Return-time statistics, Hitting-time statistics and Inducing

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Abstract. In the framework of abstract ergodic probability-preserving transformations, we prove that the limiting return-time statistics and hitting-time statistics persist if we pass from the original system to a first-return map and vice versa.

1. Introduction

The asymptotic behaviour of return-time and hitting-time distributions of small sets H in an ergodic probability-preserving dynamical system $(X, \mathcal{A}, \mu, T)$ is a well-studied circle of questions. For the open dynamical system obtained by regarding H as a *hole* in X, this is of obvious interest since the hitting time of H represents the survival time of orbits in the open system.

In the present note, we shall always assume that $(X, \mathcal{A}, \mu)$ is a probability space, and that $T : X \to X$ is an ergodic measure-preserving map thereon. Also, H, H_l and Y will always denote measurable sets of positive measure. By ergodicity and the Poincaré recurrence theorem, the measurable *(first) hitting time* function of H, $\varphi_H : X \to \overline{\mathbb{N}} := \{1, 2, \dots, \infty\}$ with $\varphi_H(x) := \inf\{n \geq 1 : T^n x \in H\}$, is finite a.e. on X. When restricted to H it is called the *(first) return time* function of our set, and it satisfies Kac' formula $\int_H \varphi_H \, d\mu_H = 1/\mu(H)$, where $\mu_H(A) := \mu(H \cap A)/\mu(H)$, $A \in \mathcal{A}$. That is, when regarded as a random variable on the probability space $(H, \mathcal{A} \cap H, \mu_H)$, the return time has a distribution with expectation $1/\mu(H)$, and we will often normalize our variable accordingly, thus passing to $\mu(H)\, \varphi_H$.

In the following, our goal is to obtain further information about this *return-time distribution* of H, in particular when H is really small, that is, we are going to study sequences $(H_l)_{l \geq 1}$ of *asymptotically rare events*, meaning that[1] $\mu(H_l) \to 0$. The sequence is said to have *(asymptotic) return time statistics* given by a random variable $\widetilde{\mathsf{R}}$ which takes values in $[0, \infty]$, if its normalized return time distributions converge, in the usual sense, to that of $\widetilde{\mathsf{R}}$, so that $\widetilde{F}_l(t) := \mu_{H_l}(\mu(H_l)\, \varphi_{H_l} \leq t) \to \widetilde{F}(t) := \Pr[\widetilde{\mathsf{R}} \leq t]$ as $l \to \infty$ whenever $t > 0$ is a continuity point of $\widetilde{F}$. In standard probabilistic notation, this is expressed, on the level of distribution functions, as $\widetilde{F}_l \Longrightarrow \widetilde{F}$.

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[1]This is obviously a property of the sequence, and not of the individual events H_l. Still, we take the liberty of following this imprecise but common terminology.

Similarly, one may study the *hitting-time distribution* of H, that is, the law of φ_H on the original probability space $(X, \mathcal{A}, \mu)$, and ask, for sequences $(H_l)_{l\geq 1}$ as above, for *(asymptotic) hitting time statistics* given by some $[0,\infty]$-valued random variable R. This means that $F_l(t) := \mu(\mu(H_l)\,\varphi_{H_l} \leq t) \to F(t) := \Pr[\mathsf{R} \leq t]$ for all continuity points $t > 0$ of F, that is $F_l \Longrightarrow F$. This situation is somewhat simpler, as the underlying measure μ remains the same.

A large amount of material on return- and hitting time statistics for specific types of dynamical systems is available. We refer the reader to [**8**], [**9**], [**12**] and the references cited there. (For systems with some hyperbolicity, and reasonably well behaved sequences $(H_l)_{l\geq 1}$ one typically gets convergence to an exponential law.)

Moreover, certain fundamental questions can be posed and answered in an abstract ergodic-theoretical setup. Most important for us, this is the case for the relation between the two types of limits introduced above, which has been clarified in [**6**]. We will recall it below, see Theorem 2.1.

