Infinite measure preserving transformations with compact first regeneration

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Roland Zweimüller

Abstract. We study ergodic infinite measure preserving transformations T possessing reference sets of finite measure for which the set of densities of the conditional distributions given a first return (or entrance) at time n is precompact in a suitable function space. Assuming regular variation of wandering rates, we establish versions of the Darling-Kac theorem and the arcsine laws for waiting times and for occupation times, which apply to transformations with indifferent orbits and to random walks driven by Gibbs-Markov maps.

1. Introduction

The present paper is devoted to stochastic properties of transformations with an *infinite invariant measure*. We establish distributional limit theorems generalizing classical results about null-recurrent Markov chains to the weakly dependent processes generated by these dynamical systems. Specifically, we improve the abstract distributional limit theorems presented in [**TZ**] (a Darling-Kac type result and two arcsine laws), significantly weakening the assumptions, and present a new limit theorem related to them. The conditions we give enable us to cover new classes of examples with this renewal-theoretic approach.

Understanding probabilistic properties of a *measure preserving transformation (m.p.t.)* T on a σ-finite measure space $(X, \mathcal{A}, \mu)$ often depends on information about the long-term behaviour of its *transfer operator* $\widehat{T} : L_1(\mu) \to L_1(\mu)$ describing the evolution of probability densities under T, that is, $\widehat{T}u := d(\nu \circ T^{-1})/d\mu$, where ν has density u w.r.t. μ. Equivalently, $\int_X (g \circ T) \cdot u \, d\mu = \int_X g \cdot \widehat{T}u \, d\mu$ for all $u \in L_1(\mu)$ and $g \in L_\infty(\mu)$, i.e. $\widehat{T}$ is dual to $g \longmapsto g \circ T$. The operator $\widehat{T}$ naturally

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extends to $\{u : X \to [0,\infty) \text{ measurable } \mathcal{A}\}$, invariance of μ means $\widehat{T}1 = 1$, and T is *conservative ergodic (c.e.)* iff $\sum_{k\geq 0} \widehat{T}^k u = \infty$ a.e. for all $u \in L_1(\mu)$ with $\int u\, d\mu > 0$.

The probability density $\widehat{T}^n u$ represents the distribution of the process $(T^n)_{n\geq 0}$ at time n, when u is its initial density. Typically, distributional limit theorems (both in finite and infinite measure preserving situations) based on the behaviour of $\widehat{T}$ require some condition related to the convergence, after suitable normalization, of $\widehat{T}^n u$ (or of $\sum_{k=0}^{n-1} \widehat{T}^k u$) to the invariant density 1_X. To verify an assumption of this type we need to study the full asymptotics of the operator.

Using a renewal-theoretic approach originating from [**T5**], we showed in [**TZ**] that, for certain questions about infinite measure preserving systems, a different kind of condition which only depends on the dynamics up to the first return (or entrance) to a suitable reference set Y suffices. For $Y \in \mathcal{A}$ with $\mu(Y) > 0$ the *first return (entrance) time* of Y is[1] $\varphi(x) = \varphi_Y(x) := \min\{n \geq 1 : T^n x \in Y\}$, $x \in X$, and we define $T_Y x := T^{\varphi(x)} x$, $x \in X$. The restricted measure $\mu \mid_{Y\cap\mathcal{A}}$ is invariant under the *first return map*, T_Y restricted to Y. In other words, $1_Y = \sum_{k\geq 1} \widehat{T}^k 1_{Y\cap\{\varphi=k\}}$ a.e. If $\mu(Y) < \infty$, we can regard φ as a random variable on the probability space $(Y, Y\cap\mathcal{A}, \mu_Y)$, $\mu_Y(E) := \mu(Y)^{-1}\mu(Y\cap E)$. Under additional properties making Y a suitable reference set, the asymptotic behaviour of its *return distribution*, i.e. that of the *(first) return probabilities* $f_k(Y) := \mu_Y(Y \cap \{\varphi_Y = k\})$, is a crucial feature determining the stochastic properties of the system. For distributional limit theorems to hold, regular variation of $f_k(Y)$ or, more generally, of the *tail probabilities* $q_n(Y) := \sum_{k>n} f_k(Y) = \mu_Y(Y \cap \{\varphi_Y > n\})$, or the *wandering rate* of Y given by $w_N(Y) := \mu(Y) \sum_{n=0}^{N-1} q_n(Y) = \mu(Y^N)$, where $Y^N := \bigcup_{n=0}^{N-1} T^{-n}Y$, $N \geq 1$, is decisive.

The distributional limit theorems of [**TZ**] which we are going to generalize here apply to (necessarily non-invertible) c.e.m.p.t.s T on a σ-finite measure space $(X, \mathcal{A}, \mu)$ for which there is a reference set $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, and some probability density H such that

$$\frac{1}{w_N(Y)} \sum_{n=0}^{N-1} \widehat{T}^n 1_{Y_n} \to H \qquad \text{uniformly on } Y \text{ as } N \to \infty. \tag{1.1}$$

Here

$$Y_0 := Y \quad \text{and} \quad Y_n := Y^c \cap \{\varphi = n\},\ n \geq 1, \tag{1.2}$$

so that $\mu(Y_n) = \mu(Y)\, q_n(Y)$, and $Y^N = \bigcup_{n=0}^{N-1} Y_n$ (disjoint), and

$$\widehat{T}^n 1_{Y_n} = \sum_{k>n} \widehat{T}^k 1_{Y\cap\{\varphi=k\}}, \qquad n \geq 0, \tag{1.3}$$

cf. (2.3) of [**TZ**]. To understand the probabilistic meaning of (1.1), observe via (1.3) that it is a generalized (averaged) version of

$$f_k(Y)^{-1} \cdot \widehat{T}^k 1_{Y\cap\{\varphi=k\}} \to H \qquad \text{uniformly on } Y \text{ as } N \to \infty. \tag{1.4}$$

Suppose our system starts with initial density $\mu(Y)^{-1} \cdot 1_Y$. The probability density H_k on the left-hand side of (1.4) then is the conditional density of our process at

[1] Whenever the set Y is understood, we suppress the dependence of φ on Y in our notation.

time k, given that this is the time of its first return to Y. In other words, having returned at step k, the process starts anew, this time with initial density H_k. Returning to Y therefore constitutes a *proper renewal* or *stochastic regeneration* iff $H_k = \mu(Y)^{-1} \cdot 1_Y$ for all $k \geq 1$, and we obtain an imbedded renewal process in this case. Interesting dynamical systems, however, do not usually come with an obvious regenerative event like that. Still, knowing the H_k or, more generally, the densities on the left-hand side of (1.1), means to know exactly how our process fails to regenerate properly as it returns to Y, and the convergence conditions (1.4) or (1.1) ensure that, asymptotically as $k \to \infty$, there is a definite way in which this happens.

As illustrated in [**TZ**], condition (1.1) can indeed be verified quite easily for a large family of examples. Still, requiring the densities $w_N(Y)^{-1} \sum_{n=0}^{N-1} \widehat{T}^n 1_{Y_n}$ to actually converge (even uniformly) is a rather strong condition, not satisfied in various other natural examples. The purpose of the present paper is to extend the method developed in [**TZ**] by showing that (1.1) may be replaced by the significantly more general assumption that

$$(1.5) \quad \left\{ \frac{1}{w_N(Y)} \sum_{n=0}^{N-1} \widehat{T}^n 1_{Y_n} \right\}_{N \geq 1} =: \mathfrak{H}_Y \qquad \text{is a (strongly) precompact set in } \mathfrak{B}$$

with $\mathfrak{B} = L_\infty(\mu)$. We thus require that the different ways in which our process regenerates (improperly) upon its first return to Y form a small set. While for the examples we are going to discuss, precompactness in $L_\infty(\mu)$ can be verified with little effort (using distortion control and the Arzela-Ascoli theorem), we will in fact show that for the arcsine laws even precompactness in $\mathfrak{B} = L_1(\mu)$ suffices.

2. Main results

The first set of results we present generalizes the abstract distributional limit theorems of [**TZ**], showing that their conclusions remain valid under suitable compactness assumptions. To formulate them, we need to recall some basic concepts. A function $a : (L, \infty) \to (0, \infty)$ is *regularly varying of index* $\rho \in \mathbb{R}$ at infinity, written $a \in \mathcal{R}_\rho$, if $a(ct)/a(t) \to c^\rho$ as $t \to \infty$ for any $c > 0$, and we shall interpret sequences $(a_n)_{n \geq 0}$ as functions on $\mathbb{R}_+$ via $t \longmapsto a_{[t]}$. *Slow variation* means regular variation of index 0. $\mathcal{R}_\rho(0)$ is the family of functions $r : (0, \varepsilon) \to \mathbb{R}_+$ regularly varying of index ρ at zero (same condition as above, but for $t \searrow 0$). Basic background information about regular variation can be found in Chapter 1 of [**BGT**]. Throughout we use the efficient convention that for $a_n, b_n \geq 0$ and $\vartheta \in [0, \infty)$,

$$a_n \sim \vartheta \cdot b_n \text{ as } n \to \infty \quad \text{means} \quad \lim_{n \to \infty} a_n / b_n = \vartheta,$$

even in case $\vartheta = 0$, where it is equivalent to the usual $a_n = o(b_n)$ as $n \to \infty$. (An analogous convention applies to $f(s) \sim \vartheta \cdot g(s)$ as $s \searrow 0$ etc.) This extension enables us to avoid tedious distinctions between the $\vartheta > 0$ and the $\vartheta = 0$ case in our notation (but requires extra care as this distinction now has to be made in the arguments). We shall heavily depend on Karamata's Tauberian theorem for discrete Laplace transforms and the Monotone Density theorem for regularly varying functions, in the versions provided by Proposition 4.2 and Lemma 4.1 of [**TZ**].

If ν is a probability measure on $(X,\mathcal{A})$ and $(R_n)_{n\geq 1}$ is a sequence of measurable real-valued functions on X, distributional convergence of $(R_n)_{n\geq 1}$ w.r.t. ν to some random variable R will be denoted by $R_n \overset{\nu}{\Longrightarrow} R$. *Strong distributional convergence* $R_n \overset{\mathcal{L}(\mu)}{\Longrightarrow} R$ on the σ-finite measure space $(X,\mathcal{A},\mu)$ means that $R_n \overset{\nu}{\Longrightarrow} R$ for all probability measures $\nu \ll \mu$. We denote the family of probability densities w.r.t. μ by $\mathcal{D}(\mu) := \{u \in L_1(\mu) : \int_X u\, d\mu = 1, u \geq 0\}$. Let $\mathfrak{H} \subseteq \mathcal{D}(\mu)$ be a collection of densities. If there is some $K \in \mathbb{N}_0$ such that $\inf_{u\in\mathfrak{H}} \inf_Y \sum_{k=0}^{K} \widehat{T}^k u > 0$, we say that $\mathfrak{H}$ is *uniformly sweeping (in K steps)* for Y.

2.1. The Darling-Kac theorem. Henceforth T denotes a c.e.m.p.t. of the σ-finite measure space $(X,\mathcal{A},\mu)$. We are interested in the asymptotic distributional behaviour of ergodic sums $\mathbf{S}_n(f) := \sum_{j=0}^{n-1} f\circ T^j$, $n \in \mathbb{N}_0$, of functions $f \in L_1(\mu)$ with $\int_X f\, d\mu \neq 0$. For $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, we define

$$a_n(Y) := \int_Y \mathbf{S}_n(1_Y)\, d\mu_Y = \sum_{k=0}^{n-1} \frac{\mu(Y\cap T^{-k}Y)}{\mu(Y)}, \quad n \geq 0$$

(which is the obvious candidate for a normalizing sequence for $(\mathbf{S}_n(1_Y))_{n\geq 0}$). We let $\mathcal{M}_\alpha$, $\alpha \in [0,1]$, denote a non-negative real random variable distributed according to the *(normalized) Mittag-Leffler distribution of order* α, which can be characterized by its moments

$$\mathbb{E}\left[\mathcal{M}_\alpha^r\right] = r!\,\frac{(\Gamma(1+\alpha))^r}{\Gamma(1+r\alpha)}\ , \quad r \in \mathbb{N}_0. \tag{2.1}$$

We are going to prove

Theorem 2.1 (**Ergodic sums of integrable functions**). *Let T be a c.e.m.p.t. on the σ-finite measure space $(X,\mathcal{A},\mu)$, and assume there is some $Y\in\mathcal{A}$, $0 < \mu(Y) < \infty$, such that*

$$\mathfrak{H}_Y = \left\{ \frac{1}{w_N(Y)} \sum_{n=0}^{N-1} \widehat{T}^n 1_{Y_n} \right\}_{N\geq 1} \quad \text{is precompact in } L_\infty(\mu) \text{ and uniformly sweeping} \tag{2.2}$$

and

$$(w_N(Y)) \in \mathcal{R}_{1-\alpha} \quad \text{for some } \alpha\in[0,1]. \tag{2.3}$$

Then

$$\frac{1}{a_n}\mathbf{S}_n(f) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mu(f)\cdot\mathcal{M}_\alpha \quad \text{for all } f\in L_1(\mu) \text{ with } \int_X f\,d\mu \neq 0,$$

where

$$a_n := \frac{1}{\mu(Y)}\,a_n(Y) \sim \frac{1}{\Gamma(1+\alpha)\Gamma(2-\alpha)}\cdot\frac{n}{w_n(Y)} \quad \text{as } n\to\infty.$$

2.2. The arcsine law for occupation times. The second limit theorem we are interested in concerns occupation times of certain subsets $A \subseteq X$ of infinite measure. As in [**ATZ**] and [**TZ**] we say that two disjoint sets $A_1, A_2 \subseteq X$ are *dynamically separated by* $Y \subseteq X$ *(under the action of* T*)* if $x \in A_1$ (resp. A_2) and $T^n x \in A_2$ (resp. A_1) imply the existence of some $k = k(x) \in \{0, \dots, n\}$ for which $T^k x \in Y$ (i.e. T-orbits can't pass from one set to the other without visiting Y). Define the following quantities related to *first returns through the set* A_1: Let $f_n(Y, A_1) := \mu_Y(Y \cap T^{-1}A_1 \cap \{\varphi_Y = n\})$, $q_n(Y, A_1) := \mu_Y(Y \cap T^{-1}A_1 \cap \{\varphi_Y > n\})$, and $w_n(Y, A_1) := \mu(Y) \sum_{k=0}^{n-1} q_n(Y, A_1) = \sum_{k=0}^{n-1} \mu(Y \cap T^{-1}A_1 \cap \{\varphi_Y > n\})$, $n \geq 1$. We are going to study the distributional behaviour of $\mathbf{S}_n(1_{A_1})$ in cases where $\mu(A_1) = \mu(A_2) = \infty$ and where Y, $0 < \mu(Y) < \infty$, is a set for which the first return (or entrance) behaviour of T through the A_i is good. (For related questions about the pointwise behaviour in such situations see [**ATZ**].)

