2 of [Z3] was exactly to ensure this property, see equation (3.16) there. This proves (26).

Now fix $r \geq 1$ and assume $(\blacklozenge_j)$ for $0 \leq j < r$. Recalling the representation (20) and using (18), (19), and (26), we find for the complete right-hand side of Lemma 1 that

$$\begin{aligned} &\sum_{j=0}^{r-1} \binom{r}{j} \int_Y \left(\sum_{k=1}^{l} k^{r-j} \, \widehat{T}^k 1_{Y \cap \{\varphi = k\}} \right) \cdot J_l^j \, d\mu + \mu(Y \cap \{\varphi > l\}) \\ &\sim \frac{1-\alpha}{\Gamma(2-\alpha)} \cdot \left(\alpha \sum_{j=0}^{r-1} \binom{r}{j} \frac{\mathbb{E}[\mathcal{J}_\alpha^j]}{r-j-\alpha} \right) \cdot l^{r-\alpha} \ell(l) \quad \text{as } l \to \infty. \end{aligned} \tag{27}$$

Likewise, the left-hand side of Lemma 1 is now seen to satisfy

$$\begin{aligned} &\int_Y \widehat{T}^l 1_{Y_l} \cdot J_l^r \, d\mu = q_l(Y) \int_Y J_l^r \cdot v_l \, d\mu \\ &\sim q_l(Y) \int_Y J_l^r \, d\mu \sim \frac{1-\alpha}{\Gamma(2-\alpha)} \cdot l^{r-\alpha} \ell(l) \int_Y \left(\frac{J_l}{l} \right)^r \, d\mu \quad \text{as } l \to \infty. \end{aligned} \tag{28}$$

Combining (27) and (28) yields $(\blacklozenge_r)$. □

4. Close visits to indifferent fixed points

We turn to our limit theorem for the distributions of waiting-times for close visits to indifferent fixed points of infinite measure preserving interval maps. To avoid undue technicalities we focus on prototypical maps T on $X := [0, 1]$ with two full branches and one indifferent fixed point at $x = 0$. Henceforth we assume that

- **(a)** for some $c \in (0,1)$ the restrictions of T to $Z_0 := (0,c)$ and $Z_1 := (c,1)$ are increasing $\mathcal{C}^2$-diffeomorphisms onto $(0,1)$ with inverses v_0 and v_1, and $T\,|_{Z_i}$ extends to a $\mathcal{C}^2$-map on $\mathrm{cl}(Z_i)$;
- **(b)** the map T is expanding except for an *indifferent fixed point* at $x = 0$, i.e. for any $\varepsilon > 0$, $|\,T'\,| \geq \rho(\varepsilon) > 1$ on $[\varepsilon, 1]$, while $T(0) = 0$ and $\lim_{x \searrow 0} T'x = 1$; moreover this fixed point is a *regular source*, i.e. T' is increasing on $(0, \delta_0)$ for some $\delta_0 > 0$.

The family of maps T satisfying **(a)-(b)** will be denoted by $\mathcal{T}$. It is well known (cf. [T1]) that any map $T \in \mathcal{T}$ is conservative and exact (hence also ergodic) w.r.t. Lebesgue measure λ, and preserves a σ-finite infinite measure $\mu \ll \lambda$ (unique up to a multiplicative constant) with a positive density h which is continuous on $(0,1]$. Let $r_T(x) := x - v_0(x)$, $x \in [0,1]$. We are going to prove

Theorem 2 (**Asymptotic hitting-time distribution for neighbourhoods of the neutral point**). *Assume that $T \in \mathcal{T}$ satisfies $r_T \in \mathcal{R}_{1+p}(0)$ for some $p \in (1,\infty]$, and let $\alpha := 1/p \in [0,1)$. Then the hitting-times of the sets $[0,\epsilon]$, $\epsilon \in (0,1]$,*

$$\tau_\epsilon(x) := \inf\{n \geq 1 : T^n x \in [0,\epsilon]\},$$

converge in distribution,

$$\frac{1}{I_T(\epsilon)} \cdot \tau_\epsilon \stackrel{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{J}_\alpha \quad \text{as } \epsilon \searrow 0, \tag{29}$$

where $I_T \in \mathcal{R}_{-1/\alpha}(0)$ is given by

$$I_T(\epsilon) := \int_\epsilon^1 \frac{dx}{r_T(x)}, \quad \epsilon \in (0,1].$$

