Ergodic structure and invariant densities of non-markovian interval maps with indifferent fixed points

Roland Zweimüller

Mathematisches Institut, Universität Erlangen-Nürnberg, Bismarckstraße $1\frac{1}{2}$, D-91054 Erlangen, FRG

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Abstract

We consider piecewise twice differentiable maps T on $[0,1]$ with indifferent fixed points giving rise to infinite invariant measures. Without assuming the existence of a Markov partition but only requiring the first image of the fundamental partition to be finite we prove that the interval decomposes into a finite number of ergodic cycles with exact powers plus a dissipative part. T is shown to be exact on components containing indifferent fixed points. We also determine the order of the singularities of the invariant densities.

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1 Introduction

Interval maps with indifferent (or neutral) fixed points have attracted the attention of mathematicians concerned with infinite ergodic theory (cf. [A0]) as well as that of scientists interested in phenomena of intermittency ([Ma], [PM]). Recent mathematical work can for example be found in [In], [Is], [Ru], and [T3]. All these papers, however, deal with piecewise surjective maps, whereas the results of [ADU] apply to Markov maps with indifferent fixed points. Transformations with finite range structure as studied in [Y1], [Y2] are most intimately related to the latter case (a Markov partition being hidden there). The present note is meant to be a first step towards a study of the general non-markovian case in one dimension.

2 Definitions and main results

To begin with, let us fix some notations. Throughout, λ will denote Lebesgue measure and $\mathcal{B}$ will be the Borel $\sigma-$field on $[0,1]$. For any interval I and any point $x \in cl(I)$, an *$I-$neighbourhood of x* is meant to be a set of the form $(x-\epsilon, x+\epsilon) \cap I$ (and thus need not contain x). The *support* of a function $h : [0,1] \to [0,\infty)$ will simply be the set $\{h > 0\}$.

Definition 1 *A map $T : [0,1] \to [0,1]$ will be called* piecewise monotonic *if there is a collection ξ (not necessarily finite) of nonempty pairwise disjoint open subintervals (the* cylinders of rank one*) with $\lambda(\bigcup \xi) = 1$ such that $T \mid_Z$ is continuous and strictly monotonic for each $Z \in \xi$. ξ_n will denote the family of* cylinders of rank n*, that is, the nonempty sets of the form $Z = \bigcap_{i=0}^{n-1} T^{-i} Z_i$ with $Z_i \in \xi_1 = \xi$, and we let $f_Z := (T^n \mid_Z)^{-1}$. Most maps will also be assumed to be twice differentiable on members of ξ and satisfy*

(A) Adler's condition*:* $T''/(T')^2$ *is bounded on* $\bigcup \xi$

and

(F) Finite image condition*:* $T\xi = \{TZ : Z \in \xi\}$ *is finite.*

If in addition T is

(U) uniformly expanding*, i.e.* $|T'| \geq \tau > 1$ *on* $\bigcup \xi$,

then we will call T an AFU-map.

We are mainly interested in maps for which the last condition may be violated at a finite number of fixed points.

Definition 2 *We consider piecewise monotonic maps $T : [0,1] \to [0,1]$ satisfying (A) and (F) for which there is a finite set $\zeta \subseteq \xi$ such that each $Z \in \zeta$ has an* indifferent fixed point x_Z *satisfying* Thaler's assumptions *as one of its endpoints, i.e.*

$$\lim_{x \to x_Z,\ x \in Z} Tx = x_Z \quad \text{and} \quad T'x_Z := \lim_{x \to x_Z,\ x \in Z} T'x = 1,$$

T is uniformly expanding outside of $Z-$neighbourhoods of $\{x_Z : Z \in \zeta\}$*, that is, letting $A_\epsilon := [0,1] \setminus \bigcup_{Z \in \zeta} ((x_Z - \epsilon, x_Z + \epsilon) \cap Z)$, we have*

$$|T'| \geq \rho(\epsilon) > 1 \quad \text{on } A_\epsilon \quad \text{for each } \epsilon > 0 \ .$$

Moreover, each x_Z, $Z \in \zeta$, is assumed to be a one-sided regular source*, that is, T' decreases on $(0, x_Z) \cap Z$, respectively increases on $(x_Z, 1) \cap Z$. T being* nonuniformly expanding *in general we call it an* AFN-map.

Some comments are in order: First, we require the x_Z to be endpoints of cylinder sets Z only for notational convenience. If this condition is not fulfilled in the first place, simply dissect Z at x_Z and replace it by the resulting intervals Z' and Z''. Clearly then $x_{Z'} = x_{Z''}$. Similarly, if the concave-convex condition for a regular source is not satisfied on all of Z but only on some $Z-$neighbourhood of x_Z, we need only regard the latter as a separate cylinder to see that T is AFN anyway. Therefore this class of transformations contains the maps investigated in [T2] and [T3] (which were assumed to be piecewise surjective).

We shall study the effects of the indifferent fixed points within the scope of conditions (A) and (F) which prevent the occurence of concurrent phenomena.

Our range condition (F) is trivially satisfied whenever the fundamental partition ξ is finite (in particular T need not be onto). Example 4 in Section 3 shows that weakening this condition can let us lose control of the invariant densities. Condition (A) bounds the distortion of the map, ensuring for example that the derivative cannot be unbounded on any cylinder, which to some extent might compensate for a neutral fixed point, as the following example due to M.Thaler illustrates.

Example 1 (Infinite slope balancing an indifferent fixed point) *The map T on $[0,1]$ given by $Tx = 1 - \sqrt{1-2x}$ for $x \in (0,1/2)$ and $Tx = \sqrt{2x-1}$ on $(1/2,1)$ has indifferent fixed points at 0 and 1, and still preserves Lebesgue measure.*

As we did not require that $\zeta \neq \emptyset$, an AFU-map is also AFN. Of course results valid for AFN-maps with some obvious modifications carry over to maps having indifferent periodic points.

A structure theorem. The purpose of this note is to prove that the basic ergodic structure of such maps is very similar to the uniformly expanding case, the only difference being that the x_Z, unless sited in dissipative regions, let the invariant measures become infinite:

Theorem 1 (Ergodic structure and invariant densities of T) *For any AFN-map T there is a finite number of pairwise disjoint open sets $X_1, \ldots, X_m$ such that $TX_i = X_i$ mod λ, and $T\mid_{X_i}$ is conservative and ergodic w.r.t. Lebesgue measure. Almost all points of $D := [0,1] \setminus \bigcup_i X_i$ are eventually mapped into one of these ergodic components. The tail-σ-field $\mathcal{B}_\infty := \bigcap_{n\geq 1} T^{-n}\mathcal{B}$ is discrete, so that each X_i admits a finite partition $X_i = X_i(1) \cup \ldots \cup X_i(l(i))$ whose members are cyclically permuted by T, and for any $j \in \{1, \ldots, l(i)\}$, $T^{l(i)}\mid_{X_i(j)}$ is exact. The sets $X_i(j)$ are finite unions of open intervals, and hence so are the X_i. Each X_i supports an absolutely continuous invariant measure μ_i (unique up to a constant factor) which has a lower semicontinuous density h_i of the form*

$$h_i(x) = 1_{X_i}(x) \cdot H_i(x) \cdot G(x) \ ,$$

where H_i satisfies $0 < C^{-1} \leq H_i \leq C$ for some constant C, and

$$G(x) := \begin{cases} \frac{x - x_Z}{x - f_Z(x)} & \text{for } x \in Z \in \zeta \\ 1 & \text{for } x \in [0,1] \setminus \bigcup \zeta \ . \end{cases}$$

In particular, μ_i is infinite iff X_i contains a $Z-$neighborhood of some x_Z, $z \in \zeta$.