The focus of the present paper is on the behaviour of the two limiting relations under the standard operation of *inducing* on a suitable reference set $Y \subseteq X$. The technique of inducing is a basic tool, both for the analysis of specific systems, and for abstract ergodic theory. In the former sense, it has been used to identify return-time statistics of certain non-uniformly hyperbolic systems in [**4**], where it is shown that return-time statistics persist under inducing, at least in situations in which X is a Riemannian manifold, (H_l) is a sequence of ε-neighbourhoods of a typical point, and where the limit law has no mass at zero. This has been exploited in further papers, see e.g. [**5**].

Here, we extend this principle to the general measure-theoretical setup and dispose of these extra conditions, see Theorem 4.1 below. We do so by first proving a corresponding abstract statement for the easier case of hitting-times (Theorem 3.1), and then transfer it to return-times via the aforementioned universal correspondence between the two. The results below were obtained independently in [**7**] and [**13**].

2. Preliminaries

Distributional convergence. A *sub-probability distribution function* on $[0,\infty)$ is a non-decreasing right-continuous function $F : [0,\infty) \to [0,1]$ (with a canonical extension to $\mathbb{R}$ which vanishes on $(-\infty,0)$). These F are in a one-to-one correspondence with the Borel probability measures Q on $[0,\infty]$, where Q corresponds to the function given by $F(t) := Q([0,t])$, $t \in [0,\infty)$, so that $Q(\{\infty\}) = 1 - \lim_{t\to\infty} F(t)$. These Q, in turn, are the distributions of random variables R taking values in $[0,\infty]$, $Q(B) = \Pr[\mathsf{R} \in B]$, $B \in \mathcal{B}_{[0,\infty]}$. R is (almost surely) real-valued iff $F(t) \to 1$ as $t \to \infty$, i.e. iff F is a proper *probability distribution function* on $[0,\infty)$.

If F, F_l are sub-probability distribution functions on $[0,\infty)$, then $F_l \Longrightarrow F$ means that $F_l(t) \to F(t)$ as $l \to \infty$ for all continuity points $t > 0$ of F. This is equivalent to the usual weak convergence $Q_l \Longrightarrow Q$ (cf. [**3**]) of the corresponding Borel probabilities on $[0,\infty]$. (This is obvious if, for example, we map $[0,\infty]$ onto $[0,1]$ by some orientation-preserving homeomorphism, and carry the measures and distribution functions along.)

In the present context we are interested in certain measurable functions $R_l : X \to [0,\infty]$, the distributions of which may be taken w.r.t. different probabilities

ν_l on $(X, \mathcal{A})$. Convergence in the above sense of these distributions to the law of some random variable R will be denoted by

$$R_l \overset{\nu_l}{\Longrightarrow} \mathsf{R} \qquad \text{as } l \to \infty, \tag{2.1}$$

that is, with $F(t) := \Pr[\mathsf{R} \leq t]$ denoting the distribution function of R, (2.1) means

$$\nu_l(R_l \leq t) \longrightarrow F(t) \quad \text{for all continuity points } t > 0 \text{ of } F.$$

As a special case, this includes the notation $R_l \overset{\nu}{\Longrightarrow} \mathsf{R}$ for distributional convergence of the R_l when regarded as random variables on the common probability space $(X, \mathcal{A}, \nu)$, that is, $\nu(R_l \leq t) \longrightarrow F(t)$ for all continuity points $t > 0$ of F.

If, in the setup of the introduction, $R_l := \mu(H_l)\, \varphi_{H_l}$, then return-time statistics refer to convergence $R_l \overset{\nu_l}{\Longrightarrow} \mathsf{R}$ with $\nu_l = \mu_{H_l}$, while hitting-time statistics concern the convergence $R_l \overset{\nu}{\Longrightarrow} \mathsf{R}$ with $\nu := \mu$.

Relation between return-time and hitting-time statistics. Given an arbitrary sequence $(H_l)_{l\geq 1}$ of asymptotically rare events, its return-time statistics and its hitting-time statistics are intimately related to each other. The following fundamental result was first established in [**6**]. For an alternative proof see [**2**].

