Asymptotic orbit complexity of infinite measure preserving transformations

Roland Zweimüller
Faculty of Mathematics, University of Vienna
Nordbergstraße 15, 1090 Vienna, Austria
roland.zweimueller@univie.ac.at

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Abstract

We determine the asymptotics of the Kolmogorov complexity of symbolic orbits of certain infinite measure preserving transformations. Specifically, we prove that the Brudno - White individual ergodic theorem for the complexity generalizes to a ratio ergodic theorem analogous to previously established extensions of the Shannon - McMillan - Breiman theorem.

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1 Introduction and statement of results

Kolmogorov complexity theory offers an interesting possibility of describing the complicated behaviour of typical orbits of ergodic transformations. It provides us with an essentially unique meaningful way of assigning, to any finite string $\omega \in \mathfrak{A}^* := \bigcup_{l \geq 0} \mathfrak{A}^l$ of symbols from a finite *alphabet* $\mathfrak{A}$, an integer $\mathrm{C}(\omega)$, the *complexity* of ω, formalizing the intuitive concept of the length of the shortest possible description of ω. We refer to [LV] for a readable account of complexity theory, the basics required for our purpose will briefly be reviewed in Section 2. For the dynamicist it is natural to apply this notion to orbits of dynamical systems via the usual symbolic coding w.r.t. a partition of the phase space. Specifically, let T be a *nonsingular transformation* of a σ-finite measure space $(X, \mathcal{A}, \mu)$ (i.e. T is measurable with $\mu \circ T^{-1} \ll \mu$), and ξ a finite measurable partition (mod μ). For a.e. $x \in X$ the member $\xi(x)$ of ξ containing x is well defined, and so are the $\xi_n(x)$, where $\xi_n := \bigvee_{k=0}^{n-1} T^{-k}\xi$, $n \geq 1$. We will identify $\xi_n(x)$ with the ξ-*coding* of the orbit section $(T^k x)_{0 \leq k < n}$, i.e. with the string

$(\xi(T^k x))_{k=0}^{n-1} \in \xi^n$ over the alphabet $\mathfrak{A} := \xi$. Using this convention, we define the *complexity of the n-orbit* $(T^k x)_{0 \le k < n}$ of $x \in X$ w.r.t. the partition ξ as $\mathrm{C}(\xi_n(x))$.

It has been shown that this approach leads to results closely related to key facts from Kolmogorov-Sinai (K-S) entropy theory. If μ is an ergodic invariant probability measure for T, it turns out that the behaviour of $\mathrm{C}(\xi_n(x))$ is similar to that of the *information function* $I_{\xi_n}(x) := -\log \mu(\xi_n(x))$. In particular, just as the Shannon-McMillan-Breiman theorem, the individual ergodic theorem for information, asserts that

$$\frac{-\log \mu(\xi_n(x))}{n} \longrightarrow h_\mu(T, \xi) \qquad \begin{array}{l} \text{as } n \to \infty \\ \text{for a.e. } x \in X, \end{array} \tag{1.1}$$

(where, of course, $h_\mu(T)$ and $h_\mu(T, \xi)$ denote the K-S-entropy of T w.r.t. μ (and ξ)), the Brudno-White-theorem (cf. [Br], [Wh]), the individual ergodic theorem for orbit complexity, ensures that also

$$\frac{\mathrm{C}(\xi_n(x))}{n} \longrightarrow h_\mu(T, \xi) \qquad \begin{array}{l} \text{as } n \to \infty \\ \text{for a.e. } x \in X. \end{array} \tag{1.2}$$

Quite recently, cf. [BBGMV], [BG], typical and average growth rates[1] of the quantity $\mathrm{C}(\xi_n(x))$ have been proposed as a means of distinguishing between some transformations for which classical entropy theory appears not to be applicable. This zoo of "weakly chaotic" systems accommodates some transformations on $[0, 1]$ which have a (σ-finite) *infinite invariant measure* μ absolutely continuous w.r.t. Lebesgue measure λ, namely piecewise $\mathcal{C}^2$ interval maps with indifferent fixed points like, for example,

$$Tx := \begin{cases} x + 2^p x^{p+1} & \text{for } x \in (0, 1/2), \\ 2x - 1 & \text{for } x \in (1/2, 1), \end{cases} \tag{1.3}$$

where[2] $p \in [1, \infty)$. Families of transformations containing these maps have been studied extensively from the viewpoint of infinite ergodic theory, see e.g. [T1]-[T3], [A0]-[A3], [Z1]-[Z2], [TZ], and references therein.

In accordance with numerical experiments (cf. Section 7 of [BBGMV]), [BG] provides pointwise estimates for maps T as in (1.3) proving that, up to a logarithmic error, $\mathrm{C}(\xi_n)$ has the same order of magnitude as the number $S_n := \sum_{k=0}^{n-1} 1_Y \circ T^k$ of visits to $Y := (1/2, 1)$. This is then used to show that the average growth rate of $\mathrm{C}(\xi_n)$ is similar to that of S_n. As already pointed out in Section 5 of [BBGMV], sharper results require a closer look at the complicated

[1] More precisely, the weaker concept of *chaos indices*, i.e. of best polynomial approximations to the actual rate, is used there.

[2] For $p \in (0, 1)$ there is an invariant probability measure $\mu \ll \lambda$, see e.g. Example 3 of [T1]. We refer to [G1], [G2], [Sa], [Yo] for recent work about this case.

pointwise ergodic behaviour of infinite measure preserving transformations. We are going to address this "very delicate point in ergodic theory" (*loc.cit.*) and clarify the a.e. behaviour of $\mathrm{C}(\xi_n)$, amending the argument used in [BG].

Let us first recall that there is a meaningful entropy theory even in situations with infinite invariant measure: Following [Kr], we may define the measure theoretic entropy of any conservative ergodic measure preserving transformation (*c.e.m.p.t.*) T on a σ-finite space $(X, \mathcal{A}, \mu)$ as

$$h_\mu(T) := \mu(Y)\, h_{\mu_Y}(T_Y),$$

where $Y \in \mathcal{A}$ is any set with $0 < \mu(Y) < \infty$, $T_Y : Y \to Y$ is the *first return map* of Y, i.e. $T_Y x := T^{\varphi(x)}x$ with $\varphi(x) := \min\{n \geq 1 : T^n x \in Y\}$, $x \in Y$, the *first return time*, and the probability measure $\mu_Y(E) := \mu(Y \cap E)/\mu(Y)$ on $(Y, Y \cap \mathcal{A})$ is invariant and ergodic under T_Y. Due to Abramov's formula for the entropy of first-return maps, cf. [Ab], the definition of $h_\mu(T)$ does not depend on the choice of Y. Combining this notion with asymptotic characteristics of an infinite m.p.t. (like wandering rate or asymptotic type) leads to useful isomorphism invariants, cf. [A1] and [T2].