Hence, the order of magnitude of h_i can be expressed in terms of the local behavior of T at x_Z exactly as in the piecewise surjective situation (cf. [T1]). Observe that the above estimate completely determines the spaces $L_1(\mu_i)$.

Example 2 *If $Tx = x + a\,|x - x_Z|^{p+1} + o\Big(|x - x_Z|^{p+1}\Big)$ as $x \to x_Z$ in Z, where $a \neq 0$ and (necessarily) $p \geq 1$, then (cf. [T1])*

$$G(x) \sim |x - x_Z|^{-p} \qquad \text{as } x \to x_Z \text{ in } Z.$$

Remark 1 *The simple geometric structure of the sets X_i and $X_i(j)$ not only will be used in the course of the proof and lead to Theorem 2, but also shows that the restrictions $T\mid_{X_i}$ in an obvious way are again isomorphic to AFN-maps on $[0,1]$. To study the behaviour of T on components, one may therefore assume w.l.o.g. that T is conservative ergodic on $[0,1]$. Observe that T is uniformly expanding on components X_i of finite invariant measure, so that $T\mid_{X_i}$ in this sense is AFU and thus belongs to the class of transformations studied in [Ry] (cf. Section 6).*

Remark 2 *If, replacing T' by $|T'|$ in the assumptions for T, we allow indifferent fixed points with slope -1, the assertions of the theorem remain true as in this case T^2 satisfies Definition 2. The point to take care of is the fact that $T'x_Z = -1$ implies that small $Z-$neighbourhoods of x_Z are being mapped to the outside of Z, so that they can no longer be controlled by the one-sided local behaviour of T at x_Z. A correct estimate is obtained if we modify $G(x)$ on the corresponding cylinders $Z \in \zeta$ to equal $(x - x_{Z'})/(x - f_{Z'}(f_{Z''}(x)))$ for $x \in Z' \cap T^{-1}Z''$, where $T'x_{Z'} = -1$, $x_{Z'} = x_{Z''}$ and $Z' \neq Z'' \in \zeta$, and $G(x) = 1$ on the rest of Z.*

Number of ergodic components and exactness. If T admits a finite fundamental partition, a simple standard argument yields an upper bound on the number of components X_i. The proof of Theorem 4 of [Go] applies to our situation and we obtain

Proposition 1 *If ξ is a finite partition, then the number m of ergodic components X_i of T does not exceed $\#\xi - 1$.*

This estimate is far from optimal even for very simple examples, and inapplicable whenever ξ is infinite. Taking the range structure of the particular map into account, a telescope argument (cf. Lemma 2 below) applied to an auxiliary map like $\widetilde{T}$ introduced in section 3 can sometimes give complete information even if $\#\xi = \infty$.

Our knowledge of the geometric structure of the tail sets $X_i(j)$ yields

Theorem 2 (Ergodicity implies exactness) *If T is an AFN-map and X an ergodic component of infinite invariant measure, then $T\mid_X$ is exact.*

Proof. By Theorem 1, we have a partition $X = X(1) \cup \ldots \cup X(l)$, such that T maps $T(j)$ onto $T(j+1 \bmod l)$. As there is some $Z \in \zeta$ for which X contains some $Z-$neighbourhood of x_Z and each $X(j)$ is a finite union of intervals, there is j_0 for which $X(j_0)$ contains some $Z-$neighbourhood U_Z of x_Z. But then $TX(j_0) \supseteq TU_Z \supseteq U_Z$, which is possible only if $l = 1$. ■

The same type of argument applies to give a bound for the cycle length of components of finite measure containing a one-sided neighbourhood of some periodic point, so that we find

Remark 3 *If T is an AFN (AFU)-map and X any ergodic component, then the cardinality l of its tail field (mod λ) is a divisor of each p for which X contains a $Z-$neighbourhood of some x with $x = T^p x$ $(:= \lim_{y \to x, y \in Z} T^p y)$ and $(T^p)'(x) > 0$.*

3 Structure and invariant densities of AFU-maps

We are going to investigate our AFN-map T by means of an auxiliary transformation $S = \widetilde{T}$ which is AFU with infinite fundamental partition even if ξ is finite. We therefore need to collect a few facts about interval maps of this type, often generalizing results well known in the case of a finite fundamental partition ξ.

Remark 4 *(See Adler's afterword to [Bo]) It is a standard fact that if a piecewise monotonic map S satisfies Adler's condition, then this carries over to any iterate S^n. In general the corresponding bounds increase with n and may tend to infinity. If however S is also uniformly expanding, then there is a common bound for all iterates and S therefore has* bounded distortion*, i.e. there exists $R > 0$ s.t. for all $n \geq 1$, $Z \in \xi_n$ and $A \in \mathcal{B}$,*

$$\frac{\lambda(S^n(Z \cap A))}{\lambda(S^n Z)} = r_Z \cdot \frac{\lambda(Z \cap A)}{\lambda(Z)} \qquad \text{with } r_Z \in (R, R^{-1}).$$

Lemma 1 *If S is a piecewise monotonic map with $S\xi$ finite, then $S^n\xi_n$ is finite for every $n \geq 1$. Thus, any iterate S^n $(n \geq 1)$ of an AFN-map (AFU-map) is an AFN-map (AFU-map).*

Proof. Simple induction using that $S^n\xi_n = S\mathcal{Z}_n$, where $\mathcal{Z}_n := \{S^{n-1}W \cap V : W \in \xi_{n-1}, V \in \xi_1\}$ contains only finitely many intervals which are not members of ξ_1. ■

We will use a telescoping argument for AFU-maps to show that our maps possess discrete tail-fields. Our approach was inspired by [Wa]. A cylinder $Z = \bigcap_{i=0}^{n-1} S^{-i} Z_i$ $(Z_i \in \xi_1)$ of rank n will be called *$\xi-$full* iff $S^{n-1}Z = Z_{n-1}$. Clearly, then $S^n Z = SZ_{n-1} \in S\xi$.

