A JOINT LIMIT THEOREM FOR COMPACTLY REGENERATIVE ERGODIC TRANSFORMATIONS

DAVID KOCHEIM AND ROLAND ZWEIMÜLLER

Abstract. We study conservative ergodic infinite measure preserving transformations satisfying a compact regeneration property introduced in [Z4]. Assuming regular variation of the wandering rate, we clarify the asymptotic distributional behaviour of the random vector (Z_n, S_n), where Z_n and S_n respectively are the time of the last visit before time n to, and the occupation time of, a suitable set Y of finite measure.

1. Introduction

Various interesting classes of conservative (i.e. recurrent) ergodic *infinite measure preserving dynamical systems*, cf. [A0], exhibit stochastic properties which parallel phenomena known from the probability theory of null-recurrent Markov chains. In fact, systems containing (as an induced map, say) some hyperbolic mechanism generalize these classical processes, while still being closely related to them on a structural level, cf. [Z6]. Lacking the clear-cut dependence structure of Markov chains, their probabilistic analysis depends on identifying ergodic properties which (can be verified and) still entail the desired stochastic features.

Let T be a *measure preserving transformation* (*m.p.t.*) on the σ-finite measure space $(X, \mathcal{A}, \mu)$ with $\mu(X) = \infty$. It is *conservative* and *ergodic* (*c.e.*) if $\mathbf{S}_n(f) \to \infty$ a.e. whenever $f \geq 0$ and $\mu(f) = \int f \, d\mu > 0$, where $\mathbf{S}_n(f) := \sum_{k=0}^{n-1} f \circ T^k$, $n \geq 0$, cf. Proposition 1.2.2 of [A0]. For specific types of such systems, distributional limit theorems for ergodic sums $\mathbf{S}_n(f)$ with $f \in L_1(\mu)$ (in Darling-Kac type theorems, cf. [DK]) and for renewal-theoretic variables like $\mathbf{Z}_n(Y) := \max\{k \, 1_Y \circ T^k : 0 \leq k \leq n\}$, $n \geq 0$ (in Dynkin-Lamperti type arcsine laws, cf. [D], [L]), are available, see [A0]-[A2], [T2], [TZ], [Z4]. The limit distributions, which depend on a single parameter $\alpha \in [0, 1]$ encoding a characteristic return rate of the system, are given in terms of *(normalized) Mittag-Leffler variables* $\mathcal{M}_\alpha$ characterized by their

Date: 17 December 2010, final version, to appear in *Studia Mathematica*.

2010 *Mathematics Subject Classification.* Primary 28D05; Secondary 37A40, 37A50, 37C30.

Key words and phrases. Infinite invariant measure, null-recurrent, limit distribution, Darling-Kac theorem, Mittag-Leffler distribution, Dynkin-Lamperti Arcsine law, fluctuation theory, indifferent fixed points, flat critical points, skew products.

moments $\mathbb{E}[\mathcal{M}_\alpha^r] = r!(\Gamma(1+\alpha))^r/\Gamma(1+r\alpha)$, $r \geq 0$, and by variables $\mathcal{Z}_\alpha$ with *generalized arcsine laws*,

$$\Pr\left(\{0 \leq \mathcal{Z}_\alpha \leq t\}\right) = \frac{\sin \pi\alpha}{\pi} \int_0^t \frac{dx}{x^{1-\alpha}(1-x)^\alpha}, \quad t \in [0,1],$$

respectively. We recall that $\mathbb{E}[\mathcal{Z}_\alpha^r] = (-1)^r \binom{-\alpha}{r}$, $r \geq 0$.

To fix notations, let ν be a probability measure on $(X, \mathcal{A})$ and $(R_n)_{n\geq 1}$ a sequence of measurable real functions on X. Then distributional convergence of $(R_n)_{n\geq 1}$ w.r.t. ν to some random variable R will be denoted by $R_n \overset{\nu}{\Longrightarrow} R$. *Strong distributional convergence* $R_n \overset{\mathcal{L}(\mu)}{\Longrightarrow} R$ on the σ-finite measure space $(X, \mathcal{A}, \mu)$ means that $R_n \overset{\nu}{\Longrightarrow} R$ for all probability measures $\nu \ll \mu$.

An approach which only relies on fairly weak conditions regarding the dynamics up to the first visit to a suitable reference set, has been developed in [T3], [TZ], and [Z4]. The purpose of the present note is to show that the same method can be used to improve on individual limit theorems by establishing convergence of their joint distributions, thus clarifying the asymptotic dependencies between them. Applied to a prominent system, our main result takes the following form:

Example 1.1 (**A joint limit theorem for Boole's transformation**). The map $T : \mathbb{R} \to \mathbb{R}$ given by $Tx := x - \frac{1}{x}$ preserves Lebesgue measure λ and is conservative ergodic, cf. [AW]. Let $Y \subseteq \mathbb{R}$ be some fixed bounded interval. By [A1] and [T2] (or, alternatively, [TZ]),

$$\frac{\mathbf{Z}_n(Y)}{n} \overset{\mathcal{L}(\lambda)}{\Longrightarrow} \mathcal{Z} \quad \text{and} \quad \frac{\mathbf{S}_n(1_Y)}{\sqrt{2n/\pi}} \overset{\mathcal{L}(\lambda)}{\Longrightarrow} \lambda(Y)\,\mathcal{M}, \tag{1.1}$$

with $\mathcal{Z} := \mathcal{Z}_{\frac{1}{2}}$ and $\mathcal{M} := \mathcal{M}_{\frac{1}{2}}$ having densities

$$\mathfrak{z}(s) := \frac{1}{\pi\sqrt{s(1-s)}},\ s \in (0,1), \quad \text{and} \quad \mathfrak{m}(t) := \frac{2}{\pi}\, e^{-\frac{t^2}{\pi}},\ t > 0.$$

