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RETURN- AND HITTING-TIME LIMITS FOR RARE EVENTS OF NULL-RECURRENT MARKOV MAPS

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Abstract. We determine limit distributions for return- and hitting-time functions of certain asymptotically rare events for conservative ergodic infinite measure preserving transformations with regularly varying asymptotic type. Our abstract result applies, in particular, to shrinking cylinders around typical points of null-recurrent renewal shifts and infinite measure preserving interval maps with neutral fixed points.

1. Introduction

Return- and hitting-time statistics for asymptotically rare events in ergodic dynamical systems have undergone some intense research in the past 15 years, as documented in [BSTV], [Coe], [Col], [CGS2], [Hi], [HSV], [Ko], [Pa], [Pi], to name just a few references. In particular, it has been shown that the emergence of exponential limit distributions is an amazingly robust phenomenon for systems possessing an invariant probability measure, that is, for positively recurrent situations.

Nevertheless, very little is known about rare events of null-recurrent systems, [BZ], [GKP], [PS1], [PS2]. The present paper contributes to the study of this situation. More precisely, given a conservative ergodic dynamical system $(X, \mathcal{A}, \mu, T)$ with infinite T-invariant measure μ and given a sequence of sets $(E_k)_{k\geq 1}$ in $\mathcal{A}$ such that $\mu(E_k) \to 0$, we are interested in the asymptotic behaviour of the *first hitting time* φ_{E_k} in E_k. Our aim is to establish convergence in distribution for φ_{E_k} with respect to any probability measure ν absolutely continuous with respect to μ. We prove, under some general hypotheses, that φ_{E_k}, suitably normalized, converges to $\mathcal{E}^{\frac{1}{\alpha}}\,\mathcal{G}_\alpha$ (for some $\alpha \in (0, 1]$ which is a characteristic of the system), where $\mathcal{E}$ and $\mathcal{G}_\alpha$ are two independent random variables, $\mathcal{E}$ being exponentially distributed ($\Pr[\mathcal{E} > t] = e^{-t}$ for $t \geq 0$) and $\mathcal{G}_\alpha$, $\alpha \in (0, 1)$, being distributed according to the one-sided stable law of order α ($\mathbb{E}[\exp(-s\mathcal{G}_\alpha)] = \exp(-s^\alpha)$ for $s \geq 0$), while $\mathcal{G}_1 = 1$. We also prove the same convergence result for the *first return time* to E_k (that is, for φ_{E_k} with respect to $\mu(\centerdot \cap E_k)/\mu(E_k)$).

Our general hypotheses rely on the new concept of $\mathcal{U}$-uniform sets and apply to shrinking cylinders around typical points (i.e. repetition times of symbolic orbits),

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for certain Markovian interval maps with fixed points. This includes, as a special case, null-recurrent renewal Markov chains.

The paper is organized as follows. In Section 2 we introduce notations and illustrate our main results by some examples. Section 3 states a limit theorem for hitting- and return-times for a concrete class of interval maps with indifferent fixed points. At the heart of this result is a more abstract limit theorem which is presented in Section 4. This is where the concept of $\mathcal{U}$-uniform sets is introduced. The remainder of the article is devoted to the proofs of these results. In Section 5 we establish the abstract limit theorem. Section 6 discusses how $\mathcal{U}$-uniformity arises in the context of suitable induced maps. The final Section 7 completes the proof of the limit theorem for interval maps.

2. General setup and examples

A baby example. As a leisurely warm-up, we mention a very simple probabilistic example (which will come in handy later on). In fact, it incorporates the very special situation we are going to leave behind us: It is a toy model given by one simple Bernoulli process, with a family of asymptotically rare events with exponential limit law, and an *independent* null-recurrent renewal process used to delay the former. For random variables we use $\Rightarrow$ and $\stackrel{d}{=}$ to indicate convergence and equality in distribution, respectively.

Example 2.1 (**Markov chain baby example**). *We fix $\alpha \in (0,1)$ and consider a continuous-state Markov chain $(\mathsf{X}_n^*)_{n\geq 0}$ on some $(\Omega, \mathcal{A}, \mathsf{P})$, constructed from an iid sequence $(\mathsf{U}_n)_{n\geq 0}$ of uniformly distributed random variables on $[0,1]$, and an independent discrete irreducible null-recurrent renewal Markov chain $(\mathsf{X}_n)_{n\geq 0}$ on $\mathbb{N}_0 = \{0,1,\ldots\}$, starting in $\{0\}$, with return distribution $(f_j)_{j\geq 1}$ (i.e. $\mathsf{P}[\mathsf{X}_{n+1} = k \parallel \mathsf{X}_n = 0] = p_{0,k} = f_{k+1}$, and $\mathsf{P}[\mathsf{X}_{n+1} = k \parallel \mathsf{X}_n = k+1] = p_{k+1,k} = 1$ for any $k \geq 0$), in the domain of attraction of $\mathcal{G}_\alpha$. Hence, the consecutive excursions from the renewal state $\{0\}$ form an iid sequence $(\mathsf{Y}_n)_{n\geq 1}$ of variables distributed according to $\mathsf{P}[\mathsf{Y}_n = j] = f_j$, and there is some normalizing sequence $(b(m))_{m\geq 1}$, regularly varying of order $1/\alpha$, such that the $\mathsf{S}_m := \sum_{i=1}^m \mathsf{Y}_i$ satisfy $b(m)^{-1}\mathsf{S}_m \Longrightarrow \mathcal{G}_\alpha$. The Markov chain (X_n) has a unique (up to a multiplicative constant) invariant measure $(r_k)_{k\geq 0}$ on $\mathbb{N}_0$, with weights $r_k := \sum_{j>k} f_j$ (and $\mathbb{E}[Y_1] = \sum_{k\geq 0} r_k = \infty$). Now use the counting function $\mathsf{N}_n := \sum_{m\geq 0} 1_{[0,n]}(\mathsf{S}_m)$ to define*

$$\mathsf{X}_n^* := (\mathsf{X}_n, \mathsf{U}_{\mathsf{N}_n}) \in \mathbb{N}_0 \times [0,1], \quad \text{for } n \geq 0.$$

We study the law of the first hitting time *of our chain in the set $\{0\} \times [0,\epsilon]$, i.e. of*

$$\varphi_\epsilon := \min\{n \geq 1 : \mathsf{X}_n = 0 \text{ and } \mathsf{U}_{\mathsf{N}_n} \in [0,\epsilon]\}. \tag{2.1}$$

Observe that we can represent this as

$$\varphi_\epsilon = \sum_{i=0}^{\tau_\epsilon - 1} \mathsf{Y}_i \quad \text{with} \quad \tau_\epsilon := \min\{i \geq 1 : \mathsf{U}_i \in [0,\epsilon]\}. \tag{2.2}$$

Since, obviously, the τ_ϵ are independent of the Y_i, and satisfy $\epsilon \cdot \tau_\epsilon \Longrightarrow \mathcal{E}$ as $\epsilon \searrow 0$, routine arguments (exploiting regular variation) enable us to conclude that

$$\frac{1}{b(1/\epsilon)}\,\varphi_\epsilon = \frac{1}{b(1/\epsilon)}\,b(\tau_\epsilon) \cdot \frac{1}{b(\tau_\epsilon)} \sum_{i=0}^{\tau_\epsilon - 1} \mathsf{Y}_i \Longrightarrow \mathcal{E}^{\frac{1}{\alpha}} \cdot \mathcal{G}_\alpha \quad \text{as } \epsilon \searrow 0. \tag{2.3}$$

That is, we observe distributional convergence of the hitting times for this simple family of asymptotically rare events to the independent product of the appropriate power $\mathcal{E}^{\frac{1}{\alpha}}$ *of the exponential variable* $\mathcal{E}$*, and the one-sided stable variable* $\mathcal{G}_\alpha$*. It is not hard to see that the Laplace transform of this limit variable* $\mathcal{E}^{1/\alpha}\mathcal{G}_\alpha$ *is*

$$\mathbb{E}\left[\exp\left(-s\,\mathcal{E}^{\frac{1}{\alpha}}\mathcal{G}_\alpha\right)\right] = \frac{1}{1+s^\alpha} \qquad \text{for } s \geq 0.$$

Note also that we naturally have

$$\mathcal{E}^{\frac{1}{\alpha}} \cdot \mathcal{G}_\alpha \stackrel{d}{=} \mathcal{G}_{\alpha,\mathcal{E}}, \tag{2.4}$$

where $(\mathcal{G}_{\alpha,t})_{t\geq 0}$ *denotes the stable subordinator of index* α *(which the normalized partial sum processes* $(b(m)^{-1}\mathsf{S}_{mt})_{t\geq 0}$ *converge to in the Skorohod* J_1*-topology as* $m \to \infty$*), so that* $\mathcal{G}_{\alpha,t} \stackrel{d}{=} t^{1/\alpha}\mathcal{G}_\alpha$ *for* $t \geq 0$*, and* $\mathcal{G}_{\alpha,\mathcal{E}}$ *is the subordinator at an independent exponential time.*

While the clear-cut dependence structure of this toy model is not typical for the situations we are going to study, we shall see that the result (2.3) is. The aim of this paper is to provide conditions on null-recurrent measure preserving transformations which ensure that natural families of asymptotically rare events exhibit hitting-time (and, in fact, also return-time) limit distributions given by $\mathcal{E}^{\frac{1}{\alpha}} \cdot \mathcal{G}_\alpha$.

General setup. Return-times and inducing. Throughout the paper, *all measures are understood to be* σ*-finite.* Given a measure space $(X, \mathcal{A}, \mu)$, a *partition mod* μ of X will be a countable family $\xi \subseteq \mathcal{A}$ of sets which, up to sets of measure zero, are pairwise disjoint and cover X. For a.e. $x \in X$ we then have $x \in \xi(x)$ for some well defined $\xi(x) \in \xi$. We study (typically non-invertible) *measure preserving transformations* T on $(X, \mathcal{A}, \mu)$, i.e. measurable maps $T : X \to X$ for which $\mu \circ T^{-1} = \mu$. The transformation T will be *ergodic* (i.e. for $A \in \mathcal{A}$ with $T^{-1}A = A$ we have $0 \in \{\mu(A), \mu(A^c)\}$) and *conservative* (meaning that $\mu(A) = 0$ for all wandering sets, that is, $A \in \mathcal{A}$ with $T^{-n}A$, $n \geq 1$, pairwise disjoint), whence *recurrent* (in that $A \subseteq \bigcup_{n\geq 1} T^{-n}A \mod \mu$ for $A \in \mathcal{A}$). Our emphasis will be on the *infinite measure case*: we assume throughout that $\mu(X) = \infty$.

For T such a conservative ergodic measure preserving transformation (*c.e.m.p.t.*) on $(X, \mathcal{A}, \mu)$, and any $Y \in \mathcal{A}$, $\mu(Y) > 0$, we define the *first entrance time* function of Y, $\varphi_Y : X \to \mathbb{N} \cup \{\infty\}$ by $\varphi_Y(x) := \min\{n \geq 1 : T^n x \in Y\}$, $x \in X$, and let $T_Y x := T^{\varphi(x)}x$, $x \in X$. When restricted to Y, φ_Y is called the *first return time* of Y, and $\mu\mid_{Y\cap\mathcal{A}}$ is invariant under the *first return map*, T_Y restricted to Y. If $\mu(Y) < \infty$, it is natural to regard φ_Y as a random variable on the probability space $(X, \mathcal{A}, \mu_Y)$, where $\mu_Y(E) := \mu(Y)^{-1}\mu(Y \cap E)$, and $\mu(X) = \infty$, the case we are interested in, is equivalent to $\int \varphi_Y\, d\mu_Y = \infty$ by Kac' formula. That is, we study *null-recurrent* situations.

Note that $\varphi_Y \circ T_Y^{i-1}$ is the time between the $(i-1)$st and the ith visit to Y. To fix a notation for the *occupation times* of a set $Y \in \mathcal{A}$, we let

$$\mathbf{S}_k(Y) := \sum_{j=0}^{k-1} 1_Y \circ T^j, \qquad k \geq 1. \tag{2.5}$$

In this setup, a sequence $(E_k)_{k\geq 1}$ in $\mathcal{A}$ with $\mu(E_k) \to 0$ will be referred to as a sequence of *asymptotically rare events*. Asking for an *asymptotic return-time* *distribution* means to look for normalizing constants $\widetilde{B}_k > 0$ and a nondegenerate limit random variable $\widetilde{\mathcal{V}}$ such that

$$\mu_{E_k}(\widetilde{B}_k \cdot \varphi_{E_k} \leq t) \longrightarrow \Pr[\widetilde{\mathcal{V}} \leq t] \quad \text{as } k \to \infty, \tag{2.6}$$

while an *asymptotic hitting-time distribution* is given by some nondegenerate $\mathcal{V}$ such that for some (and hence every, see [Z7]) fixed probability measure $\nu \ll \mu$ we have

$$\nu\left(B_k \cdot \varphi_{E_k} \leq t\right) \longrightarrow \Pr[\mathcal{V} \leq t] \quad \text{as } k \to \infty, \tag{2.7}$$

with suitable $B_k > 0$. (Here, of course, convergence is supposed to take place at continuity points t of the respective limit distribution function). In finite measure situations, a canonical choice for $\widetilde{B}_k$ is $\mu(E_k)^{-1}$, in which case the relation between (2.6) and (2.7) has been clarified in [HLV] (see also [AS]). In particular, it is known that each convergence implies the other, and $\widetilde{\mathcal{V}} \stackrel{d}{=} \mathcal{V}$ if and only if $\widetilde{\mathcal{V}} \stackrel{d}{=} \mathcal{E}$. The infinite-measure result below shows, in particular, that in null-recurrent situations there is more than one limit law which can occur simultaneously in (2.6) and (2.7) (where again we use a canonical normalization). When specialized to a prototypical standard family of infinite measure preserving maps, it takes the following form.

Example 2.2 (**Standard examples with neutral fixed points**). *Fix some* $\alpha \in (0,1)$*, set* $p := 1/\alpha$*, and define* $T : [0,1] \to [0,1]$ *by letting*

$$Tx := x + x^{1+p} \ \mathit{mod}\ 1.$$

It is well known ([T1], [T2]*) that* T *is conservative ergodic with a unique invariant density* h *(with respect to Lebesgue measure* λ*) which is continuous on* $(0,1)$ *and satisfies* $h(x) \sim x^{-p}$ *as* $x \searrow 0$*. Let* $c \in (0,1)$ *be the critical point (*$c + c^{1+p} = 1$*), and set* $Y := (c,1)$*. Then the cylinder of order* k *around* x *is the set* $\xi_k(x) := \{y : 1_Y(T^j y) = 1_Y(T^j x) \text{ for } 0 \leq j < k\}$*. When applied to* T*, Theorem 3.1 below implies that for* λ*-a.e.* $x \in [0,1]$ *the cylinders* $E_k := \xi_k(x)$ *have the same asymptotic hitting-time distribution as Example 2.1,*

$$\lambda\left[\{\kappa\, \lambda(E_k)^{-p} \cdot \varphi_{E_k} \leq t\}\right] \longrightarrow \Pr[\mathcal{E}^{\frac{1}{\alpha}}\, \mathcal{G}_\alpha \leq t] \quad \text{as } k \to \infty \tag{2.8}$$

(with $\kappa = \kappa(x) \in (0,\infty)$ *a suitable constant). Here,* λ *can be replaced by every fixed probability measure* $\nu \ll \lambda$ *on* $[0,1]$*. Moreover, the same law shows up as its asymptotic return-time distribution in the sense that*

$$\lambda_{E_k}\left[\{\kappa\, \lambda(E_k)^{-p} \cdot \varphi_{E_k} \leq t\}\right] \longrightarrow \Pr[\mathcal{E}^{\frac{1}{\alpha}}\, \mathcal{G}_\alpha \leq t] \quad \text{as } k \to \infty. \tag{2.9}$$

Reference sets. The renewal shift example. If $\mu(X) = \infty$, a good understanding of T frequently depends on its behaviour relative to a suitable *reference set* Y of finite measure, defined through some distinctive property. Specifically, the asymptotic behaviour of the *return distribution of* Y is a crucial feature determining the stochastic properties of the system. For distributional limit theorems to hold, regular variation of the *tail probabilities* $q_n(Y) := \mu_Y(\varphi_Y > n)$, $n \in \mathbb{N}_0$, or, more generally, of the *wandering rate* of Y, $(w_N(Y))_{N \geq 1}$, is decisive. Here, we let

$$w_N(Y) := \mu(Y) \sum_{n=0}^{N-1} q_n(Y) = \sum_{n=0}^{N-1} \mu(Y \cap \{\varphi_Y > n\}) = \int_Y (\varphi_Y \wedge N)\, d\mu. \tag{2.10}$$

The basic example for a suitable reference set Y is the renewal state of a Markov renewal process. Indeed, our result below effectively contains the following example.

Example 2.3 (**Null-recurrent renewal chains**). *Consider a (one-sided)* renewal shift $\mathsf{R}\langle f\rangle = (X, \mathcal{A}, \mu^{\langle f\rangle}, T, \xi')$ *with return distribution* $f = (f_k)_{k\geq 1}$. *This is the canonical shift-space representation of the renewal Markov chain* $(\mathsf{X}_n)_{n\geq 0}$ *of Example 2.1. Thus,* $X := \mathbb{N}_0^{\mathbb{N}_0} = \{x = (x_i)_{i\geq 0} : x_i \in \mathbb{N}_0\}$ *with product* σ*-field* $\mathcal{A}$ *and infinite stationary Markov measure* $\mu^{\langle f\rangle}$ *given by* $\mu^{\langle f\rangle}([s_0,\ldots,s_{k-1}]) = r_{s_0}\, p_{s_0,s_1}\cdots p_{s_{k-2},s_{k-1}}$ *for* cylinders of rank k, $[s_0,\ldots,s_{k-1}] := \{x : x_0 = s_0,\ldots,x_{k-1} = s_{k-1}\}$. *Let* $\xi' := \{[s_0] : s_0 \in \mathbb{N}_0\}$ *be the natural time-zero partition, and* T *be the* shift map *on* X, $(Tx)_i = x_{i+1}$.