THEOREM 2.1 (**Hitting-time statistics versus return-time statistics**). *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic probability-preserving system, and $(H_l)_{l\geq 1}$ a sequence of asymptotically rare events. Then*

$$\mu(H_l)\, \varphi_{H_l} \overset{\mu}{\Longrightarrow} \mathsf{R} \quad \textit{for some random variable } \mathsf{R} \textit{ in } [0, \infty] \tag{2.2}$$

iff

$$\mu(H_l)\, \varphi_{H_l} \overset{\mu_{H_l}}{\Longrightarrow} \widetilde{\mathsf{R}} \quad \textit{for some random variable } \widetilde{\mathsf{R}} \textit{ in } [0, \infty]. \tag{2.3}$$

In this case, the sub-probability distribution functions F and $\widetilde{F}$ of R and $\widetilde{\mathsf{R}}$ satisfy

$$\int_0^t (1 - \widetilde{F}(s))\, ds = F(t) \quad \textit{for } t \geq 0. \tag{2.4}$$

Through this integral equation each of F and $\widetilde{F}$ uniquely determines the other. Moreover, F is necessarily continuous and concave with $F(t) \leq t$, while $\widetilde{F}$ is always a probability distribution function satisfying $\int_0^\infty (1 - \widetilde{F}(s))\, ds \leq 1$.

It is also known that the properties recorded above (which follow easily from (2.4)) completely determine the class of all F and $\widetilde{F}$ which do occur as hitting- and return-time limits in an arbitrary aperiodic ergodic system, see [**11**] and [**10**].

Strong distributional convergence for hitting-times of rare events. It is an interesting but sometimes neglected fact that many distributional limit theorems for ergodic processes automatically hold for large collections of initial probability distributions ν on the underlying space. In Section 4 below, this principle will serve as an important technical tool. Let $(R_l)_{l\geq 1}$ be a sequence of non-negative measurable functions on $(X, \mathcal{A}, \mu)$, and R some $[0, \infty]$-valued random variable. *Strong distributional convergence* w.r.t. μ of $(R_l)_{l\geq 1}$ to R means that

$$R_l \overset{\nu}{\Longrightarrow} \mathsf{R} \qquad \text{for all probability measures } \nu \ll \mu, \tag{2.5}$$

compare [**1**]. This type of convergence is denoted by $R_l \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathsf{R}$. (The probabilistic literature sometimes uses the term *(Rényi-)mixing*.)

A situation in which one always has strong convergence for trivial reasons is that of an a.s. constant limit variable R, i.e. $\Pr[\mathsf{R} = c] = 1$ for some $c \in [0, \infty]$. There are, however, more interesting scenarios in which strong distributional convergence is automatic. A discussion of a natural and widely applicable sufficient condition for this to happen in an ergodic system can be found in [**15**]. Using it, it is not hard to see that asymptotic hitting-time distributions for rare events always behave in this way: If a limit law shows up under one particular initial distribution $\nu \ll \mu$, then it does so for all $\nu \ll \mu$. The following fact, which will be very useful for the proof of our main results, is contained in Corollary 5 of [**15**].

Proposition 2.1 (**Strong distributional convergence of hitting-times**). *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic probability-preserving system, and $\nu \ll \mu$ some probability measure. Let $(H_l)_{l \geq 1}$ be a sequence of asymptotically rare events. Then, for any random variable R in $[0, \infty]$,*

$$\mu(H_l)\, \varphi_{H_l} \overset{\nu}{\Longrightarrow} \mathsf{R} \quad \textit{implies} \quad \mu(H_l)\, \varphi_{H_l} \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathsf{R}. \tag{2.6}$$

As a consequence, we see that proving $\mu(H_l)\, \varphi_{H_l} \overset{\nu}{\Longrightarrow} \mathsf{R}$ for some particular ν is as good as proving it for any other probability measure.

We emphasize that the analogous statement for return-time statistics is false. First, for an arbitrarily chosen probability $\nu \ll \mu$, we may have $\nu(H_l) = 0$, so that the statement doesn't even make sense. But even if $\nu(H_l) > 0$, there is no hope for a universality statement like that of the Proposition. We illustrate this in the simple setup of disjoint sequences $(H_l)_{l \geq 1}$.