The result of the present paper determines the asymptotics of the joint distributions, showing that

$$\left(\frac{\mathbf{Z}_n(Y)}{n}, \frac{\mathbf{S}_n(1_Y)}{\sqrt{2n/\pi}}\right) \overset{\mathcal{L}(\lambda)}{\Longrightarrow} \left(\mathcal{Z}, \lambda(Y)\,\mathcal{M}^* \cdot \sqrt{\mathcal{Z}}\right), \tag{1.2}$$

where $\mathcal{Z}$ and $\mathcal{M}^*$ are two independent random variables, $\mathcal{Z}$ as above, and $\mathcal{M}^*$ having density

$$\mathfrak{m}^*(t) := \frac{2t}{\pi}\, e^{-\frac{t^2}{\pi}},\ t > 0.$$

Remark 1.2. Statements (1.1) and (1.2) show, in particular, that $\mathcal{M}^* \cdot \sqrt{\mathcal{Z}} \overset{d}{=} \mathcal{M}$ (in distribution). We briefly indicate how to check this directly: As $\mathcal{Z}$ has the classical arcsine distribution, we have $\Pr(\{0 \leq \sqrt{\mathcal{Z}} \leq t\}) =$

$\frac{2}{\pi}\arcsin t$, so that $\sqrt{\mathcal{Z}}$ has density $\widetilde{\mathfrak{z}}(s) := \frac{2}{\pi}\frac{1}{\sqrt{1-s^2}}$, $s \in (0,1)$. To explicitly compute the density $\mathfrak{d}$ of $\mathcal{M}^* \cdot \sqrt{\mathcal{Z}}$, that is,

$$\mathfrak{d}(t) := \int_t^\infty \mathfrak{m}^*(s)\, \widetilde{\mathfrak{z}}\left(\frac{t}{s}\right) \frac{ds}{s} = \int_t^\infty \frac{2}{\pi} \frac{s\,\mathfrak{m}(s)}{\sqrt{s^2-t^2}}\, ds, \qquad t > 0,$$

let $M(t) := \int_0^t \mathfrak{m}(s)\, ds$, $t > 0$, and observe that

$$\frac{d}{ds} M\left(\sqrt{s^2-t^2}\right) = \frac{2}{\pi} \frac{s}{\sqrt{s^2-t^2}} \frac{\mathfrak{m}(s)}{\mathfrak{m}(t)} \qquad \text{for } 0 < t < s.$$

Use this to evaluate the definite integral to obtain $\mathfrak{d}(t) = \mathfrak{m}(t)(M(\infty) - M(0)) = \mathfrak{m}(t)$ for $t > 0$, as required.

2. A joint limit theorem

Let T be a c.e.m.p.t. on the σ-finite measure space $(X, \mathcal{A}, \mu)$. For $Y \in \mathcal{A}$ with $\mu(Y) > 0$ the *first return (entrance) time* of Y is $\varphi(x) := \min\{n \geq 1 : T^n x \in Y\}$, $x \in X$. For suitably chosen Y, the asymptotics of the *return probabilities* $f_k(Y) := \mu_Y(Y \cap \{\varphi = k\})$, $k \geq 1$, determines the stochastic properties of the system. Here, $\mu_Y(A) := \mu(Y)^{-1}\mu(Y \cap A)$. Distributional limit theorems frequently depend on regular variation of the *tail probabilities* $q_n(Y) := \sum_{k>n} f_k(Y) = \mu_Y(Y \cap \{\varphi > n\})$, or (slightly more general) of the *wandering rate* of Y given by $w_N(Y) := \mu(Y)\sum_{n=0}^{N-1} q_n(Y) = \mu(Y^N)$, where $Y^N := \bigcup_{n=0}^{N-1} T^{-n}Y$, $N \geq 1$. For background material about regular variation we refer to Chapter 1 of [BGT]. We will follow the convention that for $a_n, b_n \geq 0$ and $\vartheta \in [0, \infty)$, $a_n \sim \vartheta \cdot b_n$ as $n \to \infty$ means $\lim_{n\to\infty} a_n/b_n = \vartheta$, even in case $\vartheta = 0$, where it is interpreted as the usual $a_n = o(b_n)$ as $n \to \infty$. Analogously for $f(s) \sim \vartheta \cdot g(s)$ as $s \searrow 0$ etc.

The *transfer operator* $\widehat{T} : L_1(\mu) \to L_1(\mu)$ of T is the positive linear operator characterized by $\int_X (g \circ T) \cdot u\, d\mu = \int_X g \cdot \widehat{T}u\, d\mu$ for all $u \in L_1(\mu)$ and $g \in L_\infty(\mu)$.

A good understanding of the long-term behaviour of $\widehat{T}$ when acting on $\mathcal{D}(\mu) := \{u \in L_1(\mu) : u \geq 0, \mu(u) = 1\}$ is often crucial for probabilistic studies. In the course of [T3], [TZ], and [Z4], an approach which only assumes information on $\widehat{T}$ up to the first entrance into a suitable refernce set Y has been developed. Let $Y_0 := Y$ and $Y_n := Y^c \cap \{\varphi = n\}$ for $n \geq 1$. We require some control of how large a collection of densities on Y we see when considering (averaged and normalized versions of) $\sum_{k>n} \widehat{T}^k 1_{Y\cap\{\varphi=k\}} = \widehat{T}^n 1_{Y_n}$ for $n \geq 1$. (Note that if our system starts with initial distribution μ_Y, then $f_k(Y)^{-1}\widehat{T}^k 1_{Y\cap\{\varphi=k\}}$ is its conditional density at time k, given that $\{\varphi = k\}$.) This can easily be verified for several relevant classes of examples. We'll say that a collection $\mathfrak{H}$ of densities on Y, is *uniformly sweeping* if there is some $K \in \mathbb{N}_0$ such that $\inf_{u\in\mathfrak{H}} \inf_Y \sum_{k=0}^{K} \widehat{T}^k u > 0$.

We are going to prove the following limit theorem. The assumptions on Y are exactly those of Theorem 2.1 in [Z4].