The rank k *cylinder* $E'_k = \xi'_k(x)$ *containing* x *records the first* k *states in* $\mathbb{N}_0$ *which a particular realization* x *of the Markov chain* $(\mathsf{X}_n)_{n\geq 0}$ *visits, and* φ_{E_k} *is the time one has to wait until this pattern first appears after time zero. Alternatively, we might also be interested in a coarser coding, which only distinguishes between the renewal state* $\{0\}$ *and the rest* $\mathbb{N}$. *This amounts to considering the partition* $\xi := \{Y, Y^c\}$ *of* X *with* $Y := [0]$. *Let* $E_k = \xi_k(x) := \{\widetilde{x} : 1_Y(\widetilde{x}_i) = 1_Y(x_i)$ *for* $0 \leq i < k\}$. *Our limit theorem covers both codings.*

Assume now that $f = (f_k)_{k\geq 1}$ *satisfies* $q_n = \sum_{k>n} f_k \sim cn^{-\alpha}$ *as* $n\to\infty$ *for some* $c > 0$ *and* $\alpha \in (0,1)$, *and set* $b(s) := (s/\kappa_\alpha)^{1/\alpha}$ *with* $\kappa_\alpha := c\Gamma(1-\alpha)\Gamma(1+\alpha)$.

Then, for $\mu^{\langle f\rangle}$*-a.e.* $x \in X$, *the consecutive return- and hitting-time distributions of the cylinders* $E_k := \xi_k(x)$, $k \geq 1$, *converge: For* $d \in \mathbb{N}$, $t_i > 0$, *and* $k \to \infty$,

$$\mu^{\langle f\rangle}_{E_k}\left(\bigcap_{i=0}^{d-1}\{b_T(1/\mu(E_k))^{-1}\cdot\varphi_{E_k}\circ T^i_{E_k} \leq t_i\}\right) \longrightarrow \prod_{i=0}^{d-1}\Pr[\mathcal{E}^{\frac{1}{\alpha}}\,\mathcal{G}_\alpha \leq t_i], \tag{2.11}$$

and, for every fixed probability measure $\nu \ll \mu^{\langle f\rangle}$,

$$\nu\left(\bigcap_{i=0}^{d-1}\{b_T(1/\mu(E_k))^{-1}\cdot\varphi_{E_k}\circ T^i_{E_k} \leq t_i\}\right) \longrightarrow \prod_{i=0}^{d-1}\Pr[\mathcal{E}^{\frac{1}{\alpha}}\,\mathcal{G}_\alpha \leq t_i]. \tag{2.12}$$

Both statements remain true if $E_k = \xi_k(x)$ *is replaced by* $E'_k := \xi'_k(x)$.

Remark 2.1. It is well known that every recurrent Markov shift contains many renewal shifts as factors (e.g. the return processes to its states). In this sense, these form a very basic class of processes. In fact, also the smooth dynamical systems discussed below are intimately related to renewal shifts in a similar way, see [Z9].

Recall finally that a function $a : (L,\infty) \to (0,\infty)$ is *regularly varying of index* $\rho \in \mathbb{R}$ *at infinity* (see [Ka]), written $a \in \mathcal{R}_\rho$, if a is measurable and $a(ct)/a(t) \to c^\rho$ as $t \to \infty$ for all $c > 0$. We shall interpret sequences (a_n) as functions on $\mathbb{R}_+$ via $t \longmapsto a_{[t]}$. *Slow variation* means regular variation of index 0. $\mathcal{R}_\rho(0)$ is the family of functions $r : (0,\varepsilon) \to \mathbb{R}_+$ regularly varying of index ρ at zero (same condition as above, but for $t \searrow 0$). We refer to Chapter 1 of [BGT] for a collection of basic results. We use the convention that for $a_n, b_n \geq 0$ and $C \in [0,\infty)$,

$$a_n \sim C\cdot b_n \quad \text{as } n\to\infty \quad \text{means} \quad b_n > 0 \text{ for } n \geq n_0 \text{ and } \lim_{n\to\infty}\frac{a_n}{b_n} = C, \tag{2.13}$$

even if $C = 0$ (and analogously for functions and $f(s) \sim C\cdot g(s)$ as $s \searrow 0$ etc).

3. A limit theorem for maps with indifferent fixed points

Piecewise and Markov invertible systems. Turning to a dynamical systems setup encompassing several interesting classes of examples, we consider situations in which (X, d_X) is a metric space with Borel σ-field $\mathcal{A}$, and where X comes with a partition ξ_0 into open *components* (e.g. X may be a union of open intervals). Let λ be a measure on $\mathcal{A}$. A *piecewise invertible system* on X is a quintuple $(X, \mathcal{A}, \lambda, T, \xi)$, where $\xi = \xi_1$ is a (finite or) countable partition mod λ of X into open sets, refining ξ_0, such that each *branch* of T, i.e. its restriction to any of its *cylinders* $Z \in \xi$ is a nonsingular (meaning $\lambda\,|_Z \circ T^{-1} \ll \lambda$) homeomorphism onto TZ. If the measure is T-invariant, we denote it by μ and call $(X, \mathcal{A}, \mu, T, \xi)$ a *measure preserving* system.

The system is *Markov* if $TZ \cap Z' \neq \varnothing$ for $Z, Z' \in \xi$ implies $Z' \subseteq TZ$, and *piecewise onto* if $TZ = X$ mod λ for all $Z \in \xi$. It satisfies the *finite image condition* if $T\xi := \{TZ : Z \in \xi\}$ is a finite collection of sets.

We let ξ_n denote the family of *cylinders of rank* n, that is, the sets of the form $Z = [Z_0, \ldots, Z_{n-1}] := \bigcap_{i=0}^{n-1} T^{-i} Z_i$ with $Z_i \in \xi$. Each *iterate* $(X, \mathcal{A}, \mu, T^n, \xi_n)$, $n \geq 1$, of the system is again piecewise invertible.

Markovian interval maps with indifferent fixed points. A large class of infinite measure preserving dynamical systems is given by transformations possessing neutral orbits. We focus on interval maps, for which a well developed theory is available. The most basic case is that of indifferent fixed points. In [Z1], [Z2] the large class of *AFN-maps* has been introduced and analyzed, generalizing the results of [T1]-[T3]. Here we shall focus on *Markovian AFN-maps* (or, using the terminology of [A0], on $\mathcal{C}^2$ *Markov interval maps satisfying Thaler's assumptions*).

A *piecewise monotonic system* is a piecewise invertible system $(X, \mathcal{A}, \lambda, T, \xi)$, where X is the union of some finite family ξ_0 of disjoint bounded open intervals, ξ is a collection of nonempty pairwise disjoint open subintervals of the $Z \in \xi_0$, and λ is Lebesgue measure. The Markov maps considered here will be $\mathcal{C}^2$ on each $Z \in \xi$ and satisfy the classical version of *Adler's condition*,

$$T''/(T')^2 \text{ is bounded on } \bigcup_{Z \in \xi} Z, \tag{3.1}$$

as well as the *finite image condition*,

$$T\xi = \{TZ : Z \in \xi\} \text{ is finite.} \tag{3.2}$$

Moreover, there is a finite set $\zeta \subseteq \xi$ of cylinders Z having an *indifferent fixed point* x_Z as an endpoint (i.e. $\lim_{x \to x_Z, x \in Z} Tx = x_Z$ and $\lim_{x \to x_Z, x \in Z} T'x = 1$), and each x_Z is a *one-sided regular source*, meaning that

$$\text{for } x \in Z,\ Z \in \zeta, \text{ we have } (x - x_Z)T''(x) \geq 0. \tag{3.3}$$

These maps are *uniformly expanding on sets bounded away from* $\{x_Z : Z \in \zeta\}$, in the sense that letting $X_\varepsilon := X \setminus \bigcup_{Z \in \zeta} ((x_Z - \varepsilon, x_Z + \varepsilon) \cap Z)$ we have

$$|T'| \geq \rho(\varepsilon) > 1 \quad \text{on } X_\varepsilon \qquad \text{for each } \varepsilon > 0. \tag{3.4}$$

Following [Z1], [Z2], we call (X, T, ξ) an *AFN-system* if it satisfies (3.1)-(3.4). It is called an *AFU-system* (uniformly rather than nonuniformly expanding) if $\zeta = \varnothing$, and a *basic AFN-system* in case it is conservative ergodic (with respect to λ) with $\zeta \neq \varnothing$. (See Theorem 1 in [Z1] for finite ergodic decompositions.) In the latter case the system has an (essentially unique) invariant measure $\mu \ll \lambda$ with $\mu(X) = \infty$

whose density $d\mu/d\lambda$ has a version h which has finite regularity on each $Z \in \xi \setminus \zeta$ and admits a representation $h(x) = h_0(x)G(x)$, where

$$G(x) := \begin{cases} \frac{x-x_Z}{x-f_Z(x)} & \text{for } x \in Z \in \zeta \\ 1 & \text{for } x \in X \setminus \bigcup \zeta, \end{cases}$$

and $0 < \inf_X h_0 \leq \sup_X h_0 < \infty$, and h_0 has bounded variation on each X_ε, $\varepsilon > 0$ (see Theorem 1 of [Z1] and Corollary 1 of [Z2]).

If (X, T, ξ) is an AFN-map, natural reference sets Y^N, $N \geq 1$, can be obtained as follows. We let $f_Z := (T\mid_Z)^{-1} : TZ \to Z$, $Z \in \xi$, denote the inverse branches of T. For each neutral cylinder $Z \in \zeta$ the presence of the fixed point $x_Z \in \partial Z$ ensures that $TZ \supseteq Z$, and we let $Z(n)$ be the interior of $f_Z^{n-1}(Z) \setminus f_Z^n(Z)$, $n \geq 1$, and set $Y^N := X \setminus \bigcup_{Z\in\zeta} f_Z^N(Z)$, $N \geq 1$. Then $Y^N \nearrow X$ mod μ, and each Y^N is a union (mod μ) of elements of the refined partition

$$\xi' := (\xi \setminus \zeta) \cup \{Z(n) : Z \in \zeta, n \geq 1\} \tag{3.5}$$

which is obtained from ξ by replacing each $Z \in \zeta$ by the $Z(n)$, $n \geq 1$.

The limit theorem for interval maps. We shall show that shrinking cylinders around typical points of an AFN-map, both for the original partition ξ and for its refined version ξ', exhibit nice return- and hitting time statistics. Again we will in fact prove a d-dimensional version for successive return- and hitting times $\varphi_{E_k} \circ T_{E_k}^i$, which turn out to be asymptotically iid. The proof is given in Section 7 below.

Theorem 3.1 (**Return- and hitting-time limits for maps with neutral fixed points**). *Let $(X, \mathcal{A}, \mu, T, \xi)$ be a Markovian basic AFN-map. Assume that for each $Z \in \zeta$ there are constants $a_Z \neq 0$ and $p_Z \in [1, \infty)$ for which*

$$Tx = x + a_Z\,|x - x_Z|^{1+p_Z} + o\left(|x - x_Z|^{1+p_Z}\right) \qquad \text{as } x \to x_Z \text{ in } Z. \tag{3.6}$$

Let $p := \max\{p_Z : Z \in \zeta\}$, $\alpha := 1/p$, $h_0(Z) := \lim_{x\to x_Z, x\in Z} h_0(x)$,

$$a_T(n) := \left(\Gamma(1+\alpha)\Gamma(2-\alpha) \sum_{Z\in\zeta:p_Z=p} \frac{h_0(Z)}{|a_Z|^\alpha}\right)^{-1} \cdot \begin{cases} n/\log n & \text{if } p = 1, \\ \alpha^{-\alpha}(1-\alpha)\cdot n^\alpha & \text{if } p > 1, \end{cases}$$

and $b_T : [0, \infty) \to [0, \infty)$ an increasing continuous function asymptotically inverse to a_T in that $b_T(a_T(n)) \sim n$ as $n \to \infty$.

Then, for μ-a.e. $x \in X$, the return- and hitting-time distributions of the shrinking cylinders $E_k := \xi_k(x)$, $k \geq 1$, converge: For $d \in \mathbb{N}$, $t_i > 0$, and $k \to \infty$,

$$\mu_{E_k}\left(\bigcap_{i=0}^{d-1}\{b_T(1/\mu(E_k))^{-1} \cdot \varphi_{E_k} \circ T_{E_k}^i \leq t_i\}\right) \longrightarrow \prod_{i=0}^{d-1} \Pr[\mathcal{E}^{\frac{1}{\alpha}}\,\mathcal{G}_\alpha \leq t_i], \tag{3.7}$$

as well as

$$\nu\left(\bigcap_{i=0}^{d-1}\{b_T(1/\mu(E_k))^{-1} \cdot \varphi_{E_k} \circ T_{E_k}^i \leq t_i\}\right) \longrightarrow \prod_{i=0}^{d-1} \Pr[\mathcal{E}^{\frac{1}{\alpha}}\,\mathcal{G}_\alpha \leq t_i], \tag{3.8}$$

where ν is any fixed probability measure with $\nu \ll \mu$ (for example $\nu = \mu_Y$).

The above statements remain true if $E_k = \xi_k(x)$ is replaced by $E'_k := \xi'_k(x)$.

Remark 3.1 (**Changing the measure in the return-time limit**). It is not possible to replace μ_{E_k} in (3.7) by ν_{E_k} for an arbitrary probability measure with $\nu \ll \mu$ (see Example 2.2 of [HWZ]). However, it is easily seen that in the present context any ν with a density $d\nu/d\mu$ which is positive and continuous at x gives the same result. In particular, we can replace μ_{E_k} by λ_{E_k} for a.e. $x \in [0,1]$. This observation justifies (2.9) in Example 2.2.

Remark 3.2 (**Cylinders shrinking to a neutral fixed point**). The indifferent fixed points x_Z, $Z \in \zeta$, themselves are always exceptional. Indeed, as shown in [Z8] (for earlier work in special cases with $\alpha = 1$ see [CGS1], [CG], and [CI]), the hitting-time distributions of the cylinders $\bigcap_{i=0}^{k-1} T^{-i}Z$ around each of them converge to limit laws different from the above as $k \to \infty$.

Remark 3.3 (**Decay rate of the** $\mu(E_k)$). For a better understanding of the normalization in the limit theorem, we mention that a suitable version of the ergodic theorem for the information function has been established in §4 of [T2] and §7 of [Z2]. According to that result,

$$-\log \mu(\xi_k(x))/\mathbf{S}_k(Y) \longrightarrow h_\mu(T)/\mu(Y) \quad \text{a.e.} \quad \text{as } k \to \infty, \tag{3.9}$$

where $h_\mu(T)$ denotes the Krengel entropy of T (assumed finite). When combined with the Darling-Kac limit theorem (see e.g. §5 of [Z2]), this implies distributional convergence

$$\nu\left(-\log \mu(\xi_k(x))/a_T(k) \le t\right) \longrightarrow \Pr[h_\mu(T)\, \mathcal{Y}^{(\alpha)} \le t] \quad \text{as } k \to \infty, \tag{3.10}$$

where ν is any probability with $\nu \ll \mu$, and $\mathcal{Y}^{(\alpha)}$ has a Mittag-Leffler distribution of index α, $\mathbb{E}[\exp(z\mathcal{Y}^{(\alpha)})] = \sum_{m\ge 0}[\Gamma(1+\alpha)\, z]^m/\Gamma(1+m\alpha)$. (An analogous result for Kolmogorov complexity has been given in [Z5].)

Remark 3.4 (**Related pointwise result for** $\varphi_{\xi_k(x)}(x)$). In [GKP] the almost sure growth rate of $\varphi_{\xi_k(x)}(x)$ is shown to satisfy

$$-\log \varphi_{\xi_k(x)}(x)/\mathbf{S}_k(Y) \longrightarrow h_\mu(T)/\alpha\mu(Y) \quad \text{a.e.} \quad \text{as } k \to \infty. \tag{3.11}$$

4. An abstract limit theorem for return-time distributions

In the present section we formulate the abstract core of our results.

Pointwise dual ergodicity and $\mathcal{U}$-uniform sets. The key to the analysis of the stochastic properties of a m.p.t. T on $(X, \mathcal{A}, \mu)$ often lies in the study of the long-term behaviour of its *transfer operator* $\widehat{T} : L_1(\mu) \to L_1(\mu)$, which describes the evolution of measures under the action of T on the level of densities: $\widehat{T}u := d(\nu \circ T^{-1})/d\mu$, where ν has density u with respect to μ. Equivalently, $\int_X u \cdot (v \circ T)\, d\mu = \int_X \widehat{T}u \cdot v\, d\mu$ for all $u \in L_1(\mu)$ and $v \in L_\infty(\mu)$, i.e. $v \longmapsto v \circ T$ is the dual of $\widehat{T}$. The operator $\widehat{T}$ naturally extends to $\{u : X \to [0,\infty)\ \mathcal{A}\text{-measurable}\}$. It is a linear Markov operator, $\int_X \widehat{T}u\, d\mu = \int_X u\, d\mu$ for $u \ge 0$. The m.p.t. T is conservative and ergodic if and only if $\sum_{k\ge 0} \widehat{T}^k u = \infty$ a.e. for all $u \in L_1^+(\mu) := \{u \in L_1(\mu) : u \ge 0$ and $\mu(u) > 0\}$ or (equivalently) all $u \in \mathcal{D}(\mu) := \{u \in L_1(\mu) : u \ge 0,\ \mu(u) = 1\}$. Invariance of μ under T simply means $\widehat{T}1_X = 1_X$.