Lemma 2 (First Telescope Lemma) *If S is a piecewise monotonic map differentiable on each $Z \in \xi$ and satisfying* $\inf \mid S' \mid > 2$*, then $\lambda-$almost every $x \in [0,1]$ is contained in infinitely many $\xi-$full cylinders.*

Proof. We claim that for any $n \geq 1$ and $\varepsilon > 0$ there is some $m > n$ such that the set $E(n,m)$ of points contained in some $\xi-$full cylinder of rank $r \in \{n, \ldots, m-1\}$ covers $[0,1]$ up to a set of Lebesgue measure less than ε. Below, $\xi_r(x)$ shall denote the cylinder of rank r containing the point x, and relations between sets are meant to hold mod λ.

For $n \geq 1$ let $\mathcal{R}_n := \{Z \in \xi_n : Z \text{ is not } \xi-\text{full}\}$ and $R_n := \bigcup \mathcal{R}_n = \{x \in [0,1] : \xi_n(x) \text{ is not } \xi-\text{full}\} = E(n, n+1)^c$. Let us now fix n. Since $E(n, n+k+1)^c = \{x \in [0,1] : \xi_r(x) \text{ is not } \xi-\text{full for } n \leq r \leq n+k\} \subseteq E(n, n+1)^c$, we need only consider R_n. As we wish to use a finiteness argument, we choose a finite collection $\mathcal{R}'_n \subseteq \mathcal{R}_n$ for which $\lambda(R_n \backslash R'_n) < \varepsilon/2$, where $R'_n := \bigcup \mathcal{R}'_n$, and henceforth concentrate on R'_n.

For $k \geq 1$ we inductively define $\mathcal{R}'_{n+k} := \{Z' = Z \cap S^{-(n+k-1)}W \neq: Z \in \mathcal{R}'_{n+k-1}, W \in \xi, \text{ and } Z' \text{ is not } \xi-\text{full}\} \subseteq R'_{n+k-1} \cap \xi_{n+k}$, where $R'_j := \bigcup \mathcal{R}'_j$, i.e. $\mathcal{R}'_{n+k}$ is the family of those rank $n+k$ subcylinders Z' of the non$-\xi-$full cylinders $Z \in \mathcal{R}'_{n+k-1}$ which still are not $\xi-$full. Then $R'_{n+k} = \{x \in R'_{n+k-1} :$

$\xi_{n+k}(x)$ is not $\xi-$full$\}$, (and in particular $R'_{n+k} \subseteq R'_{n+k-1}$), whence

$$\begin{aligned} R'_{n+k} &= \bigcap_{j=0}^{k} R'_{n+j} = \{x \in R'_n : \xi_r(x) \text{ is not } \xi-\text{full for } n \leq r \leq n+k\} = \\ &= R'_n \cap E(n, n+k+1)^c. \end{aligned}$$

Recalling that $E(n, n+k+1)^c \subseteq R_n$ we thus find that

$$E(n, n+k+1)^c = E(n, n+k+1)^c \cap ((R_n \backslash R'_n) \cup R'_n) \subseteq (R_n \backslash R'_n) \cup R'_{n+k},$$

and it remains to prove that $\lambda(R'_{n+k}) < \varepsilon/2$ for k sufficiently large.

Consider any fixed cylinder $Z = \bigcap_{j=0}^{n+k-2} S^{-j} Z_j \in \mathcal{R}'_{n+k-1}$. Then among the nonempty sets of the form $S^{n+k-1}Z \cap W$, $W \in \xi$, at most two are strictly smaller than the respective cylinder W. The others however correspond to $\xi-$full cylinders $Z \cap S^{-(n+k-1)}W \subseteq Z$ of rank $n+k$. Thus, $\#\mathcal{R}'_{n+k} \leq 2^k \cdot \#\mathcal{R}'_n$, and since the length of a cylinder in ξ_{n+k} does not exceed $\sigma^{-(n+k)}$, where $\sigma := \inf \mid S' \mid$, we find that

$$\lambda(R'_{n+k}) \leq 2^{-n} \cdot \#\mathcal{R}'_n \cdot \left(\frac{2}{\sigma}\right)^{n+k} < \frac{\varepsilon}{2}$$

for k sufficiently large, which proves the claim.

Accordingly there is a sequence $n_k \nearrow \infty$ of integers such that $\lambda(E(n_{k-1}, n_k)^c) < 2^{-k}$. The Borel-Cantelli Lemma then shows that in fact $\lambda-$a.e. point $x \in [0,1]$ lies in infinitely many sets $E(n_{k-1}, n_k)$. Finally, since the collection of all endpoints of cylinders is countable, the assertion follows. ■

Example 3 *One cannot drop the condition on* $\inf \mid S' \mid$ *above: For* $s \in (1,2)$ *define* $Sx := sx$ *for* $x \in (0, \frac{1}{2}) =: Z_1$ *and* $Sx := sx - s + 1$ *for* $x \in (\frac{1}{2}, 1) =: Z_2$. *Then* S *is exact with invariant density* h *positive on* $X = (1 - \frac{s}{2}, \frac{s}{2})$. *However, each* SZ_i, $i \in \{1,2\}$, *overlaps the dissipative part* $[0,1] \backslash X$. *As on the other hand* X *is forward invariant, i.e.* $SX \subseteq X$, *no cylinder contained in* X *can be* $\xi-$*full.*

Lemma 3 (Second Telescope Lemma) *If* S *is an AFU-map then there exists a constant* $\eta > 0$ *such that for any* $A \in \mathcal{B}$ *with* $\lambda(A) > 0$ *there exist* $k \geq 1$ *and* $Z \in \xi_k$ *for which* $\lambda(S^k(A \cap Z)) \geq \eta$.

Proof. There is some $N \geq 1$ such that $\inf \mid (S^N)' \mid > 2$, and S^N is an AFU-map, so that $\rho := \inf\{\lambda(S^N Z) : Z \in \xi_N\} > 0$. We choose a density point $x \in A$ of A satisfying the conclusions of the preceding Lemma (applied to S^N). Let R be as in Remark 4 and consider a $\xi-$full cylinder $Z \in \xi_{lN}$ for S^N around x of sufficiently high rank so that $\lambda(Z \cap A)/\lambda(Z) \geq (2R)^{-1}$. Then

$$\frac{\lambda(S^{lN}(Z \cap A))}{\lambda(S^{lN} Z)} \geq \frac{1}{2} \text{ , and thus } \lambda(S^{lN}(Z \cap A)) \geq \frac{1}{2}\rho =: \eta > 0 \text{ . } \blacksquare$$

As we wish to infer the ergodic structure of T from that of some associated AFU-map $S = \widetilde{T}$, we need to recall the latter.

