UNIDIRECTIONALLY COUPLED INTERVAL MAPS: BETWEEN DYNAMICS AND STATISTICAL MECHANICS

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ABSTRACT. We study infinite systems of unidirectionally coupled interval maps on the lattice $\mathbb{N}^d$. It is assumed that the single site map τ is piecewise C^2 expanding, mixing and satisfies stably a Lasota-Yorke type condition, and that the interaction is such that each lattice site i is influenced only by sites from $i+\mathbb{N}^d$. We concentrate on properties of time-invariant probability measures whose finite-dimensional conditional distributions are absolutely continuous w.r.t. the corresponding finite-dimensional Lebesgue measure. For sufficiently weak interactions we prove:

- Any $d \geq 1$: Two measures from this class are different if and only if their restrictions to the spatial tail field are different. (The existence of such measures was established previously.)
- $d = 1$: If the interaction is superexponentially decreasing and if τ is continuous, then there is a unique measure in this class. It has exponentially decreasing correlations both in time and in space.

The key to these results is a probabilistic coupling procedure for nonautonomous dynamical systems.

1. INTRODUCTION

Spatially extended systems of mutually interacting low dimensional dynamical systems can be studied from two complementary points of view: Locally, i.e. focusing only on finite sets of coordinates, its evolution looks like that of a finite-dimensional *nonautonomous dynamical system*. Globally, however, an aspect known from statistical mechanics on infinite lattices must be added: The measurable dynamics of the infinite system restricted to its spatial Borel tail field. In order to make this idea precise it is helpful to think first of situations where finite subsystems of any size can be picked in such a way that the subsystem is influenced by its infinite environment (so it is a nonautonomous dynamical system) whereas the evolution of this environment is autonomous, i.e. unaffected by the subsystem. One instance of such a situation is the case of mean field coupled systems as treated in [16]. Here we study another class of models of this type - systems with one-sided interactions.

To be more precise, we start with a piecewise $\mathcal{C}^2$expanding interval map $\tau : [0,1] \to [0,1]$, let G be some countable set, and denote

$$\Omega := [0,1]^G = \{\omega = (\omega_i)_{i\in G} : \omega_i \in [0,1]\}$$

equipped with the compact, metrizable product topology. We define the infinite direct product action of τ by itself to be

$$T_0 : \Omega \to \Omega,\ (T_0\omega)_i = \tau\omega_i\ .$$

Throughout we assume that the single site map τ

- satisfies *stably a Lasota-Yorke inequality* and
- is *mixing*,

see Definitions 2 and 3 below. (Prototypical examples of such maps are symmetric tent maps with slope $> \sqrt{2}$.) The *interaction* (of *strength* ϵ) is modeled by a map $\Phi_\epsilon : \Omega \to \Omega$ which is $\mathcal{C}^2$-close to Id_Ω in the sense of Definition 4. To prescribe which sites may influence which ones, we make

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the index set a reflexive directed graph $\mathcal{G} = (G, \to)$, $j \to i$ meaning that the configuration at site j may have some effect on that at site i, i.e. $(\Phi_\epsilon(\omega))_i$ shall not depend on ω_j unless $j \to i$. We restrict our attention to situations with one-sided interaction only, that is,

- $\mathcal{G}$ contains no loops of length larger than one,

and we assume that

- for any $i \in G$ there is only a finite number of loop-free paths in $\mathcal{G}$ starting at i.

Typical examples to be kept in mind are the graphs $\mathcal{G}_d$, $d \geq 1$, given by $G_d := \mathbb{N}^d$, with $j \to i$ if and only if $i \leq j$ in each component. Now the map representing the interacting system we want to study is

$$T_\epsilon := \Phi_\epsilon \circ T_0 : \Omega \to \Omega \ .$$

We are interested in the *ergodic properties* of this map "with respect to Lebesgue measure". Since Ω is an *infinite-dimensional* unit cube, this is not to be understood as "with respect to the infinite product Lebesgue measure on Ω". (Recall that the infinite product of any probability measure on $[0, 1]$ which has nonconstant density with respect to the Lebesgue measure on $[0, 1]$ is singular to the infinite product Lebesgue measure.) Therefore we focus on T_ϵ-invariant probability measures all of whose finite-dimensional conditional distributions are absolutely continuous with respect to the corresponding finite-dimensional product Lebesgue measure.

Under similar assumptions as above it was proved in [18, 20] (see also [14]) that there is some $\epsilon_0 > 0$ such that for all $\epsilon \in [0, \epsilon_0)$ there exists a T_ϵ-invariant measure μ with absolutely continuous finite-dimensional *marginal* distributions whose densities are of bounded variation. The methods used in those articles are, however, far from being suitable to prove uniqueness of this measure. This is what we attempt to do for the much more restricted class of one-sided interactions in this paper. (Since the term "coupling" will be used in a precise probabilistic sense later, which has nothing to do with couplings between maps, we will always call these latter ones "interactions".) Our assumptions, both on the single-site map τ and on the interaction Φ_ϵ are much weaker than those made in [2, 8, 10, 12], and it is difficult to imagine that the beautiful functional analytic results from [2, 10] can be paralleled in the setting studied in this paper. One reason for this is that in our situation (as remarked in [1, p.976]) typically there is no conjugacy between the coupled and the uncoupled system.