Example 1 (**The standard examples of indifferent fixed points**). *In the frequently studied situation with $Tx = x + ax^{1+p} + o(x^{1+p})$ as $x \searrow 0$, one finds that $I_T(\epsilon) \sim [ap\epsilon^p]^{-1}$ as $\epsilon \searrow 0$, explaining (3) above.*

We show how the abstract Theorem 1 implies this assertion.

Proof of Theorem 2. **(i)** The obvious natural reference set for T is $Y := (c,1]$. The well-known fact that the induced map T_Y is uniformly expanding with full branches and satisfies the (folklore) Adler condition $\sup \left|T''/(T')^2\right| < \infty$ means, in particular, very good distortion control in that the derivatives $w = v'$ of its inverse branches v of arbitrary order have uniformly bounded *regularity* $R(w) := \sup_Y(|w'|/w)$. Moreover, the invariant measure μ_Y of T_Y has a density of finite regularity. As a consequence, the family $\mathfrak{H}''_Y$ of probability densities is uniformly bounded away from zero, and also equicontinuous, hence precompact in $L_\infty(\mu)$ by the Arzela-Ascoli theorem.

Lemma 4 of [T2] shows that $r_T \in \mathcal{R}_{1+p}(0)$ implies $(w_N(Y)) \in \mathcal{R}_{1-\alpha}$. (In fact, these statements are equivalent if $p < \infty$.) Therefore, the assumptions of Theorem 1 are fulfilled, and we conclude that

$$\frac{1}{l}\,\mathbf{J}_l(Y) \stackrel{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{J}_\alpha \quad \text{as } l \to \infty. \tag{30}$$

(ii) Starting from $c_0 := 1$ define $c_l := v_0^l(c_0)$, $l \geq 0$ (so that $c_1 = c$), and observe that

$$(c_{l+1}, c_l] = Y^c \cap \{\varphi_Y = l\} \quad \text{for } l \geq 1.$$

Define $L : (0, c] \to \mathbb{N}$ by requiring that $c_{L(\epsilon)+1} < \epsilon \leq c_{L(\epsilon)}$. Due to the obvious inclusions between the sets involved we then see that

$$\mathbf{J}_{L(\epsilon)}(Y) \leq \tau_\epsilon \leq \mathbf{J}_{L(\epsilon)+1}(Y) \quad \text{for } \epsilon \in (0, c].$$

Hence (30) implies

$$\frac{1}{L(\epsilon)} \cdot \tau_\epsilon \stackrel{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{J}_\alpha \quad \text{as } \epsilon \searrow 0.$$

To finally obtain (29), note first that (by the monotone density theorem and Lemma 4 of [T2]) $(c_l) \in \mathcal{R}_{-\alpha}$. Together with Lemma 2 of [T1] this shows that I_T is the asymptotic inverse to $(c_l)_{l \geq 0}$ (unique up to asymptotic equivalence), and hence that $L(\epsilon) \sim I_T(\epsilon)$ as $\epsilon \searrow 0$. □

Remark 4. The interval maps above have the special property (12). Due to the more flexible assumption (9) given in Theorem 1, the same argument applies to the significantly more general family of those (not necessarily markovian) *AFN-maps T* (as studied in [Z1], [Z2]) which have the same asymptotic behaviour at all of their indifferent fixed points. (Condition (9) follows as in Theorem 8.1 of [TZ].)

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Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, 1090 Wien, Austria
E-mail address: roland.zweimueller@univie.ac.at