It is well known (cf. [T1], [T2]) that the map T from (1.3) is conservative and exact (hence also ergodic) w.r.t. λ, and preserves a σ-finite infinite measure $\mu \ll \lambda$ (unique up to a multiplicative constant) with a positive density h which is continuous on $(0, 1]$. As shown in [T2], the entropy of T satisfies Rohlin's formula

$$h_\mu(T) = \int_X \log |T'| \, d\mu, \tag{1.4}$$

and hence is nonzero and finite. It is clear that the natural partition $\xi = \{(0, 1/2), (1/2, 1)\}$ is generating. Moreover, a version of the Shannon-McMillan-Breiman theorem is available for these infinite measure preserving maps, asserting that for every $f \in L_1(\mu)$ with $\int f \, d\mu \neq 0$,

$$\frac{-\log \mu(\xi_n(x))}{\mathbf{S}_n(f)(x)} \longrightarrow \frac{h_\mu(T)}{\int f \, d\mu} \qquad \begin{array}{c} \text{as } n \to \infty \\ \text{for a.e. } x \in X, \end{array} \tag{1.5}$$

where $\mathbf{S}_n(f) := \sum_{k=0}^{n-1} f \circ T^k$, $n \geq 1$ (compare also [KS]). Notice that, by the ergodic theorem, the standard version (1.1) for finite μ can also be stated in this way. The aim of the present paper is to show that the Brudno-White result (1.2) extends to our infinite measure preserving situations in an analogous fashion. In particular, for T from (1.3), and any $f \in L_1(\mu)$ with $\int f \, d\mu \neq 0$, we show that

$$\frac{\mathrm{C}(\xi_n(x))}{\mathbf{S}_n(f)(x)} \longrightarrow \frac{h_\mu(T)}{\int f \, d\mu} \qquad \begin{array}{c} \text{as } n \to \infty \\ \text{for a.e. } x \in X. \end{array} \tag{1.6}$$

In fact, we can deal with the much larger class of AFN-maps studied in [Z1] and [Z2], generalizing earlier work from [A0], [A3], [ADU], and [T1]-[T3]. A *piecewise monotonic system* is a triple (X, T, ξ), where X is the union of some finite family of disjoint bounded open intervals, ξ is a collection of non-empty pairwise disjoint open subintervals with $\lambda(X \setminus \bigcup \xi) = 0$, and $T : X \to X$ is a map such that $T \mid_Z$ is continuous and strictly monotonic for each $Z \in \xi$. The maps of [Z1], [Z2] are $\mathcal{C}^2$ on each $Z \in \xi$ and satisfy *Adler's condition*

$$T''/(T')^2 \text{ is bounded on } \bigcup \xi, \tag{1.7}$$

as well as the *finite image condition*

$$T\xi = \{TZ : Z \in \xi\} \text{ is finite.} \tag{1.8}$$

There is a finite set $\zeta \subseteq \xi$ of cylinders Z having an *indifferent fixed point* x_Z as an endpoint (i.e. $\lim_{x \to x_Z, x \in Z} Tx = x_Z$ and $\lim_{x \to x_Z, x \in Z} T'x = 1$), and each x_Z is a *one-sided regular source*, i.e.

$$\text{for } x \in Z,\ Z \in \zeta, \text{ we have } (x - x_Z)T''(x) \geq 0. \tag{1.9}$$

The maps are *uniformly expanding on sets bounded away from* $\{x_Z : Z \in \zeta\}$, in the sense that letting $X_\varepsilon := X \setminus \bigcup_{Z \in \zeta} ((x_Z - \varepsilon, x_Z + \varepsilon) \cap Z)$ we have

$$|T'| \geq \rho(\varepsilon) > 1 \quad \text{on } X_\varepsilon \quad \text{ for each } \varepsilon > 0. \tag{1.10}$$

Given (1.7)-(1.10) we call (X, T, ξ) an *AFN-system.* If T is conservative ergodic and if $\zeta \neq \varnothing$, then the AFN-system is called *basic.* (See Theorem 1 in [Z1] for ergodic decompositions of AFN-systems into basic components.)

Any basic AFN-system (X, T, ξ) has an invariant measure $\mu \ll \lambda$ with $\mu(X) = \infty$ whose density $d\mu/d\lambda$ has a version h bounded on each X_ε. As shown in [Z2], the Krengel entropy of T always satisfies Rohlin's formula (1.4), the fundamental partition generates, and Thaler's version (1.5) of the Shannon-McMillan-Breiman theorem holds. We are going to prove an analogous ratio ergodic theorem for orbit complexity:

Theorem 1 (Ratio ergodic theorem for complexity of AFN-maps) *Let (X, T, ξ) be a basic AFN-map with finite ξ, invariant measure $\mu \ll \lambda$, and $h_\mu(T) < \infty$. Then, for any $f \in L_1(\mu)$ with $\int f \, d\mu \neq 0$,*

$$\frac{\mathrm{C}(\xi_n(x))}{\mathbf{S}_n(f)(x)} \longrightarrow \frac{h_\mu(T)}{\int f \, d\mu} \qquad \begin{array}{c} \textit{as } n \to \infty \\ \textit{for a.e. } x \in X. \end{array} \tag{1.11}$$

This result reduces (many aspects of) the asymptotic study of $\mathrm{C}(\xi_n(x))$ to that of $\mathbf{S}_n(f)(x)$. Precise results about these ergodic sums are available in case each $r_Z(x) := Tx - x$, $x \in Z \in \zeta$, is regularly varying at x_Z, e.g. if $r_Z(x) = a_Z \, |x - x_Z|^{1+p_Z} + o(|x - x_Z|^{1+p_Z})$ for $x \to x_Z$ as in example (1.3).

For results about the a.e. behaviour of ergodic sums $\mathbf{S}_n(f)$ of observables f integrable w.r.t. an infinite invariant measure see [A0], [A2], and [AD] (non-integrable functions have been studied in [ATZ]). In particular, it is possible to characterize those normalizing sequences (a_n) for which $\underline{\lim}_{n\to\infty} a_n^{-1}\mathbf{S}_n(f)$ (resp. $\overline{\lim}_{n\to\infty} a_n^{-1}\mathbf{S}_n(f)$) is positive (resp. finite) almost everywhere. Moreover, the Darling-Kac theorem for infinite measure preserving transformations (cf. [A0], [A2], [A3] and [TZ]) provides us with a distributional limit theorem for $\mathbf{S}_n(f)$, and hence also for $\mathrm{C}(\xi_n)$.

The result for interval maps is a consequence of the following abstract theorem which is the main result of the present paper.

Let T be some c.e.m.p.t. on the σ-finite space $(X, \mathcal{A}, \mu)$ and $\xi \subseteq \mathcal{A}$ a partition (mod μ) of X. Assume that Y is ξ-measurable with $0 < \mu(Y) < \infty$ and return time φ. Then there is a natural *induced partition* (mod μ_Y) on Y,

$$\xi_Y := \bigcup_{k\geq 1} \{V \cap \{\varphi = k\} : V \in Y \cap \xi_k\} \tag{1.12}$$

(ignoring empty intersections). For any probability measure ν we let $H_\nu(\eta) := -\sum_{M\in\eta} \nu(M) \log \nu(M)$ denote the entropy of a countable (mod ν) partition η w.r.t. ν. As in [ATZ] and [TZ] we say that two disjoint sets $A, B \subseteq X$ are *dynamically separated by $Y \subseteq X$ (under the action of T)* if $x \in A$ (resp. B) and $T^n x \in B$ (resp. A) imply the existence of some $k = k(x) \in \{0, \ldots, n\}$ for which $T^k x \in Y$ (i.e. T-orbits can't pass from one set to the other without visiting Y).