For $\alpha, \beta \in (0, 1)$ we let $\mathcal{L}_{\alpha,\beta}$ denote a random variable with (values in $[0, 1]$ and) distribution given by

$$\begin{aligned}\Pr(\{0 \leq \mathcal{L}_{\alpha,\beta} \leq t\}) &= \frac{b \sin \pi\alpha}{\pi} \int_0^t \frac{x^{\alpha-1}(1-x)^{\alpha-1}}{b^2 x^{2\alpha} + 2b x^\alpha (1-x)^\alpha \cos \pi\alpha + (1-x)^{2\alpha}}\, dx \\ &= \frac{1}{\pi\alpha} \operatorname{arccot}\left(\frac{((1-t)/t)^\alpha}{b \sin \pi\alpha} + \cot \pi\alpha \right), \quad t \in (0, 1],\end{aligned}$$

where $b := (1 - \beta)/\beta$, cf. [**L1**] and [**T5**]. Continuously extending this family, we let $\mathcal{L}_{\alpha,1} := 1$ and $\mathcal{L}_{\alpha,0} := 0$, $\alpha \in [0, 1]$, and $\mathcal{L}_{1,\beta} := \beta$, $\Pr(\mathcal{L}_{0,\beta} = 1) = \beta = 1 - \Pr(\mathcal{L}_{0,\beta} = 0)$. Then $\mathbb{E}[\mathcal{L}_{\alpha,\beta}] = \beta$ and $\mathrm{Var}[\mathcal{L}_{\alpha,\beta}] = (1 - \alpha)\beta(1 - \beta)$. Generally, cf. Proposition 1 of [**T5**], if $\alpha, \beta \in [0, 1]$, then

$$\mathbb{E}[\mathcal{L}^r_{\alpha,\beta}] = (-1)^r \beta \left[\sum_{j=0}^{r-1} (-1)^{j+1} \binom{\alpha}{r-j} \mathbb{E}[\mathcal{L}^j_{\alpha,\beta}] + \binom{\alpha-1}{r} \right], \quad r \in \mathbb{N}, \tag{2.4}$$

where, by convention, $\mathbb{E}[\mathcal{L}^0_{\alpha,\beta}] = 1$. We are going to prove the following generalization of [**T5**] and Theorem 3.2 in [**TZ**]:

Theorem 2.2 (**Arcsine law for occupation times**). *Let T be a c.e.m.p.t. of the σ-finite measure space $(X, \mathcal{A}, \mu)$, and assume that $X = A_1 \cup Y \cup A_2$ (measurable and pairwise disjoint) where $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, dynamically separates A_1 and A_2. Suppose that*

$$\mathfrak{H}_Y = \left\{ \frac{1}{w_N(Y)} \sum_{n=0}^{N-1} \widehat{T}^n 1_{Y_n} \right\}_{N \geq 1} \quad \text{is precompact in } L_1(\mu), \tag{2.5}$$

with

$$(w_N(Y)) \in \mathcal{R}_{1-\alpha} \quad \text{for some } \alpha \in [0, 1]. \tag{2.6}$$

If, in addition,

$$\mathfrak{H}_{Y,A_1} = \left\{ \frac{1}{w_N(Y, A_1)} \sum_{n=0}^{N-1} \widehat{T}^n 1_{A_1 \cap Y_n} \right\}_{N \geq 1} \quad \text{is precompact in } L_1(\mu), \tag{2.7}$$

and

$$\frac{w_N(Y, A_1)}{w_N(Y)} \longrightarrow \beta \in [0,1] \quad \text{as } N \to \infty, \tag{2.8}$$

then

$$\frac{1}{n}\mathbf{S}_n(1_A) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{L}_{\alpha,\beta}$$

for every $A \in \mathcal{A}$ satisfying $\mu(A \triangle A_1) < \infty$.

2.3. The arcsine law for waiting times. Let T be a c.e.m.p.t. of the σ-finite measure space $(X, \mathcal{A}, \mu)$. For $Y \in \mathcal{A}$, $\mu(Y) > 0$, we define the $\mathbb{N}_0$-valued variables $\mathbf{Z}_n(Y)$, $n \geq 0$, on X by $\mathbf{Z}_n(Y)(x) := \max(\{0\} \cup \{1 \leq k \leq n : T^k x \in Y\})$. The Dynkin-Lamperti arcsine law for waiting times describes the asymptotic behaviour of these renewal-theoretic quantities in infinite measure preserving situations: For $\alpha \in (0,1)$ we let $\mathcal{Z}_\alpha$ denote a random variable (with values in $[0,1]$) having the $B(\alpha, 1-\alpha)$*-distribution* (sometimes called the *generalized arcsine law*), i.e.

$$\Pr(\{0 \leq \mathcal{Z}_\alpha \leq t\}) = \frac{\sin \pi\alpha}{\pi} \int_0^t \frac{dx}{x^{1-\alpha}(1-x)^\alpha}, \quad t \in [0,1].$$

Continuously extending this family to $\alpha \in \{0,1\}$ we let $\mathcal{Z}_0 := 0$ and $\mathcal{Z}_1 := 1$. Since for any $\alpha \in [0,1]$, $\mathcal{Z}_\alpha$ is a bounded random variable, its distibution is determined by its moments $\mathbb{E}[\mathcal{Z}_\alpha^r] = (-1)^r \binom{-\alpha}{r}$, $r \in \mathbb{N}_0$, which satisfy the recursion formula

$$\mathbb{E}[\mathcal{Z}_\alpha^r] = \sum_{j=0}^{r-1} (-1)^{r-j+1} \binom{\alpha}{r-j} \mathbb{E}[\mathcal{Z}_\alpha^j] \quad \text{for } r \in \mathbb{N}. \tag{2.9}$$

In particular, $\mathbb{E}[\mathcal{Z}_\alpha] = \alpha$ and $\mathrm{Var}[\mathcal{Z}_\alpha] = \alpha(1-\alpha)/2$. We are going to prove

THEOREM 2.3 (**Arcsine law for waiting times**). *Let T be a c.e.m.p.t. on the σ-finite space $(X, \mathcal{A}, \mu)$, and assume there is some $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, such that*

$$\mathfrak{H}_Y = \left\{ \frac{1}{w_N(Y)} \sum_{n=0}^{N-1} \widehat{T}^n 1_{Y_n} \right\}_{N \geq 1} \quad \text{is precompact in } L_1(\mu), \tag{2.10}$$

and

$$(w_N(Y)) \in \mathcal{R}_{1-\alpha} \quad \text{for some } \alpha \in [0,1]. \tag{2.11}$$

Then

$$\frac{1}{n}\mathbf{Z}_n(Y) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{Z}_\alpha.$$

Remark 3.5 of [**TZ**], about limit theorems for other renewal-theoretic variables, also applies in the present setup. Proposition 7.1 of [**TZ**] provides a condition under which $n^{-1}\mathbf{Z}_n(Y) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{Z}_\alpha$ implies $n^{-1}\mathbf{Z}_n(E) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{Z}_\alpha$ for all $E \in Y \cap \mathcal{A}$ with $\mu(E) > 0$.

2.4. Complete excursions spent in one component. Theorem 2.3 shows that a preselected inspection time n will, for most orbits, fall into some long excursion from Y. For the situation of Theorem 2.2 this means that this last excursion really contributes to the occupation times of the component A_1. Formally we have

$$\sum_{j=1}^{n} 1_{A_1} \circ T^j = \sum_{j=1}^{Z_n} 1_{A_1} \circ T^j + (1_{Y \cap T^{-1}A_1} \circ T^{Z_n}) \cdot (n - Z_n), \quad n \in \mathbb{N}_0, \tag{2.12}$$

where $Z_n := \mathbf{Z}_n(Y)$, and the sum on the right-hand side, henceforth denoted K_n, is the amount of time spent in A_1 during excursions completed before step n. We know that $(n - Z_n)/n \overset{\mathcal{L}(\mu)}{\Longrightarrow} 1 - \mathcal{Z}_\alpha$, which is not concentrated in zero if $\alpha < 1$. Therefore it is interesting to have a closer look at K_n. For $\alpha, \beta \in [0,1]$ we let $\mathcal{K}_{\alpha,\beta}$ denote a random variable taking values in $[0,1]$, and with distribution characterized by the following recursion formula for its moments (where, by convention, we start with $\mathbb{E}[\mathcal{K}^0_{\alpha,\beta}] := 1$),

$$\mathbb{E}[\mathcal{K}^r_{\alpha,\beta}] = (-1)^r \beta \sum_{j=0}^{r-1} (-1)^{j+1} \binom{\alpha}{r-j} \mathbb{E}[\mathcal{K}^j_{\alpha,\beta}], \quad r \in \mathbb{N}. \tag{2.13}$$

In particular, $\mathbb{E}[\mathcal{K}_{\alpha,\beta}] = \alpha\beta$ and $\mathrm{Var}[\mathcal{K}_{\alpha,\beta}] = \alpha(1-\alpha)\beta/2$. Note that (2.13) is parallell to (2.4), with the rightmost term inside the square brackets missing, and that (2.13) generalizes (2.9) via the extra factor β. The fact that these are the moments of a probability distribution on $[0,1]$ is implicit in the proof of the Theorem 2.4 below which identifies $\mathcal{K}_{\alpha,\beta}$ as the limiting variable for the sequence $(K_n/n)_{n\geq 1}$. The boundary cases obviously are $\mathcal{K}_{0,\beta} = 0$ and $\mathcal{K}_{1,\beta} = \beta$ for all $\beta \in [0,1]$, while $\mathcal{K}_{\alpha,0} = 0$ and $\mathcal{K}_{\alpha,1} = \mathcal{Z}_\alpha$ in distribution for $\alpha \in [0,1]$. A tangible description of the other distributions[2] is given by

Proposition 2.1 (**Identifying the density of $\mathcal{K}_{\alpha,\beta}$**). *For $\alpha, \beta \in (0,1)$, the distribution of $\mathcal{K}_{\alpha,\beta}$ is given by*

$$\Pr(\{0 \leq \mathcal{K}_{\alpha,\beta} \leq t\}) = \frac{\sin \pi\alpha}{\pi} \int_0^t \frac{(1+b)\, x^{\alpha-1}(1-x)^\alpha}{b^2 x^{2\alpha} + 2b x^\alpha (1-x)^\alpha \cos \pi\alpha + (1-x)^{2\alpha}}\, dx$$

for $t \in [0,1]$, where $b := (1-\beta)/\beta$.

We will establish the following limit theorem.

Theorem 2.4 (**Completed excursions to a component**). *Let $(X, \mathcal{A}, \mu, T)$ and Y, A_1, A_2 be as in Theorem 2.2, satisfying (2.5) - (2.8). Then*

$$\frac{1}{n} \mathbf{S}_{\mathbf{Z}_n(A^c)}(1_A) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{K}_{\alpha,\beta}$$

for every $A \in \mathcal{A}$ satisfying $\mu(A \triangle A_1) < \infty$.

[2] Note that (2.13) does *not* just give $\mathcal{K}_{\alpha,\beta} = \beta \mathcal{Z}_\alpha$ in distribution, which the similarity to (2.9) might suggest. In particular, the variances differ.

2.5. Arcsine laws for randomly chosen excursions. Theorems 2.2 and 2.4 neither use, nor provide, further insight into details of the dependence structure of our dynamically defined process. In particular, they don't tell us if (or in what sense) our excursion-length process $(\varphi \circ T_Y^j)_{j\geq 0}$ is (or needs to be) asymptotically independent of the process $(1_{Y\cap T^{-1}A_1} \circ T_Y^j)_{j\geq 1}$ selecting the component. The following "randomized" version of these theorems (which actually is an application of them) shows that combining the dynamical excursion process $(\varphi \circ T_Y^j)_{j\geq 0}$ with an independent iid process selecting excursions, we still end up with the same limiting behaviour.

Let $\Omega := \{\omega = (\omega_i)_{i\geq 0} : \omega_i \in \{0,1\}\}$ with the usual σ-field $\mathcal{B}$ and shift map $\sigma : \Omega \to \Omega$. For $\beta \in [0,1]$ let ν_β denote the Bernoulli measure on $(\Omega, \mathcal{B})$ with $\nu_\beta([1]) = \beta$, where $[1] := \{\omega \in \Omega : \omega_0 = 1\}$. We use this system to decide, according to the value of ω_l, whether or not to count the steps within the l-th excursion of a T-orbit $(T^n x)_{n\geq 1}$ from Y:

Theorem 2.5 (**Arcsine law for occupation times of independently selected excursions**). *Let $(X, \mathcal{A}, \mu, T)$ and Y be as in Theorem 2.3, satisfying (2.10) and (2.11), and write $S_j := \sum_{i=0}^{j-1} 1_Y \circ T^i$ and $Z_n := \mathbf{Z}_n(Y)$. On the product of this system and the Bernoulli shift $(\Omega, \mathcal{B}, \nu_\beta, \sigma)$, $\beta \in [0,1]$, define*

$$\mathbf{L}_n(x,\omega) := \sum_{j=0}^{n-1} \omega_{S_j(x)} \cdot (1_{Y^c} \circ T^j)(x), \qquad n \geq 1.$$

Then

$$\frac{1}{n}\mathbf{L}_n \overset{\mathcal{L}(\mu\otimes\nu_\beta)}{\Longrightarrow} \mathcal{L}_{\alpha,\beta} \quad \text{and} \quad \frac{1}{n}\mathbf{L}_{Z_n} \overset{\mathcal{L}(\mu\otimes\nu_\beta)}{\Longrightarrow} \mathcal{K}_{\alpha,\beta}.$$

Other variants can be obtained using similar arguments.

2.6. A comment on wandering rates. Regular variation of $(w_N(Y))$ is a property of the system $(X, \mathcal{A}, \mu, T)$ rather than a property of a particular set: By Proposition 3.2 and Remark 3.6 of [**TZ**],

$$\text{if } \{w_N(Y)^{-1}\textstyle\sum_{n=0}^{N-1} \widehat{T}^n 1_{Y_n}\}_{N\geq 1} \text{ is uniformly integrable,} \\ \text{then } Y \text{ has } \textit{minimal wandering rate}, \tag{2.14}$$

meaning that $\underline{\lim}_{N\to\infty} w_N(Z)/w_N(Y) \geq 1$ for all $Z \in \mathcal{A}$, $0 < \mu(Z) < \infty$. Equivalently, $w_N(Y) \sim w_N(Z)$ provided $\mu(Z) > 0$ and $Z \subseteq Y$. This common rate is a crucial asymptotic characteristic of the system, the *wandering rate of* T, $(w_N(T))$. As our compactness conditions imply uniform integrability, we may replace $(w_N(Y))$ by $(w_N(T))$ in Theorems 2.1 to 2.5. Similarly, we may replace $(w_N(Y, A_1))$ by $(w_N(T_{Y\cup A_1}))$ in Theorems 2.2 and 2.4.

2.7. Application to specific classes of transformations. To conclude the paper, we will illustrate our abstract results by applying them to several classes of examples, thereby extending earlier results, covering new situations, and demonstrating that our conditions can readily be verified in various nontrivial situations.

In particular, this is the case for transformations with ergodic behaviour governed by some distinct (and exceptional) indifferent orbits. Examples include indifferent fixed (or periodic) points, which have already been studied in [**TZ**], but we can now remove an extra assumption that had been required there. Moreover, our compactness conditions also cover the case of the indiffererent orbits of flat critical points.

In addition, we are now also able to deal with situations where, in marked contrast to the above, an infinite invariant measure reflects the homogeneity of a system on an infinite space: To conclude the paper, we briefly discuss how our results apply to random walks driven by Gibbs-Markov maps. While there are other approaches that could be used in this case (employing functional limit theorems showing that the partial sum processes converge to a Wiener process), it is worth observing that our method works there as well.

3. Compact sets of densities, Asymptotically invariant sequences, and Strong distributional convergence

The present section contains the tools enabling us to understand the significance of the compactness properties. The following classical companion of the mean ergodic theorem, and some relatives discussed below, lie at the heart of our approach:

THEOREM 3.1 (**L_1-characterization of ergodicity**). *A m.p.t. T on a σ-finite measure space $(X,\mathcal{A},\mu)$ is ergodic iff*

$$\left\| \frac{1}{n}\sum_{k=0}^{n-1}\widehat{T}^k(u-u^*)\right\|_1 \longrightarrow 0 \qquad \text{for all } u,u^*\in\mathcal{D}(\mu).$$

See [**Kr**], Theorem 2.1.3, or [**Yo**] for the usual Hahn-Banach proof of Theorem 3.1. We emphasize that there also is a nice alternative w^*-compactness argument parallel to the proof of Lin's characterization of exactness, cf. [**Li**] or Theorem 1.3.3 in [**A0**]. We will repeatedly employ uniform versions of this and similar convergence results, obtained via the following simple principle:

REMARK 3.1 (**Equicontinuity and precompactness**). A family $\mathcal{F}$ of maps between two metric spaces (M_i,d_i), $i\in\{1,2\}$, is *equicontinuous* if for every $\varepsilon>0$ there is some $\delta>0$ s.t. for all $F\in\mathcal{F}$, $d_2(F(a),F(b))<\varepsilon$ whenever $d_1(a,b)<\delta$. If $\mathcal{F}=(F_\iota)_{\iota\in\mathfrak{I}}$, $\mathfrak{I}$ some directed index set, and if we have pointwise convergence $\lim_\iota F_\iota(a)=F(a)$ for all $a\in M_1$, then equicontinuity of $\mathcal{F}$ implies that the convergence $F_\iota\to F$ is uniform on precompact subsets K of M_1.