Notation. We collect a few remarks on the graph $\mathcal{G}$, and fix some notations. Notice that any finite set $\Lambda \subset G$ has a finite *convex hull in* $\mathcal{G}$, i.e. a smallest set $conv(\Lambda) \supseteq \Lambda$ which is *convex in* $\mathcal{G}$ in the sense that it contains any path joining two of its elements. A set $\Lambda \subset G$ will be called *accessible from* $\Lambda' \subset G$ if and only if there is some path in $\mathcal{G}$ which starts in Λ' and ends in Λ, and analogously for single vertices.

Observe that it is possible to choose an increasing sequence $(\Lambda_p)_{p \geq 1}$ (not necessarily unique) of sets $\Lambda_p \subset G$ with $|\Lambda_p| = p$ and $\bigcup_{p \geq 1} \Lambda_p = G$ such that for any $p \geq 1$, Λ_p^c is inaccessible from Λ_p. For our main example $\mathcal{G} = \mathcal{G}_d$ this is obvious, in the general case it is a simple exercise. We shall call $(\Lambda_p)_{p \geq 1}$ a *$\mathcal{G}$-exhausting sequence* in this case. For any $\Lambda \subset G$ we set $\Omega_\Lambda := [0, 1]^\Lambda$ regarded as a Cartesian factor of Ω, let $\mathcal{F}_\Lambda$ denote the product Borel σ-algebra on Ω_Λ, and $\pi_\Lambda : \Omega \to \Omega_\Lambda$ denote the canonical projection. It is easy to see that the *spatial tail field* $\mathcal{F}_{tail} := \bigcap_{p \geq 1} \pi_{\Lambda_p^c}^{-1} \mathcal{F}_{\Lambda_p^c}$ does not depend on the particular choice of $(\Lambda_p)_{p \geq 1}$.

With these notations we make more precise the notion of a "probability measure on Ω all of whose finite-dimensional conditional distributions are absolutely continuous with respect to the corresponding finite-dimensional product Lebesgue measure".

Definition 1. *The probability measure ν on $(\Omega, \mathcal{F}_G)$ belongs to the class ACC_Ω if, for each finite $\Lambda \subset G$, ν has a $\mathcal{F}_{\Lambda^c}$-measurable family $(\nu_{\Lambda,\omega} | \omega \in \Omega_{\Lambda^c})$ of conditional probability distributions on Ω_Λ such that all $\nu_{\Lambda,\omega}$ are absolutely continuous with respect to the $|\Lambda|$-dimensional Lebesgue measure on this cube. The set of T_ϵ-invariant measures in ACC_Ω will be denoted by $ACC_\Omega(T_\epsilon)$.*

Remark 1. If ν is a probability measure on $(\Omega, \mathcal{F}_G)$ for which the above condition is fulfilled for the special sets $\Lambda = \Lambda_p$, $p \geq 1$, from some $\mathcal{G}$-exhausting sequence, then $\nu \in ACC_\Omega$.

In [16, Lemma 4] it was proved that measures satisfying suitable bounds on the variation of the densities of finite-dimensional *marginal distributions* actually belong to ACC_Ω.

Our first result is:

Theorem 1. **(Existence and tail-field reduction of invariant measures in ACC_Ω)** *Let the mixing map τ satisfy stably a Lasota-Yorke inequality (see Definition 2). Suppose that $(\Phi_\epsilon)_{0\leq\epsilon<\epsilon_0}$ is a family of one-sided interactions on Ω whose $\mathcal{C}^2$-distance to Id_Ω is at most $\kappa\epsilon$ for some $\kappa>0$ (see Definition 4). Then there is $\epsilon_1\in(0,\epsilon_0)$ such that for all $\epsilon\in[0,\epsilon_1)$ the class $ACC_\Omega(T_\epsilon)$ is nonempty and such that for all $\mu,\tilde\mu\in ACC_\Omega(T_\epsilon)$ holds*

$$\mu=\tilde\mu \qquad \Longleftrightarrow \qquad \mu_{|\mathcal{F}_{tail}}=\tilde\mu_{|\mathcal{F}_{tail}}\ .$$

This result is the infinite systems analogue of the statement that in a finite-dimensional system with moderate interaction strength there exists a unique absolutely continuous invariant probability measure. See [15] for an example of such a result.

The investigation of the spatial tail field $\mathcal{F}_{tail}$ requires stronger assumptions: we restrict to the one-dimensional lattice $\mathcal{G}_1=(\mathbb{N},\geq)$ and require that the interaction *decays superexponentially* (see Definition 5).

Theorem 2. **(Uniqueness of the invariant measure for $\mathcal{G}_1=(\mathbb{N},\geq)$)** *If, in the situation of Theorem 1, we assume additionally that $\mathcal{G}=\mathcal{G}_1=(\mathbb{N},\geq)$, that τ is continuous, and that the interactions decay superexponentially, then there is $\epsilon_2\in(0,\epsilon_1)$ such that for all $\epsilon\in[0,\epsilon_2)$ there is a unique measure $\mu_\epsilon\in ACC_\Omega(T_\epsilon)$.*

Remark 2. Denote by σ the spatial left shift on $\Omega=[0,1]^{\mathbb{N}}$. If the interaction is shift invariant, i.e. if $\Phi_\epsilon\circ\sigma=\sigma\circ\Phi_\epsilon$, then also $T_\epsilon\circ\sigma=\sigma\circ T_\epsilon$, so that $\sigma\mu_\epsilon\in ACC_\Omega(T_\epsilon)$, and Theorem 2 shows that μ_ϵ is shift invariant.