Theorem 2 (Ratio ergodic theorem for orbit complexity) *Let T be some c.e.m.p.t. on the σ-finite space $(X, \mathcal{A}, \mu)$ with $h_\mu(T) < \infty$, and let $\xi \subseteq \mathcal{A}$ be a finite generating partition (mod μ). Assume that there is some ξ-measurable set Y, $0 < \mu(Y) < \infty$, with return time φ, such that Y dynamically separates any two different elements of $Y^c \cap \xi$. If*

$$H_{\mu_Y}(\xi_Y) < \infty \qquad \text{and} \qquad \int_Y \log \circ \varphi \, d\mu_Y < \infty, \tag{1.13}$$

then, for any $f \in L_1(\mu)$ with $\int f \, d\mu \neq 0$,

$$\frac{\mathrm{C}(\xi_n(x))}{\mathbf{S}_n(f)(x)} \longrightarrow \frac{h_\mu(T)}{\int f \, d\mu} \qquad \begin{array}{c} \text{as } n \to \infty \\ \text{for a.e. } x \in X. \end{array} \tag{1.14}$$

Remark 1 *According to Aaronson's ergodic theorem (cf. Section 2.4 of [A0]), it is impossible to replace the ergodic sums $\mathbf{S}_n(f)$ in (1.5) by any normalizing constants $a_n \in (0, \infty)$, $n \geq 1$.*

2 Finite strings and their complexity

We briefly review what we need to know about Kolmogorov complexity, referring to [LV] for details and more information. Given some finite *alphabet* $\mathfrak{A}$ we let $\mathfrak{A}^* := \bigcup_{l \geq 0} \mathfrak{A}^l$ with the convention that $\mathfrak{A}^0 = \{\epsilon\}$, ϵ denoting the empty string. We can w.l.o.g. identify $\mathfrak{A}$ with $\mathfrak{A}_M := \{0, \ldots, M-1\}$, $M := \#\mathfrak{A} \geq 2$. The lexicographical order on $\mathfrak{A}_M^*$ gives a natural enumeration $\mathfrak{n}_M : \mathfrak{A}_M^* \to \mathbb{N}_0$ of all finite strings (with $\mathfrak{n}_M(\epsilon) = 0$). For example, the function $\mathfrak{n}_2$ is explicitly given by

$$\mathfrak{n}_2(\omega_0, \omega_1, \ldots, \omega_{l-1}) = 2^l + \sum_{i=0}^{l-1} \omega_i 2^i - 1, \quad \omega_i \in \{0, 1\}. \tag{2.1}$$

The string $(V, \ldots, V)$ consisting of l identical symbols $V \in \mathfrak{A}$, $l \in \mathbb{N}_0$, will be denoted by (V^l). Given $\omega(1), \ldots, \omega(j) \in \mathfrak{A}^*$, we simply write $(\omega(1), \ldots, \omega(j))$ for their *concatenation*, e.g. $((0, 1^3), (0, 1, 1, 0, 0, 1), (1^0)) = (0, 1, 1, 1, 0, 1, 1, 0, 0, 1)$. The *length* of $\omega = (\omega_0, \omega_1, \ldots, \omega_{l-1}) \in \mathfrak{A}^*$ is $\mathfrak{l}_M(\omega) := l$, and its *binary length* is $\mathfrak{bl}_M(\omega) := \mathfrak{l}_2(\mathfrak{n}_2^{-1} \circ \mathfrak{n}_M(\omega))$. It is easy to see that for every $j \in \mathbb{N}$ there is some constant $\tau_M(j) \in \mathbb{N}$ such that

$$\mathfrak{bl}_M(\omega(1), \ldots, \omega(j)) \leq \sum_{i=1}^{j} \mathfrak{bl}_M(\omega(i)) + \tau_M(j) \quad \text{for all } \omega(1), \ldots, \omega(j) \in \mathfrak{A}^*. \tag{2.2}$$

In particular, for every $r \in \mathbb{N}$ there is some constant $\phi_M(r) \in \mathbb{N}$ such that

$$\mathfrak{bl}_M(\chi, \omega) \leq \mathfrak{bl}_M(\omega) + \phi_M(r) \quad \text{for all } \chi, \omega \in \mathfrak{A}^* \text{ with } \mathfrak{bl}_M(\chi) \leq r. \tag{2.3}$$

The *self-delimiting version* of ω is the string

$$\overline{\omega} := (1^{\mathfrak{l}_M(\omega)}, 0, \omega) \in \mathfrak{A}^*. \tag{2.4}$$

This provides us with a *prefix code*, i.e. if $\omega \neq \sigma$, then neither of $\overline{\omega}$ and $\overline{\sigma}$ can be an initial piece of the other. For $n \in \mathbb{N}_0$ we let $\overline{n} := \overline{\mathfrak{n}_M^{-1}(n)} \in \mathfrak{A}^*$.

The most convenient way to introduce Kolmogorov complexity here is to use *Turing machines* (*TM*s) (cf. Section 1.7 of [LV]). Intuitively, a TM Υ is a computer (with infinite memory), which follows a fixed finite set of rules to process some *input*, given by a string π from some $D_\Upsilon \subseteq \mathfrak{A}^*$, resulting in some *output* $\Upsilon(\pi) \in \mathfrak{A}^*$. Formally, TMs therefore are certain maps $\Upsilon : D_\Upsilon \subseteq \mathfrak{A}^* \to \mathfrak{A}^*$. The corresponding numerical functions $\mathfrak{n}_M \circ \Upsilon \circ \mathfrak{n}_M^{-1} : \mathfrak{n}_M(D_\Upsilon) \subseteq \mathbb{N}_0 \to \mathbb{N}_0$, usually called *recursive*, represent those functions which are regarded computable, i.e. those which can be evaluated by following some rule. (Various other attempts to formalize the intuitive idea of computability lead to exactly the same class of functions.)

For the precise definition of Turing machines or, equivalently, of recursive functions, we refer to [LV]. We will only need to know that any map $\Upsilon : D_\Upsilon \subseteq \mathfrak{A}^* \to \mathfrak{A}^*$ which can be described by some concrete algorithm based on elementary arithmetic operations (i.e. anything an actual computer could do) is a TM, and that the composition $\Sigma \circ \Upsilon$ of two TMs Σ, Υ again is a TM.