For example, if T is m.p. and ergodic on $(X,\mathcal{A},\mu)$, and $\mathfrak{H}\subseteq\mathcal{D}(\mu)$ is precompact in $L_1(\mu)$, then so is $\mathcal{W}:=\{u-u^*: u,u^*\in\mathfrak{H}\}$, and since the sequence of averaging operators $n^{-1}\sum_{k=0}^{n-1}\widehat{T}^k: L_1(\mu)\to L_1(\mu)$, $n\in\mathbb{N}$, is equicontinuous (each having operator norm $=1$), the pointwise convergence asserted by Theorem 3.1 implies uniform convergence on $\mathcal{W}$, meaning that

$$\left\| \frac{1}{n}\sum_{k=0}^{n-1}\widehat{T}^k(u-u^*)\right\|_1 \longrightarrow 0 \qquad \begin{array}{c}\text{uniformly in } u,u^*\in\mathfrak{H}\subseteq\mathcal{D}(\mu)\\ \text{if } \mathfrak{H} \text{ is precompact in } L_1(\mu).\end{array} \tag{3.1}$$

The sequences $(R_n)_{n\geq 0}$ of observables whose asymptotic distributional behaviour under T is of interest to us are *asymptotically T-invariant in measure* in the sense[3] that

$$R_n \circ T - R_n \xrightarrow{\mu} 0 \text{ as } n \to \infty. \tag{3.2}$$

We need good control of the distributional behaviour of such sequences w.r.t. initial distributions from some precompact family $\mathfrak{H} \subseteq \mathcal{D}(\mu)$. For the arcsine laws the following observation suffices.

Proposition 3.1 (**Equivalent moments principle I**). *Let T be an ergodic m.p.t. on the σ-finite measure space $(X, \mathcal{A}, \mu)$, and $(R_n)_{n\geq 0}$ a sequence of measurable functions $R_n : X \to [0, \infty)$.*

a) *Suppose that (R_n) satisfies*

$$R_n \circ T - R_n \xrightarrow{\mu} 0. \tag{3.3}$$

If

$$\{R_n\}_{n\geq 0} \quad \text{is bounded (i.e. } w^*\text{-precompact) in } L_\infty(\mu), \tag{3.4}$$

then, for every $u \in \mathcal{D}(\mu)$,

$$\|(R_n \circ T - R_n) \cdot u\|_1 \longrightarrow 0 \qquad \text{as } n \to \infty, \tag{3.5}$$

and for all $u, u^* \in \mathcal{D}(\mu)$,

$$\int_X R_n \cdot u \, d\mu - \int_X R_n \cdot u^* \, d\mu \longrightarrow 0 \quad \text{as } n \to \infty. \tag{3.6}$$

b) *If, in addition, $(\gamma_n)_{n\geq 0}$ is a sequence in $[0, \infty)$ with $G(s) := \sum_{n\geq 0} \gamma_n e^{-ns} < \infty$ for $s > 0$, and $\sum_{n\geq 0} \gamma_n = \infty$, then, for all $u, u^* \in \mathcal{D}(\mu)$, the weighed Laplace transform $R_\gamma(s) := \sum_{n\geq 0} R_n \gamma_n e^{-ns}$, $s > 0$, satisfies*

$$\int_X R_\gamma(s) \cdot u \, d\mu - \int_X R_\gamma(s) \cdot u^* \, d\mu = o(G(s)) \quad \text{as } s \searrow 0. \tag{3.7}$$

c) *Moreover, if $\mathfrak{H} \subseteq \mathcal{D}(\mu)$ is precompact in $L_1(\mu)$, then convergence in (3.5), (3.6), and (3.7) is uniform in $u, u^* \in \mathfrak{H}$.*

Proof. **(i)** We first prove our assertions concerning (3.5). For any $\varepsilon > 0$ and $u \in \mathcal{D}(\mu)$,

$$\int_X |R_n \circ T - R_n| \cdot u \, d\mu \leq \varepsilon + \sup_{j\geq 0} \|R_j \circ T - R_j\|_\infty \cdot \int_{\{|R_n \circ T - R_n| > \varepsilon\}} u \, d\mu, \tag{3.8}$$

and the rightmost integral tends to 0 as $n \to \infty$ since $R_n \circ T - R_n \xrightarrow{\mu} 0$. Hence,

$$\|(R_n \circ T - R_n) \cdot u\|_1 \longrightarrow 0 \qquad \text{for all } u \in \mathcal{D}(\mu). \tag{3.9}$$

If $\mathfrak{H} \subseteq \mathcal{D}(\mu)$ is precompact in $L_1(\mu)$, it is tight, so that there is some $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, with $\int_{Y^c} u \, d\mu < \varepsilon$ for all $u \in \mathfrak{H}$. Moreover, $\mathfrak{H}$ is uniformly integrable, i.e. there is some $M \in (0, \infty)$ for which $\int_{\{u > M\}} u \, d\mu < \varepsilon$ whenever $u \in \mathfrak{H}$. Therefore, returning to (3.8), we see that

$$\begin{aligned} \int_{\{|R_n \circ T - R_n| > \varepsilon\}} u \, d\mu &< \varepsilon + \int_{Y \cap \{|R_n \circ T - R_n| > \varepsilon\}} u \, d\mu \\ &< 2\varepsilon + M \cdot \mu(Y \cap \{|R_n \circ T - R_n| > \varepsilon\}), \end{aligned}$$

[3]Throughout, $\xrightarrow{\mu}$ for some σ-finite measure μ means $\xrightarrow{\nu}$ for all probability measures $\nu \ll \mu$.

proving that (3.9) holds uniformly in $u \in \mathfrak{H}$.

Note that, more generally, we have

$$\left\| (R_n \circ T^k - R_n) \cdot u \right\|_1 \longrightarrow 0 \qquad \text{as } n \to \infty,$$

for every $u \in \mathcal{D}(\mu)$ and $k \in \mathbb{N}$, the convergence being uniform in $u \in \mathfrak{H}$ for any fixed k. (The continuous images $\widehat{T}^j \mathfrak{H}$, $j \in \mathbb{N}$, are precompact, too.)

(ii) As a consequence of the previous step, we also have, for any $K \geq 1$,

$$\int_X R_n \cdot u \, d\mu - \int_X R_n \cdot \left(\frac{1}{K} \sum_{k=0}^{K-1} \widehat{T}^k u \right) d\mu = \frac{1}{K} \sum_{k=0}^{K-1} \int_X \left(R_n - R_n \circ T^k \right) \cdot u \, d\mu \to 0$$

as $n \to \infty$, uniformly in $u \in \mathfrak{H}$. But for any $u, u^* \in \mathcal{D}(\mu)$,

$$\left| \int_X R_n \cdot \left(\frac{1}{K} \sum_{k=0}^{K-1} \widehat{T}^k (u - u^*) \right) d\mu \right| \leq \sup_{j \geq 0} \| R_j \|_\infty \cdot \left\| \frac{1}{K} \sum_{k=0}^{K-1} \widehat{T}^k (u - u^*) \right\|_1 ,$$

and according to Theorem 3.1 the right-hand side is arbitrarily small if we choose K large in the first place. By (3.1), we see that this in fact holds uniformly in $u, u^* \in \mathfrak{H}$. Putting things together, our assertions concerning (3.6) follow.

(iii) To finally deal with the weighed transform $R_\gamma(s)$, write down

$$\int_X R_\gamma(s) \cdot (u - u^*) \, d\mu = \sum_{n \geq 0} \left(\int_X R_n \cdot (u - u^*) \, d\mu \right) \gamma_n \, e^{-ns},$$

which, due to (the uniform version of) (3.6) and $G(s) \to \infty$, implies our statements.

□

For the proof of the Darling-Kac theorem we will use a similar result applicable to variables of the type $R_n := (\mathbf{S}_n(1_Y)/a_n)^r$, $n \geq 0$, with fixed $r \geq 0$, which no longer form bounded sequences in $L_\infty(\mu)$ since $a_n = o(n)$ as $n \to \infty$. It is not too surprising that a stronger compactness assumption is called for in this case. Due to $\mathbf{S}_n(1_Y) = \mathbf{S}_{n-m}(1_Y) \circ T^m$ on Y_m for $n \geq m \geq 0$, this choice of R_n clearly satisfies condition (3.10) below. The other conditions will be verified (by induction on r) in the proof of the theorem.

Proposition 3.2 (**Equivalent moments principle II**). *Let T be a c.e.m.p.t. on the σ-finite measure space $(X, \mathcal{A}, \mu)$ and $Y \in \mathcal{A}$ with $0 < \mu(Y) < \infty$. Let $(R_n)_{n \geq 0}$ be a sequence of measurable functions $R_n : X \to [0, \infty)$.*

a) *Suppose that*

(3.10) *for all $m, n \geq 0$, the function $1_{Y_m} \cdot R_n$ is integrable, and there is some κ s.t. $R_n \leq \kappa \cdot R_{n-m} \circ T^m$ on Y_m for $n \geq m \geq 0$,*

and

(3.11) $\quad \| (R_n \circ T - R_n) \cdot u \|_1 \longrightarrow 0 \quad$ *for all $u \in L_\infty(\mu)$ supported on Y^M for some $M = M(u)$,*

If

(3.12) $\quad \{1_Y \cdot R_n\}_{n \geq 0} \quad$ *is weakly precompact in $L_1(\mu)$,* *then, for any* $u, u^* \in \mathcal{D}(\mu) \cap L_\infty(\mu)$ *which are supported on* Y*, we have*

$$\int_X R_n \cdot u \, d\mu - \int_X R_n \cdot u^* \, d\mu \longrightarrow 0 \quad \text{as } n \to \infty. \tag{3.13}$$

b) *If, in addition,* $(\gamma_n)_{n\geq 0}$ *is a sequence in* $[0, \infty)$ *with* $G(s) := \sum_{n\geq 0} \gamma_n e^{-ns} < \infty$ *for* $s > 0$*, and* $\sum_{n\geq 0} \gamma_n = \infty$*, then the weighed Laplace transform*

$$R_\gamma(s) := \sum_{n\geq 0} R_n \, \gamma_n e^{-ns}, \quad s > 0,$$

converges a.e. on Y*, is integrable there, and satisfies, for any* $u, u^* \in \mathcal{D}(\mu) \cap L_\infty(\mu)$ *supported on* Y*,*

$$\int_X R_\gamma(s) \cdot u \, d\mu - \int_X R_\gamma(s) \cdot u^* \, d\mu = o\,(G(s)) \quad \text{as } s \searrow 0. \tag{3.14}$$

c) *Moreover, if* $\mathfrak{H} \subseteq \{u \in \mathcal{D}(\mu) \cap L_\infty(\mu) : u \text{ is supported on } Y\}$ *is precompact in* $L_\infty(\mu)$*, then convergence in (3.11), (3.13), and (3.14) is uniform in* $u, u^* \in \mathfrak{H}$*.*

Proof. **(i)** Assume w.l.o.g. that $\mu(Y) = 1$. By assumption (3.12), $\{R_n \mid_Y\}_{n\geq 0}$ is weakly precompact in $L_1(\mu \mid_{Y\cap\mathcal{A}})$, which is equivalent to L_1-boundedness plus uniform integrability (cf. Theorem III.C.12 of [**Wo**], or Corollary IV.8.11 of [**DS**]). According to the Eberlein - Šmulian theorem (Theorem II.C.3 of [**Wo**], or Theorem V.6.1 of [**DS**]), weak precompactness in a Banach space is equivalent to weak sequential precompactness[4], meaning that any subsequence of $\mathbb{N}$ contains a further subsequence $n_j \nearrow \infty$ such that there is some $R^{(1)} \in L_1(\mu \mid_{Y\cap\mathcal{A}})$ with

$$\int_Y R_{n_j} \cdot g \, d\mu \longrightarrow \int_Y R^{(1)} \cdot g \, d\mu \quad \text{as } j \to \infty \quad \text{for all } g \in L_\infty(\mu \mid_{Y\cap\mathcal{A}}). \tag{3.15}$$

We are going to extend this compactness statement. We claim that for every $M \geq 1$,

$$\{R_n \mid_{Y^M}\}_{n\geq 0} \quad \text{is weakly precompact in } L_1(\mu \mid_{Y^M\cap\mathcal{A}}), \tag{3.16}$$

so that every subsequence of $\mathbb{N}$ has a limit point $R^{(M)} \in L_1(\mu \mid_{Y^M\cap\mathcal{A}})$ along some $n_j \nearrow \infty$, that is,

$$\int_{Y^M} R_{n_j} \cdot g \, d\mu \longrightarrow \int_{Y^M} R^{(M)} \cdot g \, d\mu \quad \text{as } j \to \infty \quad \text{for all } g \in L_\infty(\mu \mid_{Y^M\cap\mathcal{A}}).$$

Diagonalizing in an obvious manner, we can then conclude that for any subsequence of $\mathbb{N}$ there are a further subsequence $n_j \nearrow \infty$ and some measurable $R : X \to \overline{\mathbb{R}}$ such that for all $M \geq 1$ we have $1_{Y^M} \cdot R \in L_1(\mu)$ and

$$\int_{Y^M} R_{n_j} \cdot g \, d\mu \longrightarrow \int_{Y^M} R \cdot g \, d\mu \quad \text{as } j \to \infty \quad \text{if } 1_{Y^M} \cdot g \in L_\infty(\mu \mid_{Y^M\cap\mathcal{A}}). \tag{3.17}$$

Fix any $M \geq 1$. By the first bit of (3.10), $\{R_n \mid_{Y^M}\}_{n\geq 0}$ is a sequence in $L_1(\mu \mid_{Y^M\cap\mathcal{A}})$. To establish our claim (3.16), we first note that it is bounded, as for

[4] Recall that we can't use metrizability arguments here: the weak topology on (the closed unit ball of) $L_1(\nu)$ is metrizable iff $L_\infty(\nu)$ is separable (cf. Theorem V.5.2 in [**DS**]). The latter fails, for example, if ν is Lebesgue measure on a nondegenerate interval.

$n > M$,

$$\int_{Y^M} R_n \, d\mu = \sum_{m=0}^{M-1} \int_X 1_{Y_m} \cdot R_n \, d\mu \le \kappa \sum_{m=0}^{M-1} \int_X 1_{Y_m} \cdot R_{n-m} \circ T^m \, d\mu$$
$$= \kappa \sum_{m=0}^{M-1} \int_X \widehat{T}^m 1_{Y_m} \cdot R_{n-m} \, d\mu \le \kappa M \cdot \sup_{n\ge1} \int_Y R_n \, d\mu \quad < \infty,$$

where we use (3.10) and $\widehat{T}^m 1_{Y_m} \le 1_Y$. We also need to check uniform integrability: For any $n > M$ and $K \in (0, \infty)$, we have, again by (3.10),

$$Y^M \cap \{R_n > K\} \subseteq \bigcup_{m=0}^{M-1} Y_m \cap T^{-m} \left(Y \cap \{R_{n-m} > K/\kappa\} \right).$$

Therefore we see that

$$\int_{Y^M \cap \{R_n > K\}} R_n \, d\mu \le \sum_{m=0}^{M-1} \int_{Y_m \cap T^{-m}(Y \cap \{R_{n-m} > K/\kappa\})} R_n \, d\mu$$
$$\le \kappa \sum_{m=0}^{M-1} \int_X 1_{Y_m} \cdot \left(1_{Y \cap \{R_{n-m} > K/\kappa\}} R_{n-m} \right) \circ T^m \, d\mu$$
$$= \kappa \sum_{m=0}^{M-1} \int_{Y \cap \{R_{n-m} > K/\kappa\}} \widehat{T}^m 1_{Y_m} \cdot R_{n-m} \, d\mu$$
$$\le \kappa M \cdot \sup_{n\ge1} \int_{Y \cap \{R_n > K/\kappa\}} R_n \, d\mu \longrightarrow 0 \quad \text{as } K \to \infty,$$

since the sequence $(1_Y \cdot R_n)_{n\ge0}$ is uniformly integrable by assumption. This completes the proof of (3.16).

(ii) Suppose now that R is any weak limit point of (R_n) in the sense of (3.17). We are going to show that R is a.e. constant, which, in particular, entails

$$\int_Y R_{n_j} \cdot (u - u^*) \, d\mu \longrightarrow \int_Y R \cdot (u - u^*) \, d\mu = R \cdot \int_Y (u - u^*) \, d\mu = 0$$

for $u, u^* \in \mathcal{D}(\mu) \cap L_\infty(\mu)$ supported on Y. A straightforward subsequence-in-subsequence argument based on the compactness property established above then proves our assertion that

$$\int_Y R_n \cdot u \, d\mu - \int_Y R_n \cdot u^* \, d\mu \longrightarrow 0 \quad \text{as } n \to \infty.$$

By ergodicity, we know that R is a.e. constant as soon as $R \circ T = R$ a.e.. Note that $\{\varphi \le M\} \subseteq Y^{M+1}$ and $T(\{\varphi \le M+1\}) \subseteq Y^{M+1}$ for $M \ge 1$. Therefore, if $g \in L_\infty(\mu)$ is supported on $\{\varphi \le M+1\}$, then $\widehat{T}g \in L_\infty(\mu)$ is supported on Y^M, and we can apply (3.17) to g and $\widehat{T}g$, obtaining

$$\int_X R_{n_j} \cdot g \, d\mu \to \int_X R \cdot g \, d\mu \quad \text{and} \quad \int_X R_{n_j} \cdot \widehat{T}g \, d\mu \to \int_X (R \circ T) \cdot g \, d\mu.$$

However, due to (3.11), we have

$$\left| \int_X R_n \cdot \left(g - \widehat{T}g \right) \, d\mu \right| \le \int_X |R_n \circ T - R_n| \cdot g \, d\mu \longrightarrow 0,$$

so that the two limits coincide, i.e.

$$\int_X R \cdot g \, d\mu = \int_X (R \circ T) \cdot g \, d\mu \quad \text{for } g \in L_\infty(\mu) \text{ supported on some } \{\varphi \leq M+1\}.$$

But $\{\varphi \leq M\} \nearrow X$ as $M \to \infty$, showing that integrating against these test functions g uniquely determines a function R. Hence $R \circ T = R$ a.e. as required.