A slight modification of the argument that proves Theorem 2 provides information about the mixing properties in time and space of the measure μ_ϵ. Denote by $Lip_f(\Omega)$ the set of functions $\psi:\Omega\to\mathbb{R}$ which only depend on a finite number of coordinates and are Lipschitz in each of them with common Lipschitz constant denoted by $\mathrm{Lip}(\psi)$.

Theorem 3. **(Exponential spatiotemporal mixing for $\mathcal{G}_1=(\mathbb{N},\geq)$)** *In the situation of Theorem 2 there are $q\in(0,1)$ and $Q,K>0$ such that for any $\varphi,\psi\in Lip_f(\Omega)$ which depend only on the first M coordinates we have*

$$\left|\int(\varphi\circ T_\epsilon^l)\cdot(\psi\circ\sigma^m)\,d\mu_\epsilon-\int\varphi\,d\mu_\epsilon\cdot\int\psi\circ\sigma^m\,d\mu_\epsilon\right|\leq K\,Q^M\,L_M(\varphi,\psi)\cdot q^{l+m}$$

for $l,m\geq 0$, where $L_M(\varphi,\psi):=(\|\varphi\|_\infty\|\psi\|_\infty+\|\varphi\|_\infty M\,\mathrm{Lip}(\psi)+\|\psi\|_\infty M\,\mathrm{Lip}(\varphi))$ and σ denotes the spatial left shift on Ω.

Remark 3. The results for the one-dimensional situation $\mathcal{G}=(\mathbb{N},\geq)$ have straightforward extensions to the case $\mathcal{G}=(\mathbb{Z},\geq)$ (with all other assumptions as before). This is because the system T_ϵ on $\Omega=[0,1]^{\mathbb{Z}}$ can be exhausted by the projective family of autonomous factors $T_{\epsilon,n}$ on $\Omega_n:=[0,1]^{G_n}$, $G_n:=\{i\in\mathbb{Z}:i\geq-n\}$, $n\geq 0$, each of which is of the previously discussed type. The family of their respective invariant measures $\mu_{\epsilon,n}\in ACC_{\Omega_n}(T_{\epsilon,n})$ determines a Borel probability measure μ_ϵ on Ω with marginals $\mu_{\epsilon,n}$ which is clearly T_ϵ-invariant. (In fact, more careful arguments show that μ_ϵ has absolutely continuous conditional measures on any Ω_Λ given $\mathcal{F}_{\Lambda^c}$, $\Lambda\subset\mathbb{Z}$ finite.)

Moreover, by uniqueness of the $\mu_{\epsilon,n}$ any T_ϵ-invariant ν on Ω with marginals $\nu_n\in ACC_{\Omega_n}$ for all $n\geq 0$ must coincide with μ_ϵ since the ν_n are necessarily $T_{\epsilon,n}$-invariant.

The system $(\Omega,T_\epsilon,\mu_\epsilon)$ also shares the mixing properties of Theorem 3 since these depend on the action of the $T_{\epsilon,n}$ only.

Remark 4. None of the arguments to follow depends on the fact that T_0 is the product of a collection of *identical* single site maps τ. Everything below works just as well if $(\tau_i)_{i\in G}$ is a family of maps which satisfy the assumptions on τ uniformly in $i\in G$, and $(T_0(\omega))_i:=\tau_i(\omega_i)$.

Remark 5. E. and M. Järvenpää [11] constructed examples of coupled maps possessing infinitely many invariant measures with absolutely continuous finite-dimensional marginals. In order to compare this result to our uniqueness result in Theorem 2 we have to list carefully those ingredients of their construction which do not fit directly our assumptions and to compare our conclusion to the properties of the mesaures they provide. Since their construction works in any lattice dimension, we can restrict the discussion to the case $d = 1$.

There are essentially three features of their constuction that do not fit our assumptions.

(1) They construct examples with two-sided interactions. Their basic absolute continuity assertion is Proposition 3.2: only the conclusion of that Proposition is used in their further reasoning. For one-sided interactions the proof of that Proposition breaks down in their Lemma 3.1 because the transversality condition checked there does no longer hold. However, if the index sets Λ and $\tilde{\Lambda}$ in that Lemma and in Proposition 3.2 are assumed to be disjoint (instead of $\Lambda \subset \tilde{\Lambda}$), the absolute continuity assertion remains true. Now the case $\Lambda \subset \tilde{\Lambda}$ can be reduced to this situation by considering first the two disjoint sets Λ and $\tilde{\Lambda} \setminus \Lambda$ and arguing then that the measure in question (the one associated with Λ and $\tilde{\Lambda}$) is absolutely continuous in view of a Fubini argument.

(2) The coupling strength in [11] between sites j and l decareases like $2^{-|j-l|}$. This is not essential, however. Indeed, one can replace this weight by a sequence of strictly positive weights $w_{|i-j|}$ decaying superexponentially if the metric $d(x, y)$ in the proof of [11, Lemma 3.1] is changed accordingly.