Let T be a c.e.m.p.t. on the space $(X, \mathcal{A}, \mu)$. Recall that T is said to be *pointwise dual ergodic* (cf. [A0], [A2]) if there is some sequence (a_n) in $(0, \infty)$ such that

$$\frac{1}{a_n} \sum_{k=0}^{n-1} \widehat{T}^k u \longrightarrow \mu(u) \cdot 1_X \qquad \begin{array}{c}\text{a.e. on } X \text{ as } n \to \infty \text{, for every} \\ u \in L_1(\mu) \text{ with } \mu(u) \neq 0.\end{array} \tag{4.1}$$

In this case, (a_n) (which is uniquely determined up to asymptotic equivalence and satisfies $a_n \to \infty$) is called a *return sequence* of T. Without loss of generality we will assume throughout that $a_n = a_T(n)$ for some strictly increasing continuous $a_T : [0, \infty) \to [0, \infty)$ with $a_T(0) = 0$. In case $\mu(X) = \infty$, we always have $a_T(s) = o(s)$ as $s \to \infty$. Letting $b_T : [0, \infty) \to [0, \infty)$ denote the inverse function of a_T, we thus see that

$$s = o(b_T(s)) \quad \text{ as } s \to \infty. \tag{4.2}$$

According to Hurewicz' ratio ergodic theorem (cf. §2.2 of [A0]; also contained in the Chacon-Ornstein theorem), (4.1) is fulfilled as soon as the a.e. convergence there holds for one particular u. By Egorov's theorem, this convergence is then uniform on suitable sets (depending on u) of arbitrarily large measure, but it is sometimes desirable to actually identify particular pairs (u, Y), with $u \in \mathcal{D}(\mu)$ and $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, such that

$$\left\| 1_Y \cdot \left(\frac{1}{a_n} \sum_{k=0}^{n-1} \widehat{T}^k u - 1_X \right) \right\|_\infty \longrightarrow 0 \quad \text{ as } n \to \infty, \tag{4.3}$$

in which case we shall refer to Y as a *u-uniform set* (compare [A0], [T4]). A set Y which is $\mu(Y)^{-1} \cdot 1_Y$-uniform is called a *Darling-Kac set*, cf. [A0], [A3]. Slightly generalizing Proposition 3.7.5 of [A0], one checks that *the existence of a uniform set in fact implies pointwise dual ergodicity* (hence the a_n in (4.3) automatically form a return sequence).

The abstract distributional limit theorem at the heart of the present paper requires a refinement of this concept. We will depend on the observation that there are natural situations in which one can also achieve uniformity in u by restricting to nontrivial collections of functions $\mathcal{U}$.

Definition 4.1. For $\mathcal{U} \subseteq \mathcal{D}(\mu)$ we shall call Y a *$\mathcal{U}$-uniform set* if the $L_\infty(\mu)$-convergence asserted in (4.3) holds uniformly in $u \in \mathcal{U}$, which we may express by stating that

$$\sum_{k=0}^{n-1} \widehat{T}^k u \sim a_n \qquad \begin{array}{c}\text{as } n \to \infty \text{, uniformly mod } \mu \text{ on } Y, \\ \text{and uniformly in } u \in \mathcal{U}.\end{array} \tag{4.4}$$

A method for checking $\mathcal{U}$-uniformity will be discussed in Section 6 below. Given finitely many sets $\mathcal{U}_1, \ldots, \mathcal{U}_L \subseteq \mathcal{D}(\mu)$, it is clear that

$$\text{if } Y \text{ is } \mathcal{U}_l\text{-uniform for } 1 \leq l \leq L, \text{ then } Y \text{ is } (\mathcal{U}_1 \cup \ldots \cup \mathcal{U}_L)\text{-uniform,} \tag{4.5}$$

and it is straightforward to check that

$$\text{if } Y \text{ is } \mathcal{U}\text{-uniform, then } Y \text{ is } \overline{\text{co}}(\mathcal{U})\text{-uniform,} \tag{4.6}$$

where $\overline{\text{co}}(\mathcal{U}) := \{\sum_{m \geq 0} p_m u_m : u_m \in \mathcal{U}, p_m \geq 0, \sum_{m \geq 0} p_m = 1\}$ is the closed convex hull of $\mathcal{U}$. Moreover, if Y is $\mathcal{U}$-uniform, then it is also $\widehat{T}\mathcal{U}$-uniform.

The abstract limit theorem. The notion of $\mathcal{U}$-uniform reference sets Y is the main new ingredient which allows us to formulate an abstract version of a limit theorem for return-time distributions of asymptotically rare events in infinite measure preserving systems.

The result requires the system to be pointwise dual ergodic with regularly varying return sequence. Roughly speaking, each set E_k is supposed to grow to a macroscopic scale within a certain number z_k of steps. This number needs to be small compared to the return time function of E_k. Ideally, E_k could be good after z_k steps in the sense that $\mu(E_k)^{-1} \cdot \widehat{T}^{z_k} 1_{E_k}$ belongs to a nice class $\mathcal{U}$ of densities. In the natural examples we are going to consider, the situation is a bit more complicated, though. Not all of E_k is good after z_k steps, but parts of E_k need a random number Υ_k of extra steps until they fulfil our needs. Since the limit theorem will, in particular, show that φ_{E_k} is of order $b_T(1/\mu(E_k))$, condition (4.10) below ensures that the time delay by Υ_k (which we introduce for technical reasons) has a smaller order of magnitude than the times φ_{E_k} we wish to study.

Theorem 4.1 (Return- and hitting-times for asymptotically rare events). *Let T be a c.e.m.p.t. on $(X, \mathcal{A}, \mu)$, $\mu(X) = \infty$, pointwise dual ergodic with $a_T \in \mathcal{R}_\alpha$ for some $\alpha \in (0, 1]$. Suppose that Y is a $\mathcal{U}$-uniform set for some $\mathcal{U} \subseteq \mathcal{D}(\mu)$, and that $E_k \subseteq Y$, $k \geq 1$, are sets of positive finite measure with $\mu(E_k) \to 0$, and that $z_k \geq 0$ are integers such that*

$$z_k \cdot \mu(E_k) \to 0 \quad \text{as } k \to \infty, \tag{4.7}$$

and

$$\mu_{E_k}(\varphi_{E_k} \leq z_k) \longrightarrow 0 \quad \text{as } k \to \infty. \tag{4.8}$$

Assume, in addition, that $\widehat{T}^{z_k}(1_{E_k}/\mu(E_k)) = \sum_{\iota \geq 0} \pi_{k,\iota}\, w_{k,\iota}$ with densities $w_{k,\iota} \in \mathcal{D}(\mu)$ satisfying

$$1_Y\, \widehat{T}^j w_{k,\iota} = 0 \text{ for } 1 \leq j < \iota, \quad \text{while} \quad \widehat{T}^\iota w_{k,\iota} \in \mathcal{U}, \tag{4.9}$$

and weights $\pi_{k,\iota} \geq 0$ such that any random variables Υ_k with $\Pr[\Upsilon_k = \iota] = \pi_{k,\iota}$ satisfy

$$b_T(1/\mu(E_k))^{-1} \cdot \Upsilon_k \xrightarrow{\Pr} 0 \quad \text{as } k \to \infty. \tag{4.10}$$

Then the return-time distributions of the E_k converge in that for any $t > 0$,

$$\mu_{E_k}(b_T(1/\mu(E_k))^{-1} \cdot \varphi_{E_k} \leq t) \longrightarrow \Pr[\mathcal{E}^{\frac{1}{\alpha}}\, \mathcal{G}_\alpha \leq t] \quad \text{as } k \to \infty. \tag{4.11}$$

Moreover, the hitting-time distributions converge as well, and for any $t > 0$,

$$\mu_Y(b_T(1/\mu(E_k))^{-1} \cdot \varphi_{E_k} \leq t) \longrightarrow \Pr[\mathcal{E}^{\frac{1}{\alpha}}\, \mathcal{G}_\alpha \leq t] \quad \text{as } k \to \infty. \tag{4.12}$$

In fact, in (4.12), μ_Y can be replaced by any probability measure $\nu \ll \mu$.

Remark 4.1. Writing $F_k(v) := \sum_{i>v} \pi_{k,\iota}$, condition (4.10) becomes

$$F_k\,(\varepsilon\, b_T(1/\mu(E_k))) \longrightarrow 0 \quad \text{as } k \to \infty \quad \text{for all } \varepsilon > 0. \tag{4.13}$$

Example 4.1 (Continuation of Example 2.1). *We claim that our introductory Markov chain example satisfies the assumptions of the theorem. Assume the chain is given by its canonical shift-space representation, i.e. $\Omega = X := (\mathbb{N}_0 \times [0, 1])^{\mathbb{N}_0}$ with product σ-field $\mathcal{A}$, and X_n^* the projection $((k_j, y_j)_{j \geq 0}) \mapsto (k_n, y_n)$, so that*

$\mathsf{X}_n^* = \mathsf{X}_0^* \circ T^n$ *where* T *denotes the shift on* X*. For* $E \subseteq \mathbb{N}_0 \times [0,1]$ *we let* $[E] := E \times (\mathbb{N}_0 \times [0,1])^{\mathbb{N}} = \{\mathsf{X}_0^* \in E\}$ *denote the corresponding cylinder set. The infinite invariant distribution* (r_k) *of the renewal chain inside gives an infinite invariant measure* μ *for* T *with* $\mu \circ (\mathsf{X}_0^*)^{-1} = (\sum_k r_k \delta_k) \otimes \lambda$*. A routine argument shows that* $(X, \mathcal{A}, \mu, T)$ *is conservative ergodic. The probability describing our original chain which starts at* $\mathsf{X}_0 = 0$ *is just the restricted measure* $\mathsf{P} = \mu\mid_Y = \mu_Y$*, where* $Y := [\{0\} \times [0,1]] \subseteq X$*. For the transfer operator* $\widehat{T}$ *we find that* Y *is a* $\mathcal{U}$*-uniform set for* $\mathcal{U} := \{\widehat{T}(\lambda(B)^{-1} 1_{\{0\}\times B}) : B \in \mathcal{B}_{[0,1]} \text{ with } \lambda(B) > 0\}$*. In particular,* T *is pointwise dual ergodic, and standard results about the renewal chain enable us to conclude that* $a_T \in \mathcal{R}_\alpha$*.*

Now fix any sequence $\epsilon_k \searrow 0$ *in* $[0,\infty]$*, and let* $E_k := \{(k_j, y_j)_{j\geq 0}\colon k_0 = 0 \text{ and } y_0 \in [0,\epsilon_k]\} \in \mathcal{A}$*. Then* $\mu(E_k) = \epsilon_k$ *and* $\varphi_{\epsilon_k} = \varphi_{E_k}$ *for* $k \geq 1$*. We see that our conditions are fulfilled with* $z_k := 1$ *and* $\rho_{k,0} := 1$*, because* $\in \mathcal{U}$*. Conditions (4.7) and (4.8) are trivial for a constant sequence* (z_k)*. Therefore the theorem applies to reproduce, via (4.12), the conclusion of Example 2.1.*

The strategy. In part, the strategy of our proof of the Theorem is similar to that of [PS1] and [PSZ]. However, *we replace the refined conditional local limit theorems used there by exploiting the concept of* $\mathcal{U}$*-uniform sets introduced above.*

The distributional convergence statements can be reformulated as follows.

Lemma 4.1 (**Equivalent formulation of the results**). *Under the assumptions of Theorem 4.1, assertion (4.11) is equivalent to*

$$\mu_{E_k}(\mu(E_k)\, a_T(\varphi_{E_k}) \leq t) \longrightarrow \Pr[\mathcal{E}\, \mathcal{G}_\alpha^\alpha \leq t] \quad \text{as } k \to \infty, \tag{4.14}$$

while (4.12) is equivalent to

$$\mu_Y(\mu(E_k)\, a_T(\varphi_{E_k}) \leq t) \longrightarrow \Pr[\mathcal{E}\, \mathcal{G}_\alpha^\alpha \leq t] \quad \text{as } k \to \infty. \tag{4.15}$$

Proof. This is an immediate consequence of regular variation. □

Let $\mathcal{F} := \{F : [0,\infty) \to [0,1]$, non-decreasing and right-continuous$\}$ be the set of sub-probability distribution functions on $[0,\infty)$. For F, $F_n \in \mathcal{F}$ $(n \geq 1)$ we write $F_n \Rightarrow F$ for *vague convergence*, i.e. $F_n(t) \to F(t)$ at all continuity points of F. (If, in this case, $\sup F(t) = 1$, then this is the usual *weak convergence* of probability distribution functions.)

To prove the theorem, we are going to establish (4.14) and (4.15). Denote the relevant variables by R_k, and their distribution functions with respect to the respective measures by $\widetilde{F}_k$ and F_k, that is, we define

$$\mathsf{R}_k := \mu(E_k)\, a_T(\varphi_{E_k}), \quad \text{for } k \geq 1, \tag{4.16}$$

and

$$\widetilde{F}_k(t) := \mu_{E_k}(\mathsf{R}_k \leq t), \quad F_k(t) := \mu_Y(\mathsf{R}_k \leq t) \quad \text{for } k \geq 1,\ t \in [0,\infty). \tag{4.17}$$

The proof of the theorem consists of two main steps, summarized in the following propositions. We first prove that any weak limit points $\widetilde{F}, F \in \mathcal{F}$ of the $\widetilde{F}_k$ or F_k necessarily satisfy a certain functional equation.

Proposition 4.1 (**Functional equation satisfied by limit laws**). *Under the assumptions of Theorem 4.1, suppose that $\widetilde{F}_{k(h)} \Rightarrow \widetilde{F} \in \mathcal{F}$ along some subsequence $k(h) \nearrow \infty$. Then $\widetilde{F}$ satisfies*

$$\widetilde{F}(t) = \alpha t \int_0^1 [1 - \widetilde{F}(t(1-s)^\alpha)] \cdot s^{\alpha-1}\, ds \quad \text{for } t \in [0,\infty). \tag{4.18}$$

Likewise, if $F_{k(h)} \Rightarrow F \in \mathcal{F}$ along some subsequence $k(h) \nearrow \infty$, then F satisfies the same functional equation.

It then remains to check that there is only one $\widetilde{F} \in \mathcal{F}$ with this property, and that it corresponds to the asserted limit law.

Proposition 4.2 (**Uniqueness of limit laws**). *For every $\alpha \in (0,1]$, there is at most one function $\widetilde{F} \in \mathcal{F}$ satisfying (4.18).*

(It is easy to check that for $\alpha = 1$ the exponential law satisfies (4.18).)

The convergence theorem then follows easily:

Proof of Theorem 4.1. **(i)** Lemma 4.1 shows that (4.14) and (4.11) are equivalent, and so are (4.15) and (4.12). By the classical Helly selection theorem, any subsequence of indices k contains a further subsequence $k(h) \nearrow \infty$ such that $\widetilde{F}_{k(h)} \Rightarrow \widetilde{F}$ for some $\widetilde{F} \in \mathcal{F}$ as $h \to \infty$. According to Propositions 4.1 and 4.2 this limit point $\widetilde{F}$ is the unique function in $\mathcal{F}$ satisfying (4.18). Due to this uniqueness, we must have $\widetilde{F}_k \Rightarrow \widetilde{F}$ for the full sequence. The same argument proves $F_k \Rightarrow \widetilde{F}$. It remains to identify $\widetilde{F}$.

Since Example 2.1 satisfies our assumptions (see Example 4.1), we conclude that $\widetilde{F}$ is indeed the distribution function of the non-degenerate variable $\mathcal{E}\,\mathcal{G}_\alpha^\alpha$.

The final statement of the theorem, which extends (4.12) to all $\nu \ll \mu$ means that the variables $b_T(1/\mu(E_k))^{-1} \cdot \varphi_{E_k}$ exhibit *strong distributional convergence* in the sense of [A0]. But Corollary 5 in [Z7] guarantees that, for hitting-times, this is an automatic consequence of ordinary distributional convergence. □

Remark 4.2. Alternatively, it is possible to check directly that (4.18) characterizes the distribution function of our limit law, see [PSZ].

5. Proof of Propositions 4.1 and 4.2

We now turn to the proofs of the propositions. As a warm-up, we take a look at the functional equation (4.18). Proposition 4.2 is immediate from the following

Lemma 5.1 (**The functional equation**). *For each $\alpha \in (0,1]$,*

$$(\Phi_\alpha G)(t) := \alpha t \int_0^1 [1 - G(t(1-s)^\alpha)] \cdot s^{\alpha-1}\, ds \quad \text{for } t \in [0,\infty) \tag{5.1}$$

defines a map $\Phi_\alpha : \mathcal{F} \to \mathcal{C}[0,\infty)$, and at most one $F \in \mathcal{F}$ satisfies $\Phi_\alpha F = F$.

Proof. **(i)** To check continuity of $\Phi_\alpha G$, fix any $t_0 \in [0,\infty)$ and any sequence $t_n \to t_0$. We need to prove that $\int_0^1 g_n(s)\, ds \to \int_0^1 g_0(s)\, ds$ as $n \to \infty$, where $g_n(s) := [1 - G(t_n(1-s)^\alpha)] \cdot s^{\alpha-1}$, $n \geq 0$. Since G is non-decreasing, $t_0(1-s)^\alpha$

is a continuity point of G for almost every s. Therefore, $g_n \to g_0$ a.e. on $[0,1]$. On the other hand, for every n, $|g_n(s)| \leq s^{\alpha-1}$, and as the latter function is integrable on $[0,1]$, our claim follows via dominated convergence.