Theorem 2.1 (**Joint limit distributions for $\mathbf{Z}_n$ and $\mathbf{S}_n$**). *Let T be a c.e.m.p.t. on the σ-finite measure space $(X, \mathcal{A}, \mu)$, and assume there is some $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, such that*

$$\mathfrak{H}_Y = \left\{ \frac{1}{w_N(Y)} \sum_{n=0}^{N-1} \widehat{T}^n 1_{Y_n} \right\}_{N\geq 1} \quad \begin{array}{l}\textit{is precompact in } L_\infty(\mu) \\ \textit{and uniformly sweeping}\end{array} \tag{2.1}$$

and that there is some some $\alpha \in [0, 1]$ such that

$$(w_N(Y)) \textit{ is regularly varying of index } 1-\alpha. \tag{2.2}$$

Then, for any $f \in L_1(\mu)$ with $\mu(f) \neq 0$, we have

$$\left(\frac{\mathbf{Z}_n(Y)}{n}, \frac{\mathbf{S}_n(f)}{a_n} \right) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \left(\mathcal{Z}_\alpha, \mu(f)\mathcal{M}_\alpha^* \cdot \mathcal{Z}_\alpha^\alpha \right), \tag{2.3}$$

where

$$a_n := \frac{1}{\mu(Y)} \int_Y \mathbf{S}_n(1_Y)\, d\mu_Y \sim \frac{1}{\Gamma(1+\alpha)\Gamma(2-\alpha)} \cdot \frac{n}{w_n(Y)} \tag{2.4}$$

as $n \to \infty$. Here $\mathcal{M}_\alpha^*$ is a random variable, independent of $\mathcal{Z}_\alpha$, with density $\mathfrak{m}_\alpha^*(t) := t\, \mathfrak{m}_\alpha(t)$, $t \geq 0$, where $\mathfrak{m}_\alpha$ denotes the density of $\mathcal{M}_\alpha$.

The result immediately applies to the collection of examples discussed in [Z4], that is, to a large class of infinite measure preserving interval maps with indifferent fixed points (as in [Z1], generalizing [T1]), to S-unimodal Misiurewicz interval maps with sufficiently flat critical points (as in [Z2]), and to recurrent $\mathbb{Z}$-extensions of Gibbs-Markov maps (as in §7.3 of [Z4]). We refrain from re-stating the details here.

Remark 2.2. Theorem 2.1 in [Z4] asserts that $a_n^{-1}\mathbf{S}_n(f) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mu(f)\mathcal{M}_\alpha$ under the present assumptions. Hence $\mathcal{M}_\alpha^* \cdot \mathcal{Z}_\alpha^\alpha \overset{d}{=} \mathcal{M}_\alpha$, compare Remark 1.2. For general α, the $\mathfrak{m}_\alpha$ are not known explicitly, but this equality in law be checked by calculating moments, as was pointed out in [P] (where this observation was attributed to M.Dwass). In fact, our proof below uses the method of moments, see [ST] for a wealth of background information.

Remark 2.3. If $E \in Y \cap \mathcal{A}$ with $n^{-1}\mathbf{Z}_n(E) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{Z}_\alpha$, then, due to $\mathbf{Z}_n(E) \leq \mathbf{Z}_n(Y)$, we necessarily have $\mathbf{Z}_n(E) - \mathbf{Z}_n(Y) \overset{\mu}{\longrightarrow} 0$, so that Y can be replaced by E in (2.3).

Remark 2.4. a) A Markov-chain version of this result can be found in [P].

b) It would be desirable to establish corresponding results including the other random variables studied in [Z4] (occupation times of suitable infinite-measure sets). Alas, even in a classical Markov-chain setup, no tangible description of the prospective joint limit distributions appears to be available.

3. Analytical tools

Karamata's asymptotic theory of regularly varying functions lies at the heart of many limit theorems for null-recurrent processes, a core result being

Lemma 3.1 (Karamata's Tauberian Theorem, KTT). *Let (b_n) be a sequence in $[0,\infty)$ such that for all $s>0$, $B(s) := \sum_{n\geq 0} b_n\, e^{-ns} < \infty$. Let ℓ be slowly varying, and $\rho, \vartheta \in [0,\infty)$. Then*

$$B(s) \sim \vartheta \left(\frac{1}{s}\right)^{\rho} \ell\left(\frac{1}{s}\right) \qquad \text{as } s \searrow 0, \tag{3.1}$$

iff

$$\sum_{k=0}^{n-1} b_k \sim \frac{\vartheta}{\Gamma(\rho+1)}\, n^{\rho}\ell(n) \qquad \text{as } n \to \infty. \tag{3.2}$$

If (b_n) is eventually monotone and $\rho > 0$, then both are equivalent to

$$b_n \sim \frac{\vartheta\rho}{\Gamma(\rho+1)}\, n^{\rho-1}\ell(n) \qquad \text{as } n \to \infty. \tag{3.3}$$

We will also rely on parts of the somewhat technical Propositions 3.2 and 3.3 of [Z4]. For the reader's convenience, we re-state the relevant assertions as

Lemma 3.2. *Let T be a c.e.m.p.t. on the σ-finite measure space $(X, \mathcal{A}, \mu)$ and $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$. Suppose that $(R_n)_{n\geq 0}$ is a sequence of measurable functions $R_n : X \to [0,\infty)$ such that*

$$R_n \leq R_{n-m} \circ T^m \text{ on } Y_m \quad \text{for } n \geq m \geq 0, \tag{3.4}$$

and

$$\|(R_n \circ T - R_n) \cdot u\|_1 \longrightarrow 0 \qquad \text{for all } u \in L_\infty(\mu) \text{ supported on } Y^M \text{ for some } M = M(u), \tag{3.5}$$

and

$$\{1_{Y^M} \cdot R_n\}_{n\geq 0} \qquad \text{is weakly precompact in } L_1(\mu) \text{ for each } M \geq 1. \tag{3.6}$$