Intuitively, an object is simple if it is easy to describe it. For a given TM Υ, the Υ-*complexity* of a string $\omega \in \mathfrak{A}^*$ is the binary length of its shortest description in terms of Υ, that is, of the shortest input π which generates the output $\Upsilon(\pi) = \omega$,

$$\mathrm{C}_\Upsilon(\omega) := \min\{\mathfrak{bl}_M(\pi) : \pi \in \mathfrak{A}^*,\ \Upsilon(\pi) = \omega\}, \quad \omega \in \mathfrak{A}^*. \tag{2.5}$$

This quantity clearly depends on the choice of the TM Υ. The key to a meaningful concept is the fact that there exist *universal* Turing machines Ξ, meaning that Ξ is able to simulate any other TM, i.e. for every TM Υ on $\mathfrak{A}^*$ there is some $\sigma(\Upsilon) \in \mathfrak{A}^*$ such that

$$\Xi(\sigma(\Upsilon), \pi) = \Upsilon(\pi) \quad \text{for all } \pi \in D_\Upsilon. \tag{2.6}$$

Fixing such a Ξ, we define the *(Kolmogorov-) complexity* of ω as

$$\mathrm{C}(\omega) := \mathrm{C}_\Xi(\omega) = \min\{\mathfrak{bl}_M(\pi) : \pi \in \mathfrak{A}^*,\ \Xi(\pi) = \omega\}, \quad \omega \in \mathfrak{A}^*. \tag{2.7}$$

The crucial point here is that up to an additive constant, $\mathrm{C}(\omega)$ is independent of the choice of Ξ: Let Ξ' be another universal TM. For any $\omega \in \mathfrak{A}^*$ there is some $\pi' \in \mathfrak{A}^*$ such that $\mathfrak{bl}_M(\pi') = \mathrm{C}_{\Xi'}(\omega)$ and $\Xi'(\pi') = \omega$. But then $\pi := (\sigma(\Xi'), \pi')$ satisfies $\Xi(\pi) = \Xi'(\pi') = \omega$, and it has length $\mathfrak{bl}_M(\pi) \leq \mathfrak{bl}_M(\pi') + \kappa = \mathrm{C}_{\Xi'}(\omega) + \kappa$ with $\kappa := \phi_M(\mathfrak{bl}_M(\sigma(\Xi')))$ independent of ω, cf. (2.3). Hence $\mathrm{C}_\Xi(\omega) \leq \mathrm{C}_{\Xi'}(\omega) + \kappa$ for all ω. Interchanging the roles of Ξ and Ξ', we see that there is some constant $\kappa_{\Xi,\Xi'}$ such that

$$|\mathrm{C}_\Xi(\omega) - \mathrm{C}_{\Xi'}(\omega)| \leq \kappa_{\Xi,\Xi'} \quad \text{for all } \omega \in \mathfrak{A}^*. \tag{2.8}$$

For asymptotic questions about strings with large complexity the particular choice of Ξ therefore is immaterial.

Note that, quite trivially, there is some constant κ such that

$$\mathrm{C}(\omega) \leq \mathfrak{bl}_M(\omega) + \kappa \quad \text{for all } \omega \in \mathfrak{A}^*, \tag{2.9}$$

since $\Xi(\sigma(\Upsilon_{\mathrm{Id}}), \omega) = \omega$, with Υ_{Id} the identity on $\mathfrak{A}^*$, and $\kappa = \phi_M(\mathfrak{bl}_M(\sigma(\Upsilon_{\mathrm{Id}})))$ as in (2.3).

Due to the use of binary length in the definition of C, the particular choice of $\mathfrak{A}$ doesn't affect asymptotic properties (as long, of course, as $\mathfrak{A}$ remains fixed during all computations). To see this, take any universal TM Ξ_2 on $\mathfrak{A}_2^*$, and observe that we obtain a universal TM Ξ_M on $\mathfrak{A}_M^*$ by letting $\Xi_M := \mathfrak{n}_M^{-1} \circ \mathfrak{n}_2 \circ \Xi_2 \circ \mathfrak{n}_2^{-1} \circ \mathfrak{n}_M$. Let $\mathrm{C}^{(M)}$ denote the corresponding complexity function on $\mathfrak{A}_M^*$. Then $\mathrm{C}^{(M)} = \mathrm{C}^{(2)} \circ \mathfrak{n}_2^{-1} \circ \mathfrak{n}_M$ and an argument like the one proving (2.8) shows that for $M \leq N$, $|\ \mathrm{C}^{(M)} - \mathrm{C}^{(N)}\ |$ is bounded on $\mathfrak{A}_M^* \subseteq \mathfrak{A}_N^*$. (There are TMs $\Upsilon_{M,N}$ and $\Upsilon_{M,N}^{-1}$ on $\mathfrak{A}_2^*$ converting $\mathfrak{n}^{-1} \circ \mathfrak{n}_M(\omega)$ into $\mathfrak{n}^{-1} \circ \mathfrak{n}_N(\omega)$ and vice versa.)

3 Proof of Theorem 2

The reference set Y is the union of the finite family $Y \cap \xi \subseteq \xi$, and we can assume w.l.o.g. that $\mu(Y) = 1$. By Hopf's ratio ergodic theorem (see [KK] or [Z3] for simple proofs), it is enough to consider $f := 1_Y$, and we will abbreviate $S_n := \mathbf{S}_n(1_Y)$. Because of the evident T-invariance of all the limits involved, we need only consider $x \in Y$ (meaning that the string $\xi_n(x)$ starts with a symbol $V \in Y \cap \xi$). Therefore our goal is to prove that

$$\frac{\mathrm{C}(\xi_n(x))}{S_n(x)} \longrightarrow h_\mu(T) \qquad \begin{array}{c} \text{as } n \to \infty \\ \text{for a.e. } x \in Y. \end{array} \tag{3.1}$$

Splitting of symbolic orbits. The invariant measure μ being infinite, typical orbits of T spend most of their time in Y^c, so that their ξ-codings consist of very long blocks (Z^b) of symbols $Z \in Y^c \cap \xi$, while symbols $V \in Y \cap \xi$ are very scarce. Refining the argument of [BG], we are going to break up these symbolic sequences in a way that captures this structure:

Due to dynamical separation, ξ-codings of orbits never contain a block of the type (W^a, Z^b) with $W, Z \in Y^c \cap \xi$, $W \neq Z$ and $a, b \geq 1$. Therefore we can represent any $\omega \in (Y \cap \xi) \times \xi^*$ (i.e. $\omega \in \xi^*$ with first symbol from $Y \cap \xi$) as a sequence of (possibly void) blocks of the type (Z^b), $Z \in Y^c \cap \xi$, delimited by single symbols $V \in Y \cap \xi$, that is,

$$\omega = (V_1, Z_1^{b_1}, V_2, Z_2^{b_2}, \ldots, V_j, Z_j^{b_j}, V_{j+1}, Z_{j+1}^d) \tag{3.2}$$

with $j = j(\omega)$, $d = d(\omega) \in \mathbb{N}_0$, and $b_i = b_i(\omega) \in \mathbb{N}_0$, $V_i = V_i(\omega) \in Y \cap \xi$, $Z_i = Z_i(\omega) \in Y^c \cap \xi$ for $i \in \{1, \ldots, j\}$. This representation is unique (except, of course, for the Z_i with zero exponents b_i).