(ii) To prove uniqueness, assume that $F, G \in \mathcal{F}$ satisfy $\Phi_\alpha F = F$ and $\Phi_\alpha G = G$. By straightforward calculation,

$$\sup_{r\in[0,t]} |\Phi_\alpha F(r) - \Phi_\alpha G(r)| \leq t \cdot \sup_{r\in[0,t]} |F(r) - G(r)| \quad \text{for } t \in [0,\infty),$$

which for our special F and G immediately implies that

$$F = G \text{ on } [0,1). \tag{5.2}$$

We now extend this to all of $[0,\infty)$. We claim that for $y > 0$,

$$F = G \text{ on } [0,y) \quad \text{implies} \quad F = G \text{ on } [0,S(y)), \tag{5.3}$$

where $S : [0,\infty) \to [0,\infty)$ is given by $S(y) := (1 + y^{1/\alpha})^\alpha$. To see this, consider $t \geq y$. By assumption, $F(t(1-s)^\alpha) = G(t(1-s)^\alpha)$ for $s > 1 - (y/t)^{1/\alpha}$, so that

$$\begin{aligned} |\Phi_\alpha F(t) - \Phi_\alpha G(t)| &\leq \alpha t \int_0^{1-(y/t)^{1/\alpha}} |F(t(1-s)^\alpha) - G(t(1-s)^\alpha)| \cdot s^{\alpha-1}\, ds \\ &\leq \rho_y(t) \cdot \sup_{r\in[0,t]} |F(r) - G(r)|. \end{aligned}$$

with $\rho_y(t) := \left(t^{1/\alpha} - y^{1/\alpha}\right)^\alpha$, $t \geq y$. As $\rho_y(t)$ is increasing, we actually have

$$\sup_{r\in[0,t]} |\Phi_\alpha F(r) - \Phi_\alpha G(r)| \leq \rho_y(t) \cdot \sup_{r\in[0,t]} |F(r) - G(r)|. \tag{5.4}$$

But since $\rho_y(t) < 1$ if and only if $t < S(y)$, this proves (5.3).

An induction based on (5.2) and (5.3) then shows that $F = G$ on $[0, S^m(1))$ for all $m \geq 1$. However, S is continuous with $S(y) > y$ for $y > 0$, so that $S^m(1) \to \infty$ as $m \to \infty$. Therefore $F = G$ on $[0,\infty)$, as required. □

The main issue is the proof of Proposition 4.1. We first record an observation about $\mathcal{U}$-uniform sets Y: if $a_T \in \mathcal{R}_\alpha$ for some $\alpha \in (0,1]$, and $0 \leq c_1 < c_2$, then (4.4) is easily seen to entail

$$\sum_{j=c_1 n}^{c_2 n - 1} \widehat{T}^j u \sim (c_2^\alpha - c_1^\alpha) \cdot a_n \qquad \begin{array}{c} \text{as } n \to \infty, \text{ uniformly mod } \mu \text{ on } Y, \\ \text{and uniformly in } u \in \mathcal{U}. \end{array} \tag{5.5}$$

Below we shall need the following refined version for densities made up from bits which eventually belong to $\mathcal{U}$.

Lemma 5.2 (**Dual ergodic sums for eventually good densities**). *Let T be a c.e.m.p.t. on $(X, \mathcal{A}, \mu)$, pointwise dual ergodic with $a_T \in \mathcal{R}_\alpha$ for some $\alpha \in (0,1]$, and suppose that Y is a $\mathcal{U}$-uniform set. Let $w_k \in \mathcal{D}(\mu)$, $k \geq 1$, be convex combinations $w_k = \sum_{\iota \geq 0} \pi_{k,\iota}\, w_{k,\iota}$, $\pi_{k,\iota} \geq 0$, of densities $w_{k,\iota} \in \mathcal{D}(\mu)$ satisfying*

$$1_Y\, \widehat{T}^j w_{k,\iota} = 0 \text{ for } j < \iota, \quad \text{while} \quad \widehat{T}^\iota w_{k,\iota} \in \mathcal{U}. \tag{5.6}$$

Then, for any $0 \leq c_1 < c_2$, and any $\overline{n}_k \nearrow \infty$ such that

$$\overline{n}_k^{-1} \cdot \Upsilon_k \xrightarrow{\text{Pr}} 0 \quad \text{as } k \to \infty, \tag{5.7}$$

where the Υ_k are random variables with $\Pr[\Upsilon_k = \iota] = \pi_{k,\iota}$*, we have*

$$\sum_{j=c_1\overline{n}_k+1}^{c_2\overline{n}_k} \widehat{T}^j w_k \sim (c_2^\alpha - c_1^\alpha)\cdot a_{\overline{n}_k} \qquad \text{as } k\to\infty, \text{ uniformly mod } \mu \text{ on } Y. \tag{5.8}$$

Proof. **(i)** Obviously, it suffices to consider the case $(c_1, c_2) = (0, 1)$. Since

$$\begin{aligned} 1_Y \sum_{j=1}^{\overline{n}_k} \widehat{T}^j w_k &= \sum_{\iota\geq 0}\pi_{k,\iota}\sum_{i=0}^{\overline{n}_k-\iota} 1_Y\, \widehat{T}^i\left(\widehat{T}^\iota w_{k,\iota}\right) \\ &\leq 1_Y \sum_{i=0}^{\overline{n}_k-1}\widehat{T}^i\left(\sum_{\iota\geq 0}\pi_{k,\iota}\,\widehat{T}^\iota w_{k,\iota}\right) = 1_Y\sum_{i=0}^{\overline{n}_k-1}\widehat{T}^i W_k \end{aligned} \tag{5.9}$$

with $W_k \in \overline{\text{co}}(\mathcal{U})$, the earlier observation (4.6) shows that

$$\overline{\lim_{k\to\infty}}\,\frac{1}{a_{\overline{n}_k}}\sum_{j=1}^{\overline{n}_k}\widehat{T}^j w_k \leq 1 \quad \text{uniformly mod } \mu \text{ on } Y.$$

(ii) To prove a corresponding lower bound, fix $\varepsilon \in (0, 1/2)$. Observe that $\sum_{\iota=0}^{\lfloor\varepsilon\overline{n}_k\rfloor}\pi_{k,\iota} = \Pr[\overline{n}_k^{-1}\,\Upsilon_k \leq \varepsilon]$. Hence, (5.7) provides us with some $K \geq 1$ such that

$$\sum_{\iota=0}^{\lfloor\varepsilon\overline{n}_k\rfloor}\pi_{k,\iota} > 1-\varepsilon \quad \text{for } k\geq K.$$

Via the uniform convergence theorem for regularly varying functions (Theorem 1.5.2 of [BGT]), applied to a_T, there is some $K' \geq K$ such that $a_{\overline{n}_k-\iota} \geq (1-2\varepsilon)^\alpha a_{\overline{n}_k}$ for every $k \geq K'$ and every $\iota \in \{1, \ldots, \lfloor\varepsilon\overline{n}_k\rfloor\}$. Hence,

$$\sum_{\iota=0}^{\lfloor\varepsilon\overline{n}_k\rfloor}\pi_{k,\iota}\,\frac{a_{\overline{n}_k-\iota}}{a_{\overline{n}_k}} \geq (1-2\varepsilon)^\alpha\sum_{\iota=0}^{\lfloor\varepsilon\overline{n}_k\rfloor}\pi_{k,\iota} > (1-2\varepsilon)^{1+\alpha} \quad \text{for } k\geq K'. \tag{5.10}$$

As Y is a $\mathcal{U}$-uniform set, there is some $I = I(\varepsilon)$ such that

$$\left\|1_Y\cdot\left(\frac{1}{a_{\overline{n}_k-\iota}}\sum_{i=0}^{\overline{n}_k-\iota}\widehat{T}^i\left(\widehat{T}^\iota w_{k,\iota}\right) - 1_X\right)\right\|_\infty < \varepsilon \quad \text{if } \overline{n}_k-\iota\geq I. \tag{5.11}$$

Choose $K'' \geq K'$ so large that $(1-\varepsilon)\overline{n}_k \geq I$ for $k \geq K''$. Starting from the first identity of (5.9) we thus see, using (5.11) and (5.10), that (mod μ)

$$\begin{aligned} 1_Y\cdot\frac{1}{a_{\overline{n}_k}}\sum_{j=1}^{\overline{n}_k}\widehat{T}^j w_k &\geq 1_Y\cdot\sum_{\iota=0}^{\lfloor\varepsilon\overline{n}_k\rfloor}\pi_{k,\iota}\frac{a_{\overline{n}_k-\iota}}{a_{\overline{n}_k}}\,\frac{1}{a_{\overline{n}_k-\iota}}\sum_{i=0}^{\overline{n}_k-\iota}\widehat{T}^i\left(\widehat{T}^\iota w_{k,\iota}\right) \\ &> \left(\sum_{\iota=0}^{\lfloor\varepsilon\overline{n}_k\rfloor}\pi_{k,\iota}\frac{a_{\overline{n}_k-\iota}}{a_{\overline{n}_k}}\right)(1-\varepsilon)\cdot 1_Y \\ &> (1-2\varepsilon)^{1+\alpha}(1-\varepsilon)\cdot 1_Y \quad \text{for } k\geq K''. \end{aligned}$$

As ε was arbitrary, we conclude that indeed

$$\underline{\lim}_{k\to\infty}\,\frac{1}{a_{\overline{n}_k}}\sum_{j=1}^{\overline{n}_k}\widehat{T}^j w_k \geq 1 \quad \text{uniformly mod } \mu \text{ on } Y,$$

thus completing the proof of the lemma. □

The previous Lemma will enable us to exploit the following decomposition which generalizes the Ansatz of [DE].

Lemma 5.3 (**Decomposing according to the last visit before time** n). *Let T be a c.e.m.p.t. on $(X, \mathcal{A}, \mu)$, and $A, B \in \mathcal{A}$. Then*

$$\mu(A) = \mu(A \cap \{\varphi_B > n\}) + \sum_{l=1}^{n} \int_{B \cap \{\varphi_B > n-l\}} \widehat{T}^l 1_A \, d\mu \quad \textit{for } n \geq 0. \tag{5.12}$$

Proof. Fix any integer $n \geq 0$, and decompose A according to the last instant $l \in \{1, \ldots, n\}$ (if any) at which an orbit visits B, to obtain (mod μ)

$$A = (A \cap \{\varphi_B > n\}) \cup \bigcup_{l=1}^{n} (A \cap T^{-l}(B \cap \{\varphi_B > n-l\})) \quad \text{(disjoint)}. \tag{5.13}$$

Applying μ and using duality then gives (5.12). □

We are going to use this decomposition, with B one of the E_k, with either $A = B$ or $A = Y$, and with n chosen as follows. For $t \in [0, \infty)$ and $k \geq 1$ we define (where again b_T is inverse to a_T, and R_k is given by (4.16))

$$n_k^{[t]} := b_T\left(\frac{t}{\mu(E_k)}\right), \quad \text{so that} \quad \{\varphi_{E_k} > n_k^{[t]}\} = \{\mathsf{R}_k > t\}. \tag{5.14}$$

It will also be convenient to denote, for $t \in [0, \infty)$, $k \geq 1$, and $i \in \{0, \ldots, n_k^{[t]}\}$,

$$\vartheta_{k,i}^{[t]} := \mu(E_k) \cdot a_T(n_k^{[t]} - i), \quad \text{so that} \quad \{\varphi_{E_k} > n_k^{[t]} - i\} = \{\mathsf{R}_k > \vartheta_{k,i}^{[t]}\}. \tag{5.15}$$

Observe that, for fixed t and k, $i \mapsto \vartheta_{k,i}^{[t]}$ is non-increasing. Moreover, given $t > 0$ and $\rho \in [0, 1)$, if $(i_k)_{k \geq 1}$ is any sequence with

$$i_k \sim \rho \cdot n_k^{[t]}, \quad \text{then} \quad \vartheta_{k,i_k}^{[t]} \sim t \cdot (1-\rho)^{\alpha} \quad \text{as } k \to \infty, \tag{5.16}$$

since, due to $a_T \in \mathcal{R}_\alpha$ (specifically, the Uniform Convergence Theorem, cf. Theorem 1.5.2 of [BGT]), $\vartheta_{k,i_k}^{[t]} = \mu(E_k) \cdot a_T(n_k^{[t]}(1 - i_k/n_k^{[t]})) \sim \mu(E_k) \cdot (1 - i_k/n_k^{[t]})^{\alpha} \, a_T(n_k^{[t]})$.

With the aid of the $\vartheta_{k,i}^{[t]}$ we can now formulate a key step of our proof.

Lemma 5.4 (**Dual ergodic sums on sets which return late**). *Let $(X, \mathcal{A}, \mu, T)$, a_T, Y, $(w_k)_{k \geq 1}$, and Υ_k be as in Lemma 5.2. Let $E_k \subseteq Y$, $k \geq 1$, be sets of positive finite measure.*

Fix some $t > 0$, and abbreviate $n_k := n_k^{[t]}$ and $\vartheta_{k,i} := \vartheta_{k,i}^{[t]}$, defined as in (5.14) and (5.15). Assume that $n_k^{-1} \Upsilon_k \xrightarrow{\Pr} 0$, and let $z_k \geq 0$ be integers with $z_k = o(n_k)$.

Then, for any integer $M \geq 1$, and any $\varepsilon > 0$, there is some $K(M,\varepsilon)$ such that for all $k \geq K(M,\varepsilon)$,

$$\begin{aligned} &e^{-\varepsilon}\alpha t \sum_{m=1}^{M-1} \mu_{E_k}\left(\mathsf{R}_k > \vartheta_{k,z_k+\lfloor \frac{m}{M}(n_k-z_k)\rfloor}\right)\left(\frac{m+1}{M}\right)^{\alpha-1}\frac{1}{M} \\ \leq\ & \sum_{j=1}^{n_k-z_k} \int_{E_k\cap\{\varphi_{E_k}>n_k-z_k-j\}} \widehat{T}^j w_k\,d\mu \\ \leq\ & \frac{e^{\varepsilon}t}{M^{\alpha}} + e^{\varepsilon}\alpha t \sum_{m=1}^{M-1} \mu_{E_k}\left(\mathsf{R}_k > \vartheta_{k,z_k+\lfloor \frac{m+1}{M}(n_k-z_k)\rfloor}\right)\left(\frac{m}{M}\right)^{\alpha-1}\frac{1}{M}. \end{aligned} \tag{5.17}$$

Proof. We focus on the estimate from above, the estimate from below can be verified by an analogous argument. Writing $\overline{n}_k := n_k - z_k$, using the $\vartheta_{k,i}$, and decomposing the sum into M sections, we can rewrite the expression we are interested in as

$$\sum_{j=1}^{\overline{n}_k} \int_{E_k\cap\{\varphi_{E_k}>n_k-z_k-j\}} \widehat{T}^j w_k\,d\mu = \sum_{m=0}^{M-1} \sum_{j=\lfloor \frac{m}{M}\overline{n}_k\rfloor+1}^{\lfloor \frac{m+1}{M}\overline{n}_k\rfloor} \int_{E_k\cap\{\mathsf{R}_k>\vartheta_{k,z_k+j}\}} \widehat{T}^j w_k\,d\mu.$$

As $i \mapsto \vartheta_{k,i}$ is non-increasing, we have $\{\mathsf{R}_k > \vartheta_{k,z_k+j}\} \subseteq \{\mathsf{R}_k > \vartheta_{k,z_k+\lfloor (m+1)\overline{n}_k/M\rfloor}\}$ for $j \leq (m+1)\overline{n}_k/M$, and hence find that

$$\sum_{j=1}^{\overline{n}_k} \int_{E_k\cap\{\varphi_{E_k}>n_k-z_k-j\}} \widehat{T}^j w_k\,d\mu \leq \sum_{m=0}^{M-1} \int_{E_k\cap\{\mathsf{R}_k>\vartheta_{k,z_k+\lfloor \frac{m+1}{M}\overline{n}_k\rfloor}\}} \sum_{j=\lfloor \frac{m}{M}\overline{n}_k\rfloor+1}^{\lfloor \frac{m+1}{M}\overline{n}_k\rfloor} \widehat{T}^j w_k\,d\mu.$$

Since $a_T \in \mathcal{R}_\alpha$ and $\overline{n}_k \sim n_k \to \infty$ as $k \to \infty$ (recall $z_k = o(n_k)$), and since, by assumption, Y is a $\mathcal{U}$-uniform set, Lemma 5.2 ensures that for $m \geq 0$,

$$\sum_{j=\lfloor \frac{m}{M}\overline{n}_k\rfloor+1}^{\lfloor \frac{m+1}{M}\overline{n}_k\rfloor} \widehat{T}^j w_k \sim \left(\left(\frac{m+1}{M}\right)^{\alpha} - \left(\frac{m}{M}\right)^{\alpha}\right)\cdot a_T(n_k) \qquad \begin{array}{c}\text{as } k\to\infty,\\ \text{uniformly mod } \mu \text{ on } Y.\end{array}$$

Recalling $a_T(n_k) = t/\mu(E_k)$ and using, for $m > 0$, that $(\frac{m+1}{M})^{\alpha} - (\frac{m}{M})^{\alpha} \leq \alpha \frac{1}{M}(\frac{m}{M})^{\alpha-1}$ by the mean-value theorem, we thus get, for any $\varepsilon > 0$, some $K(M,\varepsilon) \geq 1$ such that (if we isolate the $m = 0$ term)

$$\begin{aligned} &\sum_{j=1}^{\overline{n}_k} \int_{E_k\cap\{\varphi_{E_k}>n_k-z_k-j\}} \widehat{T}^j w_k\,d\mu \\ \leq\ & \frac{e^{\varepsilon}t}{M^{\alpha}}\,\mu_{E_k}\left(\mathsf{R}_k > \vartheta_{k,z_k+\lfloor \frac{1}{M}\overline{n}_k\rfloor}\right) \\ &+e^{\varepsilon}\alpha\, t \sum_{m=1}^{M-1} \mu_{E_k}\left(\mathsf{R}_k > \vartheta_{k,z_k+\lfloor \frac{m+1}{M}\overline{n}_k\rfloor}\right)\left(\frac{m}{M}\right)^{\alpha-1}\frac{1}{M} \end{aligned}$$

for $k \geq K(M,\varepsilon)$, and hence the asserted upper bound. □

We are now ready for the

Proof of Proposition 4.1. **(i)** We can without loss of generality assume that $\widetilde{F}_k \Rightarrow \widetilde{F} \in \mathcal{F}$ along the full sequence. Fix any continuity point $t \in (0,\infty)$ of $\widetilde{F}$ with the property that for all integers $0 \leq m \leq M$, the $t(1-\frac{m}{M})^{\alpha}$ also are continuity points of $\widetilde{F}$. As the right-hand expression in (4.18) is continuous (cf. Lemma 5.1), and $\widetilde{F}$ is non-decreasing, it suffices to prove (4.18) for such t (only a countable set of points t is discarded). Again we abbreviate $n_k := n_k^{[t]}$ and $\vartheta_{k,i} := \vartheta_{k,i}^{[t]}$.