(3) Finally we assume continuity of the single site map $\tau : [0, 1] \to [0, 1]$ in our Theorem 2 whereas the examples in [11] are constructed with $\tau(x) = 3x \mod 1$. However, the proof there would work just as well with a modification of τ where the middle branch is decreasing giving rise to a continuous map.

In conclusion it seems that it is possible to modify the construction in [11] in such a way that the resulting systems meet the assumptions of our Theorem 2 (with some inessential modifications). Therefore we must focus on the absolute continuity properties of the invariant measures. Following the original point of view of [8], see also [6, 7, 18], the Järvenpääs construct measures with absolutely continuous finite dimensional marginals, whereas we assert uniqueness for measures with absolutely continuous finite dimensional conditional distributions. So the measures constructed in [11] have the property that their *singular* finite dimensional conditional distributions integrate to absolutely continuous finite dimensional marginal distributions.

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2. The single site map τ

A map $\tau : [0, 1] \to [0, 1]$ is *piecewise $\mathcal{C}^2$ expanding*, if there are $0 = a_0 < a_1 < \cdots < a_N = 1$ such that $\tau_{|(a_{i-1}, a_i)}$ extends to a $\mathcal{C}^2$map on $[a_{i-1}, a_i]$ and if there is an iterate τ^k for which $\alpha_k := \inf\{|(\tau^k)'(x)| : x \in [0, 1], \tau^k \text{ differentiable at } x\} > 2$. [1]

The ergodic theory of such maps is well understood in terms of its Perron-Frobenius (or transfer) operator $\mathbf{P}_\tau : L^1_{dx} \to L^1_{dx}$ defined by

$$\int_0^1 (\psi \circ \tau) \cdot \varphi \, dx = \int_0^1 \psi \cdot \mathbf{P}_\tau \varphi \, dx \quad \forall \varphi \in L^1_{dx} \, \forall \psi \in L^\infty_{dx} \tag{1}$$

In fact, $\mathbf{P}_\tau$ acts on the Banach space $BV[0, 1]$, the space of complex valued functions of bounded variation on $[0, 1]$ and satisfies the following inequality originally due to Lasota and Yorke [21]

$$\mathrm{var}_{[0,1]}(\mathbf{P}^k_\tau f) \leq \frac{2}{\alpha_k} \mathrm{var}_{[0,1]}(f) + C_{\tau,k} \int_0^1 |f| \, dx \tag{2}$$

where $C_{\tau,k}$ depends continuously on α_k, on the second derivatives of τ^k, and, most importantly, on $\delta^{-1}_{\tau,k}$, where $\delta_{\tau,k}$ is the length of the shortest monotonicity interval of τ^k. [2] We remark already here that, while α_k and τ'' are stable under $\mathcal{C}^2$-perturbations of τ, this is not necessarily true for

[1] It is of course sufficient to require $\inf |(\tau^j)'| > 1$ for some j, but for what follows we wish to explicitly identify an integer k with $\inf |(\tau^k)'| > 2$.

[2] In most references $\mathrm{var}_{[0,1]}(f)$ denotes the variation of f *inside* the interval $[0, 1]$. The inequality is equally valid (with possibly different $C_{\tau,k}$), if $\mathrm{var}_{[0,1]}(f)$ denotes the variation of $f \cdot 1_{[0,1]}$ on all of $\mathbb{R}$ so that the jumps from $f(0)$

$\delta_{\tau,k}^{-1}$. If, however, $\tau^\ell a_i \notin \{a_0,\dots,a_N\}$ for all $\ell = 1,\dots,k$ and all $i = 1,\dots,N$, it is easy to see that also $\delta_{\tau,k}^{-1}$ is stable. Therefore we make the following definition:

Definition 2. *The piecewise $\mathcal{C}^2$expanding map τ* satisfies stably a Lasota-Yorke inequality, *if it satisfies estimate (2) and if for the integer k involved in (2) holds: $\tau^\ell a_i \notin \{a_0,\dots,a_N\}$ for all $\ell = 1,\dots,k$ and all $i = 1,\dots,N$.*

A recent discussion of this stability problem and further references can be found in [3].

It is well known (see e.g. [5] for reference) that a map τ satisfying inequality (2) has a maximal invariant probability density h of bounded variation which decomposes into finitely many ergodic components each of which is mixing up to a finite rotation.

Definition 3. *τ is* mixing, *if it is mixing for its maximal invariant probability density h. (Simple sufficient conditions for τ to be mixing are given in* [4]*.)*

From the spectral properties of $\mathbf{P}_\tau$ it follows that if τ is mixing with respect to $h\,dx$, then it is in fact exponentially mixing. Even more, in this situation there are constants $C > 0$ and $\vartheta \in (0,1)$ such that

$$\mathrm{var}_{[0,1]}(\mathbf{P}_\tau^n\psi) \le C\,\vartheta^n\ \mathrm{var}_{[0,1]}(\psi)$$

for all $n > 0$ and all $\psi \in BV[0,1]$ with $\int_0^1 \psi(x)dx = 0$. In particular, for such ψ we have

$$\int_0^1 |\mathbf{P}_\tau^n\psi|\,dx \le C\,\vartheta^n\ \mathrm{var}_{[0,1]}(\psi). \tag{3}$$