For any fixed integer $K \geq 1$ we now split ω into four components: first define

$$\Theta_K^{(1)}(\omega) := (V_1, Z_1^{b_1 \wedge K}, V_2, Z_2^{b_2 \wedge K}, \ldots, V_j, Z_j^{b_j \wedge K}), \tag{3.3}$$

where $a \wedge b := \min(a, b)$, i.e. we throw away the last block (V_{j+1}, Z_{j+1}^d) and truncate the other $(Z_i^{b_i})$ at length K. There is an obvious algorithm turning any ω into $\Theta_K^{(1)}(\omega)$, showing that the map $\Theta_K^{(1)}$ is a Turing machine. To keep track of the information lost in passing from ω to $\Theta_K^{(1)}(\omega)$, we first record the final block, letting

$$\Theta_K^{(2)}(\omega) := (V_{j+1}, Z_{j+1}^d). \tag{3.4}$$

We single out one element $\widehat{V} \in Y \cap \xi$ and one $\widehat{Z} \in Y^c \cap \xi$. We use these special symbols to represent binary words in our alphabet ξ, recording which blocks have been truncated by defining

$$\Theta_K^{(3)}(\omega) := (\gamma_1, \ldots, \gamma_j), \tag{3.5}$$

where $\gamma_i = \gamma_i(\omega) = \widehat{Z}$ if $b_i > K$, and $\gamma_i = \widehat{V}$ otherwise. Finally, we keep track of how much has been chopped off: Let $1 \le i_1 < \ldots < i_L \le j$, $L = L(\omega) \in \mathbb{N}_0$, be those i with $b_i > K$, and define

$$\Theta_K^{(4)}(\omega) := (\widehat{V}, \widehat{Z}^{b_{i_1}-K}, \ldots, \widehat{V}, \widehat{Z}^{b_{i_L}-K}). \tag{3.6}$$

(Note that blocks $(Z_i^{b_i})$ of ω with $b_i \le K$ do not contribute at all to this string!) Evidently, the transformations $\Theta_K^{(2)}$, $\Theta_K^{(3)}$, and $\Theta_K^{(4)}$ from $(Y \cap \xi) \times \xi^*$ into ξ^* can be performed by easy algorithms and hence are Turing machines.

The obvious fact that ω contains more information than $\Theta_K^{(1)}(\omega)$ and can, on the other hand, easily be reconstructed by combining the partial information contained in $\Theta_K^{(1)}(\omega)$, $\Theta_K^{(2)}(\omega)$, $\Theta_K^{(3)}(\omega)$, and $\Theta_K^{(4)}(\omega)$, is formalized in the following lemma which will be crucial for our argument. It allows us to analyze $\mathrm{C}(\omega)$ by separately studying the complexity of these pieces.

Lemma 1 (Complexity of split strings) *For every $K \in \mathbb{N}$ there is some $\kappa_K \in \mathbb{N}$ such that for all $\omega \in (Y \cap \xi) \times \xi^*$ we have*

$$\mathrm{C}(\Theta_K^{(1)}(\omega)) - \kappa_K \le \mathrm{C}(\omega) \le \mathrm{C}(\Theta_K^{(1)}(\omega)) + 2\sum_{i=2}^{4} \mathrm{C}(\Theta_K^{(i)}(\omega)) + \kappa_K.$$

Proof. (i) The first inequality is easy: Consider the TM $\Upsilon_K := \Theta_K^{(1)} \circ \Xi$. For any $\omega \in (Y \cap \xi) \times \xi^*$ there is some $\pi \in \xi^*$ such that $\mathfrak{bl}_M(\pi) = \mathrm{C}(\omega)$ and $\Xi(\pi) = \omega$. In this case,

$$\Xi\left(\sigma(\Upsilon_K), \pi\right) = \Upsilon_K\left(\pi\right) = \Theta_K^{(1)}(\omega),$$

and the argument $\pi' := (\sigma(\Upsilon_K), \pi) \in \xi^*$ in the left-most expression has binary length $\mathfrak{bl}_M(\pi') \le \mathfrak{bl}_M(\pi) + \kappa_K$ with $\kappa_K := \phi_M(\mathfrak{bl}_M(\sigma(\Upsilon_K)))$, cf. (2.3), proving that $\mathrm{C}(\Theta_K^{(1)}(\omega)) \le \mathrm{C}(\omega) + \kappa_K$.

(ii) To prove the second inequality, we first observe that, starting from Ξ, it is possible to construct a TM Υ such that

$$\begin{aligned} &\Upsilon\left(\overline{\mathfrak{l}_M(\pi^{(2)})}, \ldots, \overline{\mathfrak{l}_M(\pi^{(4)})}, \pi^{(1)}, \ldots, \pi^{(4)}\right) \\ &= \left(\overline{\mathfrak{l}_M(\Xi(\pi^{(2)}))}, \ldots, \overline{\mathfrak{l}_M(\Xi(\pi^{(4)}))}, \Xi(\pi^{(1)}), \ldots, \Xi(\pi^{(4)})\right) \end{aligned} \tag{3.7}$$

for all $\pi^{(1)}, \ldots, \pi^{(4)} \in \xi^*$. (Including the self-delimiting versions of the lengths $\mathfrak{l}_M(\pi^{(2)}), \ldots, \mathfrak{l}_M(\pi^{(4)})$ of $\pi^{(2)}, \ldots, \pi^{(4)}$ in the left-hand argument enables a reconstruction of each of the building blocks $\pi^{(1)}, \ldots, \pi^{(4)}$ through a TM. Then apply Ξ to each of these words, and finally represent the respective results $\Xi(\pi^{(1)}), \ldots, \Xi(\pi^{(4)})$ as a single string in the same format as the input string.)

Take $\omega \in (Y \cap \xi) \times \xi^*$, and let $\omega^{(i)} := \Theta_K^{(i)}(\omega)$. Then there is an obvious algorithm recovering ω from $\omega^{(1)}, \ldots, \omega^{(4)}$. Representing the combined information of these four words by means of the string

$$\left(\overline{\mathfrak{l}_M(\omega^{(2)})}, \ldots, \overline{\mathfrak{l}_M(\omega^{(4)})}, \omega^{(1)}, \ldots, \omega^{(4)}\right) \in \xi^*,$$

(thus enabling a reconstruction of the $\omega^{(i)}$) we therefore see that there is a TM Σ_K such that

$$\Sigma_K\left(\overline{\mathfrak{l}_M(\Theta_K^{(2)}(\omega))}, \ldots, \overline{\mathfrak{l}_M(\Theta_K^{(4)}(\omega))}, \Theta_K^{(1)}(\omega), \ldots, \Theta_K^{(4)}(\omega)\right) = \omega \tag{3.8}$$

for all $\omega \in (Y \cap \xi) \times \xi^*$, and $\Lambda_K := \Sigma_K \circ \Upsilon$ again is a TM.