Lemma 5.3, for $A := B := E_k$, and $n := n_k$, gives

$$1 - \mu_{E_k}(\varphi_{E_k} > n_k) = \sum_{l=1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^l 1_{E_k} \, d\mu_{E_k}, \tag{5.18}$$

with left-hand side satisfying

$$1 - \mu_{E_k}(\varphi_{E_k} > n_k) = \widetilde{F}_k(t) \longrightarrow \widetilde{F}(t) \quad \text{as } k \to \infty,$$

by our assumptions. We thus need to understand the asymptotics of the right-hand side of (5.18). Observe that, for any $t > 0$,

$$z_k = o(n_k^{[t]}) \quad \text{as } k \to \infty, \tag{5.19}$$

which follows from (4.2) and (4.7) since $b_T \in \mathcal{R}_{1/\alpha}$, $\alpha \in (0,1]$. We now split the sum in (5.18) as $\sum_{l=1}^{n_k} = \sum_{l=1}^{z_k} + \sum_{l=z_k+1}^{n_k}$, and observe that the first part is asymptotically negligible, as by (5.12),

$$\begin{aligned} \sum_{l=1}^{z_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^l 1_{E_k} \, d\mu_{E_k} &\leq \sum_{l=1}^{z_k} \int_{E_k \cap \{\varphi_{E_k} > z_k - l\}} \widehat{T}^l 1_{E_k} \, d\mu_{E_k} \\ &= \mu_{E_k}(\varphi_{E_k} \leq z_k) \longrightarrow 0 \quad \text{as } k \to \infty \end{aligned}$$

by assumption (4.8). Therefore it suffices to prove, for $k \to \infty$, that

$$\sum_{l=z_k+1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^l 1_{E_k} \, d\mu_{E_k} \longrightarrow \alpha t \int_0^1 \left[1 - \widetilde{F}\left(t\,(1-s)^\alpha\right)\right] \cdot s^{\alpha-1} \, ds. \tag{5.20}$$

(ii) Observe that letting $w_k := \widehat{T}^{z_k}(1_{E_k}/\mu(E_k))$, we have

$$\sum_{l=z_k+1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^l 1_{E_k} \, d\mu_{E_k} = \sum_{j=1}^{n_k - z_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - z_k - j\}} \widehat{T}^j w_k \, d\mu.$$

By assumption (4.9), the w_k satisfy the assumptions of Lemmas 5.2 and 5.4. Moreover, since $b_T \in \mathcal{R}_{1/\alpha}$, we have $n_k \sim t^{1/\alpha} b_T(1/\mu(E_k))$, and condition (4.10) entails $n_k^{-1} \Upsilon_k \xrightarrow{\Pr} 0$ as $k \to \infty$. Therefore Lemma 5.4 applies.

Fix some $M \geq 1$, take any $\varepsilon \in (0,1)$, and consider the asymptotics of the upper bound given in (5.17). According to (5.16) we have (recalling (5.19)), for any $m \in \{0, \dots, M-2\}$,

$$\vartheta_{k, z_k + \lfloor \frac{m+1}{M}(n_k - z_k) \rfloor} \longrightarrow t \left(1 - \frac{m+1}{M}\right)^\alpha \quad \text{as } k \to \infty,$$

while for $m = M-1$ we have $\vartheta_{k, z_k + \lfloor \frac{m+1}{M}(n_k - z_k) \rfloor} = 0 = t\,\left(1 - \frac{m+1}{M}\right)^\alpha$ anyway. As, by our choice of t, $\widetilde{F}$ is continuous at these limit points, we infer that

$$\mu_{E_k}\left(\mathsf{R}_k > \vartheta_{k, z_k + \lfloor \frac{m+1}{M}(n_k - z_k) \rfloor}\right) \longrightarrow 1 - \widetilde{F}\left(t \left(1 - \frac{m+1}{M}\right)^\alpha\right) \tag{5.21}$$

as $k \to \infty$ whenever $m \in \{0, \ldots, M-1\}$. Combining this with the upper estimate from Lemma 5.4, we conclude (since ε was arbitrary) that

$$\begin{aligned}(5.22)\qquad &\overline{\lim_{k\to\infty}} \sum_{l=z_k+1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^l 1_{E_k}\, d\mu \\ &\leq \frac{t}{M^\alpha} + \alpha t \sum_{m=1}^{M-1} \left[1 - \widetilde{F}\left(t\left(1 - \frac{m+1}{M}\right)^\alpha\right)\right] \left(\frac{m}{M}\right)^{\alpha-1} \frac{1}{M}.\end{aligned}$$

(iii) Now $M \geq 1$ in (5.22) was arbitrary, and $t/M^\alpha \to 0$ as $M \to \infty$. On the other hand, the sum on the right-hand side is almost a Riemann sum for $\int_0^1 [1 - \widetilde{F}(t\,(1-s)^\alpha)] \cdot s^{\alpha-1}\, ds$. It is not hard to check that (although the integrand is unbounded) it really converges to this integral as $M \to \infty$:

Just decompose, for any $\delta \in (0,1)$, $\sum_{m=1}^{M-1} = \sum_{m=1}^{\lfloor \delta M \rfloor - 1} + \sum_{m=\lfloor \delta M \rfloor}^{M-1}$, and note that $0 \leq \sum_{m=1}^{\lfloor \delta M \rfloor - 1} \ldots \leq \sum_{m=1}^{\lfloor \delta M \rfloor - 1} (\frac{m}{M})^{\alpha-1} \frac{1}{M} \leq \delta^\alpha/\alpha$, while $\sum_{m=\lfloor \delta M \rfloor}^{M-1} \ldots$, up to some multiplicative constant $\kappa_M \in (\frac{\delta M - 1}{\delta M + 1}, 1)$, is a proper Riemann sum for $\int_\delta^1 [1 - \widetilde{F}(t\,(1-s)^\alpha)] \cdot s^{\alpha-1}\, ds$, and hence converges to the latter as $M \to \infty$. Letting $\delta \searrow 0$ establishes our claim.

This proves the upper half of (5.20). The lower half follows analogously.

(iv) We turn to our assertion on limit points of $(F_k)_{k\geq 1}$. Suppose without loss of generality that $F_k \Rightarrow F \in \mathcal{F}$. By passing to a subsequence, if necessary, we may assume that also $\widetilde{F}_k \Rightarrow \widetilde{F}$ for some $\widetilde{F} \in \mathcal{F}$. Now fix any continuity point $t \in (0, \infty)$ of F with the property that for all integers $0 \leq m \leq M$, the $t(1 - \frac{m}{M})^\alpha$ also are continuity points of F. As before, this only rules out countably many t, and we can thus assume that t also satisfies the corresponding condition for $\widetilde{F}$. As seen above, this implies validity of (5.21) for all $0 \leq m < M$.

Now Lemma 5.3, for $A := Y$, $B := E_k$, and $n := n_k$, gives

$$(5.23)\qquad 1 - \mu_Y(\varphi_{E_k} > n_k) = \sum_{l=1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^l \left(\frac{1_Y}{\mu(Y)}\right) d\mu,$$

with left-hand side satisfying

$$1 - \mu_Y(\varphi_{E_k} > n_k) = F_k(t) \longrightarrow F(t) \quad \text{as } k \to \infty.$$

Turning to the asymptotics of the right-hand side, we again focus on the upper bound (the argument for the lower bound being analogous). We can immediately apply Lemma 5.4 with $w_k := 1_Y/\mu(Y) \in \mathcal{U} := \{1_Y/\mu(Y)\}$, and $z_k := 0$, since Y is, in particular, a Darling-Kac set. Together with (5.21) above, this shows that for any $M \geq 1$,

$$\begin{aligned}&\overline{\lim_{k\to\infty}} \sum_{l=1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^l \left(\frac{1_Y}{\mu(Y)}\right) d\mu \\ &\leq \frac{t}{M^\alpha} + \alpha t \sum_{m=1}^{M-1} \left[1 - \widetilde{F}\left(t\left(1 - \frac{m+1}{M}\right)^\alpha\right)\right] \left(\frac{m}{M}\right)^{\alpha-1} \frac{1}{M}.\end{aligned}$$

Letting $M \to \infty$ then yields

$$\overline{\lim_{k\to\infty}} \sum_{l=1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^l \left(\frac{1_Y}{\mu(Y)} \right) d\mu \le \alpha t \int_0^1 [1 - \widetilde{F}(t(1-s)^\alpha)] \cdot s^{\alpha-1} \, ds$$

as before, and a parallel argument shows that the expression on the right-hand side also is a lower bound for the corresponding $\underline{\lim}_{k\to\infty} \sum_{l=1}^{n_k}$. Whence, in view of the conclusion about $\widetilde{F}$ obtained above,

$$F(t) = \alpha t \int_0^1 [1 - \widetilde{F}(t(1-s)^\alpha)] \cdot s^{\alpha-1} \, ds = \widetilde{F}(t)$$

for all but countably many t, and thus, in fact, for all $t \in [0, \infty)$. □

6. $\mathcal{U}$-uniform sets via induced maps

An abstract condition for $\mathcal{U}$-uniform sets. The goal of the present section is to show that suitable induced maps give rise to $\mathcal{U}$-uniform sets for natural families $\mathcal{U}$ of densities. The argument is inspired by Thaler's method for finding u-uniform sets (see [T3], [Z2]). Its abstract core is isolated in the following result.

Proposition 6.1 (**$\mathcal{U}$-uniform sets via precompactness**). *Let T be a pointwise dual ergodic c.e.m.p.t. on $(X, \mathcal{A}, \mu)$, and $Y \in \mathcal{A}$ some set with $0 < \mu(Y) < \infty$. Suppose $\mathcal{U} \subseteq \mathcal{D}(\mu)$ is a family of probability densities supported on Y, such that the sequence of maps*

$$\Theta_n : \mathcal{U} \times Y \to [0, \infty) \quad \textit{given by} \quad \Theta_n(u, x) := \frac{\sum_{k=0}^{n-1} \widehat{T}^k u(x)}{\sum_{k=0}^{n-1} \mu(1_Y \cdot \widehat{T}^k u)} \tag{6.1}$$

is precompact for uniform convergence mod μ on $\mathcal{U} \times Y$. Then Y is $\mathcal{U}$-uniform.

Proof. **(i)** We assume without loss of generality that $\mu(Y) = 1$. Let $\Theta : \mathcal{U} \times Y \to [0, \infty)$ be a limit point of (Θ_n), i.e. assume there are $n_l \nearrow \infty$ such that $\Theta_{n_l} \longrightarrow \Theta$ uniformly on $\mathcal{U} \times Y$ as $l \to \infty$. Then pointwise dual ergodicity ensures that for every $u \in \mathcal{U}$,

$$\Theta(u, x) = 1 \quad \text{for a.e. } x \in Y.$$

Hence, the limit point of (Θ_n) is uniquely determined mod μ, so that in fact

$$\Theta_n \longrightarrow \Theta \quad \text{uniformly mod } \mu \text{ on } \mathcal{U} \times Y \quad \text{as } n \to \infty. \tag{6.2}$$

Pointwise dual ergodicity also implies that

$$a_n(u) := \sum_{k=0}^{n-1} \mu(1_Y \cdot \widehat{T}^k u) \sim a_T(n) \quad \text{as } n \to \infty \quad \text{for all } u \in \mathcal{U}. \tag{6.3}$$

To prove $\mathcal{U}$-uniformity of Y, it remains to check that this asymptotic relation holds uniformly in $u \in \mathcal{U}$, as this enables us to replace the u-dependent normalization in (6.2) by the single sequence $(a_T(n))$ without losing uniformity.

(ii) We are going to verify the equivalent statement that

$$a_n(u) \sim a_n(1_Y) \quad \text{as } n \to \infty, \text{ uniformly in } u \in \mathcal{U}. \tag{6.4}$$

By duality we have $a_n(u) = \int_Y \mathbf{S}_n(Y) \cdot u \, d\mu$, $n \geq 1$, with $\mathbf{S}_n(Y) = \sum_{k=0}^{n-1} 1_Y \circ T^k$, and hence

$$\frac{a_n(u)}{a_n(1_Y)} - 1 = \int_Y Q_n \cdot u \, d\mu - \int_Y Q_n \cdot 1_Y \, d\mu,$$

where $Q_n := a_n(1_Y)^{-1}\mathbf{S}_n(Y)$, $n \geq 1$. We claim that Proposition 3.2 of [Z6] applies to our sequence $(Q_n)_{n\geq 1}$ and hence proves (6.4). Indeed, the assumptions of that Proposition can be checked by the same argument as in the proof of Theorem 2.1 of [Z6]. □

Inducing for a piecewise invertible system $(X, \mathcal{A}, \lambda, T, \xi)$. Inducing T on some ξ-measurable set Y, with partition $\xi_{Y,0}$ into connected components, yields an *induced system* $(Y, Y \cap \mathcal{A}, \lambda, T_Y, \xi_Y)$ with natural *induced partition* on Y, $\xi_Y = \xi_{Y,1} := \bigcup_{k\geq 1} \{V \cap \{\varphi_Y = k\} \cap T^{-k}M : V \in Y \cap \xi_k,\, M \in \xi_{Y,0}\}$, so that $\xi_Y(x) = \xi_{\varphi_Y(x)}(x) \cap T_Y^{-1}\xi_{Y,0}(T_Y x)$ for a.e. $x \in Y$. More generally, with $\xi_{Y,m} := \bigvee_{i=0}^{m-1} T_Y^{-i}\xi_Y$,

$$\xi_{Y,m}(x) = \xi_{\varphi_{Y,m}(x)}(x) \cap T_Y^{-m}\xi_{Y,0}(T_Y^m x) \quad \text{for } m \geq 1, \tag{6.5}$$

where $\varphi_{Y,m} := \sum_{i=0}^{m-1} \varphi_Y \circ T_Y^i$ gives the time of the mth return. When Y has a single component (i.e. $\xi_{Y,0} = \{Y\}$), this reduces to $\xi_Y(x) = \xi_{\varphi_Y(x)}(x)$. If T is Markov, then so is T_Y, and if, in that case, $Y \in \xi$, then then T_Y is piecewise onto.

Gibbs-Markov systems. We shall impose conditions on a suitable induced system $(Y, Y \cap \mathcal{A}, \lambda, T_Y, \xi_Y) = (Y, \mathcal{B}, \lambda, S, \eta)$, with (Y, d_Y) compact, calling it *uniformly expanding* if there is some $\rho = \rho(S) \in (0,1)$ such that $\mathrm{d}_Y(g_H(x), g_H(y)) \leq \rho \cdot d_Y(x,y)$ whenever $x, y \in H \in \eta$. Here, for any $m \geq 1$ and $H \in \eta_m = \bigvee_{i=0}^{m-1} S^{-i}\eta$, $g_H := (S^m \mid_H)^{-1} : S^m H \to H$ is the inverse of the branch $S^m \mid_H$. All g_H have Radon-Nikodym derivatives $\omega_H := d(\lambda \circ g_H)/d\lambda$. To ensure good ergodic properties, we will need some distortion control. As in [Z4], a real function u will be called *admissible on* $Z \subseteq Y$ if it is Lipschitz on Z with $\inf_Z u > 0$ or, equivalently, if $u > 0$ and there is some $r \in (0, \infty)$ for which $u(x)/u(y) \leq 1 + r \cdot \mathrm{d}_Y(x,y)$ for $x, y \in Z$. In this case, the inf of all such r is the *regularity* $R_Z(u)$ *of* u on Z. The constant function $u = 0$ will also be regarded *admissible* (with $R_Z(u) = 0$).

A natural version of *Adler's condition*, suitable for this setup, is that there should be some $A = A(S) \in [0, \infty)$ for which $\sup_{H\in\eta} R_{SH}(\omega_H) \leq A$. Markov systems with a uniformly expanding iterate S^m which satisfy Adler's condition plus the *big image condition* $\inf_{H\in\eta} \lambda(SH) > 0$ are called *Gibbs-Markov* (see e.g. [A0]). In this case there is an ergodic invariant probability measure $\nu \ll \lambda$, and the system is also Gibbs-Markov with respect to ν, so that we can just as well work with the measure ν. If $(Y, \mathcal{B}, \lambda, S, \eta)$ is Gibbs-Markov, and $H \in \eta$ is recurrent, then the induced system on H is Gibbs-Markov and piecewise onto.