(iii) We verify the assertions about uniform convergence in (3.11) and (3.13). In the first case we note that (3.11), which implicitely states that $1_Y \cdot (R_n \circ T - R_n) \in L_1(\mu)$ for each $n \geq 0$, implies (e.g. via the principle of uniform boundedness, cf. Theorem II.1.11 of [**DS**] or I.A.7 of [**Wo**]) that $\{1_Y \cdot (R_n \circ T - R_n)\}_{n \geq 0}$ is bounded in $L_1(\mu)$, i.e. that the linear functionals $\psi_n : L_\infty(\mu) \to \mathbb{R}$, $\psi_n(u) := \int_Y |R_n \circ T - R_n| \cdot u \, d\mu$, $n \geq 0$, are equicontinuous, and our claim follows by $L_\infty(\mu)$-precompactness of $\mathfrak{H}$, cf. Remark 3.1.

Similarly, since $\mathcal{W} := \{u - u^* : u, u^* \in \mathfrak{H}\}$ is $L_\infty(\mu)$-precompact as well, uniformity of (3.13) in $u, u^* \in \mathfrak{H}$ follows by the same type of argument, as the $\rho_n : L_\infty(\mu) \to \mathbb{R}$, $\rho_n(w) := \int_Y R_n \cdot w \, d\mu$, $n \geq 0$, due to (3.12), also are equicontinuous.

(iv) For any fixed $s > 0$ the measurable function $R_\gamma(s) : X \to [0, \infty]$ satisfies, due to monotone convergence,

$$\int_Y R_\gamma(s) \, d\mu = \sum_{n \geq 0} \left(\int_Y R_n \, d\mu \right) \gamma_n e^{-ns} \leq G(s) \cdot \sup_{n \geq 0} \int_Y R_n \, d\mu < \infty$$

($\{1_Y \cdot R_n\}_{n \geq 0}$ being bounded in $L_1(\mu)$ by (3.12)). Hence $R_\gamma(s)$ is a.e. finite and integrable on Y. The remaining statements about $R_\gamma(s)$ follow exactly as in the proof of the previous proposition. □

Remark 3.2 (**Version for L_p, $p \in (1, \infty)$**). By the same type of argument, the statement of the proposition remains true with L_1 and L_∞ replaced by L_p and L_q, respectively, for arbitrary $p \in (1, \infty)$ and $1/p + 1/q = 1$.

Remark 3.3 (**Limitations of our approach**). **a)** While our proof of the Darling-Kac theorem requires, via Proposition 3.2, condition (2.2), the weaker assumption (2.5) might still be sufficient. **b)** It is also natural to ask whether the assumptions of strong precompactness could be replaced by weak precompactness. Our proofs below depend crucially on the uniform convergence in (3.6) and (3.13), respectively. Under the assumptions of Proposition 3.1, the corresponding stronger conclusion would read

$$\int_X R_n \cdot u \, d\mu - \int_X R_n \cdot u^* \, d\mu \longrightarrow 0 \qquad \begin{array}{l} \text{as } n \to \infty, \text{ uniformly in } u, u^* \text{ from any} \\ \text{weakly } L_1(\mu)\text{-precompact } \mathfrak{H} \subseteq \mathcal{D}(\mu), \end{array}$$

which (in that situation) is equivalent to $R_n \xrightarrow{\tau} 1_X$ in the Mackey toplogy $\tau = \tau(L_\infty(\mu), L_1(\mu))$, cf. [**Sc**]. This, however, would entail $R_n \xrightarrow{\mu} 1_X$, cf. [**No**], which, for the relevant choice of R_n, is not the case in the $\alpha < 1$ situations of our theorems. (Still, we can't rule out that an argument more sensitive to the particular densities u, u^* we need to deal with could give more general results.)

We will use the previous two propositions in the following way:

REMARK 3.4 (**Applying the equivalent moments principles**). In the situation of either proposition, with $\mathfrak{H}, \mathfrak{H}^*$ precompact sets of densities as specified there, if for some $u_n \in \mathfrak{H}$, $n \geq 0$, we have

$$\varliminf_{n\to\infty} \int_X R_n \cdot u_n \, d\mu > 0,$$

then

$$\int_X R_n \cdot u \, d\mu \sim \int_X R_n \cdot u_n \, d\mu \qquad \begin{array}{c}\text{as } n \to \infty, \\ \text{uniformly in } u \in \mathfrak{H}^*.\end{array}$$

This is a uniform version of the "equivalent moments principle", Lemma 4.4, of [**TZ**]. Similarly, if there are $\kappa \in [0, \infty)$ and $u_s \in \mathfrak{H}$, $s > 0$, such that

$$\int_X R_\gamma(s) \cdot u_s \, d\mu \sim \kappa \cdot G(s) \quad \text{ as } s \searrow 0,$$

then

$$\int_X R_\gamma(s) \cdot u \, d\mu \sim \kappa \cdot G(s) \qquad \begin{array}{c}\text{as } s \searrow 0, \\ \text{uniformly in } u \in \mathfrak{H}^*.\end{array}$$

We also need the following related observation:

REMARK 3.5 (**Normalized transforms**). Let $\mathfrak{H}$ be a (strongly) precompact subset of a Banach space $\mathfrak{B}$, $(D_n)_{n\geq 0}$ a sequence in $\mathfrak{B}$, and $d_n \geq 0$, $n \geq 0$, such that $d_n^{-1} D_n \in \mathfrak{H}$ whenever $d_n > 0$. Suppose that $\beta_n \in [0, \infty)$, $n \geq 0$, are such that $\sum_{n\geq 0} \beta_n d_n \in (0, \infty)$. Then

$$u := \frac{\sum_{n\geq 0} \beta_n D_n}{\sum_{n\geq 0} \beta_n d_n} \quad \text{belongs to the closed convex hull } \overline{\mathrm{co}}_{\mathfrak{B}}(\mathfrak{H}) \text{ of } \mathfrak{H} \text{ in } \mathfrak{B},$$

and (by Mazur's theorem, cf. Theorem V.2.6 of [**DS**]) $\overline{\mathrm{co}}_{\mathfrak{B}}(\mathfrak{H})$ is compact. We will consider sets $\mathfrak{H}$ of probability densities as specified in Proposition 3.1 (or 3.2), precompact in $\mathfrak{B} = L_q(\mu)$, $q = 1$ (or $q = \infty$). Assume that $D_n : X \to [0, \infty)$, $n \geq 0$, are integrable functions such that

$$\frac{D_n}{\int_X D_n \, d\mu} \in \mathfrak{H} \quad \text{ for } n \geq 0 \text{ with } \int_X D_n \, d\mu > 0,$$

and that $(\gamma_n)_{n\geq 0}$ is a sequence in $[0, \infty)$ such that

$$\sum_{n\geq 0} \left(\int_X D_n \, d\mu \right) \gamma_n \, e^{-ns} \in (0, \infty) \quad \text{ for } s > 0.$$

Then each of the functions $u_s : X \to [0, \infty)$, $s > 0$, defined by

$$u_s := \frac{\sum_{n\geq 0} D_n \, \gamma_n \, e^{-ns}}{\sum_{n\geq 0} \left(\int_X D_n \, d\mu \right) \gamma_n \, e^{-ns}},$$

belongs to the (compact) closed convex hull $\overline{\mathrm{co}}_q(\mathfrak{H}) \subseteq \mathcal{D}(\mu)$ of $\mathfrak{H}$ in $L_q(\mu)$.

With the aid of these results we are able to generalize the crucial analytic Lemmas 4.2 and 4.3 of [**TZ**].

Proposition 3.3 (**Integrating transforms**). *Let T be a c.e.m.p.t. on the σ-finite measure space $(X, \mathcal{A}, \mu)$, $Y \in \mathcal{A}$ with $0 < \mu(Y) < \infty$, and assume that $\mathfrak{H} \subseteq \mathcal{D}(\mu)$ and $R_n : X \to [0, \infty)$, $n \geq 0$, satisfy the assumptions of either Proposition 3.1 or Proposition 3.2.*

Let $v_n : Y \to [0, \infty)$, $n \geq 0$, be bounded measurable functions with $\int_Y \sum_{n\geq 0} v_n \, d\mu > 0$, and $(b_n)_{n\geq 0}$ be a sequence in $[0, \infty)$ such that $B(s) := \sum_{n\geq 0} b_n \, e^{-ns} \in (0, \infty)$ for $s > 0$. Assume that

$$\frac{\sum_{k=0}^{n} v_k}{\sum_{k=0}^{n} \int_Y v_k \, d\mu} \in \mathfrak{H} \tag{3.18}$$

for all $n \in \mathbb{N}_0$ for which the denominator is positive, and that for some $\vartheta \in [0, \infty)$,

$$\sum_{k=0}^{n} \int_Y v_k \, d\mu \sim \vartheta \cdot \sum_{k=0}^{n} b_k \quad \text{as } n \to \infty. \tag{3.19}$$

Let $(\gamma_n)_{n\geq 0}$ be a sequence in $[0, \infty)$ with $\sum_{n\geq 0} \gamma_n = \infty$ and such that $G(s) := \sum_{n\geq 0} \gamma_n \, e^{-ns} \in (0, \infty)$ for $s > 0$, and consider the weighed Laplace transform $R_\gamma(s) := \sum_{n\geq 0} R_n \, \gamma_n e^{-ns}$.

a) *Suppose that for some $\kappa \in [0, \infty)$,*

$$\int_Y \left(\sum_{n\geq 0} v_n \, e^{-ns} \right) \cdot R_\gamma(s) \, d\mu \sim \kappa\vartheta \cdot B(s) \, G(s) \quad \text{as } s \searrow 0. \tag{3.20}$$

If $\vartheta > 0$, then, for any $u \in \mathcal{D}(\mu)$ resp. $u \in \mathcal{D}(\mu) \cap L_\infty(\mu)$ with $\int_Y u \, d\mu = 1$,

$$\int_X R_\gamma(s) \cdot u \, d\mu \sim \kappa \cdot G(s) \quad \text{as } s \searrow 0. \tag{3.21}$$

b) *Assume that $r = 0$, or that $r \in \mathbb{N}$ and $B \in \mathcal{R}_{-\rho}(0)$ for some $\rho \in [0, \infty)$. Suppose also that for some $\kappa \in [0, \infty)$ and some $u \in \mathcal{D}(\mu)$ resp. $u \in \mathcal{D}(\mu) \cap L_\infty(\mu)$ with $\int_Y u \, d\mu = 1$,*

$$\int_X R_\gamma(s) \cdot u \, d\mu \sim \kappa \cdot G(s) \quad \text{as } s \searrow 0, \tag{3.22}$$

then, as $s \searrow 0$,

$$\int_Y \left(\sum_{n\geq 0} n^r v_n \, e^{-ns} \right) \cdot R_\gamma(s) \, d\mu \sim \kappa\vartheta \cdot (-1)^r r! \binom{-\rho}{r} \left(\frac{1}{s} \right)^r B(s) \, G(s). \tag{3.23}$$

c) *If, in the situation of b), $v_n \searrow 0$ a.e. on Y as $n \to \infty$, so that $v_n = \sum_{k>n} w_k$ with $w_n \geq 0$, $n \geq 1$, measurable, then, for all $r \geq 1$, as $s \searrow 0$,*

$$\int_Y \left(\sum_{n\geq 1} n^r w_n \, e^{-ns} \right) \cdot R_\gamma(s) \, d\mu \sim \kappa\vartheta \cdot (-1)^{r-1} r! \binom{1-\rho}{r} \left(\frac{1}{s} \right)^{r-1} B(s) \, G(s). \tag{3.24}$$

Proof. **a)** Due to (3.19) we have

$$\sum_{n\geq 0} \left(\int_Y v_n \, d\mu \right) e^{-ns} \sim \vartheta \cdot B(s) \quad \text{as } s \searrow 0, \tag{3.25}$$

and the relation (3.20) thus shows that

$$\int_Y R_\gamma(s) \cdot u_s \, d\mu \sim \kappa \cdot G(s) \quad \text{as } s \searrow 0,$$

where

$$u_s := \frac{\sum_{n\geq 0} v_n \, e^{-ns}}{\sum_{n\geq 0}(\int_Y v_n \, d\mu) \, e^{-ns}} = \frac{\sum_{n\geq 0} \left(\sum_{k=0}^n v_k\right) e^{-ns}}{\sum_{n\geq 0} \left(\sum_{k=0}^n \int_Y v_k \, d\mu\right) e^{-ns}}, \quad s > 0, \tag{3.26}$$

so that (cf. Remark 3.5) $u_s \in \overline{\mathrm{co}}_q(\mathfrak{H})$, $s > 0$, with $q = 1$ (or $q = \infty$) in the situation of Proposition 3.1 (or 3.2). We can thus apply the respective proposition to obtain the desired conclusion (cf. Remark 3.4).

b) Suppose first that $r = 0$ (but not necessarily $B \in \mathcal{R}_{-\rho}(0)$). Recalling (3.25) we see that (3.23) essentially restates part c) of the relevant proposition, which (cf. Remark 3.4) guarantees that

$$\int_Y R_\gamma(s) \cdot u_s \, d\mu \sim \kappa \cdot G(s) \quad \text{as } s \searrow 0. \tag{3.27}$$

By Remark 3.5, all the densities u_s belong to the $L_q(\mu)$-compact set $\overline{\mathrm{co}}_q(\mathfrak{H})$.

Assume now that $r \geq 1$ and $B \in \mathcal{R}_{-\rho}(0)$. We let $V_n := \sum_{k=0}^n v_k$, $B_n := \sum_{k=0}^n b_k$, $n \geq 0$. On Y we have, for $s > 0$,

$$\sum_{n\geq 0} n^r v_n \, e^{-ns} = (1-e^{-s}) \sum_{n\geq 0} (n+1)^r V_n \, e^{-ns} - \sum_{n\geq 0} \left((n+1)^r - n^r\right) V_n \, e^{-ns}. \tag{3.28}$$

Since

$$\frac{V_n}{\int_Y V_n \, d\mu} \in \mathfrak{H} \quad \text{for all } n \in \mathbb{N}_0 \text{ with } \textstyle\int_Y V_n \, d\mu > 0, \tag{3.29}$$

(and using $(n+1)^r \sim n^r$), the $r = 0$ case discussed above, applied to $n^r V_n$, therefore yields, for $s \searrow 0$,

$$\int_Y \left(\sum_{n\geq 0} (n+1)^r V_n \, e^{-ns} \right) \cdot R_\gamma(s) \, d\mu \sim \kappa \cdot \left(\sum_{n\geq 0} n^r \left(\int_Y V_n \, d\mu \right) e^{-ns} \right) \cdot G(s).$$

Now $\sum_{n\geq 0} B_n e^{-ns} \sim B(s)/s \in \mathcal{R}_{-(\rho+1)}(0)$, and by (3.19) and part b) of Lemma 4.1 in [**TZ**] (the differentiation lemma),

$$\sum_{n\geq 0} n^r \left(\int_Y V_n \, d\mu \right) e^{-ns} \sim \vartheta \cdot \sum_{n\geq 0} n^r B_n \, e^{-ns} \sim \vartheta \cdot c_{\rho+1,r} \left(\frac{1}{s} \right)^{r+1} B(s)$$

as $s \searrow 0$, where $c_{\rho,r} := (-1)^r r! \binom{-\rho}{r}$ for $\rho \in \mathbb{R}$ and $r \geq 0$, so that

$$(1-e^{-s}) \int_Y \left(\sum_{n\geq 0} (n+1)^r V_n \, e^{-ns} \right) \cdot R_\gamma(s) \, d\mu \sim \kappa\vartheta \cdot c_{\rho+1,r} \left(\frac{1}{s} \right)^r B(s) \, G(s). \tag{3.30}$$

Due to (3.29) and $((n+1)^r - n^r) \sim r \, n^{r-1}$ as $n \to \infty$, we obtain analogously that

$$\int_Y \left(\sum_{n\geq 0} \left((n+1)^r - n^r\right) V_n \, e^{-ns} \right) \cdot R_\gamma(s) \, d\mu \sim \kappa\vartheta \cdot r \, c_{\rho+1,r-1} \left(\frac{1}{s} \right)^r B(s) \, G(s) \tag{3.31}$$

as $s \searrow 0$. Recalling (3.28), we can combine (3.30) and (3.31) to find

$$\int_Y \left(\sum_{n\geq 1} n^r v_n \, e^{-ns} \right) \cdot R_\gamma(s) \, d\mu \sim \kappa\vartheta \cdot \underbrace{(c_{\rho+1,r} - r \, c_{\rho+1,r-1})}_{=c_{\rho,r}} \left(\frac{1}{s}\right)^r B(s) \, G(s)$$

as $s \searrow 0$. This completes the proof of (3.23).

c) We need to sharpen (3.23) to get (3.24) for $r \geq 1$. As for the case $r \geq 1$ above, this is done by exactly the same argument as in the proof of part b) of Lemma 4.3 in [**TZ**]. □

We finally review another important principle which (though, unfortunately, not too widely known) has proved very useful for establishing distributional limit theorems for ergodic transformations, and is of considerable interest in itself. The final versions of our limit theorems which assert strong distributional convergence crucially depend on it.