Since the iteration of T_ϵ at a single site can be looked at as a composition of a sequence of small deterministic perturbations of τ, we need the following lemma:

Lemma 1. *If the piecewise $\mathcal{C}^2$expanding map τ is mixing and satisfies stably a Lasota Yorke inequality, then there are constants $K, C_0 > 0$, $\gamma \in (0,1)$ and $\epsilon^* > 0$ such that the following holds: If $\sigma_1, \sigma_2, \dots : [0,1] \to [0,1]$ are $\mathcal{C}^2$and such that $\|\sigma_i - \mathrm{Id}_{[0,1]}\|_{\mathcal{C}^2} < \epsilon^*$ for all $i = 1,2,\dots$, then the Perron-Frobenius operator $\mathbf{P}^{(n)}$ of the composition $\sigma_n\tau\dots\sigma_2\tau\sigma_1\tau$ satisfies the Lasota Yorke type inequality*

$$\mathrm{var}_{[0,1]}(\mathbf{P}^{(n)}f) \le K\,\gamma^n\ \mathrm{var}_{[0,1]}(f) + C_0\,\|f\|_1 \quad \textit{for all } f \in BV[0,1]$$

and

$$\int_0^1 |\mathbf{P}^{(n)}\psi|\,dx \le \mathrm{var}_{[0,1]}(\mathbf{P}^{(n)}\psi) \le K\,\gamma^n\ \mathrm{var}_{[0,1]}(\psi) \quad \textit{for all } \psi \in BV[0,1] \textit{ with } \int_0^1 \psi(x)dx = 0.$$

Proof: Since the constant ϑ in (3) is in general completely unrelated to the contraction coefficient $\frac{2}{\alpha_k}$ from (2), the uniformity is not obvious. Denote $\|f\|_1 = \int_0^1 |f|\,dx$, $\|f\|_{BV} = \mathrm{var}_{[0,1]}(f) + \|f\|_1$ and define the following norm on operators P from BV to BV: $|||P||| = \sup\{\|Pf\|_1 : f \in BV, \|f\|_{BV} \le 1\}$.

Before we are entering the estimates observe the following. From the discussion preceding Definition 2 it follows that if ϵ^* is sufficiently small, then inequality (2) holds with *uniform constants* $\frac{2}{\alpha_{(k)}} \in \left(\frac{2}{\alpha_k}, 1\right)$ and $C_{(k)} > C_k$ whenever $\mathbf{P}_\tau^k$ in (2) is replaced by the Perron-Frobenius operator of some composition $\sigma_{l+k}\tau\dots\sigma_{l+1}\tau$. Iterating the inequality we see that

$$\mathrm{var}_{[0,1]}(\mathbf{P}^{(rk)}f) \le \left(\frac{2}{\alpha_{(k)}}\right)^r \mathrm{var}_{[0,1]}(f) + \tilde{C}\|f\|_1 \quad \text{with } \tilde{C} := \frac{C_{(k)}}{1 - \frac{2}{\alpha_{(k)}}} \tag{4}$$

for all $r \in \mathbb{N}$. A routine argument using (2) for $k = 1$ (when $\frac{2}{\alpha_{(1)}}$ may be larger than 1) extends this estimate to iterates which are not multiples of k. This proves the first estimate.

Let C and ϑ be the constants from (3). Fix $\delta > 0$ such that

$$\delta + \sqrt{\delta\tilde{C}} < 1$$

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and $f(1)$ to 0 at the endpoints of the interval are also taken into account. This interpretation of the variation is consistent with our later use for functions of several variables.

and choose $r \in \mathbb{N}$ such that

$$2C\vartheta^{rk} \leq \delta \quad \text{and} \quad \left(\frac{2}{\alpha_{(k)}}\right)^r \leq \delta\ .$$

For $n \geq 0$ let

$$l_n := \|\mathbf{P}^{(rkn)}\psi\|_1 \quad \text{and} \quad v_n := \mathrm{var}_{[0,1]}(\mathbf{P}^{(rkn)}\psi)$$

and observe that $l_n \leq v_n$ since $\int \mathbf{P}^{(rkn)}\psi\, dx = \int \psi\, dx = 0$. Therefore $\|\mathbf{P}^{(rkn)}\psi\|_{BV} \leq 2v_n$. Denote by $\mathbf{Q}_{n+1}$ the Perron-Frobenius operator of the composition $\sigma_{rk(n+1)}\tau \ldots \sigma_{rkn+2}\tau\sigma_{rkn+1}\tau$. Then

$$l_{n+1} \leq \|(\mathbf{Q}_{n+1} - \mathbf{P}_\tau^{rk})(\mathbf{P}^{(rkn)}\psi)\|_1 + \|\mathbf{P}_\tau^{rk}\mathbf{P}^{(rkn)}\psi\|_1\ .$$

In order to bound the first term on the right hand side of this inequality we use Lemma 13 from [13] which asserts that $|||\mathbf{Q}_{n+1} - \mathbf{P}_\tau^{rk}||| = O(\epsilon)$. (In fact, some elementary extra considerations are necessary to deduce this from that lemma.) Choosing ϵ^* small enough this term can thus be bounded by $\frac{\delta}{2}v_n$. Because of (3), the second term in the inequality is bounded by $C\,\vartheta^{rk}\,v_n \leq \frac{\delta}{2}v_n$. Hence

$$l_{n+1} \leq \delta v_n\ .$$

Combined with (4) this yields

$$v_{n+1} \leq \left(\frac{2}{\alpha_{(k)}}\right)^r v_n + \tilde{C}\,l_n \leq \delta v_n + \delta\tilde{C}v_{n-1}$$