Observe that if $(Y, \mathcal{B}, \lambda, S, \eta)$ is a Markov system, then

$$S^m\eta_m = \{S^m H : H \in \eta_m\} \subseteq S\eta \quad \text{for } m \geq 1. \tag{6.6}$$

In particular, if the system satisfies the finite image condition $\#S\eta < \infty$, then so do all iterates $(Y, \mathcal{B}, \lambda, S^m, \eta_m)$. We will also use the folklore fact that if $(Y, \mathcal{B}, \nu, S, \eta)$ is a probability preserving Gibbs-Markov system, then there are constants $\kappa, \overline{\rho} \in (0, \infty)$ such that

$$\nu(H) \leq \kappa e^{-\overline{\rho} m} \quad \text{for } m \geq 1 \text{ and } H \in \eta_m. \tag{6.7}$$

$\mathcal{U}$-uniform sets for piecewise invertible systems. We verify the conditions of Proposition 6.1 for good subsets of infinite measure preserving Markov systems.

Consider a probability preserving Gibbs–Markov system $(Y, \mathcal{B}, \nu, S, \eta)$ on the compact metric space (Y, d_Y), without loss of generality with $\mathrm{diam}(Y) \leq 1$. The transfer operator $\widehat{S}$, and all of its powers, allow explicit representations as

$$\widehat{S}^m u = \sum_{H \in \eta_m} (u \circ g_H) \cdot \omega_H, \quad \text{for } m \geq 1 \text{ and } u \geq 0 \text{ measurable,} \tag{6.8}$$

where again $g_H = (S^m \mid_H)^{-1} : S^m H \to H$ is the inverse of $S^m \mid_H$ and $\omega_H = d(\lambda \circ g_H)/d\lambda$. We shall always work with the versions obtained using admissible versions of the ω_H. For $I \subseteq Y$ we let $\mathcal{C}_r(I) := \{u : Y \to [0, \infty) : u$ is supported and admissible on I with $R_I(u) \leq r\}$, $r > 0$. Note that this is a *positive convex cone* of functions, that is, $tu, u + v \in \mathcal{C}_r(I)$ whenever $u, v \in \mathcal{C}_r(I)$ and $t \geq 0$. It is easily seen that the following countable version holds,

$$\text{if } u_i \in \mathcal{C}_r(I),\ i \geq 1, \text{ and } u := \textstyle\sum_{i \geq 1} u_i < \infty \text{ on } I, \text{ then } u \in \mathcal{C}_r(I). \tag{6.9}$$

We let $\mathcal{D}_r(I) := \{u \in \mathcal{C}_r(I) : \int_Y u \, d\nu = 1\}$ denote the set of probability densities in $\mathcal{C}_r(I)$. Then,

$$\nu(I) \inf\nolimits_I u \leq 1 \leq \sup\nolimits_I u \leq (1 + r) \inf\nolimits_I u \quad \text{for } u \in \mathcal{D}_r(I), \tag{6.10}$$

(in particular, each $u \in \mathcal{D}_r(I)$ is strictly positive on I), and so

$$|u(x) - u(y)| \leq r(1 + r)/\nu(I) \cdot \mathrm{d}_Y(x, y) \quad \text{for } x, y \in I \text{ and } u \in \mathcal{D}_r(I). \tag{6.11}$$

Our goal is to prove

Proposition 6.2 (**$\mathcal{U}$-uniform sets via induced Gibbs-Markov-maps**). *Let $(X, \mathcal{A}, \mu, T, \xi)$ be a c.e.m.p. piecewise invertible system and $Y \in \mathcal{A}$ (with $0 < \mu(Y) < \infty$) a ξ-measurable set on which the system induces a Gibbs-Markov-system $(Y, Y \cap \mathcal{A}, \mu_Y, T_Y, \xi_Y)$ with $\# T_Y \xi_Y < \infty$. Suppose that $I \subseteq Y$ is ξ_Y-measurable. Then, for every $r > 0$,*

$$Y \text{ is a } \mathcal{U}\text{-uniform set for } \mathcal{U} := \mathcal{D}_r(I). \tag{6.12}$$

We first need to recall, as a warm-up, some of the well-understood distortion properties of Gibbs-Markov maps, expressed in terms of $\widehat{S}$.

Lemma 6.1 (**Distortion properties of Gibbs-Markov-maps**). *Let $(Y, \mathcal{B}, \nu, S, \eta)$ be a probability preserving Gibbs-Markov-system. Then there is some $\mathbf{r}(S) \in (0, \infty)$ such that if $r \geq \mathbf{r}(S)$, $m \geq 1$, $E \subseteq F \in \eta_m$, then*

$$\widehat{S}^m (1_E u) \in \mathcal{C}_r(S^m E) \quad \text{whenever } 1_F u \in \mathcal{C}_r(F), \tag{6.13}$$

where we use the specific versions of $\widehat{S}$ given by (6.8) with admissible ω_H.

Proof. This is verified by routine calculations like, for example, those in [Z4]. □

We now turn to the

Proof of Proposition 6.2. We verify the sufficient condition given in Proposition 6.1. To this end, fix some $r \geq \mathbf{r}(T_Y)$, let $\mathcal{U} := \mathcal{D}_r(I)$, define Θ_n as in (6.1), and $a_n(u)$ as in (6.3). Assume without loss of generality that $\mu(Y) = 1$.

(i) Let $S\eta = T_Y\xi_Y = \{B_1, \ldots, B_L\}$ be the finite collection of cylinder images of the induced Gibbs-Markov system $(Y, Y \cap \mathcal{A}, \mu_Y, T_Y, \xi_Y) =: (Y, \mathcal{B}, \lambda, S, \eta)$.

Writing $\varphi_{Y,m} := \sum_{i=0}^{m-1} \varphi_Y \circ T_Y^i$, $m \geq 1$, for the m-th return time to Y under T (which is constant on each $H \in \xi_{Y,m}$), and $\varphi_{Y,0} := 0$, we decompose for $k \geq 0$,

$$Y \cap T^{-k}Y = \bigcup_{m=0}^{k} \bigcup_{H\in\xi_{Y,m}:\, \varphi_{Y,m}(H)=k} H \quad \text{(disjoint)}.$$

This shows that for any $k \geq 0$ and measurable $u : Y \to [0, \infty)$,

$$1_Y \cdot \widehat{T}^k u = \sum_{m=0}^{k} \sum_{H\in\xi_{Y,m}:\, \varphi_{Y,m}(H)=k} \widehat{S}^m(1_H u). \tag{6.14}$$

Define $\eta(k, l, m) := \{H \in \xi_{Y,m}\colon \varphi_{Y,m}(H) = k \text{ and } T_Y^m H = B_l\}$. In view of (6.14) and (6.6) we can represent the relevant dual ergodic sums as

$$1_Y \sum_{k=0}^{n-1} \widehat{T}^k u = \sum_{l=1}^{L} u_{n,l} \quad \text{with } u_{n,l} := \sum_{0\leq m\leq k<n} \sum_{H\in\eta(k,l,m)} \widehat{S}^m(1_H u). \tag{6.15}$$

According to Lemma 6.1 we always have $\widehat{S}^m(1_H u) \in \mathcal{C}_r(B_l)$ for $H \in \eta(k, l, m)$. Via (6.9) this shows that

$$u_{n,l} \in \mathcal{C}_r(B_l) \quad \text{for } n \geq 1 \text{ and } 1 \leq l \leq L. \tag{6.16}$$

The main step of our argument will be to show the following. Set $\overline{u}_{n,l} := \int u_{n,l}\, d\mu_Y$, and define maps

$$\Theta_{n,l} : \mathcal{U} \times Y \to [0, \infty) \quad \text{by} \quad \Theta_{n,l}(u, x) := \begin{cases} u_{n,l}(x)/\overline{u}_{n,l} & \text{if } \overline{u}_{n,l} > 0, \\ 0 & \text{if } \overline{u}_{n,l} = 0. \end{cases}$$

We claim that for each $l \in \{1, \ldots, L\}$ the sequence

$$\left(\Theta_{n,l}\right)_{n\geq1} \text{ is precompact for uniform convergence (mod } \mu) \text{ on } \mathcal{U} \times Y. \tag{6.17}$$

It is easy to see how (6.17) implies the precompactness property of Proposition 6.1. Take any (strictly increasing) subsequence (n_j) of indices. Due to (6.17) there is some further subsequence $(n'_j) \subseteq (n_j)$ such that, for every $l \in \{1, \ldots, L\}$, $(\Theta_{n'_j,l})_{j\geq1}$ converges uniformly (mod μ) to some Θ_l^* on $\mathcal{U} \times Y$. Since $\overline{u}_{n,l}/a_n(u) \in [0, 1]$ for all n, l there is yet another subsequence $(n''_j) \subseteq (n'_j)$ such that, for each l, $\overline{u}_{n''_j,l}/a_{n''_j}(u)$ converges to some $s_l \in [0, 1]$ as $j \to \infty$. But in view of

$$\Theta_n(u, .) = \sum_{l=1}^{L} \frac{\overline{u}_{n,l}}{a_n(u)} \cdot \Theta_{n,l}(u, .)$$

this proves that $\Theta_{n''_j} \longrightarrow \sum_{l=1}^{L} s_l\, \Theta_l^*$ uniformly (mod μ) on $\mathcal{U} \times Y$, as required. Therefore we need only prove (6.17). Note that $\Theta_{n,l}(u, x) = 0$ for $x \in B_l^c$, so that can also regard $\Theta_{n,l}$ as a function on $\mathcal{U} \times B_l = \mathcal{D}_r(I) \times B_l$ without losing any information.

(ii) For bounded functions $u, v : Y \to \mathbb{R}$ and $x, y \in Y$ we set $\mathrm{d}_\times((u, x), (v, y)) := \|u - v\|_\infty + \mathrm{d}_Y(x, y)$. Now fix any $l \in \{1, \ldots, L\}$. To validate (6.17) we will show in this step that $\left(\Theta_{n,l}\right)_{n\geq1}$ is uniformly bounded and equicontinuous on $\mathcal{D}_r(I) \times B_l$, equipped with the metric $\mathrm{d}_\times$. The first property is clear since, by (6.16), $\Theta_{n,l}(u,.) \in \mathcal{D}_r(B_l)$, so that (6.10) ensures

$$0 \leq \Theta_{n,l} \leq (1+r)/\mu_Y(B_l) \quad \text{for all } n \geq 1. \tag{6.18}$$

Letting $\kappa := \mu(B_l)^{-2}\max(2,r)(1+r)^2$ we claim that for $u,v \in \mathcal{D}_r(I)$, $x,y \in B_l$, and all $n \geq 1$,

$$|\Theta_{n,l}(u,x) - \Theta_{n,l}(v,y)| \leq \kappa \cdot \mathrm{d}_\times((u,x),(v,y)). \tag{6.19}$$

Indeed, combining (6.11) with $\Theta_{n,l}(u,.) \in \mathcal{D}_r(B_l)$, we first see that

$$|\Theta_{n,l}(u,x) - \Theta_{n,l}(u,y)| \leq \mu_Y(B_l)^{-1} r(1+r) \cdot \mathrm{d}_Y(x,y) \quad \text{for } n \geq 1. \tag{6.20}$$

To quantify the dependence of $\Theta_n(u,y)$ on u, note that for $u,v \in \mathcal{D}_r(I)$,

$$u \leq v + \|u-v\|_\infty 1_I \leq (1+(1+r)\|u-v\|_\infty)\, v, \tag{6.21}$$

by (6.10). Hence, since $\widehat{S}$ is a positive linear operator, we have $\widehat{S}^m(1_B u) \leq (1 + (1+r)\|u-v\|_\infty)\, \widehat{S}^m(1_B v)$ whenever $m \geq 0$ and $B \in \mathcal{B}$. Define $u_{n,l}$, $\overline{u}_{n,l}$ as above, and $v_{n,l}$, $\overline{v}_{n,l}$ in the same manner, using v instead of u. Then this estimate shows

$$u_{n,l} \leq (1+(1+r)\|u-v\|_\infty)\, v_{n,l} \quad \text{for } n \geq 1. \tag{6.22}$$

Invoking this (and (6.10) again), we find that

$$\|u_{n,l} - v_{n,l}\|_\infty \leq \mu_Y(B_l)^{-1}(1+r)^2 \|u-v\|_\infty \max(\overline{u}_{n,l}, \overline{v}_{n,l}). \tag{6.23}$$

Therefore (assuming without loss of generality that $\max(\overline{u}_{n,l}, \overline{v}_{n,l}) = \overline{v}_{n,l}$),

$$|\overline{u}_{n,l} - \overline{v}_{n,l}|\; u_{n,l} \leq \mu_Y(B_l)^{-2}(1+r)^3\; \|u-v\|_\infty \overline{v}_{n,l}\, \overline{u}_{n,l}, \tag{6.24}$$

and similarly

$$\overline{u}_{n,l}\; |v_{n,l} - u_{n,l}| \leq \mu_Y(B_l)^{-1}(1+r)^2 \|u-v\|_\infty\; \overline{u}_{n,l}\, \overline{v}_{n,l}. \tag{6.25}$$

Together these two estimates entail

$$|\Theta_{n,l}(u,y) - \Theta_{n,l}(v,y)| \leq 2\mu_Y(B_l)^{-2}(1+r)^3\; \|u-v\|_\infty \quad \text{for } n \geq 1. \tag{6.26}$$

Combining this with (6.20) proves our claim (6.19).

(iii) Now $(\mathrm{cl}(I), \mathrm{d}_Y)$, the closure of I, is a compact metric space. Hence the Arzelà-Ascoli theorem can be used in the Banach space $(\mathcal{C}(\mathrm{cl}(I)), \|.\|_\infty)$ of continuous real functions on $\mathrm{cl}(I)$ with the uniform norm. Note that any Lipschitz function on I has a unique Lipschitz extension to $\mathrm{cl}(I)$ (respecting the same Lipschitz constant and having the same uniform norm). In this way, $\mathcal{D}_r(I)$ can be identified with a certain subset $\mathcal{D}$ of $\mathcal{C}(\mathrm{cl}(I))$. By (6.10) and (6.11), $\mathcal{D}$ is uniformly bounded and equicontinuous, and hence precompact in $(\mathcal{C}(\mathrm{cl}(I)), \|.\|_\infty)$. In fact, $\mathcal{D}$ is compact, since it is also closed in the complete space $(\mathcal{C}(\mathrm{cl}(I)), \|.\|_\infty)$.

Likewise, $\mathrm{cl}(B_l)$ is compact, and by the same extension principle for Lipschitz functions, we may regard each $\Theta_{n,l}(u,.)$ as a Lipschitz element of $\mathcal{C}(\mathrm{cl}(B_l))$, and hence identify each $\Theta_{n,l}$ with a map $\Theta^*_{n,l} : \mathcal{D} \times \mathrm{cl}(B_l) \to \mathbb{R}$ which, by continuity, still satisfies all the estimates of step (ii). In particular, $(\Theta^*_{n,l})_{n\geq 1}$ is uniformly bounded and equicontinuous with respect to $\mathrm{d}_\times$ on $\mathcal{D} \times \mathrm{cl}(B_l)$. But $(\mathcal{D} \times \mathrm{cl}(B_l), \mathrm{d}_\times)$ is a compact metric space, since $\mathrm{d}_\times$ induces the product topology. Applying the Arzelà-Ascoli theorem again, we thus conclude that $(\Theta^*_{n,l})_{n\geq 1}$ is precompact in $(\mathcal{C}(\mathcal{D} \times \mathrm{cl}(B_l)), \|.\|_\infty)$. This implies (6.17). □

7. Proof of Theorem 3.1

Preparatory observations. The following easy lemma on the pointwise order of magnitude of ergodic sums applies to a large class of infinite measure preserving systems. Recall the definitons (2.5) and (2.10) of $\mathbf{S}_k(Y)$ and $w_N(Y)$, respectively.

Lemma 7.1 (**Pointwise bounds for ergodic sums and return-times**). *Let T be a c.e.m.p.t. on $(X,\mathcal{A},\mu)$, and $Y \in \mathcal{A}$ (with $0<\mu(Y)<\infty$) such that $(w_N(Y)) \in \mathcal{R}_{1-\alpha}$ for some $\alpha \in (0,1]$. Then,*

$$\mathbf{S}_k(Y)/\log k \longrightarrow \infty \quad \text{a.e.} \quad \text{as } k \to \infty. \tag{7.1}$$

Moreover, if $\gamma > 1/\alpha$, then

$$\varphi_Y \circ T_Y^m = o(m^\gamma) \quad \text{a.e.} \quad \text{as } m \to \infty. \tag{7.2}$$

Proof. In view of Theorem 2.4.1 of [A0], (7.1) follows as soon as $\int_Y \log \circ \varphi_Y \, d\mu_Y < \infty$. Via the Monotone Density Theorem (see Theorem1.7.2 of [BGT]), the tail of φ_Y is seen to satisfy

$$q_n(Y) \sim (1-\alpha) w_n(Y)/(n\mu(Y)) \quad \text{as } n \to \infty \tag{7.3}$$

(recall convention (2.13)), and as $(w_N(Y)/N) \in \mathcal{R}_{-\alpha}$, $\log \circ \varphi_Y$ is μ_Y-integrable.