Example 2.2. *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic probability-preserving system, and $(H_l)_{l \geq 1}$ a sequence of pairwise disjoint asymptotically rare events. Assume that, for some non-constant random variable $\widetilde{\mathsf{R}}$, we have $\widetilde{R}_l := \mu(H_l)\, \varphi_{H_l} \overset{\mu_{H_l}}{\Longrightarrow} \widetilde{\mathsf{R}}$. Then there is some probability $\nu \ll \mu$ such that $\nu(H_l) > 0$ for $l \geq l^*$ but $\widetilde{R}_l \overset{\nu_{H_l}}{\not\Longrightarrow} \widetilde{\mathsf{R}}$.*

To see this, let $t^* \in (0, \infty)$ be a continuity point of $\widetilde{F}(t) := \Pr[\widetilde{\mathsf{R}} \leq t]$ such that $0 < \widetilde{F}(t^*) < 1$. Define $H_l^* := H_l \cap \{\widetilde{R}_l \leq t^*\}$, $l \geq 1$. By assumption, $\mu_{H_l}(H_l^*) \to \widetilde{F}(t^*)$, so that $\mu(H_l^*) > 0$ for $l \geq l^*$. Let $Z := \bigcup_{l \geq l^*} H_l^*$ and $\nu := \mu_Z$. Since the H_l are pairwise disjoint, we find that $\nu_{H_l}(\widetilde{R}_l \leq t^*) = \nu(H_l^*)/\nu(H_l) = \mu(H_l^*)/\mu(H_l^*) = 1$ for $l \geq l^*$. Our claim follows.

3. Hitting-time statistics via inducing

As before, we let $(X, \mathcal{A}, \mu, T)$ be an ergodic probability-preserving system. Now fix some $Y \in \mathcal{A}$, $\mu(Y) > 0$. It is a well-known classical result that the *first-return map* $T_Y : Y \to Y$ defined by

$$T_Y\, x := T^{\varphi_Y(x)} x, \quad x \in Y,$$

is a measure-preserving ergodic map on the probability space $(Y, \mathcal{A} \cap Y, \mu_Y)$. (For measure-preservation by more general versions of induced maps, see e.g. [**14**].) When studying specific systems, one often tries to find some good reference set Y such that T_Y is more convenient a map than T. In this case, it often pays to prove a relevant property first for T_Y, and to transfer it back to T afterwards - if possible. Here, we show that this strategy can be employed to deal with hitting-time statistics. The conclusion is pleasantly simple, as the respective limit laws for T

and T_Y coincide.

In the following, we let $\varphi_H^Y : Y \to \overline{\mathbb{N}}$ denote the hitting time of $H \in \mathcal{A} \cap Y$ under the first-return map T_Y, that is,

$$\varphi_H^Y(x) := \inf\{j \geq 1 : T_Y^j x \in H\}, \quad x \in Y. \tag{3.1}$$

The hitting time functions φ_H and φ_H^Y are naturally related to each other in that

$$\varphi_H = \sum_{j=0}^{\varphi_H^Y - 1} \varphi_Y \circ T_Y^j \quad \text{on } Y. \tag{3.2}$$

This can be exploited in a fairly straightforward manner to obtain

THEOREM 3.1 (**Hitting-time statistics via inducing**). *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic probability-preserving system, and $Y \in \mathcal{A}$, $\mu(Y) > 0$. Assume that $(H_l)_{l\geq 1}$ is a sequence of asymptotically rare events in $\mathcal{A} \cap Y$, and that R is any random variable with values in $[0, \infty]$. Then*

$$\mu_Y(H_l)\, \varphi_{H_l}^Y \overset{\mu_Y}{\Longrightarrow} \mathsf{R} \quad \text{as } l \to \infty \tag{3.3}$$

iff

$$\mu(H_l)\, \varphi_{H_l} \overset{\mu}{\Longrightarrow} \mathsf{R} \quad \text{as } l \to \infty. \tag{3.4}$$