Lemma 4 (Basic ergodic structure of AFU-maps) *Let S be an AFU-map. Then there is a finite number of pairwise disjoint open sets $X_1, \ldots, X_m$ such that $SX_i \subseteq X_i$ mod λ, and $S\mid_{X_i}$ is conservative and ergodic w.r.t. Lebesgue measure. Almost all points of $D := [0,1] \setminus \bigcup_i X_i$ are eventually mapped into one of these ergodic components. Each X_i supports a unique absolutely continuous invariant probability measure μ_i which has a lower semicontinuous density h_i of bounded variation.*

Proof. This is immediate from the much more powerful results of [Ry] (See Corollary 1 in Section 6). (Recall that being the difference of two nondecreasing functions a function of bounded variation has some lower semicontinuous version whose support must be an open set.) In fact, we need to appeal to [Ry] only to ensure the existence of invariant densities of bounded variation as Lemma 3 and Lemma 6 below can then be used to give an elementary proof of these assertions. ■

To derive the estimates of Theorem 1, we will need to know more about invariant densities and their supports. For the rest of this section we fix one ergodic component $X := X_i$ of S, and let $h := h_i$ and $\mu := \mu_i$. As h is lower semicontinuous, its support $X = \{h > 0\}$ is the union of an at most countable family of pairwise disjoint open intervals which we denote by $\mathcal{S} = \mathcal{S}(h)$. In fact $\mathcal{S}$ will turn out to be finite. We write $\mathcal{S}' := \mathcal{S} \cap \xi = \{I \cap Z \neq \emptyset : I \in \mathcal{S},\ Z \in \xi\}$, $\mathcal{D} := \{I \in \mathcal{S} : I \cap \partial\xi \neq \emptyset\}$, and $\mathcal{D}' := \mathcal{D} \cap \xi = \{I \cap Z \neq \emptyset : I \in \mathcal{D},\ Z \in \xi\}$.

Lemma 5 *Each SJ' , $J' \in \mathcal{S}'$, is contained in a unique member of $\mathcal{S}$, which we denote by $\langle SJ' \rangle$.*

Proof. All intervals under consideration being open, this is immediate from $h(x) > 0 \Longrightarrow h(Tx) > 0$, which follows from Kuzmin's equation, $h = \sum_{Z\in\xi}(h \circ f_Z) \mid f'_Z \mid 1_{SZ} \geq (h \circ f_{Z_0}) \mid f'_{Z_0} \mid 1_{SZ_0}$. ■

Lemma 6 (Support of the invariant density of an AFU-map) *Let S be an AFU-map and let h be a lower semicontinuous invariant density for S. Then $\{h > 0\}$ is a finite union of pairwise disjoint open intervals.*

Proof. (This is a modification of the proof of Theorem 5 in [Go] where the case of finite ξ is considered.) We consider the class $\mathcal{G} := \{I \in \mathcal{S} : I$ contains some SJ', $J' \in \mathcal{D}'\}$ and first show that $\mathcal{G}$ is finite:

Let $J' \in \mathcal{D}'$. SJ' necessarily contains the image of a one-sided neighborhood of some $d \in \partial\xi$. Consequently, the interval $I = \langle SJ' \rangle \in \mathcal{S}$ (cf. Lemma 5) contains an $I-$neighborhood of d', the corresponding one-sided limit of Sx as $x \to d$. By our assumption, however, there are only finitely many points d', and so there are only finitely many $\langle SJ' \rangle$, hence $\mathcal{G}$ is finite.

Now let $s := \min\{\lambda(I) : I \in \mathcal{G}\} > 0$, $\mathcal{F} := \{I \in \mathcal{S} : \lambda(I) \geq s\} \supseteq \mathcal{G}$, and $F := \bigcup \mathcal{F} \supseteq \bigcup \mathcal{G}$. We claim that $SF \subseteq F$ mod λ: Let $I \in \mathcal{F}$. If $I \in \mathcal{D}$, then (up to countably many points) I is a union of sets $J' \in \mathcal{D}'$, whence $SI \subseteq \bigcup \mathcal{G} \subseteq F$ mod λ. Otherwise S is continuous, monotonic and hence expanding on I, so that $\lambda(SI) > \lambda(I) \geq s$, whence $\langle SI \rangle \in \mathcal{F}$ in this case, too.

We finally show that in fact $\mathcal{F} = \mathcal{S}$, which completes the proof since $\mathcal{F}$ clearly is finite. Assume that $\mathcal{S} \backslash \mathcal{F}$ is nonempty and choose one element I_0 of maximal length $l < s$. The same argument as in the preceding paragraph shows that $SI_0 \subseteq F$, hence $I_0 \subseteq S^{-1}F \backslash F$ (both mod λ). But since μ is invariant under S and $F \subseteq S^{-1}F$, we find that

$$\mu\left(S^{-1}F \backslash F\right) = \mu\left(S^{-1}F\right) - \mu\left(F\right) = 0 \ ,$$

so that $\mu(I_0) = 0$, which contradicts the fact that $I_0 \in \mathcal{S}$. ■

Our knowledge of the structure of the support enables us to go further.

Lemma 7 (Lower bound for invariant densities of AFU-maps) *Let S be an AFU-map and let h be a lower semicontinuous invariant density for S. Then there is some constant C such that $0 < C^{-1} \leq h \leq C$ on $\{h > 0\}$.*

Proof. (We adapt the method used in [Ke] for the case of finite ξ.)

1. Let y be an endpoint of an arbitrary interval $I \in \mathcal{S}$. Then (by Kuzmin's equation again) there is an $I-$neighbourhood W of y which is covered by some member SJ' of the finite family of (open) intervals $\{SJ' : J' \in \mathcal{S}'\}$, and, by Lemma 5, $SJ' \subseteq \langle SJ' \rangle = I$. In this case we write

   $$(J', x) \rightsquigarrow (I, y)$$

   where x is the endpoint of J' mapped onto y by the continuous extension of S to $\overline{J'}$. Thus,

   for any pair (I, y) with $I \in \mathcal{S}$, $y \in \partial I$, there is a pair (J', x) with $J' \in \mathcal{S}'$, $x \in \partial J'$, such that $(J', x) \rightsquigarrow (I, y)$.

   Moreover,

   $$\text{if in this situation} \lim_{t \to y, t \in I} h(t) = 0, \text{ then also} \lim_{r \to x, r \in J'} h(r) = 0,$$

   since by Kuzmin's equation $h(r) \leq h(t) \cdot \mid S'(r) \mid$ if $t = S\,r$, and Adler's condition ensures that S' is bounded on each cylinder. Observe also that (as h is lower semicontinuous) $\lim_{r \to x, r \in J} h(r) = 0$ furthermore implies that x is an endpoint of some interval from $\mathcal{S}$ (and not just of $\mathcal{S}'$). Therefore, if we let $\mathcal{K} := \{(I, y) : I \in \mathcal{S},\ y \in \partial I,\ \lim_{t \to y, t \in I} h(t) = 0\}$, we have:

   $$(I, y) \in \mathcal{K} \text{ and } (J', x) \rightsquigarrow (I, y) \qquad \Longrightarrow \qquad (J, x) \in \mathcal{K},$$

   where J is the member of $\mathcal{S}$ containing J'. Extending the relation "$\rightsquigarrow$" to $\mathcal{K} \times \mathcal{K}$ in an obvious way, we then write $(J, x) \rightsquigarrow (I, y)$.