Now fix any $\omega \in (Y \cap \xi) \times \xi^*$. There are $\pi^{(1)}, \ldots, \pi^{(4)} \in \xi^*$ such that $\mathfrak{bl}_M(\pi^{(i)}) = \mathrm{C}(\Theta_K^{(i)}(\omega))$ and $\Xi(\pi^{(i)}) = \Theta_K^{(i)}(\omega)$ for $i \in \{1, \ldots, 4\}$. Due to the latter property, (3.7) and (3.8) together show that

$$\Lambda_K\left(\overline{\mathfrak{l}_M(\pi^{(2)})}, \ldots, \overline{\mathfrak{l}_M(\pi^{(4)})}, \pi^{(1)}, \ldots, \pi^{(4)}\right) = \omega,$$

and hence

$$\Xi\left(\sigma(\Lambda_K), \overline{\mathfrak{l}_M(\pi^{(2)})}, \ldots, \overline{\mathfrak{l}_M(\pi^{(4)})}, \pi^{(1)}, \ldots, \pi^{(4)}\right) = \omega.$$

Now (2.2) yields an upper bound for the binomial length of the input string $\pi \in \xi^*$ on the left-hand side,

$$\mathfrak{bl}_M(\pi) \leq \mathfrak{bl}_M(\pi) + \sum_{i=2}^{k}\left(\mathfrak{bl}_M\left(\overline{\mathfrak{l}_M(\pi^{(i)})}\right) + \mathfrak{bl}_M\left(\pi^{(i)}\right)\right) + \mathfrak{bl}_M(\sigma(\Lambda_K)) + \tau_M(8).$$

Using the obvious (and very crude) estimate

$$\mathfrak{bl}_M\left(\overline{\mathfrak{l}_M(\pi)}\right) \leq \mathfrak{bl}_M(\pi) + \overline{\kappa}(M) \qquad \text{for all } \pi \in \mathfrak{A}^*$$

with $\overline{\kappa}(M)$ a suitable constant, our assertion follows. ■

The splitting introduced above will be the key tool for our proof. Due to our lemma it is enough to show that

$$\lim_{K\to\infty}\lim_{n\to\infty}\frac{\mathrm{C}\left(\Theta_K^{(1)}(\xi_n(x))\right)}{S_n(x)} = h_\mu(T) \qquad \text{for a.e. } x \in Y, \tag{3.9}$$

while, for $i \in \{2, 3, 4\}$,

$$\lim_{K\to\infty}\overline{\lim_{n\to\infty}}\frac{\mathrm{C}\left(\Theta_K^{(i)}(\xi_n(x))\right)}{S_n(x)} = 0 \qquad \text{for a.e. } x \in Y. \tag{3.10}$$

Dynamical meaning of the splitting. To establish (3.9) as well as (3.10) for $i \in \{2,3,4\}$, we explicitly express the objects involved in terms of the dynamics. Assume that $\omega := \xi_n(x)$ for some $x \in Y$ and $n \geq 1$, with block structure given by (3.2), and components as in (3.3)-(3.6). Letting $Z_n(x) := \max\{0 \leq k < n : T^k x \in Y\}$, we then have

$$j = j(\omega) = S_{Z_n(x)}(x) = S_n(x) - 1, \tag{3.11}$$

and

$$b_i = b_i(\omega) = \varphi \circ T_Y^{i-1}(x) - 1 \quad \text{for } i \in \{1, \dots, j\}. \tag{3.12}$$

The above splitting of symbolic orbits corresponds to viewing the induced system through coarser partitions. For any countable partition (mod μ_Y) ϱ_Y of Y one has $h_{\mu_Y}(T_Y, \varrho_Y) \leq H_{\mu_Y}(\varrho_Y)$, and ϱ_Y*-codings* will always be understood w.r.t. T_Y, that is, $\varrho_{Y,j} := \bigvee_{i=0}^{j-1} T_Y^{-i} \rho_Y$, $j \geq 1$. We need to consider, for $K \geq 1$,

$$\eta_Y^K := \{Y \cap \{\varphi > K\}, Y \cap \{\varphi \leq K\}\}. \tag{3.13}$$

Clearly $\mu_Y(Y \cap \{\varphi > K\}) \to 0$ as $K \to \infty$, and hence $H_{\mu_Y}(\eta_Y^K) \to 0$ as well. Consequently,

$$\lim_{K \to \infty} h_{\mu_Y}(T_Y, \eta_Y^K) = 0. \tag{3.14}$$

Next, observe that due to our assumption on dynamical separation,

$$\begin{aligned} \xi_Y = &\{V \cap \{\varphi = 1\} : V \in Y \cap \xi\} \cup \\ &\{V \cap T^{-1}Z \cap \{\varphi = k\} : V \in Y \cap \xi,\, Z \in Y^c \cap \xi,\, k > 1\} \end{aligned} \tag{3.15}$$

(disregarding empty intersections). We define finite truncated versions of ξ_Y by letting

$$\begin{aligned} \xi_Y^K := &\{V \cap \{\varphi = 1\} : V \in Y \cap \xi\} \cup \\ &\{V \cap T^{-1}Z \cap \{\varphi = k\} : V \in Y \cap \xi,\, Z \in Y^c \cap \xi,\, k \in \{2, \dots, K\}\} \cup \\ &\{V \cap T^{-1}Z \cap \{\varphi > K\} : V \in Y \cap \xi,\, Z \in Y^c \cap \xi\}, \end{aligned} \tag{3.16}$$

which trivially satisfy $h_{\mu_Y}(T_Y, \xi_Y^K) \leq h_{\mu_Y}(T_Y, \xi_Y)$. Due to $H_{\mu_Y}(\xi_Y) < \infty$ we have $h_{\mu_Y}(T_Y, \xi_Y) \leq h_{\mu_Y}(T_Y, \xi_Y^K) + H_{\mu_Y}(\xi_Y \mid \xi_Y^K)$, see e.g. Lemma 3.2.15 of [Ke], and $H_{\mu_Y}(\xi_Y \mid \xi_Y^K) \to 0$ as $K \to \infty$. Therefore,

$$\lim_{K \to \infty} h_{\mu_Y}(T_Y, \xi_Y^K) = h_{\mu_Y}(T_Y, \xi_Y) = h_{\mu_Y}(T_Y), \tag{3.17}$$

since ξ_Y is easily seen to be a generator for T_Y.

Proof of (3.9). Let us first fix some $K \geq 1$ and consider $\Theta_K^{(1)}(\xi_n(x))$, which is a string of the form

$$\widetilde{\omega} = (V_1, Z_1^{k_1}, V_2, Z_2^{k_2}, \dots, V_j, Z_j^{k_j}) \in (Y \cap \xi) \times \xi^* \quad \text{with } 0 \leq k_i \leq K. \tag{3.18}$$

Successively applying

$$\psi(VZ^k) := \begin{cases} V \cap \{\varphi = 1\} & \text{if } k = 0 \\ V \cap T^{-1}Z \cap \{\varphi = k+1\} & \text{if } k \in \{1, \dots, K-1\} \\ V \cap T^{-1}Z \cap \{\varphi \geq K+1\} & \text{if } k = K \end{cases}$$

to its basic building blocks, we obtain a representation

$$\Psi(\widetilde{\omega}) := (\psi(V_1 Z_1^{k_1}), \dots, \psi(V_j Z_j^{k_j})) \in (\xi_Y^K)^*$$

of $\widetilde{\omega}$ in terms of the truncated induced partition ξ_Y^K. Since there are easy algorithms turning $\widetilde{\omega}$ into $\Psi(\widetilde{\omega})$ and vice versa, the usual argument shows that there is some constant κ'_K such that

$$\left| \mathrm{C}\left(\Theta_K^{(1)}(\omega)\right) - \mathrm{C}\left(\Psi(\Theta_K^{(1)}(\omega))\right) \right| \leq \kappa'_K \quad \text{ for all } \omega \in (Y \cap \xi) \times \xi^*. \tag{3.19}$$