Since $(\varphi_Y \circ T_Y^m)_{m\geq 0}$ is a stationary sequence on $(Y, Y\cap\mathcal{A}, \mu_Y)$, the tail behaviour (7.3) implies (7.2) via a straightforward application of the Borel-Cantelli Lemma, as $\sum_{m\geq 0} \mu_Y(\varphi_Y > \delta m^\gamma) < \infty$ for every $\delta > 0$. □

In the concrete application of the abstract Theorem 4.1 below, condition (4.8) actually follows from the simpler condition (4.7) because the E_k there exhibit exponential return-time statistics for the induced map T_Y. To make this precise below we let, for $E \subseteq Y$, $\varphi_E^Y : Y \to \mathbb{N} \cup \{\infty\}$ denote the first hitting time of E under the first-return map T_Y, that is,

$$\varphi_E^Y(x) := \min\{j \geq 1 : T_Y^j x \in E\}, \quad x \in Y. \tag{7.4}$$

(For finite measure preserving situations relations between φ_E and φ_E^Y have been studied in [BSTV] and [HWZ].) Without imposing any extra conditions on the system, we then have the following useful observation.

Lemma 7.2 (**Using return-time statistics of the induced system**). *Let T be a c.e.m.p.t. on $(X,\mathcal{A},\mu)$. Suppose that Y and $E_k \subseteq Y$, $k \geq 1$, are sets of positive finite measure with $\mu(E_k) \to 0$, and that $z_k \geq 1$ are integers such that*

$$z_k \cdot \mu(E_k) \to 0 \quad \text{as } k \to \infty. \tag{7.5}$$

Assume that for every $t > 0$,

$$\mu_{E_k}(\mu_Y(E_k)\, \varphi_{E_k}^Y \leq t) \longrightarrow \Pr[\mathcal{E} \leq t] \quad \text{as } k \to \infty. \tag{7.6}$$

Then

$$\mu_{E_k}(\varphi_{E_k} \leq z_k) \longrightarrow 0 \quad \text{as } k \to \infty. \tag{7.7}$$

Proof. We start by recording some preparatory observations. First, we recall the natural duality (on Y) between consecutive return times $\varphi_{Y,m} = \sum_{i=0}^{m-1} \varphi_Y \circ T_Y^i$, $m \geq 1$, and occupation times $\mathbf{S}_l(Y) := \sum_{j=0}^{l-1} 1_Y \circ T^j$, $l \geq 1$, given by

$$\mathbf{S}_l(Y) > m \quad \text{if and only if} \quad \varphi_{Y,m} < l. \tag{7.8}$$

Second, for $E \subseteq Y$, the function φ_E can be expressed in terms of φ_E^Y and φ_Y as

$$\varphi_E = \sum_{j=0}^{\varphi_E^Y - 1} \varphi_Y \circ T_Y^j = \varphi_{Y,\varphi_E^Y} \quad \text{on } Y. \tag{7.9}$$

Next, due to (7.6) there are $t_0 > 0$ and $k_0 \geq 1$ such that

$$\mu_{E_k}(\mu_Y(E_k)\,\varphi_{E_k}^Y \leq t_0) < \varepsilon \quad \text{for } k \geq k_0. \tag{7.10}$$

Finally, as $\mathbf{S}_l(Y) \leq l$ and $z_k \geq 1$, we see that

$$\mu_Y(E_k)\,\mathbf{S}_{\lfloor z_k+1 \rfloor}(Y) > t_0 \quad \text{implies} \quad z_k\,\mu(E_k) > \mu(Y)\,t_0/2.$$

Hence, by (7.5), there is some $k_1 \geq 1$ for which

$$\mu_{E_k}(\mu_Y(E_k)\,\mathbf{S}_{\lfloor z_k+1 \rfloor}(Y) > t_0) = 0 \quad \text{for } k \geq k_1. \tag{7.11}$$

Combining these facts we conclude that

$$\begin{aligned}
\mu_{E_k}(\varphi_{E_k} \leq z_k) &\leq \mu_{E_k}(\varphi_{Y,\varphi_{E_k}^Y} \leq \lfloor z_k + 1 \rfloor) \\
&= \mu_{E_k}(\mathbf{S}_{\lfloor z_k+1 \rfloor}(Y) > \varphi_{E_k}^Y) \\
&= \mu_{E_k}(\mu_Y(E_k)\,\varphi_{E_k}^Y < \mu_Y(E_k)\,\mathbf{S}_{\lfloor z_k+1 \rfloor}(Y)) \\
&\leq \mu_{E_k}(\mu_Y(E_k)\,\varphi_{E_k}^Y < t_0) + \mu_{E_k}(\mu_Y(E_k)\,\mathbf{S}_{\lfloor z_k+1 \rfloor}(Y) > t_0) \\
&< \varepsilon + 0 \quad \text{for } k \geq \max(k_0, k_1),
\end{aligned}$$

which proves the assertion of the lemma. □

Remark 7.1. The proof of the lemma shows that (7.6) can be replaced by

$$\lim_{t \searrow 0} \overline{\lim}_{k \to \infty} \mu_{E_k}(\mu_Y(E_k)\,\varphi_{E_k}^Y \leq t) = 0. \tag{7.12}$$

This lemma will be put to use via another auxiliary observation.

Lemma 7.3 (**Exponential return-time statistics for AFU-maps**). *Suppose that $(Y, \mathcal{B}, \nu, S, \eta)$ is an ergodic probability preserving AFU-map. Then, for ν-a.e. $x \in Y$, and every sequence (E_k) of open intervals with $Y \supseteq E_k \searrow \{x\}$ we have*

$$\nu(E_k)\,\tau_{E_k} \overset{\mu_{E_k}}{\Longrightarrow} \mathcal{E} \quad \text{as } k \to \infty, \tag{7.13}$$

where $\tau_E(x) := \min\{n \geq 1 : S^n x \in E\}$, the first return time of E under S.

Proof. Any AFU-map belongs to the class of Rychlik-maps studied in [Ry] (see Corollary 1 of [Z1]). Theorem 3.2 of [BSTV] thus guarantees (7.13) in case the E_k are symmetric ε-neighbourhoods of a μ-typical point x. However, the proof given there does not depend on the symmetry property, and applies to a.e. point and any sequence (E_k) of intervals around x, with diameters shrinking to zero. □

The above considerations allow us to complete the

Proof of Theorem 3.1. **(i)** Since $Y^N \nearrow X$, it suffices to prove the asserted distributional convergence for a.e. $x \in Y$, where $Y := Y^N$ with $N \geq 1$ arbitrary but fixed. Recall that Y is ξ'-measurable (but not necessarily ξ-measurable). It has been shown in §4 of [Z2] that the system $(Y, Y \cap \mathcal{A}, \mu_Y, T_Y, \xi'_Y)$ which $(X, \mathcal{A}, \mu, T, \xi')$ induces on Y is an AFU-system. It is Markovian since Y is ξ'-measurable, and the original system is Markovian. Hence, this induced system is probability preserving, Gibbs-Markov, and satisfies the finite image condition $\# T_Y \xi'_Y < \infty$. Set $\mathbf{r} := \mathbf{r}(T_Y)$, as in Lemma 6.1.

According to Theorem 3 of [Z2], assumption (3.6) ensures that $(w_N(Y)) \in \mathcal{R}_{1-\alpha}$. By Theorem 1 of [Z2], T is pointwise dual ergodic, and by Theorem 4 of [Z2], $a_T(n)$ as defined in our theorem is indeed a return sequence for T. Its inverse function b_T belongs to $\mathcal{R}_p$, $p := 1/\alpha \in [1, \infty)$.

(ii) The main part of the proof (steps (iii)-(xi)) is devoted to establishing the $d = 1$ case of the theorem. That is, we show

$$\mu_{E_k}\left(\{b_T(1/\mu(E_k))^{-1} \cdot \varphi_{E_k} \le t\}\right) \longrightarrow \Pr[\mathcal{E}^{\frac{1}{\alpha}}\, \mathcal{G}_\alpha \le t] \quad \text{as } k \to \infty, \tag{7.14}$$

and

$$\mu_Y\left(\{b_T(1/\mu(E_k))^{-1} \cdot \varphi_{E_k} \le t\}\right) \longrightarrow \Pr[\mathcal{E}^{\frac{1}{\alpha}}\, \mathcal{G}_\alpha \le t] \quad \text{as } k \to \infty, \tag{7.15}$$

plus the analogous assertions for (E'_k).

Using Theorem 4.1, we are going to prove the asserted convergence (7.14) and (7.15) for every $x \in Y$ for which all $\xi_k(x) = E_k$ and $\xi'_k(x) = E'_k$ are defined, and which satisfies the conclusions of both Lemma 7.1 and Lemma 7.3. *Henceforth we fix such a point* x. The conclusions of Lemma 7.1 give

$$\mathbf{S}_k(Y)(x)/\log k \longrightarrow \infty \quad \text{as } k \to \infty \tag{7.16}$$

and

$$\varphi_Y \circ T_Y^m(x) = o(m^{2/\alpha}) \quad \text{as } m \to \infty. \tag{7.17}$$

Moreover, both (E_k) and (E'_k) are sequences of open intervals shrinking to x. The conclusion of Lemma 7.3 thus means that for every $s > 0$,

$$\mu_{E_k}\left(\mu_Y(E_k)\, \varphi^Y_{E_k} \le s\right) \longrightarrow \Pr[\mathcal{E} \le s] \quad \text{as } k \to \infty, \tag{7.18}$$

and likewise with E'_k in place of E_k.

Below, the argument for the sequence (E_k) is given in full detail. Some care is required since Y need not be ξ-measurable. The sequence (E'_k) is easier, exactly because Y is measurable (mod μ) with respect to ξ'. An outline of this case is given at the end of the proof, but we do not provide every detail where the argument is an easier version of one explained before.

(iii) To get started, let $Z_i := \xi(T^i x)$, $i \ge 0$, so that $E_k = [Z_0, \ldots, Z_{k-1}]$ for $k \ge 1$, and define the z_k as follows. If $Z_{k-1} \in \xi \setminus \zeta$, henceforth referred to as (case A), set $z_k := k-1$. Otherwise (case B), $Z_{k-1} \in \zeta$ and we let $z_k := \max\{i \in \{0, \ldots, k-2\} : Z_i \ne Z_{k-1}\}$, which is well defined for $k \ge k_0$ since x is none of the x_Z, $Z \in \zeta$. Note that, due to $TZ(n+1) = Z(n)$ for $Z \in \zeta$, we have

$$Z_{z_k+1} = \ldots = Z_{k-1} \quad \text{for } k \ge k_0. \tag{7.19}$$

In either case, $Z_{z_k} \subseteq Y(1) \subseteq Y$. We set $j_k := k - 1 - z_k$.

To relate E_k to the induced partition ξ'_Y, let $s_k := \mathbf{S}_{z_k}(Y)(x)$ and define $F_k := \xi'_{Y,s_k}(x) \in \xi'_{Y,s_k}$. For $k \ge k_0$ we then have

$$E_k = F_k \cap T_Y^{-s_k}(Z_{z_k} \cap T^{-1}G_k) \text{ and } T^{z_k}E_k = T_Y^{s_k}E_k = Z_{z_k} \cap T^{-1}G_k, \tag{7.20}$$

where $G_k := \bigcap_{i=0}^{j_k-1} T^{-i} Z_{k-1}$ (so that $G_k = X$ if $z_k = k-1$).

(iv) Conditions (4.7) and (4.8) are easily verified. Since $E_k \subseteq F_k \in \xi'_{Y,s_k}$, (6.7) gives

$$\mu(E_k) \leq \kappa e^{-\overline{\rho}\, s_k} \quad \text{for } k \geq k_0, \tag{7.21}$$

for suitable constants $\kappa, \overline{\rho} \in (0,\infty)$. In view of (7.19) and our choice of Y, we have

$$s_k \geq \mathbf{S}_k(Y)(x) - N \quad \text{for } k \geq k_0. \tag{7.22}$$

Combining this with (7.21) and (7.16), we find that

$$z_k\, \mu(E_k) \leq k\, \mu(E_k) \leq \kappa e^{\overline{\rho}\, N}\, k\, e^{-\overline{\rho}\, \mathbf{S}_k(Y)(x)} \longrightarrow 0 \text{ as } k \to \infty, \tag{7.23}$$

proving (4.7). Lemma 7.2 enables us to combine (4.7) with (7.18) to obtain (4.8).

For later use we record that (7.21) and (7.22) together with regular variation of index $p \geq 1$ of b_T ensure that there is some $\kappa^* \in (0,\infty)$ such that, letting $\overline{\rho}^* := p\overline{\rho}/2 > 0$, we have

$$b_T(1/\mu(E_k)) \geq \kappa^* e^{\overline{\rho}^* \mathbf{S}_k(Y)(x)} \quad \text{for } k \geq 1. \tag{7.24}$$

(v) As the induced system $(Y, Y\cap\mathcal{A}, \mu_Y, T_Y, \xi'_Y)$ is Gibbs-Markov, our earlier results will allow us to identify Y as a $\mathcal{U}$-uniform set with $\mathcal{U} \subseteq \mathcal{D}(\mu)$ rich enough.

Recall that $\mathbf{r} := \mathbf{r}(T_Y)$, as in Lemma 6.1. According to Proposition 6.2, Y is a $\mathcal{U}'$-uniform set for $\mathcal{U}'$ any of the following, $\mathcal{U}' = \mathcal{D}_{\mathbf{r}}(T_Y W)$ for some $W \in \xi'_Y$, or $\mathcal{U}' = \mathcal{D}_{\mathbf{r}}(Z(i))$ for some $Z \in \zeta$ and $i \in \{1,\ldots,N\}$. Note that these are only finitely many different collections of densities. In view of (4.5) and (4.6), Y is therefore a $\mathcal{U}$-uniform set for

$$\mathcal{U} := \overline{\mathrm{co}} \left(\bigcup_{W\in\xi'_Y} \mathcal{D}_{\mathbf{r}}(T_Y W) \cup \bigcup_{Z\in\zeta, 1\leq i\leq N} \mathcal{D}_{\mathbf{r}}(Z(i)) \right). \tag{7.25}$$

(vi) We need to have a closer look at the local return distributions to Y. Take any $Z \in \zeta$. By well-known arguments (Corollary on p.82 of [T2]), the local asymptotics (3.6) at the neutral fixed point x_Z implies that the length of the higher-order cylinders $f_Z^{j-1}(Z) = \bigcap_{i=0}^{j-1} T^{-i} Z$ shrinks at a definite rate,

$$\lambda(f_Z^{j-1}(Z)) \sim (a_Z p_Z\, j)^{-\alpha} \quad \text{as } j \to \infty. \tag{7.26}$$

As a consequence, we see that there is some constant $K > 0$ such that, for every $Z \in \zeta$,

$$\lambda(f_Z^{v-1}(Z))/\lambda(f_Z^{j-1}(Z)) \leq K\, j^\alpha v^{-\alpha} \text{ for } v \geq j \geq 0. \tag{7.27}$$

Next, Adler's condition (3.1) guarantees (again by standard arguments) that the inverse branches $f_{Z'}$ for $Z' \in \xi \setminus \zeta$ have uniformly bounded distortion, which gives some $K_* > 0$ such that $\lambda(f_{Z'} f_Z^{v-1}(Z))/\lambda(f_{Z'} f_Z^{j-1}(Z)) \leq K_* j^\alpha v^{-\alpha}$ for $v \geq j \geq 1$, $Z \in \zeta$ and $Z' \in \xi \setminus \zeta$ such that $Z \cap TZ' \neq \varnothing$. Finally, since the invariant density h of T is of finite regularity on every $Z' \in \xi \setminus \zeta$, there is yet another constant $K^* > 0$ for which

$$\frac{\mu(f_{Z'} f_Z^v(Z))}{\mu(f_{Z'} f_Z^j(Z))} \leq K^*\, j^\alpha v^{-\alpha} \qquad \begin{array}{c}\text{for } v \geq j \geq 1, \text{ and } Z \in \zeta, \\ Z' \in \xi \setminus \zeta \text{ such that } Z \cap TZ' \neq \varnothing.\end{array} \tag{7.28}$$

By similar arguments, the uniform distortion control for the $f_{Z'}$, $Z' \in \xi\setminus\zeta$, together with (7.26), implies that (for some $K^{**} > 0$)

$$\frac{\sum_{Z\in\zeta}\mu(f_{Z'}f_Z^v(Z))}{\mu(Z')} \leq K^{**}v^{-\alpha} \quad \text{for } v \geq 1 \text{ and } Z' \in \xi\setminus\zeta. \tag{7.29}$$

(vii) We first focus on (case A). Here, $\widehat{T}^{z_k}(1_{E_k}) = \widehat{T}_Y^{s_k}(1_{F_k\cap T_Y^{-s_k}Z_{z_k}}1_{F_k}) \in \mathcal{C}_{\mathbf{r}}(T_Y^{s_k}E_k) = \mathcal{C}_{\mathbf{r}}(Z_{k-1})$ by Lemma 6.1. To obtain the desired representation $\widehat{T}^{z_k}(1_{E_k}/\mu(E_k)) = \sum_{\iota\geq 0}\pi_{k,\iota}\,w_{k,\iota}$, we let $\rho_{k,0} := 0$ and

$$\pi_{k,\iota} := \mu_{E_k}(E_k \cap T^{-z_k}(Y\cap\{\varphi_Y = \iota\})), \quad \iota \geq 1, \tag{7.30}$$

and define probability densities

$$w_{k,\iota} := (\pi_{k,\iota}\mu(E_k))^{-1}\cdot 1_{Y\cap\{\varphi_Y=\iota\}}\widehat{T}^{z_k}(1_{E_k}) \quad \text{whenever } \pi_{k,\iota} > 0. \tag{7.31}$$

It is clear that $1_Y\,\widehat{T}^j w_{k,\iota} = 0$ for $1 \leq j < \iota$, while $\widehat{T}^\iota w_{k,\iota}$ is supported on Y. Since Y is ξ'-measurable (mod μ), each set $Y\cap\{\varphi_Y = \iota\}$ is the union of some family $\eta(\iota) \subseteq \xi'_Y$. As ξ'_Y refines ξ' and $Z_{k-1} \in \xi'$, we see that

$$\pi_{k,\iota}\mu(E_k)\,\widehat{T}^\iota w_{k,\iota} = \sum_{W\in\eta(\iota)}\widehat{T}_Y\left(1_W\cdot\widehat{T}^{z_k}1_{E_k}\right), \tag{7.32}$$

with each $\widehat{T}_Y(1_W\cdot\widehat{T}^{z_k}1_{E_k}) \in \mathcal{C}_{\mathbf{r}}(T_YW)$ by Lemma 6.1. As a consequence, normalizing and convexly combining these functions we conclude that $\widehat{T}^\iota w_{k,\iota} \in \mathcal{U}$.