Proposition 3.4 (**Strong distributional convergence of asymptotically invariant sequences**). *Let T be an ergodic m.p.t. on the σ-finite measure space $(X, \mathcal{A}, \mu)$, and assume that $R_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, are measurable functions satisfying*

$$R_n \circ T - R_n \overset{\mu}{\longrightarrow} 0 \quad \text{or} \quad \frac{R_n \circ T}{R_n} \overset{\mu}{\longrightarrow} 1 \quad \text{as } n \to \infty.$$

Then the following statements hold:

a) *If R is a random variable taking values in $\overline{\mathbb{R}}$ and $R_n \overset{\nu}{\Longrightarrow} R$ for some $\nu \ll \mu$, then $R_n \overset{\mathcal{L}(\mu)}{\Longrightarrow} R$.*

b) *There are $n_k \nearrow \infty$ and some random variable R taking values in $\overline{\mathbb{R}}$, such that $R_{n_k} \overset{\mathcal{L}(\mu)}{\Longrightarrow} R$.*

See [**Ea**] for the probability preserving case. For ergodic sums, this result can be found in [**A1**] or [**A0**], §3.6, but (as pointed out in [**T4**]), the same argument applies for general asymptotically invariant sequences (R_n). This proof first establishes the compactness property b) and then derives a) by a subsequence-in-subsequence argument. We take the opportunity to offer an alternative direct proof, showing that the proposition follows naturally from Theorem 3.1 via Proposition 3.1.a).

Proof of Proposition 3.4. **a)** Assume that $R_n \overset{\nu}{\Longrightarrow} R$ for some $\nu \ll \mu$. Letting $u := d\nu/d\mu \in \mathcal{D}(\mu)$ this is equivalent to saying that

$$\int_X g(R_n) \cdot u \, d\mu \longrightarrow \mathbb{E}[g(R)] \quad \text{as } n \to \infty$$

for all $g \in \mathcal{C}_L := \{g \in \mathcal{C}(\overline{\mathbb{R}}, \mathbb{R}) : g \mid_{\mathbb{R}} \text{ is Lipschitz}\}$. For g of this type, let $\mathrm{Lip}(g)$ denote a suitable Lipschitz constant. We claim that we can apply part a) of Proposition 3.1 to the sequence $(g(R_n))_{n\in\mathbb{N}}$: we have $\|g(R_n)\|_\infty \leq \|g\|_\infty$ for all $n \in \mathbb{N}$, and

$$\{|g(R_n) \circ T - g(R_n)| > \varepsilon\} \subseteq \{|R_n \circ T - R_n| > \varepsilon/\mathrm{Lip}(g)\},$$

so that $g(R_n) \circ T - g(R_n) \xrightarrow{\mu} 0$, provided that $R_n \circ T - R_n \xrightarrow{\mu} 0$. If we merely assume $R_n \circ T / R_n \xrightarrow{\mu} 1$, continuity of g in $\pm\infty$ shows that we may focus on the subset of X where $R_n, R_n \circ T$ are in some compact real interval, where the same argument applies.

The proposition thus implies

$$\int_X g(R_n) \cdot u^* \, d\mu \longrightarrow \mathbb{E}[g(R)] \quad \text{as } n \to \infty \text{ for all } u^* \in \mathcal{D}(\mu),$$

and hence, since $g \in \mathcal{C}_L$ was arbitrary, our claim.

b) The second statement follows at once from a) and the classical Helly compactness theorem. □

Remark 3.6. Theorem 3.1, Proposition 3.1, and hence also Proposition 3.4, remain valid if T is only assumed to be nonsingular (meaning that $\mu \circ T^{-1} \ll \mu$) instead of measure preserving.

4. Proof of the Darling-Kac theorem

We are now ready for the proof of our version of the Darling-Kac theorem.

Proof of Theorem 2.1. **(i)** As a consequence of Hopf's ratio ergodic theorem (cf. [**A0**], [**KK**], or [**Z5**]), it is enough to consider a single function f. We choose $f := 1_Y$ and let $S_n := \sum_{j=1}^{n} 1_Y \circ T^j$, $n \geq 0$. Due to $a_n \to \infty$, the sequence $(a_n^{-1} S_n)_{n \geq 1}$ clearly is asymptotically invariant in measure, showing that we need only consider a single probability measure ν, cf. Proposition 3.4. As the limit distribution is determined by its moments, it is enough to verify that

$$\int_Y \left(\frac{S_n}{a_n} \right)^r d\mu_Y \longrightarrow \mu(Y)^r \, \mathbb{E}[\mathcal{M}_\alpha^r] = \mu(Y)^r \, r! \, \frac{(\Gamma(1+\alpha))^r}{\Gamma(1+r\alpha)} \,, \quad r \geq 0. \tag{4.1}$$

This will be done by inductively proving that, for all $r \geq 0$,

$$\sum_{n \geq 0} \left(\int_Y S_n^r \, d\mu_Y \right) e^{-ns} \sim \frac{r!}{s} \left(\frac{1}{s \, Q_Y(s)} \right)^r \qquad \text{as } s \searrow 0, \tag{$\blacklozenge_r$}$$

where $Q_Y(s) := \sum_{n \geq 0} q_n(Y) e^{-ns}$, $s > 0$. Due to our assumption on the wandering rate (and KTT) we have, for $s > 0$,

$$Q_Y(s) = \left(\frac{1}{s} \right)^{1-\alpha} \ell\left(\frac{1}{s} \right), \quad \text{and} \quad w_n(Y) \sim \frac{\mu(Y) n^{1-\alpha} \ell(n)}{\Gamma(2-\alpha)} \quad \text{as } n \to \infty \tag{4.2}$$

with ℓ slowly varying at infinity. Since $(\int_Y S_n^r \, d\mu_Y)_{n \geq 0}$ is non-decreasing, ($\blacklozenge_r$) implies (4.1) for any sequence (a_n) satisfying

$$a_n \sim \frac{1}{\Gamma(1+\alpha)\Gamma(2-\alpha)} \cdot \frac{n}{w_n(Y)} \qquad \text{as } s \searrow 0,$$

and the $r = 1$ case of (4.1) shows that we may take $a_n := \mu(Y)^{-1} a_n(Y)$, $n \geq 0$.

For $r = 0$, we have $S_n^r = 1$, hence $\int_X S_n^0 \cdot u \, d\mu = 1$ and therefore $\sum_{n \geq 0} (\int_X S_n^0 \cdot u \, d\mu) e^{-ns} \sim 1/s$ as $s \searrow 0$ for any $u \in \mathcal{D}(\mu)$, showing, in particular, ($\blacklozenge_0$).

A simple argument (compare (5.9) of [**TZ**]) shows that our uniform sweeping assumption implies that for every $r \geq 0$,

$$\int_Y S_n^r \cdot u \, d\mu \asymp \int_Y S_n^r \, d\mu_Y \quad \text{ as } n \to \infty \text{ uniformly in } u \in \overline{\mathrm{co}}_\infty(\mathfrak{H}_Y), \tag{4.3}$$

which trivially entails an analogous statement for the corresponding Laplace transforms.

(ii) Before proving ($\blacklozenge_r$) for $r \geq 1$, we first establish the weaker statement that for all $r \geq 0$,

$$\sum_{n\geq 0} \left(\int_Y S_n^r \, d\mu_Y \right) e^{-ns} \asymp \frac{1}{s} \left(\frac{1}{s\, Q_Y(s)} \right)^r \qquad \text{as } s \searrow 0. \tag{$\lozenge_r$}$$

For $r = 0$ this is clear from ($\blacklozenge_0$). For the inductive step we assume ($\lozenge_j$) for $0 \leq j < r$. Lemma 5.1 of [**TZ**] shows that, for $s > 0$,

$$\begin{aligned} &\int_Y \left(\sum_{n\geq 0} \widehat{T}^n 1_{Y_n} \, e^{-ns} \right) \cdot \left(\sum_{n\geq 0} S_n^r \, e^{-ns} \right) d\mu \\ &= \frac{1}{1-e^{-s}} \sum_{j=0}^{r-1} \binom{r}{j} \int_Y \left(\sum_{n\geq 1} \widehat{T}^n 1_{Y\cap\{\varphi=n\}} \, e^{-ns} \right) \cdot \left(\sum_{n\geq 0} S_n^j \, e^{-ns} \right) d\mu. \end{aligned} \tag{4.4}$$

We consider the right-hand side of (4.4): Note (cf. (5.5) of [**TZ**]) that

$$1_Y - \sum_{n\geq 1} \widehat{T}^n 1_{Y\cap\{\varphi=n\}} \, e^{-ns} = (1 - F_Y(s)) \, \frac{\sum_{n\geq 0} \widehat{T}^n 1_{Y_n} \, e^{-ns}}{Q_Y(s)} \quad \text{ a.e.}$$

with $F_Y(s) := \sum_{k\geq 1} f_k(Y) \, e^{-ks} \to 1$ as $s \searrow 0$, while, due to (2.2) and Remark 3.5, $\{u_s\}_{s>0} \subseteq \overline{\mathrm{co}}_\infty(\mathfrak{H}_Y)$ is a precompact (hence bounded) subset of $L_\infty(\mu)$, where

$$u_s := \frac{\sum_{n\geq 0} \widehat{T}^n 1_{Y_n} \, e^{-ns}}{\sum_{n\geq 0} (\int_Y \widehat{T}^n 1_{Y_n} \, d\mu) \, e^{-ns}} = \frac{\sum_{n\geq 0} \widehat{T}^n 1_{Y_n} \, e^{-ns}}{\mu(Y) Q_Y(s)}, \quad s > 0.$$

Consequently,

$$\sum_{n\geq 1} \widehat{T}^n 1_{Y\cap\{\varphi=n\}} \, e^{-ns} \longrightarrow 1_Y \quad \text{ uniformly as } s \searrow 0, \tag{4.5}$$

which implies that for any $j \geq 0$, as $s \searrow 0$,

$$\int_Y \left(\sum_{n\geq 1} \widehat{T}^n 1_{Y\cap\{\varphi=n\}} \, e^{-ns} \right) \cdot \left(\sum_{n\geq 0} S_n^j \, e^{-ns} \right) d\mu \sim \int_Y \left(\sum_{n\geq 0} S_n^j \, e^{-ns} \right) d\mu.$$

Observe now that for $0 \leq j < r-1$ we have $\int_Y S_n^j \, d\mu = o(\int_Y S_n^{r-1} \, d\mu)$ as $n \to \infty$ (since $S_n \to \infty$ a.e. on X), so that the term with $j = r-1$ dominates the asymptotics of the right-hand side of (4.4). We therefore conclude that for any $r \geq 0$,

$$\int_Y \left(\sum_{n\geq 0} \widehat{T}^n 1_{Y_n} \, e^{-ns} \right) \cdot \left(\sum_{n\geq 0} S_n^r \, e^{-ns} \right) d\mu \sim \frac{r}{s} \int_Y \left(\sum_{n\geq 0} S_n^{r-1} \, e^{-ns} \right) d\mu \tag{4.6}$$

as $s \searrow 0$. Combining this with (4.3), we obtain

$$\begin{aligned}(4.7)\qquad \sum_{n\geq 0}\left(\int_Y S_n^r\,d\mu\right)e^{-ns} &\asymp \mu(Y)\int_Y\left(\sum_{n\geq 0}S_n^r\,e^{-ns}\right)\cdot u_s\,d\mu \\ &\sim \frac{r}{sQ_Y(s)}\sum_{n\geq 0}\left(\int_Y S_n^{r-1}\,d\mu\right)e^{-ns}\end{aligned}$$

as $s \searrow 0$, and $(\lozenge_r)$ follows.

(iii) Having established $(\lozenge_r)$, an easy argument (cf. the second part of Lemma 5.3 in [**TZ**]) shows that for all $r \geq 0$,

$$\int_Y S_n^r\,d\mu = O\left(\left(\frac{n}{Q_Y(1/n)}\right)^r\right) = O\left(\left(\frac{n^\alpha}{\ell(n)}\right)^r\right) \quad \text{as } n \to \infty.$$

Letting (a_n) be any sequence in $(0,\infty)$ with

$$a_n \sim \frac{n^\alpha}{\mu(Y)\Gamma(1+\alpha)\ell(n)} \sim \frac{1}{\Gamma(1+\alpha)\Gamma(2-\alpha)}\cdot\frac{n}{w_n(Y)} \quad \text{as } n \to \infty,$$

we therefore see that the sequence of moments

$$(4.8)\qquad \left\{\int_Y\left(\frac{S_n}{a_n}\right)^r d\mu_Y\right\}_{n\geq 0} \quad \text{is bounded for every } r \geq 0,$$

and since $S_n = S_{n-k}\circ T^k$ on $Y^c\cap\{\varphi = k\}$, the same is true with Y and μ_Y replaced by Y^M and μ_{Y^M}, with $M \geq 1$ arbitrary but fixed. Moreover, as $|S_n\circ T - S_n| \leq 1$, we can use the mean-value theorem to ensure that for arbitrary $r \geq 0$, as $n \to \infty$,

$$(4.9)\quad \int_{Y^M}\left|\left(\frac{S_n}{a_n}\right)^r\circ T - \left(\frac{S_n}{a_n}\right)^r\right| d\mu_{Y^M} = O\left(\frac{1}{a_n}\int_{Y^M}\left(\frac{S_n}{a_n}\right)^{r-1} d\mu_{Y^M}\right) \longrightarrow 0.$$

(iv) We can now improve the argument to tackle the inductive step for the proof of $(\blacklozenge_r)$: Assume $(\blacklozenge_j)$ for $0 \leq j < r$ and recall (4.6) to see that

$$\int_Y\left(\sum_{n\geq 0}S_n^r\,e^{-ns}\right)\cdot u_s\,d\mu \sim \frac{r!}{s}\left(\frac{1}{s\,Q_Y(s)}\right)^r \qquad \text{as } s \searrow 0.$$

Our previous considerations show that choosing $v_n := \widehat{T}^n 1_{Y_n}$, $R_n := (S_n/a_n)^r$, and $\gamma_n := a_n^r$, we are in the situation of part a) of Proposition 3.3 (precompactness being immediate from (4.8)), so that

$$\int_Y\left(\sum_{n\geq 0}S_n^r\,e^{-ns}\right)\cdot u_s\,d\mu \sim \int_Y\left(\sum_{n\geq 0}S_n^r\,e^{-ns}\right)d\mu_Y \qquad \text{as } s \searrow 0,$$

completing the proof of $(\blacklozenge_r)$. □

5. Proof of the Arcsine laws

Our new tools enable us to prove the generalized arcsine laws following a strategy parallel to the argument used above.