Moreover, let $v_n : Y \to [0,\infty)$, $n \geq 0$, be bounded measurable functions with $\int_Y v_n \, d\mu > 0$, and $(b_n)_{n\geq 0}$ be a sequence in $[0,\infty)$ such that $\sum_{n\geq 0} b_n \, e^{-ns} =: B(s) \in (0,\infty)$ for $s > 0$. Assume that

$$\left\{ \frac{\sum_{k=0}^{n} v_k}{\sum_{k=0}^{n} \int_Y v_k \, d\mu} \right\}_{n\geq 0} \quad \text{is precompact in } L_\infty(\mu), \tag{3.7}$$

and that for some $\vartheta \in [0,\infty)$,

$$\sum_{k=0}^{n} \int_Y v_k \, d\mu \sim \vartheta \cdot \sum_{k=0}^{n} b_k \qquad \text{as } n \to \infty. \tag{3.8}$$

Let $(\gamma_n)_{n\geq 0}$ be a sequence in $[0,\infty)$ with $\sum_{n\geq 0} \gamma_n = \infty$ and such that $G(s) := \sum_{n\geq 0} \gamma_n \, e^{-ns} \in (0,\infty)$ for $s > 0$, and consider the weighed Laplace transform $R_\gamma(s) := \sum_{n\geq 0} R_n \, \gamma_n e^{-ns}$.

***a)** Suppose that for some $\kappa \in [0,\infty)$,*

$$\int_Y \left(\sum_{n\geq 0} v_n \, e^{-ns} \right) \cdot R_\gamma(s) \, d\mu \sim \kappa\vartheta \cdot B(s) \, G(s) \quad \text{as } s \searrow 0. \tag{3.9}$$

If $\vartheta > 0$, then, for any $u \in \mathcal{D}(\mu) \cap L_\infty(\mu)$ with $\int_Y u \, d\mu = 1$,

$$\int_X R_\gamma(s) \cdot u \, d\mu \sim \kappa \cdot G(s) \qquad \text{as } s \searrow 0. \tag{3.10}$$

***b)** Assume that $r = 0$, or that $r \in \mathbb{N}$ and $B \in \mathcal{R}_{-\rho}(0)$ for some $\rho \in [0,\infty)$. Suppose also that for some $\kappa \in [0,\infty)$ and some $u \in \mathcal{D}(\mu) \cap L_\infty(\mu)$ with $\int_Y u \, d\mu = 1$,*

$$\int_X R_\gamma(s) \cdot u \, d\mu \sim \kappa \cdot G(s) \quad \text{as } s \searrow 0, \tag{3.11}$$

then, as $s \searrow 0$,

$$\int_Y \left(\sum_{n\geq 0} n^r v_n \, e^{-ns} \right) \cdot R_\gamma(s) \, d\mu \sim \kappa\vartheta \cdot (-1)^r r! \binom{-\rho}{r} \left(\frac{1}{s} \right)^r B(s) \, G(s). \tag{3.12}$$

***c)** If, in the situation of b), $v_n \searrow 0$ a.e. on Y as $n \to \infty$, so that $v_n = \sum_{k>n} w_k$ with $w_n \geq 0$, $n \geq 1$, measurable, then, for all $r \geq 1$, as $s \searrow 0$,*

$$\int_Y \left(\sum_{n\geq 1} n^r w_n \, e^{-ns} \right) \cdot R_\gamma(s) \, d\mu \sim \kappa\vartheta \cdot (-1)^{r-1} r! \binom{1-\rho}{r} \left(\frac{1}{s} \right)^{r-1} B(s) \, G(s). \tag{3.13}$$

4. Proof of the Theorem

The argument to follow uses the machinery developed in [TZ] and [Z4], and determines the asymptotic behaviour of mixed moments of the variables under consideration.

Proof of Theorem 2.1. We assume w.l.o.g. that $\mu(Y)=1$.

(i) Let $S_n := \sum_{j=1}^{n} 1_Y \circ T^j$, and $Z_n := \mathbf{Z}_n(Y)$, $n \geq 0$. Distributional convergence

$$\left(\frac{Z_n}{n}, \frac{S_n}{a_n}\right) \overset{\mu_Y}{\Longrightarrow} (\,\mathcal{Z}_\alpha, \mathcal{M}_\alpha^* \cdot \mathcal{Z}_\alpha^\alpha\,)\,, \tag{4.1}$$

with $\mathcal{M}_\alpha^*$ and $\mathcal{Z}_\alpha$ independent, will be established by the method of moments, i.e. we prove that, for all integers $l, r \geq 0$,

$$\int_Y \left(\frac{Z_n}{n}\right)^r \left(\frac{S_n}{a_n}\right)^l d\mu \longrightarrow \mathbb{E}[\mathcal{Z}_\alpha^r\,(\mathcal{M}_\alpha^* \cdot \mathcal{Z}_\alpha^\alpha)^l] \quad \text{as } n \to \infty. \tag{4.2}$$

The moments on the right-hand side have been identified in Corollary 3.2 of [P] (the result being attributed to M. Dwass), where it is shown that

$$\mathbb{E}[\mathcal{Z}_\alpha^r\,(\mathcal{M}_\alpha^* \cdot \mathcal{Z}_\alpha^\alpha)^l] = (-1)^r \, \frac{r!\, l!\, (\Gamma(1+\alpha))^l}{\Gamma(1+r+\alpha l)} \binom{-\alpha(l+1)}{r} \quad \text{for } l, r \geq 0. \tag{4.3}$$

It is not hard to see that these moments determine the distribution (see [ST]) of the random vector $(\mathcal{Z}_\alpha, \mathcal{M}_\alpha^* \cdot \mathcal{Z}_\alpha^\alpha)$, compare the proof of Theorem 3.1 of [P].