PROOF. **(i)** Let $F : [0, \infty) \to [0, 1]$ be the (sub-)distribution function of R. Due to Proposition 2.1 we know that in (3.4) convergence $\overset{\mu}{\Longrightarrow}$ w.r.t. μ is equivalent to convergence $\overset{\mu_Y}{\Longrightarrow}$ w.r.t. μ_Y. It is therefore sufficient to prove that

$$\mu_Y\left(\mu(Y)^{-1}\mu(H_l)\, \varphi_{H_l}^Y \leq t\right) \longrightarrow F(t) \quad \text{for all continuity points } t > 0 \tag{3.5}$$

iff

$$\mu_Y\left(\mu(H_l)\, \varphi_{H_l} \leq t\right) \longrightarrow F(t) \quad \text{for all continuity points } t > 0. \tag{3.6}$$

To this end, we fix any continuity point $t > 0$ of F, and any $\varepsilon > 0$. Next, we choose $\delta > 0$ so small that

$$F(t) - \varepsilon/4 < F(e^{-\delta} t) \leq F(e^{\delta} t) < F(t) + \varepsilon/4. \tag{3.7}$$

Since F is continuous on a dense set, we may also assume that both $e^{-\delta} t$ and $e^{\delta} t$ are continuity points of F.

(ii) By the Ergodic theorem and Kac' formula, we have

$$m^{-1} \sum_{j=0}^{m-1} \varphi_Y \circ T_Y^j \longrightarrow \mu(Y)^{-1} \quad \text{a.e. on } Y. \tag{3.8}$$

This implies that the increasing sequence of sets given by

$$E_M := \{\textstyle\sum_{j=0}^{m-1} \varphi_Y \circ T_Y^j \geq e^{-\delta} \mu(Y)^{-1} m \quad \text{for all } m \geq M\} \in \mathcal{A} \cap Y$$

satisfies $\mu_Y(E_M^c) \to 0$ as $M \to \infty$. Now fix some M such that $\mu_Y(E_M^c) < \varepsilon/4$.

Next, let $F_l := \{\varphi_{H_l}^Y \geq M\} \in \mathcal{A} \cap Y$, $l \geq 1$. Then, $F_l = Y \cap \bigcap_{j=1}^{M-1} T_Y^{-j} H_l^c$, and hence $\mu_Y(F_l^c) \leq \sum_{j=1}^{M-1} \mu_Y(T_Y^{-j} H_l) \leq M\, \mu_Y(H_l) \to 0$ as $l \to \infty$, since T_Y preserves μ_Y, and the sequence $(H_l)_{l\geq 1}$ is asymptotically rare. Therefore there is some $L \geq 1$ such that $\mu_Y(F_l^c) < \varepsilon/4$ for $l \geq L$.

(iii) Now recall (3.2). By definition of E_M and F_l we have

$$\varphi_{H_l} = \sum_{j=0}^{\varphi_{H_l}^Y - 1} \varphi_Y \circ T_Y^j \geq e^{-\delta}\, \mu(Y)^{-1} \varphi_{H_l}^Y \quad \text{on } F_l \cap E_M.$$

Therefore, for any $s > 0$,

$$F_l \cap E_M \cap \{\mu(H_l)\, \varphi_{H_l} \leq s\} \subseteq F_l \cap E_M \cap \{\mu(Y)^{-1}\mu(H_l)\, \varphi_{H_l}^Y \leq e^{\delta} s\},$$

and hence

$$\begin{aligned} \mu_Y\left(\mu(H_l)\, \varphi_{H_l} \leq s\right) &\leq \mu_Y\left(\mu(Y)^{-1}\mu(H_l)\, \varphi_{H_l}^Y \leq e^{\delta} s\right) + \mu_Y((F_l \cap E_M)^c) \\ &< \mu_Y\left(\mu(Y)^{-1}\mu(H_l)\, \varphi_{H_l}^Y \leq e^{\delta} s\right) + \varepsilon/2 \quad \text{for } l \geq L. \end{aligned} \tag{3.9}$$

(iv) Assume (3.5). Since $e^{\delta} t$ is a continuity point of F, we can pick $L' \geq 1$ such that $\mu_Y(\mu(Y)^{-1}\mu(H_l)\, \varphi_{H_l}^Y \leq e^{\delta} t) < F(e^{\delta} t) + \varepsilon/4$ for $l \geq L'$. Combining this with (3.7) and with (3.9) for $s := t$, we obtain

$$\mu_Y\left(\mu(H_l)\, \varphi_{H_l} \leq t\right) < F(t) + \varepsilon \quad \text{for } l \geq \max(L, L'). \tag{3.10}$$