Proof of Theorem 2.2. **(i)** We write $L_n := \sum_{k=1}^{n} 1_{A_1} \circ T^k,\ n \geq 0$, and prove inductively that for all $u \in \mathcal{D}(\mu)$ and all $r \geq 0$,

$$(\clubsuit_r) \qquad \sum_{n\geq 0} \left(\int_X L_n^r \cdot u \, d\mu \right) e^{-ns} \sim r! \, \mathbb{E}[\mathcal{L}_{\alpha,\beta}^r] \left(\frac{1}{s} \right)^{r+1} \qquad \text{as } s \searrow 0.$$

By KTT and monotonicity of the sequences $(\int_Y L_n^r \cdot u \, d\mu)_{n\geq 1}$, the asymptotic equations $(\clubsuit_r)$, $r \geq 0$, imply

$$\int_X L_n^r \cdot u \, d\mu \sim \mathbb{E}[\mathcal{L}_{\alpha,\beta}^r] \cdot n^r \qquad \text{as } n \to \infty,$$

which entails $n^{-1} L_n \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{L}_{\alpha,\beta}$, since the distribution of $\mathcal{L}_{\alpha,\beta}$ is determined by its moments. Note that $(L_n/n)_{n\geq 1}$ is asymptotically T-invariant in measure (so that, as in the proof of the DK-theorem, it would be enough to check convergence of the moments for μ_Y if we appeal to Proposition 3.4). Moreover, for every $r \geq 1$, the sequence $R_n := (L_n/n)^r$, $n \geq 1$, satisfies

$$R_n \circ T - R_n \overset{\mu}{\longrightarrow} 0 \text{ as } n \to \infty.$$

This follows from the $r = 1$ case, using the mean-value theorem and the fact that $L_n, L_n \circ T \leq n$.

For $r = 0$, we have $L_n^r = 1$, and therefore $\sum_{n\geq 0} (\int_Y L_n^0 \cdot u \, d\mu) e^{-ns} \sim 1/s$ for any $u \in \mathcal{D}(\mu)$, that is $(\clubsuit_0)$.

(ii) For the inductive step, assume $(\clubsuit_j)$ for $0 \leq j < r$. According to Lemma 6.1 of [**TZ**] we have, for $s > 0$,

(5.1)

$$\begin{aligned} &(1 - e^{-s}) \int_Y \left(\sum_{n\geq 0} \left(\sum_{k>n} \widehat{T}^k 1_{Y\cap\{\varphi=k\}} \right) e^{-ns} \right) \left(\sum_{n\geq 0} L_n^r e^{-ns} \right) d\mu \\ &= e^{-s} \sum_{j=0}^{r-1} \binom{r}{j} \int_Y \left(\sum_{n\geq 1} n^{r-j} \, \widehat{T}^{n+1} 1_{Y\cap T^{-1}A_1\cap\{\varphi=n+1\}} \, e^{-ns} \right) \left(\sum_{n\geq 0} L_n^j e^{-ns} \right) d\mu \\ &+ \mu(Y)(-1)^r Q_{Y,A_1}^{(r)}(s), \end{aligned}$$

where $Q_{Y,A_1}(s) := \sum_{n\geq 1} q_n(Y, A_1) e^{-ns}$. We are going to turn this into an asymptotic recursion formula for $\sum_{n\geq 0} (\int_Y L_n^r \, d\mu_Y) e^{-ns}$. By our assumptions on the wandering rates involved, we see via KTT and the differentiation lemma (Proposition 4.2 and Lemma 4.1 of [**TZ**]) that there is some $\ell \in \mathcal{R}_0$ such that

$$(5.2)\qquad Q_Y(s) = \sum_{n\geq 1} q_n(Y)e^{-ns} = \left(\frac{1}{s}\right)^{1-\alpha} \ell\left(\frac{1}{s}\right) \qquad \text{for } s>0,$$

$$Q_{Y,A_1}(s) \sim \beta\, Q_Y(s) = \beta\left(\frac{1}{s}\right)^{1-\alpha} \ell\left(\frac{1}{s}\right) \qquad \text{as } s \searrow 0, \text{ and}$$

$$Q^{(r)}_{Y,A_1}(s) \sim \beta\, r! \binom{\alpha-1}{r}\left(\frac{1}{s}\right)^{r+1-\alpha} \ell\left(\frac{1}{s}\right) \qquad \text{as } s \searrow 0 \text{ for all } r \geq 0.$$

We consider the right-hand side of (5.1): For fixed j, we take $\rho := 1-\alpha$, $\vartheta := \mu(Y)$, $\kappa := \mathbb{E}[\mathcal{L}^j_{\alpha,\beta}]$, $b_n := q_n(Y, A_1)$, as well as $w_n := \widehat{T}^{n+1} 1_{Y\cap T^{-1}A_1 \cap \{\varphi = n+1\}}$, $\gamma_n := n^j$ (so that $G(s) \sim j!(1/s)^{j+1}$ as $s \searrow 0$), and $R_n := (L_n/n)^j$, $n \geq 1$. As shown in (6.5) of [**TZ**], $v_{n-1} = \sum_{k>n-1} w_k = \widehat{T}^n 1_{A_1\cap Y_n}$. Due to our compactness assumption (2.7), and ($\clubsuit_j$), we can apply part c) of Proposition 3.3 to obtain

$$\int_Y \left(\sum_{n\geq 1} n^{r-j}\, \widehat{T}^{n+1} 1_{Y\cap T^{-1}A_1\cap\{\varphi=n+1\}}\, e^{-ns}\right)\left(\sum_{n\geq 0} L_n^j e^{-ns}\right) d\mu$$
$$\sim \mu(Y) r! \binom{r}{j}^{-1} (-1)^{r-j-1} \binom{\alpha}{r-j} \mathbb{E}[\mathcal{L}^j_{\alpha,\beta}] \cdot \left(\frac{1}{s}\right)^r Q_{Y,A_1}(s) \qquad \text{as } s \searrow 0.$$

Summing over j, and using (5.2), we find for the complete right-hand side of (5.1) that

$$e^{-s} \sum_{j=0}^{r-1} \binom{r}{j} \int_Y \left(\sum_{n\geq 1} n^{r-j}\, \widehat{T}^{n+1} 1_{Y\cap T^{-1}A_1\cap\{\varphi=n+1\}}\, e^{-ns}\right)\left(\sum_{n\geq 0} L_n^j e^{-ns}\right) d\mu$$
$$+ (-1)^r Q^{(r)}_{Y,A_1}(s)$$
$$\sim \mu(Y) r!\, (-1)^r \beta \left[\sum_{j=0}^{r-1} (-1)^{j+1} \binom{\alpha}{r-j} \mathbb{E}[\mathcal{L}^j_{\alpha,\beta}] + \binom{\alpha-1}{r}\right] \left(\frac{1}{s}\right)^{r+1-\alpha} \ell\left(\frac{1}{s}\right)$$

as $s \searrow 0$. Plugging this into (5.1), and recalling (2.4), we get

$$\int_Y \left(\sum_{n\geq 0} \widehat{T}^n 1_{Y_n}\, e^{-ns}\right) \cdot \left(\sum_{n\geq 0} L_n^r\, e^{-ns}\right) d\mu \sim \mu(Y) r!\, \mathbb{E}[\mathcal{L}^r_{\alpha,\beta}] \cdot \left(\frac{1}{s}\right)^{r+1} Q_Y(s)$$

as $s \searrow 0$, and we may employ part a) of Proposition 3.3 with $\vartheta := \mu(Y)$, $R_n := (L_n/n)^r$, $\gamma_n := n^r$, $\kappa := \mathbb{E}[\mathcal{L}^r_{\alpha,\beta}]$, and $v_n := \widehat{T}^n 1_{Y_n}$, to derive the desired relation ($\clubsuit_r$), thus completing the inductive step. □

The proof of the third limit theorem proceeds along similar lines.

Proof of Theorem 2.3. **(i)** We prove inductively that for all $u \in \mathcal{D}(\mu)$ and all $r \geq 0$,

$$(\spadesuit_r) \qquad \sum_{n\geq 0}\left(\int_Y Z_n^r \cdot u \, d\mu\right) e^{-ns} \sim r!\, \mathbb{E}[\mathcal{Z}_\alpha^r] \left(\frac{1}{s}\right)^{r+1} \qquad \text{as } s \searrow 0,$$

which, by KTT and monotonicity of the sequences $(\int_Y Z_n^r \cdot u \, d\mu)_{n\geq 1}$, is equivalent to

$$\int_Y Z_n^r \cdot u \, d\mu \sim \mathbb{E}[\mathcal{Z}_\alpha^r] \cdot n^r \qquad \text{as } n \to \infty.$$

The assertion of the theorem then follows by the method of moments. According to Lemma 1 of [**T4**], $(Z_n/n)_{n\geq 1}$ is asymptotically T-invariant in measure. We will use that, in fact, for every $r \geq 1$, the sequence $R_n := (Z_n/n)^r$, $n \geq 1$, satisfies

$$R_n \circ T - R_n \xrightarrow{\mu} 0 \text{ as } n \to \infty$$

(same argument as in the proof of Theorem 2.2).

For $r = 0$, we have $Z_n^0 = 1_{\{\varphi \leq n\}} \nearrow 1$ a.e., hence $\int_X Z_n^0 \cdot u \, d\mu \nearrow 1$ and therefore $\sum_{n\geq 0}(\int_X Z_n^0 \cdot u \, d\mu) e^{-ns} \sim 1/s$ for any $u \in \mathcal{D}(\mu)$, that is $(\spadesuit_0)$.

(ii) For the inductive step, assume $(\spadesuit_j)$ for $0 \leq j < r$. According to Lemma 7.1 of [**TZ**], we have, for $s > 0$,

$$(5.3) \quad \int_Y \left(\sum_{n\geq 0} \widehat{T}^n 1_{Y_n} e^{-ns}\right) \cdot \left(\sum_{n\geq 0} Z_n^r e^{-ns}\right) d\mu$$
$$= \frac{1}{1-e^{-s}} \sum_{j=0}^{r-1} \binom{r}{j} \int_Y \left(\sum_{n\geq 1} n^{r-j} \widehat{T}^n 1_{Y\cap\{\varphi=n\}} e^{-ns}\right) \cdot \left(\sum_{n\geq 0} Z_n^j e^{-ns}\right) d\mu.$$

Due to our assumption on the wandering rate (and KTT) we have $Q_Y \in \mathcal{R}_{\alpha-1}(0)$. We first consider the right-hand side of (5.3): For fixed j, we take $\rho := 1-\alpha$, $\vartheta := \mu(Y)$, $b_n = q_n(Y)$, as well as $w_n := \widehat{T}^n 1_{Y\cap\{\varphi=n\}}$ (so that $v_n := \sum_{k>n} w_k = \widehat{T}^n 1_{Y_n}$), $\gamma_n := n^j$, and $R_n := (Z_n/n)^j$, $n \geq 0$. Note that $\mathbb{E}[\mathcal{Z}_\alpha^j]$ may or may not be positive. In either case, due to $(\spadesuit_j)$, we can apply part c) of Proposition 3.3 to obtain

$$\int_Y \left(\sum_{n\geq 1} n^{r-j}\, \widehat{T}^n 1_{Y\cap\{\varphi=n\}}\, e^{-ns}\right) \cdot \left(\sum_{n\geq 0} Z_n^j e^{-ns}\right) d\mu$$
$$\sim \mu(Y) r! \binom{r}{j}^{-1} (-1)^{r-j-1} \binom{\alpha}{r-j} \mathbb{E}[\mathcal{Z}_\alpha^j] \cdot \left(\frac{1}{s}\right)^r Q_Y(s) \qquad \text{as } s \searrow 0.$$

Summing over j we find for the complete right-hand side of (5.3) that

$$\frac{1}{1-e^{-s}} \sum_{j=0}^{r-1} \binom{r}{j} \int_Y \left(\sum_{n\geq 1} n^{r-j} \widehat{T}^n 1_{Y\cap\{\varphi=n\}} e^{-ns}\right) \cdot \left(\sum_{n\geq 0} Z_n^j e^{-ns}\right) d\mu$$
$$\sim \mu(Y) r! \left[\sum_{j=0}^{r-1} (-1)^{r-j+1} \binom{\alpha}{r-j} \mathbb{E}[\mathcal{Z}_\alpha^j]\right] \left(\frac{1}{s}\right)^{r+1} Q_Y(s) \qquad \text{as } s \searrow 0.$$

Plugging this into (5.3), and recalling (2.9), we get

$$\int_Y \left(\sum_{n\geq 0} \widehat{T}^n 1_{Y_n} \, e^{-ns} \right) \cdot \left(\sum_{n\geq 0} Z_n^r \, e^{-ns} \right) \, d\mu \sim \mu(Y) r! \, \mathbb{E}[\mathcal{Z}_\alpha^r] \cdot \left(\frac{1}{s} \right)^{r+1} Q_Y(s)$$

as $s \searrow 0$. We may then employ part a) of Proposition 3.3 with $\vartheta := \mu(Y)$, $R_n := (Z_n/n)^r$, $\gamma_n := n^r$, $\kappa := \mathbb{E}[\mathcal{Z}_\alpha^r]$, and $v_n := \widehat{T}^n 1_{Y_n}$, to conclude that $(\spadesuit_r)$ holds. This completes the inductive step. □

6. Proof of Proposition 2.1 and Theorems 2.4 and 2.5

We turn to the remaining abstract results of Section 2, first offering two different ways of determining the explicit form of the limit laws $\mathcal{K}_{\alpha,\beta}$. A quick argument, pointed out by the referee, uses what we already know about densities and moments of the $\mathcal{L}_{\alpha,\beta}$:

First proof of Proposition 2.1. Fix $\alpha, \beta \in (0,1)$, abbreviate $\mathcal{L} := \mathcal{L}_{\alpha,\beta}$, and let $\mathcal{K}$ denote a random variable with the density given in the conclusion of proposition. Using the explicit densities, it is easy to see that

$$\mathbb{E}\left[\mathcal{K}^r\right] = \frac{1}{1-\beta} \left(\mathbb{E}\left[\mathcal{L}^r\right] - \mathbb{E}\left[\mathcal{L}^{r+1}\right] \right) \qquad \text{for } r \in \mathbb{N}_0.$$

Now use (2.4) to see that the $\mathbb{E}\left[\mathcal{K}^r\right]$ satisfy (2.13). □

Alternatively, it is possible to directly determine the density of $\mathcal{K}_{\alpha,\beta}$ starting from the moment recursion (2.13). We follow the approach indicated in [**L1**] in connection with the $\mathcal{L}_{\alpha,\beta}$:

Second proof of Proposition 2.1. We fix $\alpha, \beta \in (0,1)$, and write $\mathcal{K} := \mathcal{K}_{\alpha,\beta}$. Observe that the recursion formula (2.13) is equivalent to saying that

$$\sum_{r\geq 0} \mathbb{E}[\mathcal{K}^r] \, y^r = \frac{1}{1-\beta+\beta(1-y)^\alpha}. \tag{6.1}$$

Letting ν denote the distribution of $\mathcal{K}$ on $([0,1], \mathcal{B}_{[0,1]})$, (6.1) enables us to identify the *Stieltjes transform* $\widehat{\nu}$ of the latter, which is defined as

$$\widehat{\nu}(w) := \int_0^1 \frac{d\nu(x)}{w+x} \qquad \text{for } w \in \mathbb{C}^- := \mathbb{C} \setminus (-\infty, 0],$$

see e.g. Chapter 8 of [**W1**], or Section 5.13 of [**W2**]. We have

$$\frac{1}{y} \, \widehat{\nu} \left(\frac{1}{y} \right) = \int_0^1 \frac{d\nu(x)}{1+yx} = \mathbb{E}\left[\frac{1}{1+y\mathcal{K}} \right] = \sum_{r\geq 0} \mathbb{E}[\mathcal{K}^r] \, (-y)^r,$$

and hence, due to (6.1),

$$\widehat{\nu}(w) = \frac{w^{\alpha-1}}{(1-\beta)w^\alpha + \beta(1+w)^\alpha} \qquad \text{for } w \in \mathbb{C}^-, \tag{6.2}$$

where the holomorphic power functions are defined using the principal branch of the logarithm on $\mathbb{C}^-$. We use the following variant of Stieltjes' complex inversion formula,

$$\lim_{y\nearrow\pi}\int_a^b \frac{\widehat{\nu}\left(xe^{-iy}\right)-\widehat{\nu}\left(xe^{iy}\right)}{2\pi i}\,dx = \frac{G(b^+)-G(b)}{2} - \frac{G(a^+)-G(a)}{2} \tag{6.3}$$

for $a,b\in[0,1]$, where G denotes the distribution function of ν. This can be derived from the standard inversion formula, see e.g. Theorem 7a on p. 339 of [**W1**], by an argument parallell to that used in the proof of Corollary V.14.1 of [**W2**] (which is (6.3) in the special case of measures with a continuous density). We claim that in our case the limit for the integrand exists,

$$\lim_{y\nearrow\pi}\frac{\widehat{\nu}\left(xe^{-iy}\right)-\widehat{\nu}\left(xe^{iy}\right)}{2\pi i} = \frac{\sin\pi\alpha}{\pi}\frac{(1+b)\,x^{\alpha-1}(1-x)^{\alpha}}{b^2x^{2\alpha}+2bx^{\alpha}(1-x)^{\alpha}\cos\pi\alpha+(1-x)^{2\alpha}},$$

and that convergence is uniform in $x\in[\varepsilon,1-\varepsilon]$ for any fixed $\varepsilon\in(0,1/2)$. It is then clear from (6.3) that this limit function is in fact the density of ν on $(0,1)$.