(ii) Due to KTT and our assumption on the wandering rate we have, for $s > 0$,

$$Q_Y(s) := \sum_{n \geq 0} q_n(Y)\, e^{-ns} = \left(\frac{1}{s}\right)^{1-\alpha} \ell\left(\frac{1}{s}\right) \quad \text{for } s > 0, \tag{4.4}$$
$$\text{and} \quad w_n(Y) \sim \frac{n^{1-\alpha}\ell(n)}{\Gamma(2-\alpha)} \quad \text{as } n \to \infty,$$

with some fixed slowly varying function ℓ. Via the crucial structural assumption (2.1), this information on the return distribution was exploited in [Z4] to obtain information about the moment asymptotics for the individual sequences (S_n) and (Z_n). Indeed, the dissection identities

$$Z_n = \begin{cases} k + Z_{n-k} \circ T^k & \text{on } Y \cap \{\varphi = k\},\ 1 \leq k \leq n, \\ 0 & \text{on } Y \cap \{\varphi > n\}, \end{cases} \quad \text{for } n \geq 0, \tag{4.5}$$

and

$$S_n = \begin{cases} 1 + S_{n-k} \circ T^k & \text{on } Y \cap \{\varphi = k\},\ 1 \leq k \leq n, \\ 0 & \text{on } Y \cap \{\varphi > n\}, \end{cases} \quad \text{for } n \geq 0, \tag{4.6}$$

were used to show (see the proofs of Theorems 2.1 and 2.3 of [Z4]), that for all $r, l \geq 0$,

$$(\spadesuit_r) \qquad \sum_{n\geq 0}\left(\int_Y Z_n^r\,d\mu\right)e^{-ns} \sim (-1)^r\,r!\binom{-\alpha}{r}\left(\frac{1}{s}\right)^{r+1} \quad \text{as } s\searrow 0,$$

$$(\blacklozenge_l) \qquad \sum_{n\geq 0}\left(\int_Y S_n^l\,d\mu\right)e^{-ns} \sim l!\left(\frac{1}{s}\right)^{1+\alpha l}\ell\left(\frac{1}{s}\right)^{-l} \quad \text{as } s\searrow 0.$$

As $\int_Y S_n\,d\mu \sim a_n$ is non-decreasing (since (S_n) is), KTT shows that $(\blacklozenge_1)$ entails (2.4). Analogously, by KTT and monotonicity of $(\int_Y Z_n^r\,d\mu)_{n\geq 1}$ and $(\int_Y S_n^l\,d\mu)_{n\geq 0}$, the relations $(\spadesuit_r)$ and $(\blacklozenge_l)$ are equivalent to

$$(4.7) \qquad \int_Y\left(\frac{Z_n}{n}\right)^r d\mu \longrightarrow \mathbb{E}[\mathcal{Z}_\alpha^r] \quad \text{and} \quad \int_Y\left(\frac{S_n}{a_n}\right)^l d\mu \longrightarrow \mathbb{E}[\mathcal{M}_\alpha^l]$$

as $n\to\infty$, proving distributional convergence of each coordinate variable in (4.1).

(iii) We are going to extend this argument to deal with the joint moments in (4.2), observing first that (4.4) and (2.4) give

$$a_n \sim \frac{n^\alpha}{\Gamma(1+\alpha)\ell(n)} \quad \text{as } n\to\infty.$$

This implies that for $l, r \geq 0$,

$$(4.8) \qquad G^{(r,l)}(s) := \sum_{n\geq 0} n^r a_n^l\,e^{-ns} \sim \frac{\Gamma(1+r+\alpha l)}{\Gamma(1+\alpha)^l}\left(\frac{1}{s}\right)^{1+r+\alpha l}\ell\left(\frac{1}{s}\right)^{-l}$$

as $s\searrow 0$. Now, since each of the sequences $(n^r a_n^l)_{n\geq 0}$ and $(\int_Y Z_n^r\,S_n^l\,d\mu)_{n\geq 0}$ is non-decreasing, KTT now ensures that, for arbitrary $l, r\geq 0$, our claim (4.2) is equivalent to

$$\sum_{n\geq 0}\left(\int_Y Z_n^r\,S_n^l\,d\mu\right)e^{-ns} \sim \mathbb{E}[\mathcal{Z}_\alpha^r\,(\mathcal{M}_\alpha^*\cdot\mathcal{Z}_\alpha^\alpha)^l]\cdot G^{(r,l)}(s)$$

as $s\searrow 0$. Substituting the explicit expression (4.3) for the moments, and (4.8), we thus see that our goal is to prove, for all $l, r\geq 0$, that

$$(\dagger_{r,l}) \quad \sum_{n\geq 0}\left(\int_Y Z_n^r\,S_n^l\,d\mu\right)e^{-ns}$$
$$\sim (-1)^r\,r!\,l!\binom{-\alpha(l+1)}{r}\left(\frac{1}{s}\right)^{1+r+\alpha l}\ell\left(\frac{1}{s}\right)^{-l} \quad \text{as } s\searrow 0.$$

The special cases $(\dagger_{0,l})$ and $(\dagger_{r,0})$ of this assertion coincide with $(\blacklozenge_l)$ and $(\spadesuit_r)$, respectively.

(iv) It has been shown in [Z4] that, for any $l, r \geq 0$, the sequences

$$(R_n) = \left(\left(\frac{Z_n}{n} \right)^r \right) \text{ and } (R_n) = \left(\left(\frac{S_n}{a_n} \right)^l \right) \text{ satisfy (3.4) - (3.6).} \tag{4.9}$$

To prepare the analytic argument below, we need to improve on this, and show that for any $l, r \geq 0$, the sequence given by

$$R_n = R_n^{(r,l)} := \left(\frac{Z_n}{n} \right)^r \left(\frac{S_n}{a_n} \right)^l, \quad n \geq 1,$$

also satisfies the assumptions of Lemma 3.2. Note first that $0 \leq Z_n/n \leq 1$ and

$$\left| \left(\frac{Z_n}{n} \right)^r \circ T - \left(\frac{Z_n}{n} \right)^r \right| \leq r \, \frac{|Z_n \circ T - Z_n|}{n} \xrightarrow{\mu} 0 \quad \text{as } n \to \infty \tag{4.10}$$

(by the mean-value theorem and Lemma 1 of [T2]). Next,

$$Z_n \leq Z_{n-m} \circ T^m \text{ and } S_n = S_{n-m} \circ T^m \quad \text{on } Y_m \text{ for } n \geq m \geq 0, \tag{4.11}$$

and as (a_n) is non-decreasing, we see that (3.4) is satisfied.