If, on the other hand, we start from (3.6), then $\mu_Y\left(\mu(H_l)\, \varphi_{H_l} \leq e^{-\delta} t\right) > F(e^{-\delta} t) - \varepsilon/4$ for $l \geq L'$ with L' large enough, as $e^{-\delta} t$, too, is a continuity point of F. Combining this with (3.7) and with (3.9) for $s := e^{-\delta} t$, we get

$$\mu_Y\left(\mu(Y)^{-1}\mu(H_l)\, \varphi_{H_l}^Y \leq t\right) > F(t) - \varepsilon \quad \text{for } l \geq \max(L, L'). \tag{3.11}$$

Versions of (3.10) and (3.11) providing the corresponding estimate in the opposite direction are obtained in exactly the same way. As $\varepsilon > 0$ was arbitrary, our claim follows. □

In the proof, Proposition 2.1, was used at the very start to ensure that

$$\mu(H_l)\, \varphi_{H_l} \stackrel{\mu}{\Longrightarrow} \mathsf{R} \quad \text{iff} \quad \mu(H_l)\, \varphi_{H_l} \stackrel{\mu_Y}{\Longrightarrow} \mathsf{R}. \tag{3.12}$$

This special case of strong distributional convergence can also be verified directly. We indicate an argument which is somewhat more elementary than the theory behind Proposition 2.1.

Direct proof of (3.12). **(i)** Let F denote the (necessarily continuous, see Theorem 2.1) sub-probability distribution function of R. To prove that for every $t > 0$ (henceforth fixed),

$$\mu\left(\varphi_{H_l} \leq s_l\right) \to F(t) \quad \text{iff} \quad \mu_Y\left(Y \cap \{\varphi_{H_l} \leq s_l\}\right) \to F(t) \tag{3.13}$$

with $s_l := t/\mu(H_l)$, we show that for $\varepsilon \in (0, t)$ there is some $L_0 \geq 1$ s.t.

$$\mu_Y\left(Y \cap \{\varphi_{H_l} \leq s_l^-\}\right) - \varepsilon < \mu\left(\varphi_{H_l} \leq s_l\right) < \mu_Y\left(Y \cap \{\varphi_{H_l} \leq s_l^+\}\right) + \varepsilon, \tag{3.14}$$

for $l \geq L_0$, where $s_l^{\pm} := (t \pm \varepsilon)/\mu(H_l)$. Below we focus on the second of these estimates. The other one is obtained by an analogous argument.

(ii) To switch from μ on X to its restriction to Y in (3.14), we will show that $\mu\left(\varphi_{H_l} \leq s_l\right) \approx \int_Y \varphi_Y \cdot 1_{\{\varphi_{H_l} \leq s_l\}}\, d\mu$ (which only involves μ on Y). This uses the well-known canonical representation (from the theory of induced transformations)

$$\mu(A) = \sum_{n \geq 0} \mu\left(Y \cap \{\varphi_Y > n\} \cap T^{-n} A\right) \quad \text{for } A \in \mathcal{A}. \tag{3.15}$$

As μ is finite, there is some N^* such that

$$\sum_{n > N^*} \mu\left(Y \cap \{\varphi_Y > n\}\right) < \varepsilon/4. \tag{3.16}$$

We observe an approximate invariance property of the variables φ_{H_l}. Note that

$$\varphi_{H_l} \circ T^n = \varphi_{H_l} - n \quad \text{on } \{\varphi_{H_l} > n\}. \tag{3.17}$$

As (H_l) is asymptotically rare, there is some L_1 s.t. $\mu(\varphi_{H_l} \leq N^*) < \varepsilon/4(N^*+1)$ for $l \geq L_1$. Moreover, there is some L_2 s.t. $s_l + N^* \leq (t \pm \varepsilon/2)/\mu(H_l) =: s_l^*$ whenever $l \geq L_2$. For $n \in \{0, \ldots, N^*\}$ and $l \geq L_1 \vee L_2$ we then find that

$$\begin{aligned}
&\mu\left(Y \cap \{\varphi_Y > n\} \cap T^{-n}\{\varphi_{H_l} \leq s_l\}\right) \\
&\leq \mu\left(Y \cap \{\varphi_{H_l} > N^*\} \cap \{\varphi_Y > n\} \cap \{\varphi_{H_l} \leq s_l + n\}\right) + \mu\left(\{\varphi_{H_l} \leq N^*\}\right) \\
&< \mu\left(Y \cap \{\varphi_Y > n\} \cap \{\varphi_{H_l} \leq s_l^*\}\right) + \varepsilon/4(N^*+1).
\end{aligned}$$