(At this point we consider strings over two (since K is fixed) different alphabets ξ and ξ_Y^K.) Recalling that $k_i = b_i \wedge K = (\varphi \circ T_Y^{i-1}(x) - 1) \wedge K$, that is, $k_i = k$, $0 \leq k < K$, iff $x \in Y \cap \{\varphi = k+1\}$, and $k_i = K$ iff $x \in Y \cap \{\varphi > K\}$, we see that our new string $\Psi(\Theta_K^{(1)}(\omega))$ is the ξ_Y^K-coding of the j-orbit $(T_Y^k x)_{k=0}^{j-1}$ of x under T_Y, i.e.

$$\Psi(\Theta_K^{(1)}(\xi_n(x))) = \xi^K_{Y,S_n(x)-1}(x). \tag{3.20}$$

According to the Brudno -White theorem (1.2) for T_Y and ξ_Y^K, we have

$$j^{-1} \cdot \mathrm{C}(\xi^K_{Y,j}(x)) \longrightarrow h_{\mu_Y}(T_Y, \xi_Y^K) \quad \text{ as } j \to \infty \text{ for a.e. } x \in Y.$$

Therefore, using (3.19), we obtain

$$\lim_{n\to\infty} \frac{\mathrm{C}\left(\Theta_K^{(1)}(\xi_n(x))\right)}{S_n(x)} = \lim_{n\to\infty} \frac{\mathrm{C}\left(\xi^K_{Y,S_n(x)-1}(x)\right)}{S_n(x)-1} = h_{\mu_Y}(T_Y, \xi_Y^K)$$

for a.e. $x \in Y$. Combining this with (3.17), our assertion (3.9) follows. ■

We turn to the remaining estimates from above.

Proof of (3.10) for $i = 3$. Fix any $K \geq 1$ and consider the string $\Theta_K^{(3)}(\xi_n(x)) = (\gamma_1, \dots, \gamma_j)$ recording which excursions from Y take more than $K+1$ steps. We have $j = j(\omega) = S_n(x) - 1$ and

$$\gamma_i = \begin{cases} \widehat{Z} & \text{if } T_Y^{i-1}(x) \in Y \cap \{\varphi > K+1\} \\ \widehat{V} & \text{if } T_Y^{i-1}(x) \in Y \cap \{\varphi \leq K+1\}. \end{cases}$$

Via the map $\Sigma_K : \{\widehat{Z}, \widehat{V}\}^* \to (\eta_Y^{K+1})^*$ which identifies the symbols $\widehat{Z}$ and $\widehat{V}$ with $Y \cap \{\varphi > K+1\}$ and $Y \cap \{\varphi \leq K+1\}$ respectively, we thus obtain the coding of the j-orbit $(T_Y^k x)_{k=0}^{j-1}$ of x under T_Y with respect to the partition η_Y^{K+1} of Y,

$$\Sigma_K \circ \Theta_K^{(3)}(\xi_n(x)) = \eta_{Y,S_n(x)-1}^{K+1}(x), \tag{3.21}$$

and it is clear that the complexities of $\eta_{Y,S_n(x)-1}^{K+1}(x)$ and $\Theta_K^{(3)}(\xi_n(x))$ coincide. Applying the Brudno -White theorem (1.2) to T_Y and η_Y^{K+1}, we obtain

$$j^{-1} \cdot \mathrm{C}(\eta_{Y,j}^{K+1}(x)) \longrightarrow h_\mu(T_Y, \eta_Y^{K+1}) \quad \text{as } j \to \infty \text{ for a.e. } x \in Y,$$

and hence

$$\lim_{n\to\infty} \frac{\mathrm{C}\left(\Theta_K^{(3)}(\xi_n(x))\right)}{S_n(x)} = \lim_{n\to\infty} \frac{\mathrm{C}\left(\eta_{Y,S_n(x)-1}^{K+1}(x)\right)}{S_n(x)-1} = h_\mu(T_Y, \eta_Y^{K+1})$$

for a.e. $x \in Y$. Letting $K \to \infty$, our claim (3.10) follows by (3.14). ■

Proof of (3.10) for $i = 4$. Fix any $K \geq 1$. To estimate the complexity of the string $\widetilde{\omega} := \Theta_K^{(4)}(\xi_n(x)) = (\widehat{V}, \widehat{Z}^{l_1}, \ldots, \widehat{V}, \widehat{Z}^{l_L})$ with $l_r = b_{i_r} - K \geq 1$ we use a specific one-to-one coding. We are going to employ a binary prefix code, replacing each block $(\widehat{V}, \widehat{Z}^{l_r})$ in $\widetilde{\omega}$ by the binary self-delimiting version of $\mathfrak{n}_2^{-1}(l_r)$, i.e. by $(1^{\mathfrak{l}_2(\mathfrak{n}_2^{-1}(l_r))}, 0, \mathfrak{n}_2^{-1}(l_r))$. That is, we define a map Ψ into $\{0,1\}^*$ by letting

$$\Psi(\widetilde{\omega}) := \left(\overline{\mathfrak{n}_2^{-1}(l_1)}, \ldots, \overline{\mathfrak{n}_2^{-1}(l_L)}\right).$$

Since there is an easy algorithm which, for every $\omega \in (Y \cap \xi) \times \xi^*$, reconstructs $\widetilde{\omega}$ from $\Psi(\widetilde{\omega})$, the usual argument shows that there is some constant κ_K'' such that for all $\omega \in (Y \cap \xi) \times \xi^*$,

$$\mathrm{C}\left(\Theta_K^{(4)}(\omega)\right) \leq \mathrm{C}\left(\Psi(\Theta_K^{(4)}(\omega))\right) + \kappa_K'' \leq \mathfrak{bl}_2\left(\Psi(\Theta_K^{(4)}(\omega))\right) + \kappa_K'' + \kappa. \tag{3.22}$$

(Note that we again encounter complexity functions on different domains here.) Moreover, it is straightforward to check (cf. (6) in [BG]) that the length of the encoded string $\Psi(\Theta_K^{(4)}(\omega)) \in \{0,1\}^*$ is given by

$$\mathfrak{bl}_2(\Psi(\widetilde{\omega})) = \sum_{r=1}^{L} \left(2\lfloor \log_2(l_r+1) \rfloor + 1\right).$$

Recalling, for $\omega = \xi_n(x)$, the dynamical meaning of the quantities in question, we see that $\{i_1, \ldots, i_L\}$ is the (ordered) set of those indices $i \in \{1, \ldots, j\}$ for which $T_Y^{i-1}x \in Y \cap \{\varphi > K+1\}$, and $l_k = b_{i_k} - K = \varphi(T_Y^{i_k-1}x) - K$. Therefore,

$$\mathfrak{bl}_2\left(\Psi(\Theta_K^{(4)}(\xi_n(x)))\right) = \sum_{i=1}^{S_n(x)-1} F_K \circ T_Y^{i-1}(x),$$

with $F_K : Y \to [0,\infty)$, $K \in \mathbb{N}$, defined by

$$F_K := 1_{Y \cap \{\varphi > K+1\}} \cdot \left(2 \left\lfloor \log_2((\varphi - K)^+)\right\rfloor + 1\right).$$