Due to (4.7), each moment sequence $(\int_Y (S_n/a_n)^{l+1} d\mu)_{n \geq 1}$ is bounded, so that $(1_Y (S_n/a_n)^l)_{n \geq 1}$ is uniformly integrable for every l. Now (4.11) proves that in fact

$$\left(1_{Y^M} \left(\frac{S_n}{a_n} \right)^l \right)_{n \geq 1} \text{ is uniformly integrable for all } M \geq 1, \tag{4.12}$$

since indeed $\int_{Y_m \cap \{S_n^l > t a_n^l\}} (S_n/a_n)^l \, d\mu \leq \int_{Y \cap \{S_{n-m}^l > t a_{n-m}^l\}} (S_{n-m}/a_{n-m})^l \, d\mu$. Boundedness of the Z_n/n thus guarantees that $(1_{Y^M} R_n)_{n \geq 1}$ is uniformly integrable for all $M \geq 1$, which is (3.6).

We finally have to check (3.5). Fixing M and u, we have

$$\begin{aligned} \|(R_n \circ T - R_n) \cdot u\|_1 \leq & \left\| \left(\left(\frac{S_n}{a_n} \right)^l \circ T - \left(\frac{S_n}{a_n} \right)^l \right) \cdot u \right\|_1 \\ & + r \, \|u\|_\infty \int_{Y^M} \left(\frac{S_n}{a_n} \right)^l \frac{|Z_n \circ T - Z_n|}{n} \, d\mu. \end{aligned}$$

The first expression on the right-hand side tends to zero by (4.9). For the second bit, note that for any $\delta > 0$,

$$\begin{aligned} \int_{Y^M} \left(\frac{S_n}{a_n} \right)^l \frac{|Z_n \circ T - Z_n|}{n} \, d\mu \leq & \, 2 \int_{Y^M \cap \{|Z_n \circ T - Z_n| > \delta n\}} \left(\frac{S_n}{a_n} \right)^l d\mu \\ & + \delta \int_{Y^M} \left(\frac{S_n}{a_n} \right)^l d\mu. \end{aligned}$$

Take any $\varepsilon > 0$. By (4.12), we have $\sup_{n \geq 1} \int_{Y^M} (S_n/a_n)^l \, d\mu < \infty$, so the rightmost expression here is is less than $\varepsilon/2$ for $\delta = \delta_\varepsilon$ small enough. Fixing such a δ_ε, we have $\mu(Y^M \cap \{|Z_n \circ T - Z_n| > \delta_\varepsilon n\}) \to 0$ as $n \to \infty$ by (4.10), and hence $2\int_{Y^M \cap \{|Z_n \circ T - Z_n| > \delta_\varepsilon n\}} (S_n/a_n)^l \, d\mu < \varepsilon/2$ for $n \geq n_\varepsilon$, again by uniform integrability (4.12). This proves (3.5).

It is a trivial matter to check that (3.4) - (3.6) remain valid with $R_n^{(r,l)}$ replaced by some $R_n^{*(r,l)} := (Z_n/n)^r (\Phi_n/a_n)^l$ with measurable functions Φ_n satisfying $S_n \leq \Phi_n \leq 1 + S_n$, $n \geq 1$.

Observe also that due to $S_n \nearrow \infty$ and $1 + S_n \sim S_n$ a.e. we have $\int_Y Z_n^j \Phi_n^l \, d\mu \sim \int_Y Z_n^j S_n^l \, d\mu$ as $n \to \infty$ for such Φ_n. The Laplace transforms of these sequences then satisfy

$$\int_Y \left(\sum_{n \geq 0} Z_n^j \Phi_n^l \, e^{-ns} \right) d\mu \sim \int_Y \left(\sum_{n \geq 0} Z_n^j \, S_n^l \, e^{-ns} \right) d\mu \quad \text{as } s \searrow 0. \tag{4.13}$$

(v) To validate $(\dagger_{r,l})$, we are going to use an inductive argument. Since the statements $(\dagger_{0,l})$, $l \geq 0$, and $(\dagger_{r,0})$, $r \geq 0$, are already contained in $(\blacklozenge_l)$, $l \geq 0$, and $(\spadesuit_r)$, $r \geq 0$, respectively, we need only justify the inductive step. Proving that for any pair (r,l) with $r, l \geq 1$,

$$\left.\begin{array}{ll} (\dagger_{k,i}) & \text{for } 0 \leq i < l \text{ and } k \geq 0, \text{ and} \\ (\dagger_{j,l}) & \text{for } 0 \leq j < r, \end{array}\right\} \quad \text{together imply } (\dagger_{r,l}), \tag{4.14}$$

will therefore complete the proof of (4.1).

To this end, we need to understand the relationship between mixed moments of different orders. The dissection identities (4.5) and (4.6) imply that for any l and r, and any $n \geq 1$,

$$1_Y \, Z_n^r \, S_n^l = \sum_{k=1}^{n} 1_{Y \cap \{\varphi = k\}} ((1 + S_{n-k})^l \circ T^k) \sum_{j=0}^{r} \binom{r}{j} k^{r-j} \cdot (Z_{n-k}^j \circ T^k).$$

Integrating and taking the $j = r$ terms to the left-hand side, we obtain

$$\begin{aligned} &\int_Y \left(Z_n^r \, S_n^l - \sum_{k=1}^{n} \widehat{T}^k 1_{Y \cap \{\varphi = k\}} (1 + S_{n-k})^l Z_{n-k}^r \right) d\mu \\ &\quad = \sum_{j=0}^{r-1} \binom{r}{j} \int_Y \sum_{k=1}^{n} k^{r-j} \, \widehat{T}^k 1_{Y \cap \{\varphi = k\}} \cdot (1 + S_{n-k})^l Z_{n-k}^j \, d\mu. \end{aligned} \tag{4.15}$$