Combining the preceding considerations, we then obtain

$$\begin{aligned}
\mu\left(\varphi_{H_l} \leq s_l\right) &< \textstyle\sum_{n=0}^{N^*} \mu\left(Y \cap \{\varphi_Y > n\} \cap T^{-n}\{\varphi_{H_l} \leq s_l\}\right) + \varepsilon/4 \\
&< \textstyle\sum_{n=0}^{N^*} \mu\left(Y \cap \{\varphi_Y > n\} \cap \{\varphi_{H_l} \leq s_l^*\}\right) + 2\varepsilon/4 \\
&\leq \textstyle\sum_{n\geq 0} \mu\left(Y \cap \{\varphi_Y > n\} \cap \{\varphi_{H_l} \leq s_l^*\}\right) + \varepsilon/2 \\
&= \textstyle\int_Y \varphi_Y \cdot 1_{\{\varphi_{H_l} \leq s_l^*\}} \, d\mu + \varepsilon/2.
\end{aligned} \tag{3.18}$$

(iii) We now verify that $\int_Y \varphi_Y \cdot 1_{\{\varphi_{H_l} \leq s_l\}} \, d\mu \approx \mu(Y)^{-1} \mu\left(Y \cap \{\varphi_{H_l} \leq s_l\}\right)$. To this end, we use that T_Y is measure-preserving, and write

$$\textstyle\int_Y \varphi_Y \cdot 1_{\{\varphi_{H_l} \leq s_l^*\}} \, d\mu = N^{-1} \sum_{n=0}^{N-1} \int_Y \varphi_Y \circ T_Y^n \cdot 1_{\{\varphi_{H_l} \leq s_l^*\}} \circ T_Y^n \, d\mu. \tag{3.19}$$

Due to ergodicity of T_Y and the L^1-ergodic theorem (plus Kac' formula), we can choose this N in such a way that

$$\int_Y \left| N^{-1} \textstyle\sum_{n=0}^{N-1} \varphi_Y \circ T_Y^n - \mu(Y)^{-1} \right| \, d\mu < \varepsilon/4, \tag{3.20}$$

and hence, a fortiori,

$$N^{-1} \textstyle\sum_{n=0}^{N-1} \int_Y (\varphi_Y \circ T_Y^n) \, 1_{\{\varphi_{H_l} \leq s_l^+\}} \, d\mu < \mu_Y\left(Y \cap \{\varphi_{H_l} \leq s_l^+\}\right) + \varepsilon/4. \tag{3.21}$$

The left-hand expression here differs from the right-hand side of (3.19). However, to switch from $1_{\{\varphi_{H_l} \leq s_l^*\}} \circ T_Y^n$ to $1_{\{\varphi_{H_l} \leq s_l^+\}}$ we can again exploit approximate invariance of the φ_{H_l}. We first record that there is some $\delta > 0$ such that

$$\textstyle\int_A \varphi_Y \circ T_Y^n \, d\mu < \varepsilon/4 \quad \text{for all } n \geq 0 \text{ and } A \in \mathcal{A} \cap Y \text{ with } \mu(A) < \delta. \tag{3.22}$$

Indeed, since T_Y is measure-preserving on Y, it is immediate that the sequence $(\varphi_Y \circ T_Y^n)_{n\geq 0}$ is uniformly integrable on Y, whence (3.22).