2. To prove that $\mathcal{K}$ is empty, let us assume the contrary. By the foregoing discussion, each element of $\mathcal{K}$ has at least one predecessor for the relation "$\rightsquigarrow$" in $\mathcal{K}$. On the other hand, it is clear that it can have at most one successor. $\mathcal{K}$ being finite by Lemma 6, we conclude that the relation is bijective on $\mathcal{K}$. Hence this set consists of disjoint "cycles"

   $$(I_0, y_0) \rightsquigarrow (I_1, y_1) \rightsquigarrow \cdots \rightsquigarrow (I_n, y_n) = (I_0, y_0) \ ,$$

   where $(I_j, y_j) \in \mathcal{K}$ and $n \geq 1$. Let us consider a fixed cycle of this type. We can choose an I_0-neighbourhood U of y_0 so small that $S^j U$ is a I_j-neighbourhood of y_j for each $j \in \{1, \ldots, n\}$. Clearly then $\lambda(U) \leq \alpha^{-n}\lambda(V)$, where $V := S^n U \subseteq I_0$. However, we are going to show that for sufficiently small U we have $\mu(V) = \mu(U)$, which clearly contradicts $\lambda(V \setminus U) > 0$. Hence $\mathcal{K}$ must in fact be empty.

Let $(I, y) \in \mathcal{K}$. Step 1 shows that whenever W is a sufficiently small $I-$nbd of y, then $S^{-1}W \cap (\bigcup \mathcal{S}')$ is contained in a unique $J' \in \mathcal{S}'$ (namely that for which (J, x) is the unique predecessor of (I, y) in $\mathcal{K}$). Applying this argument to each edge of our cycle we find that for U sufficiently small we have $S^{-n}V \cap (\bigcup \mathcal{S}') = U$, and thus $\mu(V) = \mu(S^{-n}V \cap (\bigcup \mathcal{S}')) = \mu(U)$ as claimed.

Having thus proved that $\inf_I h > 0$ for each I in the finite set $\mathcal{S}$, the assertion follows as h is bounded above anyway. ■

We emphasize that the preceding result is no longer correct if we only require that $\inf\{\lambda(SZ) : Z \in \xi\} > 0$. Therefore it is not possible to replace our range condition (F) by the corresponding weaker assumption for T either without loosing control of the magnitude of invariant densities:

Example 4 (A uniformly expanding piecewise affine Markov map with long branches whose invariant density is not bounded away from zero) *We let $p_0 := 0$, and $p_n := 1 - \frac{2}{3^n}$ for $n \geq 1$, and define S to map the cylinder $Z_n := (p_n, p_{n+1})$ affinely onto $(0, p_{n+2})$, $n \geq 0$. Then $\inf |S'| > 2$ and S trivially satisfies Adler's condition as well as $\inf\{\lambda(SZ) : Z \in \xi\} > 0$. By the results of [Ry] (cf. Section 6), S admits an invariant probability density $h \in BV([0,1])$ which we may assume to be lower semicontinuous. Hence, $\{h > 0\}$ contains some open interval, and according to Lemma 2 also some $\xi-$full cylinder $Z \in \xi_m$. Therefore h must be positive on $Z_1 \subseteq S^m Z$. As $S^k Z_1 \supseteq \bigcup_{j=1}^{k+1} Z_j$ for $k \geq 1$, we conclude that $h > 0$ on $(0,1)$. By Kuzmin's equation, however, for $x \in Z_{n+1}$,*

$$h(x) = \sum_{k \geq n} h(f_{Z_k}(x)) \left| f'_{Z_k}(x) \right| \leq const \cdot \|h\|_\infty \sum_{k \geq n} \frac{1}{3^k} \ ,$$

whence $\lim_{x \to 1} h(x) = 0$.

4 More Preparations

We omit the elementary proof of the following simple observation.

Remark 5 *Let T be a measurable transformation on the $\sigma-$finite space $(X, \mathcal{A}, \nu)$ such that both T and T^{-1} preserve sets of measure zero. If h is an invariant density for T and $M := \{h > 0\}$, then $TM = M$ mod ν. If moreover T is ergodic w.r.t. ν, then $\nu(X \setminus \bigcup_{n \geq 0} T^{-n}M) = 0$, i.e. $\nu-$almost every point of X is eventually mapped into M.*

An analytic lemma. To deal with the indifferent fixed points we will rely on the following estimate (cf. [T1]) which applies to the local inverses f_Z of T near x_Z.

Lemma 8 (Thaler's inequality) *Let $a, e_1, e_2 \in R$, $e_i \geq 0$, $e_1 + e_2 > 0$, $E := (a - e_1, a + e_2)$ and let $f : E \to E$ be an increasing and differentiable function such that $|f(x) - a| < |x - a|$ for $x \in E \backslash \{a\}$. Assume also that f' increases on $(a - e_1, a)$ and decreases on $(a, a + e_2)$. Then*

$$\frac{f^{-1}(x) - a}{f^{-1}(x) - x} \leq \sum_{n \geq 0} (f^n)'(x) \leq \frac{x - a}{x - f(x)} \qquad \text{for } x \in f(E) \backslash \{a\},$$

where f^n denotes the nth iterate of f. Moreover, these expressions are asymptotically equivalent for $x \to a$, $x \in E$.

Remark 6 *As the x_Z, $Z \in \zeta$, are one-sided regular sources, for each $Z \in \zeta$, $f_Z : TZ \to Z$ satisfies the assumptions of the preceding lemma. Hence, with G as in Theorem 1, $\sum_{n \geq 0} (f_Z^n)' \leq G$ on $\bigcup \zeta$.*

Jump transformations. The auxiliary map $\widetilde{T}$. Our auxiliary transformation $\widetilde{T}$ will be constructed from T in the manner considered in the following Lemma (see [Sc] and [T2], pp. 68 for proofs).

Lemma 9 (Relations between $\widetilde{T}$ and T) *Let $(X, \mathcal{A})$ be a measurable space, let $T : X \to X$ and $\varphi : X \to \{1, 2, \ldots\}$ be measurable and consider the map $\widetilde{T}$ defined by $\widetilde{T}x := T^{\varphi(x)}x$ for $x \in [0, 1]$. Then*

1. *Any $T-$invariant set is also $\widetilde{T}-$invariant.*
2. *Any $T-$wandering set is also $\widetilde{T}-$wandering.*
3. *If $\widetilde{\mu}$ is an invariant measure for $\widetilde{T}$ on $\mathcal{A}$, then an invariant measure for T is obtained by*

$$\mu(A) := \widetilde{\mu}(A) + \sum_{n \geq 1} \widetilde{\mu}\left(T^{-n}A \cap \{\varphi \geq n + 1\}\right), \quad A \in \mathcal{A}.$$