(vi) To tackle (4.14), we now consider the Laplace transform $\sum_{n \geq 0} \cdots e^{-ns}$ of the expression in (4.15), regarded as a sequence in n. Starting from the bit on the right-hand side of (4.15), we obtain the transform

$$\sum_{j=0}^{r-1} \binom{r}{j} \int_Y \left(\sum_{n \geq 1} n^{r-j} \, \widehat{T}^n 1_{Y \cap \{\varphi = n\}} \, e^{-ns} \right) \left(\sum_{n \geq 0} (1 + S_n)^l Z_n^j \, e^{-ns} \right) d\mu. \tag{4.16}$$

Take any $j \in \{0, 1, \ldots, r-1\}$. From (4.13) we see (taking $\Phi_n := 1 + S_n$) that

(4.17)
$$\int_Y \left(\sum_{n\geq 0} (1+S_n)^l Z_n^j \, e^{-ns} \right) d\mu \sim \int_Y \left(\sum_{n\geq 0} Z_n^j \, S_n^l \, e^{-ns} \right) d\mu \quad \text{as } s \searrow 0,$$

the asymptotics of which is given by $(\dagger_{j,l})$. We can now apply Lemma 3.2.c with $R_n := R_n^{*(j,l)}$ obtained from $\Phi_n := 1 + S_n$, $\gamma_n := a_n^l n^j$ (which, by step (iv) above satisfies all assumptions), and $w_n := \widehat{T}^n 1_{Y \cap \{\varphi = n\}}$ (which due to (2.1) has the required properties), $b_n := q_n(Y)$, $\vartheta := 1$, $B := Q_Y$, $\rho := 1 - \alpha$, and $G := G^{(j,l)}$, to obtain

$$\begin{aligned} &\binom{r}{j} \int_Y \left(\sum_{n\geq 1} n^{r-j} \, \widehat{T}^n 1_{Y \cap \{\varphi = n\}} \, e^{-ns} \right) \left(\sum_{n\geq 0} (1+S_n)^l Z_n^j \, e^{-ns} \right) d\mu \\ &\sim (-1)^{r+1} r! \, l! \binom{-\alpha(l+1)}{j} \binom{\alpha}{r-j} \cdot \left(\frac{1}{s} \right)^{1+r+\alpha(l-1)} \ell \left(\frac{1}{s} \right)^{1-l} \end{aligned}$$

as $s \searrow 0$. Summing over j, we see that the expression in (4.16) is asymptotically equivalent, as $s \searrow 0$, to

$$(-1)^r r! \, l! \left[\binom{-\alpha(l+1)}{r} - \binom{-\alpha l}{r} \right] \cdot \left(\frac{1}{s} \right)^{1+r+\alpha(l-1)} \ell \left(\frac{1}{s} \right)^{1-l} . \tag{4.18}$$

(vii) Turning to the left-hand side of (4.15), we again take the Laplace transform, which equals

$$\begin{aligned} &\int_Y \left(\sum_{n\geq 0} Z_n^r S_n^l \, e^{-ns} - \left(\sum_{n\geq 1} \widehat{T}^n 1_{Y \cap \{\varphi = n\}} \, e^{-ns} \right) \left(\sum_{n\geq 0} (1+S_n)^l Z_n^r \, e^{-ns} \right) \right) d\mu \\ &\qquad = \int_Y \left(1 - \sum_{n\geq 1} \widehat{T}^n 1_{Y \cap \{\varphi = n\}} \, e^{-ns} \right) \left(\sum_{n\geq 0} Z_n^r S_n^l \, e^{-ns} \right) d\mu \\ &\qquad\quad - l \int_Y \left(\sum_{n\geq 0} Z_n^r \Phi_{l,n}^{l-1} \, e^{-ns} \right) \left(\sum_{n\geq 1} \widehat{T}^n 1_{Y \cap \{\varphi = n\}} \, e^{-ns} \right) d\mu. \end{aligned} \tag{4.19}$$

Here, for any $l, n \geq 1$, we let $\Phi_{l,n} : X \to [0, \infty)$ denote the measurable function with $(1 + S_n)^l - S_n^l = l \, \Phi_{l,n}^{l-1}$ and $S_n \leq \Phi_{l,n} \leq S_n + 1$, which is provided by the mean-value theorem. Focusing on the rightmost term, we find that

$$
\begin{aligned}
(4.20)\qquad & l \int_Y \left(\sum_{n\geq 0} Z_n^r \, \Phi_{l,n}^{l-1} \, e^{-ns} \right) \left(\sum_{n\geq 1} \widehat{T}^n 1_{Y\cap\{\varphi=n\}} \, e^{-ns} \right) \, d\mu \\
& \sim l \int_Y \left(\sum_{n\geq 0} Z_n^r \, \Phi_{l,n}^{l-1} \, e^{-ns} \right) \, d\mu \\
& \sim l \int_Y \left(\sum_{n\geq 0} Z_n^r \, S_n^{l-1} \, e^{-ns} \right) \, d\mu \\
& \sim (-1)^r r! \, l! \binom{-\alpha l}{r} \cdot \left(\frac{1}{s} \right)^{1+r+\alpha(l-1)} \ell\left(\frac{1}{s} \right)^{1-l} \qquad \text{as } s \searrow 0,
\end{aligned}
$$

where we first use the fact that

$$
\sum_{n\geq 1} \widehat{T}^n 1_{Y\cap\{\varphi=n\}} \, e^{-ns} \nearrow 1_Y \qquad \text{uniformly as } s \searrow 0
$$

(which, under the present assumptions, has been proved as statement (4.5) in [Z4]), and then (4.13) above. The third step uses $(\dagger_{r,l-1})$, which holds by assumption.