The assumption $\delta + \sqrt{\delta\tilde{C}} < 1$ guarantees that the characteristic roots of this linear difference (in)equation are strictly inside the unit circle. Since finally $v_1 \leq \delta v_0 + \tilde{C}l_0 \leq (\delta + \tilde{C})v_0$, it follows that there are constants $K > 0$ and $0 < \gamma < 1$ such that $v_n \leq K\gamma^n v_0$. Again the extension of the estimate for $\mathrm{var}_{[0,1]}(\mathbf{P}^{(rkn)}\psi)$ to iterates which are not multiples of rk is straightforward.

□

Remark 6. Alternatively one could derive this uniform exponential mixing from a recent spectral perturbation result for Perron-Frobenius operators [19]. In both proofs the mixing rate γ is not explicitly determined. For a more restricted class of maps explicit mixing rates were derived in [17]. For example, if τ is a tent map with slope $\alpha \in [1.65, 2]$, then $\gamma \leq \frac{1}{\alpha}\sqrt{\frac{5\alpha-2}{2\alpha-1}} < 1$, and this estimate is stable under small $\mathcal{C}^2$-perturbations.

3. The interaction Φ_ϵ

In the following we consider Ω as a subset of $\mathbb{R}^\mathbb{N}$ with its norm $|.|_\infty$. $\mathbb{N}\times\mathbb{N}$ matrices A are interpreted as linear operators on $(\mathbb{R}^\mathbb{N}, |.|_\infty)$, and $|A|_\infty$ denotes the adapted matrix norm (maximal row sums).

Definition 4. *An* interaction (on Ω directed according to $\mathcal{G}$) *is a map $\Phi_\epsilon : \Omega \to \Omega$ all of whose second partial derivatives exist and which is* $\kappa\epsilon$-close (for some $\kappa > 0$) to Id_Ω *in the following sense. For each $\omega \in \Omega$*

$$\begin{aligned} |\Phi_\epsilon(\omega) - \omega|_\infty &\leq \kappa\epsilon \\ |D\Phi_\epsilon(\omega) - E|_\infty &\leq \kappa\epsilon \qquad (5) \\ |D_j D\Phi_\epsilon(\omega)|_\infty &\leq \kappa\epsilon\ . \end{aligned}$$

Here $D\Phi_\epsilon = (\frac{\partial\Phi_{\epsilon,i}}{\partial\omega_j})_{ij}$ denotes the matrix of partial derivatives of Φ_ϵ and $D_jD\Phi_\epsilon$ the j-th partial derivative of this matrix valued function. E is the identity matrix over $\mathbb{R}^\mathbb{N}$. Moreover we require that $\frac{\partial\Phi_{\epsilon,i}}{\partial\omega_j} = 0$ unless $j \to i$.

We note that this definition makes no assumption on the spatial decay rate of the interaction, only summability of the site-to-site influences is implicitly required. For the reduction of the uniqueness problem to $\mathcal{F}_{tail}$ this is sufficient. In order to prove that there is indeed a unique measure in $ACC_\Omega(T_\epsilon)$ we need to restrict to an essentially one-dimensional graph, and for simplicity we just consider the one-dimensional lattice $\mathcal{G}_1 = (\mathbb{N}, \geq)$ in the remainder of this section. Moreover, we need to assume that the interaction obeys the following condition.

Definition 5. *The interaction Φ_ϵ on $\Omega = [0,1]^{\mathbb{N}}$* decays superexponentially, *if there are constants $c > 0$ and $p_k \geq 0$ with $\sum_{k\in\mathbb{Z}} p_k = 1$ such that*

$$\left|\frac{\partial \Phi_{\epsilon,i}}{\partial \omega_j}\right| \leq c((1-\epsilon)\delta_{i,j} + \epsilon p_{j-i})$$

and such that for all $\lambda > 0$

$$s(\lambda) := \sum_{k\in\mathbb{Z}} p_k e^{\lambda|k|} < \infty \ .$$

The simplest one-sided superexponentially decaying interactions are affine interactions (with appropriate p_k) like

$$(\Phi_\epsilon(\omega))_i = (1-\epsilon)\omega_i + \epsilon \sum_{j\geq i} p_{j-i}\omega_j \ .$$

Remark 7. In the course of the proofs of Theorems 2 and 3 we will see that it suffices that $s(\lambda_0) < \infty$ for some large λ_0 which depends in a relatively complicated way on τ and Φ_ϵ.

The superexponential decay assumption allows for a large deviations type control over the interaction strength in time.

Lemma 2. *For a probability vector $(p_k)_{k\in\mathbb{Z}}$ define the Markov transition matrix $(q_{i,j})_{i,j\in\mathbb{Z}}$ by $q_{i,j} := (1-\epsilon)\delta_{i,j} + \epsilon p_{j-i}$. Let $\alpha, \lambda > 0$ and assume that $s(\lambda) < \infty$. Then the n-step transition kernel $q_{i,j}^{(n)}$ satisfies*

$$\sum_{k\geq\alpha n} q_{i,i+k}^{(n)} \leq e^{-(\alpha\lambda - \gamma(\epsilon,\lambda))\cdot n}$$

where $\gamma(\epsilon,\lambda) := \log(1+\epsilon s(\lambda)) \to 0$ as $\epsilon \to 0$.