By integrability of $\log \circ \varphi$ and $0 \leq F_K \leq \mathit{const} \cdot (\log \circ \varphi) + 1$, we know that $F_K \in L_1(\mu \mid_{\mathcal{A} \cap Y})$. Applying the ergodic theorem to T_Y and F_K, we thus see that

$$\lim_{n\to\infty} \frac{\mathfrak{bl}_2\left(\Psi(\Theta_K^{(4)}(\xi_n(x)))\right)}{S_n(x)} = \int_Y F_K \, d\mu \quad \text{for a.e. } x \in Y.$$

In view of (3.22), we therefore obtain

$$\overline{\lim_{n\to\infty}} \frac{\mathrm{C}\left(\Theta_K^{(4)}(\xi_n(x))\right)}{S_n(x)} \leq \int_Y F_K \, d\mu \quad \text{for a.e. } x \in Y.$$

But $K \geq 1$ was arbitrary, and since $F_K \searrow 0$ a.e. as $K \to \infty$, we have $\int_Y F_K \, d\mu \searrow 0$, and (3.10) follows. ■

Proof of (3.10) for $i = 2$. By the type of argument met before, we can represent $\Theta_K^{(2)}(\omega) = (V_{j+1}, Z_{j+1}^d)$ by $(V_{j+1}, Z_{j+1}, \mathfrak{n}_M^{-1}(d))$, with

$$\begin{aligned} \mathrm{C}(V_{j+1}, Z_{j+1}^d) &\leq \mathrm{C}(V_{j+1}, Z_{j+1}, \mathfrak{n}_M^{-1}(d)) + \kappa \leq \mathrm{C}(\mathfrak{n}_M^{-1}(d)) + \widetilde{\kappa} \\ &\leq \mathfrak{bl}_M(\mathfrak{n}_M^{-1}(d)) + \widetilde{\widetilde{\kappa}} \leq C \cdot \log d + \widetilde{\widetilde{\kappa}}. \end{aligned}$$

Since $d = d(\omega) < n$ we therefore have

$$\frac{\mathrm{C}\left(\Theta_K^{(2)}(\xi_n(x))\right)}{S_n(x)} \leq \frac{C \cdot \log n + \widetilde{\widetilde{\kappa}}}{S_n(x)}.$$

Due to $\int_Y \log \circ \varphi \, d\mu_Y < \infty$, however, we can apply Theorem 2.4.1 of [A0] with $a(t) := \log t$, $t > 1$, to finally see that for all $f \in L_1(\mu)$ with $\int_X f \, d\mu > 0$,

$$\lim_{n\to\infty} \frac{\mathbf{S}_n(f)}{\log n} = \infty \quad \text{a.e. on } X, \tag{3.23}$$

and our assertion follows. ■

This completes the proof of the theorem.

4 Proof of Theorem 1

Proof. Let (X, T, ξ) be as in the statement of the theorem. We apply the abstract Theorem 2. If $W \subseteq Z \in \xi_n$, we let $f_W := (T^n \mid_W)^{-1}$ be the inverse of the branch $T^n \mid_W$. To ensure that different cylinders of infinite measure are separated by some reference set of finite measure, we first pass to a refinement ξ' of ξ. For $Z \in \zeta$ we let $Z' := f_Z(Z) \subseteq Z$, $Z(1) := Z \setminus Z'$, and use this to define $\zeta' := \{Z' : Z \in \zeta\}$ and $\xi' := (\xi \setminus \zeta) \cup \zeta' \cup \{Z(1) : Z \in \zeta\}$. Then (X, T, ξ') is a basic AFN system with indifferent cylinders $Z' \in \zeta'$, and $Y := \bigcup_{V \in \xi' \setminus \zeta'} V$ is a ξ'-measurable set with $0 < \mu(Y) < \infty$ which dynamically separates different members of ζ'. For the proof of the theorem we may consider ξ' instead of ξ, since ξ_2 refines ξ', so that $\xi'_n(x) \subseteq \xi_n(x) \subseteq \xi'_{n-1}(x)$ for $n \geq 2$ and a.e. $x \in X$.

By the same type of argument we may assume w.l.o.g. that for any $V \in \xi'_Y$ the image $T_Y V$ only intersects one of the finitely many connected components of Y. In this case, our definition of ξ'_Y coincides with the one used in [Z2], and Remark 13 there guarantees that $h_\mu(T) < \infty$ implies $H_{\mu_Y}(\xi'_Y) < \infty$, verifying the first condition of (1.13).

It remains to check integrability of $\log \circ \varphi$. We claim that for AFN-maps this, too, is automatic when $h_\mu(T) < \infty$. Extend φ to all of X by letting $\varphi(x) := \min\{j \geq 1 : T^j x \in Y\}$, $x \in X$. Fix any $Z' \in \zeta'$, and observe that since $Z' \cap \{\varphi = k+1\} = f_{Z'}(Z' \cap \{\varphi = k\})$ with $|f'_{Z'}| \leq 1$, $\lambda(Z' \cap \{\varphi = k\})$ is decreasing with $\sum_{k \geq 1} \lambda(Z' \cap \{\varphi = k\}) = \lambda(Z') < \infty$. Therefore,

$$\lambda(Z' \cap \{\varphi = k\}) = o(1/k) \quad \text{as } k \to \infty. \tag{4.1}$$

A fortiori, $\lambda(Y \cap T^{-1}(Z' \cap \{\varphi = k\})) = \lambda(Y \cap T^{-1}Z' \cap \{\varphi = k+1\})) = o(1/k)$ for $V \in Y \cap \xi'$, and since the density $h\,|_Y = d\mu_Y/d\lambda$ is of bounded variation ([Z2], Corollary 1), hence bounded, we also have $\mu_Y(V \cap T^{-1}Z' \cap \{\varphi = k\}) = o(1/k)$ as $k \to \infty$, and therefore

$$\log k < -\log \mu_Y(V \cap T^{-1}Z' \cap \{\varphi = k\}) \quad \text{for } k \geq k_0(V, Z').$$

Recalling that ξ' and ζ' are finite, and using (3.15) for ξ'_Y, we thus see that summability of

$$\begin{aligned} H_{\mu_Y}(\xi'_Y) = &\sum_{V \in \xi' \setminus \zeta'} \mu_Y(V \cap \{\varphi = 1\}) \cdot (-\log \mu_Y(V \cap \{\varphi = 1\})) + \\ &\sum_{\substack{V \in \xi' \setminus \zeta', \\ Z' \in \zeta',\, k \geq 2}} \mu_Y(V \cap T^{-1}Z' \cap \{\varphi = k\}) \cdot (-\log \mu_Y(V \cap T^{-1}Z' \cap \{\varphi = k\})) \end{aligned}$$

implies finiteness of

$$\int_Y \log \circ \varphi \, d\mu_Y = \sum_{V \in \xi' \setminus \zeta'} \mu_Y(V \cap \{\varphi = 1\}) + \sum_{\substack{V \in \xi' \setminus \zeta', \\ Z' \in \zeta',\, k \geq 2}} \mu_Y(V \cap T^{-1}Z' \cap \{\varphi = k\}) \cdot k,$$

as required. This completes the proof of the theorem. ∎

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