(viii) Recall that (4.16) and (4.19) coincide (being the Laplace transforms of the two expressions in (4.15)). Since (4.20) is of the same order as (4.18), we can combine these two observations to conclude that

$$
\begin{aligned}
(4.21)\qquad & s \int_Y \left(\sum_{n\geq 1} \widehat{T}^n 1_{Y_n} \, e^{-ns} \right) \left(\sum_{n\geq 0} Z_n^r \, S_n^l \, e^{-ns} \right) \, d\mu \\
& \sim (-1)^r r! \, l! \binom{-\alpha(l+1)}{r} \cdot \left(\frac{1}{s} \right)^{1+r+\alpha(l-1)} \ell\left(\frac{1}{s} \right)^{1-l} \qquad \text{as } s \searrow 0.
\end{aligned}
$$

Here we have used identity (5.3) of [TZ] which states that

$$
1_Y - \sum_{n\geq 1} \widehat{T}^n 1_{Y\cap\{\varphi=n\}} \, e^{-ns} = (1-e^{-s}) \sum_{n\geq 0} \widehat{T}^n 1_{Y_n} \, e^{-ns} \qquad \text{a.e.}
$$

Letting $R_n := R_n^{(r,l)} = (Z_n/n)^r (S_n/a_n)^l$, $\gamma_n := n^r a_n^l$, $G := G^{(r,l)}$, $v_n := \widehat{T}^n 1_{Y_n}$, and b_n, B, ϑ, ρ as before, we can now apply Lemma 3.2.a to the integral on the left-hand side of (4.21) to obtain $(\dagger_{r,l})$. This completes the proof of (4.1).

(ix) As a straightforward consequence of Hopf's ratio ergodic theorem (see [A0] or [Z3]), we may replace S_n by $\mu(f)^{-1}\mathbf{S}_n(f)$ in (4.1) whenever $f \in L_1(\mu)$ and $\mu(f) \neq 0$.

According to Theorem 1 of [Z5], the now established distributional convergence $\overset{\mu_Y}{\Longrightarrow}$ w.r.t. μ_Y entails strong distributional convergence $\overset{\mathcal{L}(\mu)}{\Longrightarrow}$ if

$$\left| \left(\frac{Z_n}{n}, \frac{\mathbf{S}_n(f)}{a_n} \right) \circ T - \left(\frac{Z_n}{n}, \frac{\mathbf{S}_n(f)}{a_n} \right) \right| \xrightarrow{\mu} 0,$$

and the latter is immediate from the corresponding asymptotic invariance statements for (Z_n) and (S_n) already recorded in [Z4], cf. (4.9). This completes the proof of the theorem. □

References

[A0] J. Aaronson: *An Introduction to Infinite Ergodic Theory.* AMS 1997.

[A1] J. Aaronson: *The asymptotic distributional behaviour of transformations preserving infinite measures.* Journal d'Analyse Mathématique **39** (1981), 203-234.

[A2] J. Aaronson: *Random f-expansions.* Ann. Probab. **14** (1986), 1037-1057.

[AW] R.L. Adler, B. Weiss: *The ergodic infinite measure preserving transformation of Boole.* Isr. J. Math. **16** (1973), 263-278.

[BGT] N.H. Bingham, C.M. Goldie, J.L. Teugels: *Regular Variation.* Cambridge University Press 1989.

[DK] D.A. Darling, M. Kac: *On occupation times for Markoff processes.* Trans. Amer. Math. Soc. **84** (1957), 444-458.

[D] E.B. Dynkin: *Some limit theorems for sums of independent random variables with infinite mathematical expectation.* Selected Transl. in Math. Statist. and Probability **1** (1961), 171-189.

[L] J. Lamperti: *Some limit theorems for stochastic processes.* J. Math. Mech. **7** (1958), 433-448.

[P] S.C. Port: *Some Theorems on Functionals of Markov Chains.* Ann. Math. Stat. **35** (1964), 1275-1290.

[ST] J.A. Shohat, J.D. Tamarkin: *The Problem of Moments.* AMS Mathematical Surveys **2** (1943).

[T1] M. Thaler: *Transformations on [0,1] with infinite invariant measures.* Isr. J. Math. **46** (1983), 67-96.

[T2] M. Thaler: *The Dynkin-Lamperti Arc-Sine Laws for Measure Preserving Transformations.* Trans. Amer. Math. Soc. **350** (1998), 4593-4607.

[T3] M. Thaler: *A limit theorem for sojourns near indifferent fixed points of one-dimensional maps.* Ergod. Th. & Dynam. Sys. **22** (2002), 1289-1312.

[TZ] M. Thaler, R. Zweimüller: *Distributional limit theorems in infinite ergodic theory.* Probab. Theory Relat. Fields **135** (2006), 15-52.

[Z1] R. Zweimüller: *Ergodic properties of infinite measure preserving interval maps with indifferent fixed points.* Ergod. Th. & Dynam. Sys. **20** (2000), 1519-1549.

[Z2] R. Zweimüller: *S-unimodal Misiurewicz maps with flat critical points.* Fundamenta Math. **181** (2004), 1-25.

[Z3] R. Zweimüller: *Hopf's ratio ergodic theorem by inducing.* Colloq. Math. **101** (2004), 289-292.

[Z4] R. Zweimüller: *Infinite measure preserving transformations with compact first regeneration.* Journal d'Analyse Mathématique **103** (2007), 93-131.

[Z5] R. Zweimüller: *Mixing limit theorems for ergodic transformations.* J. Theor. Probab. **20** (2007), 1059-1071.

[Z6] R. Zweimüller: *Measure preserving transformations similar to Markov shifts.* Isr. J. Math. **173** (2009), 421-443.

Faculy of Mathematics, University of Vienna, Nordbergstrasse 15, 1090 Vienna, A

E-mail address: david.kocheim@univie.ac.at

Department of Mathematics, University of Surrey, Guildford GU2 7XH, UK

E-mail address: rzweimue@member.ams.org