Proof: $(q_{i,i+k}^{(n)})_{k\in\mathbb{Z}} = (q_{0,k}^{(n)})_{k\in\mathbb{Z}}$ is a probability vector with Laplace transform $(\psi_\epsilon(t))^n$ where $\psi_\epsilon(t) := \sum_{k\in\mathbb{Z}} q_{0,k}e^{tk} = (1-\epsilon) + \epsilon\sum_{k\in\mathbb{Z}} e^{tk}p_k$. By assumption, $\sum_{k\in\mathbb{Z}} e^{\lambda k}p_k \leq s(\lambda)$, and therefore $\psi_\epsilon(\lambda) \leq 1 + \epsilon s(\lambda)$. Consequently,

$$\begin{aligned}\sum_{k\geq\alpha n} q_{0,k}^{(n)} &\leq \sum_{k\geq\alpha n} q_{0,k}^{(n)} \cdot e^{\lambda k - \lambda\alpha n} \leq e^{-\lambda\alpha n}(\psi_\epsilon(\lambda))^n \\ &\leq \left(\frac{1+\epsilon s(\lambda)}{e^{\lambda\alpha}}\right)^n = e^{-(\alpha\lambda - \gamma(\epsilon,\lambda))\cdot n}.\end{aligned}$$

□

In order to apply this to the interacting system on Ω, we notice the following consequence.

Proposition 1. *Let $T_\epsilon = \Phi_\epsilon \circ T$ be as in the introduction. Suppose that τ is continuous, that $|\tau'| \leq R$ for some $R > 0$ and that Φ_ϵ decays superexponentially as in Definition 5. For any $n \geq 1$, $\alpha, \lambda > 0$ and any $\chi \in Lip_f(\Omega)$ depending on M coordinates ω_i, $i \in I \subseteq \{1, \ldots, N\}$ we have*

$$|\chi(T_\epsilon^n(\omega)) - \chi(T_\epsilon^n(\omega^*))| \leq M \ \mathrm{Lip}(\chi) \cdot (cR)^n \cdot e^{-(\alpha\lambda-\gamma(\epsilon,\lambda))n}.$$

provided $\omega_j = \omega_j^*$ for $j < N + \alpha n$. In particular, given α and $r > 0$ there are some $\epsilon_1 = \epsilon_1(\alpha, r) > 0$ and $Q = Q(\alpha, r) > 0$ such that for $\epsilon \in (0, \epsilon_1)$ and for any $\chi \in Lip_f(\Omega)$ as above,

$$|\chi(T_\epsilon^n(\omega)) - \chi(T_\epsilon^n(\omega^*))| \leq M\, Q^N \ \mathrm{Lip}(\chi) \cdot r^n \quad \textit{for } n \geq 1$$

whenever $\omega_j = \omega_j^*$ for $j < \alpha n$.

Proof: Observe that $|(T_\epsilon^n(\omega))_i - (T_\epsilon^n(\omega^*))_i| \le \sum_{k\ge\alpha n}\left|\frac{\partial(T_\epsilon^n)_i}{\partial\omega_{i+k}}\right|$ by continuity of τ, and $\left|\frac{\partial(T_\epsilon^n)_i}{\partial\omega_{i+k}}\right| \le R^n c^n q_{i,i+k}^{(n)}$ with $q_{i,i+k}^{(n)}$ as in the preceding Lemma and c from Definition 5. For any $n \ge 1$, $\alpha, \lambda > 0$ we therefore have

$$|\chi(T_\epsilon^n(\omega)) - \chi(T_\epsilon^n(\omega^*))| \le M\ \mathrm{Lip}(\chi)\cdot(cR)^n\cdot e^{-(\alpha\lambda-\gamma(\epsilon,\lambda))n}$$

with $\gamma(\epsilon,\lambda)$ as in the preceding Lemma if $\omega_j = \omega_j^*$ for $j < N + \alpha n$.

Choose λ so large that $cR\cdot e^{-\alpha\lambda} < r$. Since $\gamma(\epsilon,\lambda)\to 0$ as $\epsilon\to 0$, there is some $\epsilon_1 = \epsilon_1(\alpha, r) > 0$ such that for $\epsilon\in(0,\epsilon_1)$ we also have $cR\cdot e^{-\alpha\lambda+\gamma(\epsilon,\lambda)} < r$. □

4. Some preliminaries on skew products and transfer operators

In the sequel we will often work with Cartesian products of subsets of Euclidean spaces. If U is a factor of some Cartesian product of sets, π_U will denote the canonical projection onto U.

Let U, V be measurable subsets of euclidean spaces. The integration with Lebesgue measure on U and V is denoted by $\int \ldots du$ and $\int \ldots dv$ respectively, the integration on $U\times V$ by $\int\int \ldots dudv$. Given any nonnegative $F \in L^1_{dudv}$, it has a *standard factorization* $F(u,v) = F_{U,v}(u)\cdot F_V(v)$, where $F_V(v) = \int F(u,v)\,du$ and where each $F_{U,v}$ is a probability density on U if $F_V(v) > 0$ and $F_{U,v} = 0$ otherwise.