From now on we let T denote a fixed AFN-map.We first construct a refined partition $\widetilde{\xi}$ of $[0, 1]$ as follows: Fix $Z \in \zeta$ and let $Z(1)$ denote the open subinterval $Z \backslash T^{-1}(cl(Z))$. For $n \geq 1$ we define $Z(n) := (T\mid_Z)^{-1}(Z(n-1)) = (T\mid_Z)^{-n+1}(Z(1))$, thus decomposing Z into a sequence of pairwise disjoint open intervals. Observe that for $n, j \geq 1$, $T^n Z(n + j) = Z(j)$. Now let $\widetilde{\xi} := (\xi \backslash \zeta) \cup \{Z(n) : Z \in \zeta \text{ and } n \geq 1\}$. We define

$$\varphi(x) := \begin{cases} n & \text{if } x \in Z(n),\, Z \in \zeta,\, n \geq 1 \\ 1 & \text{otherwise} \end{cases}$$

and finally let $\widetilde{T}(x) := T^{\varphi(x)}(x)$ for $x \in [0, 1]$. Henceforth objects associated to the system $(\widetilde{T}, \widetilde{\xi})$ will notationally be identified by a tilde, e.g. $\widetilde{\xi}_n = \bigvee_{i=0}^{n-1} \widetilde{T}^{-i}\widetilde{\xi}$, and $\widetilde{f}_Z = (\widetilde{T}^n\mid_Z)^{-1}$ for $Z \in \widetilde{\xi}_n$. $\widetilde{T}$ is *Schweiger's jump transformation* with respect to the set $\bigcup_{W \in \xi \backslash \zeta} W \cup \bigcup_{Z \in \xi} Z(1)$ (cf. Ch. 19 of [Sc]).

Lemma 10 *$\widetilde{T}$ is an AFU-map.*

Proof. Clearly $\widetilde{T}$ is twice differentiable on each $\widetilde{Z} \in \widetilde{\xi}$. It is uniformly expanding since for every $Z \in \zeta$ and $n \geq 1$, $\inf_{Z(n)} \mid \widetilde{T}' \mid \geq \inf_{Z(1)} |T'| > 1$. Also, observe that for each $Z \in \zeta$, all new cylinders $Z(n)$ have the same image $TZ(1)$ under $\widetilde{T}$, so that $\widetilde{T}(\widetilde{\xi})$ is a finite family.

Finally we show that Adler's condition is satisfied: we need only check boundedness of $\widetilde{T}''/(\widetilde{T}')^2$ on each $\bigcup_{n\geq 1} Z(n)$. As $\mid \widetilde{T}''/(\widetilde{T}')^2 \mid \circ \widetilde{f}_{Z(n)} = \mid \widetilde{f}''_{Z(n)}/\widetilde{f}'_{Z(n)} \mid$ on $\widetilde{T}Z(n) = TZ(1)$, we have

$$\sup_{Z(n)} \left| \frac{\widetilde{T}''}{(\widetilde{T}')^2} \right| = \sup_{\widetilde{T}Z(n)} \left| \frac{\widetilde{T}''}{(\widetilde{T}')^2} \right| \circ \widetilde{f}_{Z(n)} = \sup_{TZ(1)} \left| \frac{\widetilde{f}''_{Z(n)}}{\widetilde{f}'_{Z(n)}} \right| .$$

Let $A(T)$ be an upper bound for $T''/(T')^2$. Now $\widetilde{f}_{Z(n)} = f_Z^n$ on $TZ(1)$, and logarithmic differentiation yields

$$\left| \frac{\widetilde{f}''_{Z(n)}}{\widetilde{f}'_{Z(n)}} \right| = \sum_{j=0}^{n-1} \left| \frac{f''_Z \circ f_Z^j}{f'_Z \circ f_Z^j} \right| \cdot (f_Z^j)' \leq A(T) \cdot \sum_{j=0}^{n-1} (f_Z^j)' \quad \text{on } TZ(1).$$

However, $TZ(1)$ has positive distance from x_Z, whence $\sum_{n\geq 0} (f_Z^n)'$ is bounded on $TZ(1)$ by Remark 6, thus ensuring the existence of an upper bound for $\mid (\widetilde{f}''_{Z(n)})/(\widetilde{f}'_{Z(n)}) \mid$ independent of n. ■

5 Proof of Theorem 1

1. We apply Lemma 4 to $S := \widetilde{T}$ and denote the ergodic components thus obtained by $\widetilde{X}_j$, $j \in \{1, \ldots, m\}$. Let $\widetilde{Y}_j := \bigcup_{k\geq 0} S^{-k}\widetilde{X}_j$, then these sets are invariant and cover $[0,1]$ up to a set of Lebesgue measure zero. (Also, $\widetilde{T} \mid_{\widetilde{Y}_j}$ is ergodic w.r.t. λ.) By part 1 of Lemma 9, the interval $[0,1]$ decomposes into a finite number of $T-$invariant sets $Y_1, \ldots, Y_l$, each of which is the union of some (in fact only one) $\widetilde{Y}_j$'s, and $T \mid_{Y_i}$ is ergodic for $i \in \{1, \ldots, l\}$. Let us fix i and choose an index $j \in \{1, \ldots, m\}$ for which $\widetilde{Y}_j \subseteq Y_i$. From now on we restrict our attention to these fixed components and omit the subscripts i and j as this should not lead to any confusions.

   According to part 3 of Lemma 9 we obtain a $T-$invariant measure μ on Y from $\widetilde{\mu}$, the absolutely continuous $\widetilde{T}-$invariant probability measure on $\widetilde{Y}$. Below we will identify the density h of μ, thereby proving that $\mu \ll \lambda$. Hence μ is ergodic. We let $X = X_i := \{h > 0\}$. By Remark 5, applied to $T \mid_Y$ we find that $TX = X \mod \lambda$, and therefore $\lambda-$almost all points of $D := Y \setminus X$ are eventually mapped into X by the second part of that remark.

2. The $T-$invariant measure μ has a density h given by

$$h := \widetilde{h} + \sum_{Z\in\zeta} 1_Z \cdot \sum_{n\geq 1} \left(\widetilde{h} \circ f_Z^n \right) \cdot (f_Z^n)' ,$$

   since $T^n Z(k) \subseteq Z$ for $k > n \geq 1$ implies that

$$\{\varphi \geq n+1\} \cap T^{-n}A = \bigcup_{Z\in\zeta} \bigcup_{k>n} (T^{-n}A \cap Z(k)) = \bigcup_{Z\in\zeta} f_Z^n(A \cap Z) ,$$

and thus

$$\begin{aligned}\mu(A) &= \widetilde{\mu}(A) + \sum_{n\geq 1}\sum_{Z\in\zeta}\widetilde{\mu}\left(f_Z^n(A\cap Z)\right) = \int_A \widetilde{h}\,d\lambda + \sum_{n\geq 1}\sum_{Z\in\zeta}\int_{f_Z^n(A\cap Z)}\widetilde{h}\,d\lambda \\ &= \int_A \left(\widetilde{h} + \sum_{Z\in\zeta} 1_Z \cdot \sum_{n\geq 1}\left(\widetilde{h}\circ f_Z^n\right)\cdot (f_Z^n)'\right) d\lambda \ .\end{aligned}$$