Reviewing the idea which gave (3.17), we see that

$$\varphi_{H_l} \circ T_Y^n = \varphi_{H_l} - \textstyle\sum_{j=0}^{n-1} \varphi_Y \circ T_Y^j \quad \text{on } Y \cap \{\varphi_{H_l}^Y > n\}. \tag{3.23}$$

Let $\Phi := \sum_{j=0}^{N-1} \varphi_Y \circ T_Y^j$ which is finite a.e. on Y. Take M so large that $\mu(Y \cap \{\Phi > M\}) < \delta/2$. Next, choose L_3 such that $\mu(Y \cap \{\varphi_{H_l}^Y \leq N\}) < \delta/2$ and $s_l^* + M \leq s_l^+$ whenever $l \geq L_3$. Then $A_l := Y \cap \{\Phi \leq M\} \cap \{\varphi_{H_l}^Y > N\}$ satisfies $\mu(Y \setminus A_l) < \delta$ for $l \geq L_3$. For such l and $n \in \{0, \ldots, N\}$, we thus get

$$\begin{aligned}
&\textstyle\int_Y \varphi_Y \circ T_Y^n \cdot 1_{\{\varphi_{H_l} \leq s_l^*\}} \circ T_Y^n \, d\mu \\
&\leq \textstyle\int_{A_l} \varphi_Y \circ T_Y^n \cdot 1_{\{\varphi_{H_l} \leq s_l^* + M\}} \, d\mu + \int_{Y \setminus A_l} \varphi_Y \circ T_Y^n \, d\mu \\
&< \textstyle\int_Y \varphi_Y \circ T_Y^n \cdot 1_{\{\varphi_{H_l} \leq s_l^+\}} \, d\mu + \varepsilon/4.
\end{aligned} \tag{3.24}$$

(iv) The proof of the right-hand half of (3.14) is completed by letting $L_0 := L_1 \vee L_2 \vee L_3$, and combining (3.18) with (3.24) and (3.21). □

4. Return-time statistics via inducing

We finally extend the applicability of the inducing method to return-time statistics from the setup of [**4**] to the general abstract framework of measure-theoretic ergodic theory. With Theorem 2.1 and Proposition 2.1 at our disposal, we can easily pass from the fully general Theorem 3.1 to a corresponding result for return-time statistics which does not require any further assumptions. This is a significant improvement of Theorem 2.1 of [**4**], since the latter **a)** only deals with a slightly restricted class of limit laws where the limiting distribution $\tilde{F}$ of the return times satisfies $\lim_{t\to 0^+} \tilde{F}(t) = 0$, i.e. there is no point mass at $t = 0$, **b)** assumes that X is a Riemannian manifold and **c)** only applies to sequences (H_l) of ε-neighbourhoods of typical (but not arbitrary) points of X.

Theorem 4.1 (**Return-time statistics via inducing**). *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic probability-preserving system, and $Y \in \mathcal{A}$, $\mu(Y) > 0$. Assume that $(H_l)_{l\geq 1}$ is a sequence of asymptotically rare events in $\mathcal{A} \cap Y$, and that $\widetilde{\mathsf{R}}$ is any random variable with values in $[0, \infty]$. Then*

$$\mu_Y(H_l)\, \varphi^Y_{H_l} \overset{\mu_{H_l}}{\Longrightarrow} \widetilde{\mathsf{R}} \quad \textit{as } l \to \infty \tag{4.1}$$

iff

$$\mu(H_l)\, \varphi_{H_l} \overset{\mu_{H_l}}{\Longrightarrow} \widetilde{\mathsf{R}} \quad \textit{as } l \to \infty. \tag{4.2}$$

Proof. Applying Theorem 2.1 to T_Y, we see that (4.1) is equivalent to

$$\mu_Y(H_l)\, \varphi^Y_{H_l} \overset{\mu_Y}{\Longrightarrow} \mathsf{R} \quad \text{as } l \to \infty, \tag{4.3}$$

with R and $\widetilde{\mathsf{R}}$ related by the integral equation (2.4) for their respective distribution functions F and $\widetilde{F}$. Due to Theorem 3.1, (4.3) is equivalent to

$$\mu(H_l)\, \varphi_{H_l} \overset{\mu}{\Longrightarrow} \mathsf{R} \quad \text{as } l \to \infty. \tag{4.4}$$

But then we can again apply Theorem 2.1, this time to T, to validate that (4.4) is indeed equivalent to (4.2), as claimed. □

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