(viii) Turning to (4.10), we note that since $\widehat{T}^{z_k}(1_{E_k}/\mu(E_k)) \in \mathcal{D}_{\mathbf{r}}(Z_{k-1})$, we have $\widehat{T}^{z_k}(1_{E_k}/\mu(E_k)) \leq e^{\mathbf{r}}\mu(Z_{k-1})^{-1}1_{Z_{k-1}}$, and hence

$$\begin{aligned} \pi_{k,\iota} &= \mu_{E_k}\left(E_k\cap T_Y^{-s_k}(Z_{k-1}\cap\{\varphi_Y=\iota\})\right) \\ &= \int \widehat{T}_Y^{s_k}(\mu(E_k)^{-1}1_{E_k})\cdot 1_{Z_{k-1}\cap\{\varphi_Y=\iota\}}\,d\mu \\ &\leq e^{\mathbf{r}}\,\mu_{Z_{k-1}}(Z_{k-1}\cap\{\varphi_Y=\iota\}). \end{aligned} \tag{7.33}$$

Due to our choice of Y we have, for $v \geq 1$,

$$\begin{aligned} Z_{k-1}\cap\{\varphi_Y > v\} &= \bigcup_{Z\in\zeta} f_{Z_{k-1}}(f_Z\{\varphi_Y \geq v\}) \\ &= \bigcup_{Z\in\zeta} f_{Z_{k-1}}(f_Z^{v+N-1}Z). \end{aligned} \tag{7.34}$$

Therefore (7.29) shows that

$$\begin{aligned} F_k(v) = \sum_{\iota>v}\pi_{k,\iota} &\leq e^{\mathbf{r}}\,\mu_{Z_{k-1}}(Z_{k-1}\cap\{\varphi_Y > v\}) \\ &\leq e^{\mathbf{r}}K^{**}(v+N-1)^{-\alpha} \quad \text{for } v \geq 1. \end{aligned} \tag{7.35}$$

In view of (7.24), the crucial condition (4.10) follows once we show that, for every $\varepsilon > 0$,

$$F_k(\varepsilon\, e^{\overline{\rho}^*\mathbf{S}_k(Y)(x)}) \longrightarrow 0 \quad \text{as } k\to\infty. \tag{7.36}$$

This, however, is clear from (7.16) and (7.35). We have thus checked the assumptions of Theorem 4.1 for E_k and all indices k of (case A).

(ix) Now consider (case B), where $\widehat{T}^{z_k}(1_{E_k}) = \widehat{T}_Y^{s_k}(1_{F_k\cap T_Y^{-s_k}(Z_{z_k}\cap T^{-1}G_k)}1_{F_k}) \in \mathcal{C}_{\mathbf{r}}(T_Y^{s_k}E_k) = \mathcal{C}_{\mathbf{r}}(Z_{z_k}\cap T^{-1}G_k)$ by Lemma 6.1. Again we define $\pi_{k,\iota}$ and $w_{k,\iota}$ via

(7.30) and (7.31), respectively, so that $1_Y \widehat{T}^j w_{k,\iota} = 0$ for $1 \le j < \iota$, and $\widehat{T}^\iota w_{k,\iota}$ is supported on Y.

Note that $T^{-1}G_k = \bigcup_{i \ge j_k} T^{-1}Z_{k-1}(i)$ (disjoint), and since $Y = Y^N$,

$$T_Y^{s_k} E_k \cap \{\varphi_Y = 1\} = \bigcup_{i=j_k}^N Z_{z_k} \cap T^{-1}Z_{k-1}(i), \tag{7.37}$$

while

$$T_Y^{s_k} E_k \cap \{\varphi_Y = \iota\} = Z_{z_k} \cap T^{-1}Z_{k-1}(N-1+\iota) \quad \text{for } \iota \ge 2. \tag{7.38}$$

Set $\sigma_{k,i} := \int_{Z_{z_k} \cap T^{-1}Z_{k-1}(i)} w_{k,1}\, d\mu$, and $v_{k,i} := \sigma_{k,i}^{-1} 1_{Z_{z_k} \cap T^{-1}Z_{k-1}(i)} w_{k,1}$ (tacitly suppressing indices for which $\sigma_{k,i} = 0$), then $v_{k,i} \in \mathcal{D}_{\mathbf{r}}(Z_{z_k} \cap T^{-1}Z_{k-1}(i))$. Observe next that for every $i \ge 1$, $Z_{z_k} \cap T^{-1}Z_{k-1}(i)$ is ξ'-measurable. (Obvious if $Z_{z_k} \in \xi \setminus \zeta \subseteq \xi'$. Otherwise, $Z_{z_k} \in \zeta$ but due to $Z_{z_k} \ne Z_{k-1}$ we then have $Z_{z_k} \cap T^{-1}Z_{k-1}(i) = Z_{z_k}(1) \cap T^{-1}Z_{k-1}(i) \in \xi_2'$.) We can therefore appeal to Lemma 6.1 to see that $\widehat{T} v_{k,i} = \widehat{T}_Y v_{k,i} \in \mathcal{D}_{\mathbf{r}}(T_Y(Z_{z_k} \cap T^{-1}Z_{k-1}(i))) = \mathcal{D}_{\mathbf{r}}(Z_{k-1}(i))$. Consequently, $\widehat{T}^1 w_{k,1} = \sum_{i=j_k}^N \sigma_{k,i} \widehat{T} v_{k,i} \in \mathcal{U}$. By a simpler version of the same argument, $\widehat{T}^\iota w_{k,\iota} \in \mathcal{U}$ for all $\iota \ge 2$.

(x) To tackle (4.10) in (case B), use $\widehat{T}^{z_k}(1_{E_k}/\mu(E_k)) \in \mathcal{D}_{\mathbf{r}}(Z_{z_k} \cap T^{-1}G_k)$, to see that $\widehat{T}_Y^{s_k}(1_{E_k}/\mu(E_k)) \le e^{\mathbf{r}} \mu(Z_{z_k} \cap T^{-1}G_k)^{-1} 1_{Z_{z_k} \cap T^{-1}G_k}$, and hence

$$\begin{aligned} \digamma_k(v) &= \mu_{E_k}(E_k \cap T_Y^{-s_k}(Z_{z_k} \cap \{\varphi_Y > v\})) \\ &= \int \widehat{T}_Y^{s_k}(\mu(E_k)^{-1} 1_{E_k}) \cdot 1_{Z_{z_k} \cap \{\varphi_Y > v\}}\, d\mu \\ &\le e^{\mathbf{r}}\, \mu_{Z_{z_k} \cap T^{-1}G_k}\left(Z_{z_k} \cap T^{-1}G_k \cap \{\varphi_Y > v\}\right). \end{aligned} \tag{7.39}$$

Now observe that $Z_{z_k} \cap T^{-1}G_k = f_{Z_{z_k}}(f_{Z_{k-1}}^{j_k-1}(Z_{k-1}))$ and

$$Z_{z_k} \cap T^{-1}G_k \cap \{\varphi_Y > v\} = f_{Z_{z_k}}(f_{Z_{k-1}}^{\max(j_k, v+N)-1}(Z_{k-1})). \tag{7.40}$$

Therefore (7.28) shows that

$$\digamma_k(v) \le e^{\mathbf{r}} K^*\, j_k^\alpha v^{-\alpha} \qquad \text{for } k, v \text{ such that } v \ge j_k. \tag{7.41}$$

Once again we need to check (7.36). Fix any $\varepsilon > 0$. Since $j_k \le k$, (7.16) ensures that $j_k \le \varepsilon\, e^{\overline{\rho}^* s_k}$ for $k \ge k_1$. For such k we can then appeal to (7.41) to see that

$$\digamma_k(\varepsilon\, e^{\overline{\rho}^* \mathbf{S}_k(Y)(x)}) \le e^{\mathbf{r}} \varepsilon^{-\alpha} K^*\, k^\alpha e^{-\alpha \overline{\rho}^* \mathbf{S}_k(Y)(x)} \longrightarrow 0 \qquad \text{as } k \to \infty. \tag{7.42}$$

This validates the assumptions of Theorem 4.1 for E_k and all indices k of (case B). Therefore the proof for the sequence $(E_k)_{k \ge 1}$ is complete.

(xi) The argument for (E_k') is essentially the same. Let $Z_i' := \xi'(T^i x)$, $i \ge 0$, so that $E_k' = [Z_0', \ldots, Z_{k-1}']$, define $z_k' := \max\{i \in \{0, \ldots, k-1\} : Z_i' \subseteq Y\} = \max\{i \in \{0, \ldots, k-1\} : T^i x \in Y\}$, and set $j_k' := k - 1 - z_k'$. Then either $z_k' = k-1$ (case A') or $z_k' < k-1$ (case B'). In (case A') we can argue exactly as in (case A) above.

In (case B') note first that $Z_{k-1} \in \zeta$ and that

$$Z_{z_k+1}', \ldots, Z_{k-1}' = Z_{k-1}(l_k' + N - 1), \ldots, Z_{k-1}(l_k' + N - j_k'), \tag{7.43}$$

where $l_k' = \varphi_Y(T_Y^{s_k} x)$. Hence, $E_k' = F_k \cap T_Y^{-s_k}(Z_{z_k}' \cap T^{-1}G_k')$ where we denote $G_k' := Z_{k-1}(l_k' + N - 1)$, and

$$T^{z_k'} E_k' = T_Y^{s_k} E_k' = Z_{z_k}' \cap T^{-1}G_k'. \tag{7.44}$$

Conditions (4.7) and (4.8) are then verified as in step (iv) above. Moreover, (7.24) holds with E'_k in place of E_k. Next, define $\pi'_{k,\iota}$ and $w'_{k,\iota}$ by $\pi'_{k,0} := 0$,

$$\pi'_{k,\iota} := \mu_{E'_k}(E'_k \cap T^{-z'_k}(Y \cap \{\varphi_Y = \iota\})), \quad \iota \geq 1, \tag{7.45}$$

and

$$w'_{k,\iota} := (\pi'_{k,\iota}\mu(E'_k))^{-1} \cdot 1_{Y\cap\{\varphi_Y=\iota\}}\widehat{T}^{z'_k}(1_{E'_k}) \quad \text{whenever } \pi'_{k,\iota} > 0. \tag{7.46}$$

Again, it is immediate that $1_Y\,\widehat{T}^j w'_{k,\iota} = 0$ for $1 \leq j < \iota$, while $\widehat{T}^\iota w'_{k,\iota}$ is supported on Y. Now, $T_Y^{s_k} E'_k \cap \{\varphi_Y = 1\} = \bigcup_{i=j_k}^N Z'_{z'_k} \cap T^{-1}Z_{k-1}(i)$, whereas

$$T_Y^{s_k} E'_k \cap \{\varphi_Y = \iota\} = Z'_{z_k} \cap T^{-1}Z_{k-1}(N-1+\iota) \quad \text{for } \iota \geq 2. \tag{7.47}$$

Arguing as in step (ix) we then see that $\widehat{T}^\iota w_{k,\iota} \in \mathcal{U}$ for all $\iota \geq 1$.

Finally, we need to check (4.10) in (case B'). Observe that (7.44) and (7.47) together show that for $\iota \geq 2$,

$$\pi'_{k,\iota} = \begin{cases} 1 & \text{if } \iota = l'_k, \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$F'_k(v) := \textstyle\sum_{\iota > v} \pi'_{k,\iota} = 0 \quad \text{if } v > l'_k + N,$$

and (4.10) follows once we prove that for every $\varepsilon > 0$ there is some k_1 such that

$$\varepsilon\, e^{\overline{\rho}^* s_k} > l'_k = \varphi_Y(T_Y^{s_k} x) \quad \text{for } k \geq k_1. \tag{7.48}$$

However, since $s_k \to \infty$, this is immediate from (7.17). This completes the proof of the $d = 1$ case of the theorem.

(xii) The inductive step allowing us to pass from d to $d+1$ in (3.7) and (3.8) is easy. We provide the details for (3.7), both in case (E_k) and (E'_k). To obtain (3.8) a straightforward variation of the same argument is used.

Fix any $t_0, \ldots, t_{d-1} \in (0, \infty)$ and abbreviate $B_k := b_T(1/\mu(E_k))^{-1}$, and $M_k := E_k \cap \bigcap_{i=0}^{d-1}\{B_k \cdot \varphi_{E_k} \circ T_{E_k}^i \leq t_i\}$. Below we prove convergence of the conditional distribution function,

$$\mu_{M_k}\left(\{B_k\, \varphi_{E_k} \circ T_{E_k}^d \leq t\}\right) \longrightarrow \Pr[\mathcal{E}^{\frac{1}{\alpha}}\, \mathcal{G}_\alpha \leq t], \tag{7.49}$$

for all $t > 0$. (Here we use the d-dimensional limit theorem to see that the conditioning event M_k has positive measure for $k \geq k^*$.) To do so, note that the left-hand expression equals

$$\int \widehat{T}_{E_k}^d(\mu\,(M_k)^{-1}\, 1_{M_k}) \cdot 1_{\{B_k\, \varphi_{E_k} \leq t\}}\, d\mu.$$

As (7.14) has already been established, (7.49) will follow once we check that

$$\left\| \mu(E_k)\, \widehat{T}_{E_k}^d(\mu\,(M_k)^{-1}\, 1_{M_k}) - 1_{E_k} \right\|_{L_\infty(\mu_{E_k})} \longrightarrow 0 \quad \text{as } k \to \infty \tag{7.50}$$

(and likewise with E'_k replacing E_k). Assume that

$$\widehat{T}_{E_k}^d(\mu\,(M_k)^{-1}\, 1_{M_k}) \in \mathcal{D}_{\mathbf{r}}(E_k) \quad \text{for } k \geq k^*, \tag{7.51}$$

then $u_k := \mu(E_k)\,\widehat{T}^d_{E_k}(\mu(M_k)^{-1}1_{M_k}), u^*_k := 1_{E_k} \in \mathcal{C}_r(E_k)$, and since $\mu(u_k) = \mu(u^*_k)$, we have $\inf_{E_k} u_k \leq 1 \leq \sup_{E_k} u_k$. Hence, by the definition of regularity,

$$(1+\mathbf{r}\,\mathrm{diam}(E_k))^{-1} \leq u_k \leq 1+\mathbf{r}\,\mathrm{diam}(E_k) \quad \text{on } E_k. \tag{7.52}$$

But the cylinders E_k (like the E'_k) trivially satisfy $\mathrm{diam}(E_k) \to 0$. We thus see that (7.50) is immediate if we validate (7.51) plus its (E'_k)-version.

(xiii) The classical distortion bound for good subsets of an AFN-system (which goes back to [T1]), can be rephrased as follows:

$$\begin{gathered}\text{if } V, W \subseteq Y \text{ are intervals, } m \in \mathbb{N}, \text{ and } T^m \text{ maps } V \\ \text{homeomorphically onto } W, \text{ then } \widehat{T}^m 1_V \in \mathcal{C}_{\mathbf{r}}(W).\end{gathered} \tag{7.53}$$

Since E_k is a cylinder from the Markov partition ξ for T, the induced map T_{E_k} is piecewise onto for the partition ξ_{E_k}, and φ_{E_k} is ξ_{E_k}-measureable. This automatically implies analogous properties of $T^i_{E_k}$, $\xi_{E_k,i}$ and $\varphi_{E_k} \circ T^i_{E_k}$. In particular, the conditioning event M_k above is $\xi_{E_k,d-1}$-measurable, $M_k = \bigcup_{V \in \eta^{\#}_k} V$ for some $\eta^{\#}_k \subseteq \xi_{E_k,d-1}$. But on each $V \in \eta^{\#}_k$, $T^d_{E_k}$ coincides with some branch $T^{m_V}\mid_V : V \to W =: E_k$ of the type considered in (7.53). Hence,

$$\widehat{T}^d_{E_k} 1_{M_k} = \sum\nolimits_{V \in \eta^{\#}_k} \widehat{T}^d_{E_k} 1_V = \sum\nolimits_{V \in \eta^{\#}_k} \widehat{T}^{m_V} 1_V \in \mathcal{C}_{\mathbf{r}}(W),$$

and (7.51) follows. Exactly the same argument works for (E'_k). □

To conclude, a short comment suffices to clarify the Example 2.3

Proof of the claims made in Example 2.3. It is well known that $\mathsf{R}\langle f\rangle$ can be represented as a piecewise affine interval map on $[0,1]$ with an indifferent fixed point at $x = 0$, such that $\mu^{\langle f\rangle}$ corresponds to the unique absolutely continuous invariant measure. This is not exactly an AFN-map (as it is not smooth near $x = 0$), but it is even simpler (the invariant density is constant on members of ξ', and the family of ξ'-measurable densities is invariant under the transfer operator). Most important, it shares all properties used in the proof of Theorem 3.1 above. Calculating the asymptotics of a_T (and hence b_T) starting from the tail behaviour is routine. □

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