The density h is lower semicontinuous since it is the limit of an increasing sequence of functions sharing this property. Consequently, $X = \{h > 0\}$ is open. By the formula above, as $f_Z^n(Z) \subseteq \{\varphi \geq n+1\}$,

$$X = \widetilde{X} \cup \bigcup_{n\geq 1} T^n\left(\widetilde{X}\cap\{\varphi \geq n+1\}\right) \ .$$

Now $\widetilde{X}$ is the union of a finite number of open intervals. Thus, for each $Z \in \zeta$, $\widetilde{X}$ either contains some $Z-$neighbourhood of x_Z or there is some $Z-$neighbourhood of x_Z disjoint from $\widetilde{X}$. Hence, there exists some $N \geq 1$ such that for $n \geq N$

$$T^n\left(\widetilde{X}\cap\{\varphi \geq n+1\}\right)\cap Z = T^N\left(\widetilde{X}\cap\{\varphi \geq N+1\}\right)\cap Z =: U_Z \ ,$$

and U_Z is a $Z-$neighbourhood of x_Z if the first alternative holds, and empty otherwise. The above representation for X_i thus turns out to be a finite union, and since $\widetilde{X}$ is a finite union of open intervals, so, as claimed, is X.

The foregoing observation also gives the lower estimate for h:

$$C\cdot h \geq \left[1_{\widetilde{X}} + \sum_{Z\in\zeta}\sum_{n=1}^{N} 1_{Z\cap T^n\left(\widetilde{X}\cap\{\varphi\geq n+1\}\right)}(f_Z^n)'\right] + \sum_{Z\in\zeta} 1_{U_Z}\cdot\sum_{n>N}(f_Z^n)' \ .$$

The first expression has a positive lower bound on X, since each $(f_Z^n)'$ is bounded away from zero on $TZ \supseteq Z$ by Adler's condition, whereas the second shows that

$$h \geq const \cdot \left(\sum_{Z\in\zeta} 1_{U_Z}\right)\cdot G \ .$$

The proposed estimate from above is immediate from Remark 6.

Let us show that μ is infinite if X contains a $Z-$neighbourhood of some x_Z (cf. [T2]). For $x \in Z$ sufficiently close to x_Z, $f_Z(x) = x + \frac{1}{2}f_Z''(y_x)\cdot(x-x_Z)^2$ for some y_x between x and x_Z. As Adler's condition implies that f_Z'' is bounded, this yields $|f_Z(x) - x| \leq c\cdot|x - x_Z|^2$, and hence

$$G(x) = \frac{x - x_Z}{x - f_Z(x)} \geq c^{-1}\cdot|x - x_Z|^{-1}$$

in some $Z-$neighbourhood $U \subseteq X$ of x_Z, so that indeed $\int_X G\,d\lambda = \infty$.

3. To prove that $T\mid_X$ is conservative, let $W \subseteq X$ be any wandering set for T. Clearly, for each $k \geq 0$, $T^{-k}W$ is wandering, too, and by Lemma 9 these sets are also wandering for $\widetilde{T}$. By construction of μ, however $\mu(W) > 0$ implies that $\widetilde{\mu}(W) > 0$ for some k, which is impossible since $\widetilde{T}$ is conservative.

4. To show that the tail field is discrete, assume that $A \in \mathcal{B}_\infty$, $A \subseteq \{h > 0\} = X$ with $\lambda(A) > 0$. We apply Lemma 3 to $S = \widetilde{T}$, thus obtaining $k \geq 1$ and $\widetilde{Z} \in \widetilde{\xi}_k$ for which

$$\lambda\left(\widetilde{T}^k\left(\widetilde{Z} \cap A\right)\right) \geq \eta\ ,$$

where $\eta > 0$ does not depend on A. Now $\widetilde{T}^k\mid_{\widetilde{Z}}$ equals the restriction of some fixed power T^m of T to $\widetilde{Z}$. As $h \geq \mathit{const} > 0$ on X and $T^m A \subseteq X$ mod λ by Remark 5, we conclude that $\mu\left(T^m A\right) \geq \mathit{const} \cdot \eta =: \eta' > 0$, so that finally

$$\mu\left(A\right) = \mu\left(T^{-m}T^m A\right) = \mu\left(T^m A\right) \geq \eta'\ ,$$

which shows that $\mathcal{B}_\infty$ must be discrete. This in turn implies the proposed cyclic structure, as both T and T^{-1} modulo λ preserve atoms of $\mathcal{B}_\infty$ and T is conservative. Finally, as each member $X(j) \in \mathcal{B}_\infty$ of the ergodic cycle X is an ergodic component of a suitable power T^l, this too is a finite union of open intervals. ■

6 Appendix: A characterization of Rychlik's class of piecewise monotonic maps

Having repeatedly cited [Ry], we devote this section to a discussion of the scope of this reference which, regrettably, sometimes is not adequately appreciated. In particular we show that the maps we claimed to belong to the class studied by Rychlik in fact do.

Throughout we work in the framework proposed there, thus considering a map T on some totally ordered, order-complete space X for which there is some finite or countable collection ξ of open intervals such that $U := \bigcup \xi$ is dense in X and $T\mid_Z: Z \to TZ$ is a monotonic homeomorphism with inverse f_Z for each $Z \in \xi$. We assume that m is some probability measure on the Borel $\sigma-$field $\mathcal{B}$ of X with $m(U) = 1$, and that T is nonsingular with respect to m. (X, T, ξ, m) will then be called a *nonsingular piecewise monotonic system.* Let $\omega_Z \in L_1(m)$, $Z \in \xi$ be determined by $m(Z \cap T^{-1}B) = \int_B \omega_Z\, dm$ for $B \in \mathcal{B}$, let $g := \sum_{Z \in \xi}(\omega_Z \circ T) \cdot 1_Z$ be the *weight function,* and call the system *uniformly expanding* if $\|g\|_\infty < 1$.

We must carefully distinguish between genuine functions on X (which will be notationally identified by an asterisk) and their a.e.-equivalence classes. If A is a subinterval of X, a *subdivision* $\mathcal{P} = \{p_0, \ldots, p_r\}$ of A will be a finite ordered subset. For a function $f^* : X \to R$ we let $V(f^*, \mathcal{P}) := \sum_{i=1}^r \mid f^*(p_i) - f^*(p_{i-1}) \mid$, and define the *variation of* f^* *on* A as $V_A(f^*) := \sup\{V(f^*, \mathcal{P}) : \mathcal{P}$ is a subdivision of $A\}$. If $V_A(f^*) < \infty$, then all one-sided limits of f^* which make sense exist. Also, we will frequently use that in this case $\sup_A f^* - \inf_A f^* \leq$

$V_A(f^*)$. For an equivalence class $f \in L_1(m)$ we let $v_A(f) := \inf\{V_A(f^*) : f^*$ is a version of $f\}$, and observe that this infimum is in fact attained.