To validate our assertion about the uniform limit, let $z=z(y):=e^{-iy}$, so that $z,\overline{z}\to-1$ as $y\nearrow\pi$. In particular, as the power functions are holomorphic on $\mathbb{C}^-$, we get $(1+xz)^{\alpha},(1+x\overline{z})^{\alpha}\to(1-x)^{\alpha}$, uniformly in $x\in[\varepsilon,1-\varepsilon]$. The principal branch also gives $z^{\alpha}\overline{z}^{\alpha}=(z\overline{z})^{\alpha}=1$ and hence $z^{\alpha-1}\overline{z}^{\alpha}-\overline{z}^{\alpha-1}z^{\alpha}=\overline{z}-z\to 0$. On the other hand, as $y\nearrow\pi$, our z and $\overline{z}$ approach the cut $(-\infty,0]$ from different sides, and we obtain

$$\begin{aligned} z^{\alpha}(1+x\overline{z})^{\alpha}-\overline{z}^{\alpha}(1+xz)^{\alpha} &\sim (1-x)^{\alpha}(z^{\alpha}+\overline{z}^{\alpha}) \\ &= 2(1-x)^{\alpha}\cos y\alpha \longrightarrow 2(1-x)^{\alpha}\cos\pi\alpha, \end{aligned}$$

and analogously, $z^{\alpha-1}(1+x\overline{z})^{\alpha}-\overline{z}^{\alpha-1}(1+xz)^{\alpha}\to 2i(1-x)^{\alpha}\sin\pi\alpha$, again uniformly in $x\in[\varepsilon,1-\varepsilon]$. Putting things together reveals the explicit form of the density. □

Theorem 2.4 will be established by the same method as Theorems 2.1 - 2.3. To get started, we observe, splitting orbits at their first return to Y, that $K_n := \sum_{j=1}^{Z_n} 1_{A_1}\circ T^j$, $n\geq 0$, satisfies a *dissection identity* similar to that for $\mathbf{S}_n(1_{A_1})$ (compare Lemma 6.1 of [**TZ**]): We have, for $n\geq 0$,

$$K_n = \begin{cases} k-1+K_{n-k}\circ T^k & \text{on } Y\cap T^{-1}A_1\cap\{\varphi=k\},\ 1\leq k\leq n, \\ K_{n-k}\circ T^k & \text{on } Y\cap T^{-1}A_1^c\cap\{\varphi=k\},\ 1\leq k\leq n, \\ 0 & \text{on } Y\cap\{\varphi>n\}, \end{cases} \tag{6.4}$$

which results in

Lemma 6.1 (**Splitting moments at the first return**). *Let T be a c.e.m.p.t. of $(X,\mathcal{A},\mu)$, and assume that $X=A_1\cup Y\cup A_2$ (measurable and pairwise disjoint) such that $Y\in\mathcal{A}$, $0<\mu(Y)<\infty$, dynamically separates A_1 and A_2. Let $K_n := \sum_{j=1}^{Z_n} 1_{A_1}\circ T^j$, $n\geq 0$, then, for $r\geq 1$ and $s>0$,*

$$(1-e^{-s})\int_Y\left(\sum_{n\geq 0}\widehat{T}^n 1_{Y_n}\,e^{-ns}\right)\left(\sum_{n\geq 0}K_n^r\,e^{-ns}\right)d\mu \tag{6.5}$$

$$= e^{-s}\sum_{j=0}^{r-1}\binom{r}{j}\int_Y\left(\sum_{n\geq 1}n^{r-j}\,\widehat{T}^{n+1}1_{Y\cap T^{-1}A_1\cap\{\varphi=n+1\}}\,e^{-ns}\right)\left(\sum_{n\geq 0}K_n^j\,e^{-ns}\right)d\mu.$$

PROOF. As in the proof of Lemma 6.1 of [**TZ**] or Lemma 1 of [**T5**]. □

Observe that this is like (5.1) with the last term missing.

PROOF OF THEOREM 2.4. We prove inductively that for all $u \in \mathcal{D}(\mu)$ and all $r \geq 0$,

$$(\bigstar_r) \qquad \sum_{n\geq 0} \left(\int_Y K_n^r \cdot u \, d\mu \right) e^{-ns} \sim r! \, \mathbb{E}[\mathcal{K}_{\alpha,\beta}^r] \left(\frac{1}{s} \right)^{r+1} \qquad \text{as } s \searrow 0.$$

By KTT and monotonicity of the $(\int_Y K_n^r \cdot u \, d\mu)_{n\geq 1}$, the $(\bigstar_r)$, $r \geq 0$, imply $\int_Y K_n^r \cdot u \, d\mu \sim \mathbb{E}[\mathcal{K}_{\alpha,\beta}^r] \cdot n^r$ as $n \to \infty$, and hence $n^{-1} K_n \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{K}_{\alpha,\beta}$.

Observe that $(K_n/n)_{n\geq 1}$ is asymptotically T-invariant in measure, which follows easily from the asymptotic T-invariance of $(Z_n/n)_{n\geq 1}$ already used in the proof of Theorem 2.3. For the proof of $(\bigstar_r)$ we need $R_n \circ T - R_n \overset{\mu}{\longrightarrow} 0$ for $R_n := (K_n/n)^r$, $n \geq 1$. To validate this, note that $K_n, K_n \circ T \leq n$, so that by the mean-value theorem, $|R_n \circ T - R_n| \leq r \, |(K_n \circ T)/n - K_n/n|$.

To start the induction, note that $\sum_{n\geq 0} (\int_Y K_n^0 \cdot u \, d\mu) e^{-ns} \sim 1/s$ for any $u \in \mathcal{D}(\mu)$, which is $(\bigstar_0)$. For the inductive step, assume $(\bigstar_j)$ for $0 \leq j < r$. Then argue as in the proof of Theorem 2.2, using Lemma 6.1 instead of (5.1). □

We conclude this section proving our arcsine law for randomly chosen excursions.

PROOF OF THEOREM 2.5. We are going to show that Theorems 2.2 and 2.4 apply to the following simple skew product in which $\mathbf{L}_n(x, \omega)$ corresponds to the occupation times of a nice component A_1. Let $X^* := X \times \Omega$, equipped with product σ-field $\mathcal{A}^* := \mathcal{A} \otimes \mathcal{B}$ and measure $\mu^* := \mu \otimes \nu$. Define $T^* : X^* \to X^*$ by

$$T^*(x, \omega) := \begin{cases} (Tx, \sigma\omega) & \text{if } x \in Y, \\ (Tx, \omega) & \text{if } x \in Y^c. \end{cases}$$

It is immediate that T^* preserves μ^*. Let $Y^* := Y \times \Omega$, then $\mu^*(Y^*) < \infty$ and $X^* = \bigcup_{n\geq 1} (T^*)^{-n} Y^* \mod \mu^*$, so that Maharam's recurrence theorem (cf. Theorem 1.1.7 of [**A0**]) ensures conservativity of T^*. To show that T^* is also ergodic, we need only prove ergodicity of the induced map $T^*_{Y^*}$ on $(Y^*, \mathcal{A}^*, \mu^*_{Y^*})$. But $T^*_{Y^*}$ is ergodic since $T^*_{Y^*} = T_Y \otimes \sigma$, a product of two probability preserving maps, one ergodic, and one exact. Letting $A_1^* := Y^c \times [1]$ and $A_2^* := Y^c \times [0]$, we see that these sets are dynamically separated by Y^*, and that $X^* = Y \cup A_1 \cup A_2$ (disjoint). Moreover,

$$\mathbf{L}_n(x, \omega) = \sum_{j=1}^{n} (1_{[1]} \circ \sigma^{S_j(x)}(\omega))(1_{Y^c} \circ T^j)(x) = \sum_{j=1}^{n} 1_{A_1^*} \circ (T^*)^j \, (x, \omega).$$

We denote the first-return time of Y^* under T^* by φ^*, and write $w_N^*(Y^*)$ for the corresponding wandering rate etc. For $k \geq 1$ we then have $Y^* \cap \{\varphi^* = k\} = (Y \cap \{\varphi = k\}) \times \Omega$, and therefore

$$(6.6) \qquad \widehat{T^*}^k 1_{Y^* \cap \{\varphi^* = k\}} = \widehat{T^k \otimes \sigma} \, (1_{Y \cap \{\varphi = k\}} \otimes 1_\Omega) = \widehat{T}^k 1_{Y \cap \{\varphi = k\}} \otimes 1_\Omega.$$

Similarly, for $k \geq 2$, $Y^* \cap (T^*)^{-1} A_1^* \cap \{\varphi^* = k\} = (Y \cap \{\varphi = k\}) \times \sigma^{-1}[1]$, and thus

$$\widehat{T^*}^k 1_{Y^* \cap (T^*)^{-1} A_1^* \cap \{\varphi^* = k\}} = \widehat{T^k \otimes \sigma} \left(1_{Y \cap \{\varphi = k\}} \otimes 1_{\sigma^{-1}[1]}\right) = \widehat{T}^k 1_{Y \cap \{\varphi = k\}} \otimes \widehat{\sigma} 1_{\sigma^{-1}[1]} = \widehat{T}^k 1_{Y \cap \{\varphi = k\}} \otimes 1_{[1]}. \tag{6.7}$$

As a consequence we see that $\mu^*(Y^* \cap \{\varphi^* = k\}) = \mu(Y \cap \{\varphi = k\})$ and $\mu^*(Y^* \cap (T^*)^{-1} A_1^* \cap \{\varphi^* = k\}) = \mu(Y \cap \{\varphi = k\}) \cdot \nu([1])$, implying $(w_N^*(Y^*)) \in \mathcal{R}_{1-\alpha}$ as well as $w_N^*(Y^*, A_1^*)/w_N^*(Y^*) = \beta$, and, using (1.3), that the sequences

$$\left\{ \frac{1}{w_N^*(Y^*)} \sum_{n=0}^{N-1} \widehat{T^*}^n 1_{Y_n^*} \right\}_{N \geq 1} \quad \text{and} \quad \left\{ \frac{1}{w_N^*(Y^*, A_1^*)} \sum_{n=0}^{N-1} \widehat{T^*}^n 1_{A_1^* \cap Y_n^*} \right\}_{N \geq 1}$$

inherit the required precompactness property from $\{w_N(Y)^{-1} \sum_{n=0}^{N-1} \widehat{T}^n 1_{Y_n}\}_{N \geq 1}$. All assumptions of Theorems 2.2 and 2.4 being satisfied, our assertions follow. □

7. Application to various classes of transformations

A *fibred system* (or *piecewise invertible system*) is a quintuple $(X, \mathcal{A}, m, T, \xi)$, where $(X, \mathcal{A}, m, T)$ is a nonsingular transformation on a σ-finite measure space, and ξ is a (finite or countable) partition mod m such that every *cylinder* $Z \in \xi$ has positive measure and the *branches* $T\mid_Z \colon Z \longrightarrow TZ$, $Z \in \xi$, are bijective and bimeasurable. More specifically, we deal with situations in which (X, d) is a metric space with Borel σ-field $\mathcal{A}$, so that X comes with a partition ξ_0 into connected components, and we then assume that ξ refines ξ_0 and that each branch is a homeomorphism onto its image. ξ is a *Markov partition* if each TZ, $Z \in \xi$, is measurable ξ. We let ξ_n denote the family of *cylinders of rank n*, that is, those sets of the form $Z = [Z_0, \ldots, Z_{n-1}] := \bigcap_{i=0}^{n-1} T^{-i} Z_i$ with $Z_i \in \xi$ which have positive measure. If the measure is invariant, we denote it by μ and call $(X, \mathcal{A}, \mu, T, \xi)$ a measure preserving fibred system.

Assume that Y is ξ-measurable with return time φ, and with partition $\xi_{Y,0}$ into connected components. Then there is a natural *induced partition* (mod μ_Y) on Y, $\xi_Y := \bigcup_{k \geq 1} \left\{ V \cap \{\varphi = k\} \cap T^{-k} M : V \in Y \cap \xi_k, \, M \in \xi_{Y,0} \right\}$, and each set $Y \cap \{\varphi = k\}$ (and also $Y \cap T^{-1} A_1 \cap \{\varphi = k\}$ if A_1 is measurable ξ) then is measurable ξ_{k+1} and a union of cylinder sets for the induced map T_Y.

One way to express distortion properties of T or T_Y is in terms of the regularity of the image densities $d(\mu \circ (T^n\mid_Z)^{-1})/d\mu = \widehat{T^n} 1_Z$ defined on the image sets $T^n Z$, $Z \in \xi_n$, $n \geq 1$. The *regularity* of a function $v : B \to (0, \infty)$, $B \subseteq Y$, is $R_B(v) := \inf\{\rho > 0 : v(x) \leq (1 + \rho \cdot d(x, y)) v(y) \text{ for } x, y \in B\}$, cf. [**Z3**], and $R_B(v)$ implies an upper bound for the Lipschitz constant of $v / \int_B v \, \mu_B$. Therefore, if a family $\mathcal{V}_B$ of positive functions on B has uniformly bounded regularity, then $\{v / \int_B v \, \mu_B : v \in \mathcal{V}_B\}$ is precompact in the topology of uniform convergence on B by the Arzela-Ascoli theorem, and so is the union of finitely many families of this type. For the examples below, uniformly bounded regularity of the $\{\widehat{T_Y} 1_W : W \in \xi_Y\}$ follows from standard bounded distortion results, and finiteness of the family of images $B = T_Y W$, $W \in \xi_Y$ then ensures that our compactness assumptions are satisfied in a very strong sense: We get precompactness in $L_\infty(\mu)$ of the family

$$\mathfrak{H}''_Y := \left\{ f_k(Y)^{-1} \cdot \widehat{T}^k 1_{Y \cap \{\varphi = k\}} : k \geq 1 \text{ with } f_k(Y) > 0 \right\}, \tag{7.1}$$

and hence of $\mathfrak{H}'_Y := \{q_n(Y)^{-1} \cdot \widehat{T}^n 1_{Y_n}\}_{n\geq 1}$ and of $\mathfrak{H}_Y$, both of which are contained in $\overline{\mathrm{co}}_\infty(\mathfrak{H}''_Y)$. Analogously, $\mathfrak{H}''_{Y,A_1} := \{f_k(Y,A_1)^{-1} \cdot \widehat{T}^k 1_{Y\cap T^{-1}A_1 \cap\{\varphi=k\}} : k \geq 1$ with $f_k(Y,A_1) > 0\}$ is precompact in $L_\infty(\mu)$ for the relevant sets A_1 below, and so are $\mathfrak{H}'_Y := \{q_n(Y,A_1)^{-1} \cdot \widehat{T}^n 1_{Y_n \cap A_1}\}_{n\geq 1}$ and $\mathfrak{H}_{Y,A_1}$.

Having generalized the abstract distributional limit theorems of [**TZ**], we first illustrate the strength of our results in the light of the main class of examples discussed there.

7.1. Interval maps with indifferent fixed points. In [**Z1**] and [**Z2**] a large class of infinite measure preserving interval maps T with *indifferent fixed points*, called *AFN-maps*, has been studied, generalizing earlier results from [**A0**], [**A2**], [**T1**]-[**T3**]. (See [**Z2**] or [**TZ**] for definitions and notation.) In [**TZ**] we showed that these systems often satisfy condition (1.1), the basic assumption in the abstract distributional limit theorems derived there.