If ξ is finite, bounded variation techniques apply to yield strong results on the *Perron-Frobenius operator* $\mathbf{P}$ as soon as $v_X(g) < \infty$ (equivalently $\sum_{Z\in\xi} v_Z(g) < \infty$), cf. [HK]. In [Ry] such results are obtained also for the case of countable ξ under the assumption that g has a version g^* with $V_X(g^*) < \infty$ which vanishes on U^c. In this case P_{g^*} is a version of $\mathbf{P}$, where we let $P_{g^*}h^*(x) := \sum_{y\in T^{-1}\{x\}} g^*(y)\cdot h^*(y)$, $x \in X$. The property $g^* \mid_{U^c} = 0$ makes it possible to fit the countably many pieces of P_{g^*} together without loosing control of the variation. The following rather simple result, which apparently is not too widely known, gives a criterion for Rychlik's condition to hold which is visualizable and frequently easy to check.

Proposition 2 (Characterization of Rychlik's maps) *A nonsingular uniformly expanding piecewise monotonic system* (X, T, ξ, m) *with weight* g *satisfies Rychlik's assumptions iff*

$$\sum_{Z\in\xi} v_Z(g) < \infty \qquad \textit{and} \qquad \sum_{Z\in\xi} \frac{m(Z)}{m(TZ)} < \infty. \qquad (*)$$

The first condition generalizes the standard one for finite ξ. If $(X, m) = ([0, 1], \lambda)$ and each $T \mid_Z$ is twice differentiable, it is clearly fulfilled whenever T satisfies Adler's condition (A), since then $g = \mid T' \mid^{-1}$, and thus $v_Z(g) = \int_Z \mid g' \mid d\lambda = \int_Z \mid T''/(T')^2 \mid d\lambda$. The second is a rather weak range condition and is trivially fulfilled if, for example, $\inf\{m(TZ) : Z \in \xi\} > 0$. In particular we have:

Corollary 1 *Any AFU-map satisfies Rychlik's assumptions.*

Proof of Proposition 2. The key to this is the straightforward observation that due to $m(Z) = \int_{TZ} (g \circ f_Z)\, dm$ for any $Z \in \xi$, for any version g^* of g we have

$$\inf_Z g^* \le \frac{m(Z)}{m(TZ)} \le \sup_Z g^*.$$

1. Assume that $(*)$ holds. For each $Z \in \xi$ let g^*_Z be some version of $g \mid_Z$ minimizing variation, and let $g^* := \sum_{Z\in\xi} g^*_Z \cdot 1_Z$, which gives a version of g vanishing on U^c and satisfying $V_Z(g^*) = v_Z(g)$ for all Z. We are going to prove that $V_X(g^*) < \infty$. Let $\mathcal{P} = \{p_0, \ldots p_r\}$ be any subdivision of X, and consider $V(g^*, \mathcal{P}) = \sum_{i=1}^r \Delta_{\mathcal{P}}(i)$, where $\Delta_{\mathcal{P}}(i) := \mid g^*(p_i) - g^*(p_{i-1}) \mid$. For each index i one of the following possibilities holds:

   (a) p_{i-1} and p_i lie in the same cylinder $Z \in \xi$

   (b) p_{i-1} and p_i lie in different cylinders $Z_1, Z_2 \in \xi$

   (c) one of p_{i-1} and p_i lies in U^c while the other lies in some $Z \in \xi$

   (d) p_{i-1} and p_i lie in U^c.

   Grouping the indices according to these alternatives we have $V(g^*, \mathcal{P}) = \sum_a + \sum_b + \sum_c + \sum_d$. We first notice that $\sum_a \le \sum_{Z\in\xi} V_Z(g^*)$, which is easily seen by considering the sum of those type (a) differences $\Delta_{\mathcal{P}}(i)$

belonging to the same Z as a variational sum for $g^* \mid_Z$. In case (b), we have $\Delta_{\mathcal{P}}(i) \leq g^*(p_{i-1}) + g^*(p_i) \leq \sum_{j=1,2}(m(Z_j)/m(TZ_j) + V_{Z_j}(g^*))$, and since each Z occurs as a Z_j for at most two indices i of type (b), we find that $\sum_b \leq 2\sum_{Z\in\xi} m(Z)/m(TZ) + 2\sum_{Z\in\xi} V_Z(g^*)$. Similarly, in case (c) we have $\Delta_{\mathcal{P}}(i) \leq \sup_Z g^* \leq m(Z)/m(TZ) + V_Z(g^*)$, whence, as before, $\sum_c \leq 2\sum_{Z\in\xi} m(Z)/m(TZ) + 2\sum_{Z\in\xi} V_Z(g^*)$. Finally, it is clear that $\sum_d = 0$, and we thus end up with a finite upper bound for $V(g^*, \mathcal{P})$ which does not depend on $\mathcal{P}$:

$$V(g^*, \mathcal{P}) \leq 5 \sum_{Z\in\xi} v_Z(g) + 4 \sum_{Z\in\xi} \frac{m(Z)}{m(TZ)}.$$

2. Suppose that g has a version $g^* : X \to [0,1)$ with $V_X(g^*) < \infty$ and $g^* \mid_{U^c} = 0$. Since $v_Z(g) \leq V_Z(g^*)$ for any Z, and $V_X(g^*) = \sum_{Z\in\xi} V_Z(g^*)$, the first part of $(*)$ follows immediately. We are now going to show that the assumption $\sum_{Z\in\xi} m(Z)/m(TZ) = \infty$ leads to a contradiction.

   Indeed, in this case we can choose some finite family $\xi^+ \subseteq \xi$ for which $\sum_{Z\in\xi^+} m(Z)/m(TZ) > V_X(g^*)$. A fortiori $\sum_{Z\in\xi^+} \sup_Z g^*$ also exceeds $V_X(g^*)$. Recalling that variation is additive for decompositions of X into finitely many closed intervals with disjoint interiors, and using the properties of variation mentioned before, we obtain a contradiction: (We write $Z = (a_Z, b_Z)$, and let $g^*(a^+)$, $g^*(b^-)$ denote the obvious one-sided limits.)

$$\begin{aligned}
V_X(g^*) \quad &\geq \quad \sum_{Z\in\xi^+} V_{cl(Z)}(g^*) = \sum_{Z\in\xi^+} \left(V_Z(g^*) + g^*(a_Z^+) + g^*(b_Z^-)\right) \geq \\
&\geq \quad \sum_{Z\in\xi^+} \left(V_Z(g^*) + 2\left(\sup_Z g^* - V_Z(g^*)\right)\right) = \\
&= \quad 2\sum_{Z\in\xi^+} \sup_Z g^* - \sum_{Z\in\xi^+} V_Z(g^*) > V_X(g^*). \qquad \blacksquare
\end{aligned}$$

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