Still, we weren't able to cover all cases: While the behaviour at every single indifferent fixed point is good (Lemma 8.1 of [**TZ**] or Lemma 2 of [**T5**]), there are simple examples with $(w_n(T)) \in \mathcal{R}_{1-\alpha}$ which violate (for the usual type of reference sets) the condition (1.1) required in [**TZ**]:

EXAMPLE 7.1. *Consider AFN-maps $T : [0,1] \to [0,1]$ with two full branches and indifferent fixed points at $x = 0$ and $x = 1$, i.e. $\xi = \zeta = \{Z_1, Z_2\}$, and $TZ_i = (0,1)$. Take $A_i := Z_i \cap T^{-1}Z_i$, and $Y := (A_1 \cup A_2)^c$. As a consequence of Theorem 8.1 of* [**TZ**]*, we have, for $i \in \{1,2\}$,*

$$\frac{1}{w_N(Y,A_i)} \sum_{n=0}^{N-1} \widehat{T}^n 1_{Y_n \cap T^{-1}A_i} \to D_i \qquad \text{uniformly on } Y \text{ as } N \to \infty, \tag{7.2}$$

where the D_i have disjoint supports, $\{D_i > 0\} = Z_i \cap T^{-1}(Z_i^c) \subseteq Y$. Fix any $\alpha \in (0,1)$. It is well known (see e.g. Lemma 4.8.6 and the proof of Theorem 4.8.7 of [**A0**]*), that we can choose T in such a way that $w_N(Y,A_i) \sim N^{1-\alpha}\ell_i(N)$ as $N \to \infty$, for any prescribed $\ell_i \in \mathcal{R}_0$, $i \in \{1,2\}$. Now take $\ell_1(t) := 1$ and $\ell_2(t) := \exp[(\log t)^{1/3}\cos((\log t)^{1/3})]$, $t > 0$, so that*

$$\varliminf_{N\to\infty} \frac{w_N(Y,A_1)}{w_N(Y,A_2)} = 0, \quad \textit{while} \quad \varlimsup_{N\to\infty} \frac{w_N(Y,A_1)}{w_N(Y,A_2)} = \infty.$$

Then, although $(w_N(Y))_{N\geq 1} = (w_N(Y,A_1) + w_N(Y,A_2))_{N\geq 1} \in \mathcal{R}_{1-\alpha}$, the sequence $(w_N(Y)^{-1} \sum_{n=0}^{N-1} \widehat{T}^n 1_{Y_n})_{N\geq 1}$ does not converge on Y. Note, however, that it still forms a precompact set in $L_\infty(\mu)$, and its accumulation points are the convex combinations of the D_i.

We are now in a position to remove the (mild) additional assumption (8.8) of [**TZ**], which we imposed in order to exclude such counterexamples. Whereas convergence (1.1) may be lost, precompactness (1.5) never is. Simply observe that the induced map T_Y (for the special reference set $Y = Y(T)$ defined in (8.5) of [**TZ**]) only has a finite number of different images, and inverse branches with uniformly bounded regularity of derivatives (cf. [**Z2**], [**Z3**]). Therefore we no longer need the central convergence lemma which originally motivated the present approach to the limit theorems.

For a basic AFN-map T, our distributional limit theorems therefore apply as soon as the general condition of regular variation $(w_n(T)) \in \mathcal{R}_{1-\alpha}$ of the wandering rate (plus the balance condition (2.8) in case of Theorem 2.2) is satisfied. The relation between regular variation of $(w_n(T))$ and the local behaviour of T at its indifferent fixed points has earlier been discussed in [**T4**].

7.2. Interval maps with flat critical points. Another family of interval maps whose ergodic properties are governed by some distinct *indifferent orbit* is that of maps T with *flat critical points*, i.e. points c at which all derivatives of T vanish. The dynamical effect of such a point is that orbits getting close to c, say $\left|T^k x - c\right| < \delta$, follow the critical orbit for a very long time, $\left|T^{k+j} x - T^j c\right| < \delta$ for $1 \leq j \leq J(\delta)$ with $J(\delta)$ growing faster than $-\log\delta$ as $\delta \searrow 0$. We briefly discuss how our abstract results apply in this setup.

Continuing earlier work of [**BM**], [**Z4**] was devoted to flat S-unimodal maps T on an interval $X := [a,b]$ satisfying the *Misiurewicz condition*, meaning that there is some open subinterval Y around c to which the orbit of c does not return, $c_n := T^n c \notin Y$ for $n \geq 1$. Assume for simplicity that the graph of T is symmetric under the involution $I(x) := b - (x-a)$ in the sense that $T = T \circ I$. Such a map always possesses a conservative ergodic invariant measure $\mu \ll \lambda$ and μ is infinite iff $\int \log \mid T' \mid \, d\lambda = -\infty$. Suppose that $V(e^{-t})$ is regularly varying of index $-\alpha$ at 0, $\alpha \in [0,1]$, where V is the inverse of $U(t) := T(c) - T(c-t)$, $t \in [0, c-a]$, and that, in addition, the *postcritical Lyapunov exponent* of T,

$$\lim_{n\to\infty} \frac{1}{n} \log \mid (T^n)'(c_1) \mid \in (0,\infty)$$

exists. Then the wandering rate $(w_n(T))$ belongs to $\mathcal{R}_{1-\alpha}$. Moreover, the reference set Y can be chosen arbitrarily small and in such a way that $T^k(Y \cap \{\varphi = k\}) = Y$ or $= \varnothing$ for all $k \geq 1$, and that T_Y has branches with uniformly bounded distortion, so that $\{f_k(Y)^{-1} \cdot \widehat{T}^k 1_{Y\cap\{\varphi=k\}}\}_{k\geq 1}$ is precompact in $L_\infty(\mu)$. (See [**Z4**] for proofs of these statements.) We are thus in the situation of our abstract results, and recover the Darling-Kac theorem and the Dynkin-Lamperti arcsine law for T (which in [**Z4**] had been obtained via pointwise dual ergodicity, the proof of which requires extra work).

Our third limit theorem, the arcsine law for occupation times, turns up more naturally if we consider a variation of the previous situation which allows for two distinct flat critical points. Since the arguments are similar to the unimodal case, we content ourselves with a sketch of the main points in the simplest case:

Example 7.2 (**Arcsine law for shadowing times of flat critical points**). *Let T satisfy all the properties mentioned above. We consider another map S for which the involution I is a dynamical symmetry, $S \circ I = I \circ S$, defined by $Sx := Tx$ for $x < c$ and $Sx := 1 - S(1-x)$ for $x > c$. This map has two distinct one-sided critical points c^- and c^+, with orbits $c_n^\pm := T^n c^\pm$, $n \geq 0$, clearly satisfying $I(c_n^-) = c_n^+$. Observing that $T \circ T = T \circ S$, i.e. that T is a continuous 2-to-1 factor of S, we see that S, too, is a Misiurewicz map, $c_n^\pm \notin Y$ for $n \geq 1$, and that S preserves the infinite conservative ergodic measure $\widetilde{\mu} = (\mu + \mu\circ I^{-1})/2 \ll \lambda$ with $w_n(S) \sim w_n(T) \in \mathcal{R}_{1-\alpha}$ as $n \to \infty$.*

Typical orbits $(S^n x)_{n\geq 0}$ *spend most of their time closely shadowing either* c^- *or* c^+. *We fix any* $\delta \in (0, \mathrm{diam}(Y)/4)$ *and say that the orbit of* x *is* δ-shadowing *the left critical point* c^- *during the time interval* $\{k, \ldots, k+J\}$ *if* $\left|S^{k+j}x - S^j c\right| < \delta$ *for* $1 \leq j \leq J$ *(hence* $S^k x \in Y$ *and* $S^{k+j}x \notin Y$ *for* $1 \leq j \leq J$*). Denote the total number of such steps* $k+j < n$ *by* $\mathbf{s}_\delta^-(n)$. *We claim that the proportion of time spent* δ*-shadowing* c^- *satisfies the following arcsine law,*

$$\frac{\mathbf{s}_\delta^-(n)}{n} \overset{\mathcal{L}(\lambda)}{\Longrightarrow} \mathcal{L}_{\alpha,1/2}. \tag{7.3}$$

To verify this statement, we construct a conservative ergodic measure preserving extension $(M, \mathcal{B}, \widetilde{\nu}, R)$ *of* $(X, \mathcal{A}, \widetilde{\mu}, S)$. *Let* Y^- *and* Y^+ *denote the left- and right-hand halves of* Y*, respectively, and define* $M := (Y^- \times \{-1\}) \cup (Y^+ \times \{1\}) \cup (Y^c \times \{-1, 1\}) \subseteq X \times \{-1, 1\}$*, and* $\mathcal{B}$ *the trace of the product* σ*-field. Let* π *and* σ *be the natural projections of* M *onto* X *and* $\{-1, 1\}$*, and set* $\widetilde{\nu}(E) := \widetilde{\mu}(\pi(E))$ *if* $\pi(E) \in Y \cap \mathcal{B}$ *and* $\widetilde{\nu}(E) := \widetilde{\mu}(\pi(E))/2$ *if* $\pi(E) \in Y^c \cap \mathcal{B}$. *Finally, define* $R : M \to M$ *by*

$$R(x,s) := \begin{cases} (Sx, \sigma(Sx)) & \text{if } Sx \in Y, \\ (Sx, s) & \text{if } Sx \in Y^c, \end{cases}$$

so that R *essentially acts like* S*, but* R*-orbits always remember which part of* Y *they visited last. We have a decomposition* $X = A_1 \cup Z \cup A_2$*, where* $Z := \pi^{-1}Y$*,* $A_1 := Y^c \times \{-1\}$*, and* Z *dynamically separates* A_1 *and* A_2. *Moreover,* $R^k(Z \cap T^{-1}A_1 \cap \{\tau = k\}) \in \{\varnothing, Z\}$ *for all* $k \geq 1$*, where* $\tau = \varphi \circ \pi$ *is the first return time of* Z*, the branches of* R_Z *have bounded distortion, and by symmetry our new system also satisfies* $w_N(Z, A_1) = w_N(Z)/2 \in \mathcal{R}_{1-\alpha}$. *We can therefore apply Theorem 2.2 to see that*

$$\frac{1}{n}\sum_{k=0}^{n-1} 1_{A_1} \circ R^k \overset{\mathcal{L}(\widetilde{\nu})}{\Longrightarrow} \mathcal{L}_{\alpha,1/2}. \tag{7.4}$$

However, the distribution of the time an orbit takes between escaping a δ*-shadow and entering* Y*, has exponentially small tail. Therefore it is not hard to see that (7.4) implies (7.3), compare Section 5 of* [**Z4**].

7.3. Random walks driven by Gibbs-Markov maps. Let $(X, \mathcal{A}, \mu, T, \xi)$ be an ergodic probability preserving fibred system given by a *Gibbs-Markov map* (cf. [**A0**], [**AD2**]) with finite image partition, i.e. with $\#T\xi < \infty$. Any ξ-measurable map $\phi : X \longrightarrow \mathbb{Z}$ defines a $\mathbb{Z}$-*extension* T_ϕ of T, that is the m.p.t. on the σ-finite infinite measure space $(X \times \mathbb{Z}, \mathcal{A} \otimes \mathfrak{P}(\mathbb{Z}), \mu \otimes \mu_{\mathbb{Z}})$, $\mu_{\mathbb{Z}}$ denoting counting measure on $\mathbb{Z}$, given by

$$T_\phi(x, g) := (Tx, g + \phi(x)).$$

Writing $\phi_n := \sum_{k=0}^{n-1} \phi \circ T^k$, $n \geq 0$, we see that $T_\phi^n(x,g) = (T^n x, g + \phi_n(x))$, $n \geq 0$. We henceforth assume (see [**AD2**] for definitions) that ϕ is aperiodic, and that either $\phi \in L_2(\mu)$ with $\int_X \phi \, d\mu = 0$, or that the μ-distribution of ϕ is in the strict domain of attraction of a nondegenerate stable distribution of order $p \in (1, 2)$. The local limit theory for $(\phi_n)_{n\geq 0}$ developed in [**AD2**] (see also [**AD1**], [**ADSZ**], [**GH**]) implies that T_ϕ is conservative and exact, and pointwise dual ergodic with return sequence $(a_n(T))_{n\geq 1} \in \mathcal{R}_\alpha$ with $\alpha := 1/2$ or $\alpha := 1 - 1/p \in (0, 1/2)$, respectively.

A natural candidate for a good reference set Y of finite measure is the $g = 0$ section $Y := X \times \{0\}$. The first return time function φ_Y simply is $\varphi_Y(x,i) = \min\{n \geq 1 : |\phi_n(x)| = 0\}$, and as ϕ is measurable ξ, we see that $Y \cap \{\varphi_Y = k\}$ is the union of some subcollection Φ_k of ξ_k. The transfer operator $\widehat{T}_\phi$ of T_ϕ satisfies

$$\widehat{T}_\phi^n(1_Z\, u \otimes 1_{\{g\}}) = \widehat{T}^n(1_Z\, u) \otimes 1_{\{g+\phi_n(Z)\}} \qquad \text{for } n \geq 1,\ Z \in \xi_n, \text{ and } u \in L_1(\mu).$$

Due to the standard strong distortion control available for Gibbs-Markov maps, and $\#T\xi < \infty$, $(X, \mathcal{A}, \mu, T, \xi)$ actually has the property that

$$\text{(7.5)} \qquad \mathfrak{H}_\xi := \left\{ \mu(Z)^{-1} \widehat{T}^n 1_Z : n \geq 1, Z \in \xi_n \right\} \qquad \text{is precompact in } L_\infty(\mu)$$

(the $\widehat{T}^n 1_Z : n \geq 1, Z \in \xi_n$ have uniformly bounded regularity). Hence $\overline{\text{co}}(\mathfrak{H}_\xi)$, which contains $\mathfrak{H}''_Y$, is compact in $L_\infty(\mu)$. Moreover, finiteness of $T\xi$ together with bounded regularity are easily seen to imply that any family of functions from $\overline{\text{co}}(\mathfrak{H}_\xi)$ which are supported on Y is uniformly sweeping. According to Proposition 3.8.7 of [**A0**], our previous remarks about pointwise dual ergodicity imply that $w_n(T_\phi) \in \mathcal{R}_{1-\alpha}$ (Y being a uniform set for T_ϕ). We thus see that Theorems 2.1, 2.3, and 2.5 apply in the present situation.

To also illustrate the arcsine law for occupation times in this context, we focus on the special case of symmetric and bounded ϕ, $|\phi| \leq 2M$ (so that the above applies with $\alpha = 1/2$). (The classical arcsine theory for random walks, see e.g. Theorem 20.2 of [**S2**], shows that, for symmetric ϕ and $(\phi \circ T^n)_{n\geq 1}$ independent, one still has $n^{-1}\sum_{j=0}^{n-1} 1_{X\times\mathbb{N}} \circ T_\phi^j \Longrightarrow \mathcal{L}_{1/2,1/2}$ even if $w_n(T_\phi) \in \mathcal{R}_{1-\alpha}$ with $\alpha < 1/2$. Therefore it is *a priori* clear that occupation times of half-lines cannot be covered by our Theorem 2.2 for general unbounded ϕ.) Consider $A_1 := X \times (M + \mathbb{N})$ which $Y^* := X \times \{-M, \ldots, M\}$ dynamically separates from $A_2 := X \times (-M - \mathbb{N})$. As before, we see that $\mathfrak{H}''_{Y^*}$ and $\mathfrak{H}''_{Y^*,A_1}$ are precompact in $L_\infty(\mu)$, and by symmetry, $w_N(Y^*, A_1) = w_N(Y^*)/2$. We are thus in the situation of Theorems 2.2 and 2.4 with $\beta = 1/2$.

In particular, Theorems 2.1 to 2.4 show that for bounded ϕ with symmetric distribution, and any probability measure $\nu \ll \mu \otimes \mu_{\mathbb{Z}}$, we have, as $n \to \infty$,

$$\nu\left(\left\{\frac{\#\{j \leq n : \phi_j = 0\}}{\sqrt{n}} \leq t\right\}\right) \longrightarrow \frac{2}{\pi}\int_0^t e^{-\frac{y^2}{\pi}}\, dy, \quad t \geq 0,$$

while

$$\nu\left(\left\{\frac{\#\{j \leq n : \phi_j > 0\}}{n} \leq t\right\}\right) \longrightarrow \frac{2}{\pi}\arcsin\sqrt{t}, \quad t \in [0,1],$$

and

$$\nu\left(\left\{\frac{\max(\{i \leq n : \phi_i = 0\})}{n} \leq t\right\}\right) \longrightarrow \frac{2}{\pi}\arcsin\sqrt{t}, \quad t \in [0,1],$$

all of which are parallel to classical results about the coin-tossing random walk, and that

$$\nu\left(\left\{\frac{\#\{j \leq \max(\{i \leq n : \phi_i = 0\}) : \phi_j > 0\}}{n} \leq t\right\}\right) \longrightarrow \frac{2}{\pi}\left(\arcsin\sqrt{t} + \sqrt{t(1-t)}\right), \quad t \in [0,1].$$

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Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, 1090 Wien, Austria

E-mail address: roland.zweimueller@univie.ac.at