If Θ is a measurable map on a subset U of some euclidean space and if Θ is nonsingular in the sense that pre-images of Lebesgue null sets are Lebesgue null sets, then its Perron-Frobenius operator $\mathbf{P}_\Theta$ on L^1_{du} is defined as in (1).

Lemma 3. *Let Θ be a nonsingular skew product on $U\times V$, $\Theta(u,v) = (\phi_v(u), \theta(v))$, and let $F(u,v) = F_{U,v}(u)\cdot F_V(v)$ be in L^1_{dudv}. Then*

$$\mathbf{P}_\Theta F(u,v) = \mathbf{P}_\theta\left[v' \longmapsto \mathbf{P}_{\phi_{v'}}F_{U,v'}(u)\cdot F_V(v')\right](v)\ .$$

Proof: To check this identity it is enough to integrate both sides against test functions of the type $\varphi(u)\psi(v)$ where $\varphi\in L^1_{du}$ and $\psi\in L^1_{dv}$ are nonnegative. By definition,

$$\int_U \varphi(\phi_{v'}(u'))\cdot F_{U,v'}(u')\,du' = \int_U \varphi(u)\cdot\mathbf{P}_{\phi_{v'}}F_{U,v'}(u)\,du\ .$$

Also, writing $R_u(v) := \mathbf{P}_{\phi_v}F_{U,v}(u)\cdot F_V(v)$ we have

$$\mathbf{P}_\theta\left[v'\longmapsto \int_U \varphi(u)\cdot R_u(v')\,du\right](v) = \int_U \varphi(u)\cdot\mathbf{P}_\theta\left[v'\longmapsto R_u(v')\right](v)\,du$$

and thus

$$\begin{aligned}
&\int_V\int_U \varphi(u)\,\psi(v)\cdot\mathbf{P}_\Theta F(u,v)\,du\,dv\\
&= \int_V\int_U \varphi\otimes\psi(\Theta(u',v'))F(u',v')\cdot\,du'\,dv'\\
&= \int_V\int_U \varphi(\phi_{v'}(u'))\,\psi(\theta(v'))\cdot F_{U,v'}(u')\cdot F_V(v')\,du'\,dv'\\
&= \int_V\left(\int_U \varphi(\phi_{v'}(u'))\cdot F_{U,v'}(u')\cdot F_V(v')\,du'\right)\psi(\theta(v'))\,dv'\\
&= \int_V \mathbf{P}_\theta\left[v'\longmapsto\int_U \varphi(\phi_{v'}(u'))\cdot F_{U,v'}(u')\,du'\cdot F_V(v')\right](v)\,\psi(v)\,dv\\
&= \int_V \mathbf{P}_\theta\left[v'\longmapsto\int_U \varphi(u)\cdot R_u(v')\,du\right](v)\,\psi(v)\,dv\\
&= \int_V\int_U \mathbf{P}_\theta\left[v'\longmapsto R_u(v')\right](v)\ \cdot\varphi(u)\,\psi(v)\,du\,dv
\end{aligned}$$

as claimed. □

Remark 8. If in particular θ is a piecewise invertible map whose transfer operator can be represented as a finite sum $\mathbf{P}_\theta f(v) = \sum_{v'\in\theta^{-1}\{v\}} g(v')\, f(v')$ with a suitable weight function g, then for any $v \in V$ the conditional density $F^{(1)}_{U,v}$ from the standard factorization $\mathbf{P}_\Theta F(u,v) = F^{(1)}_{U,v}(u)\cdot F^{(1)}_V(v)$ is either zero or some finite convex combination of the conditional densities $\mathbf{P}_{\phi_{v'}} F_{U,v'}$, $v' \in \theta^{-1}\{v\}$ pushed forward by the respective fiber maps.

Lemma 4. *Let Θ be a nonsingular skew product on $U \times V$, $\Theta(u,v) = (\phi_v(u), \theta(v))$, and let $F(u,v) = F_{U,v'}(u')\cdot F_V(v)$ be in L^1_{dudv}. Then*

$$\|\mathbf{P}_\Theta F\|_{L^1_{du\,dv}} \le \int_V \|\mathbf{P}_{\phi_v} F_{U,v}(u)\|_{L^1_{du}} \cdot |\, F_V(v)\,|\; dv.$$

Proof: By Lemma 3 we have $\mathbf{P}_\Theta F(u,v) = \mathbf{P}_\theta R_u(v)$ where again $R_u(v') = \mathbf{P}_{\phi_{v'}} F_{U,v'}(u)\cdot F_V(v')$, and therefore

$$\begin{aligned}
\|\mathbf{P}_\Theta F\|_{L^1_{du\,dv}} &= \int_V\int_U |\, \mathbf{P}_\theta(R_u)(v)\,|\; du\,dv \\
&\le \int_V\int_U \mathbf{P}_\theta(|\, R_u\,|)(v)\,du\,dv \\
&= \int_V \mathbf{P}_\theta\left(\int_U |\, R_u\,|\; du\right)(v)\,dv \\
&= \int_V \mathbf{P}_\theta\left[v' \longmapsto \left\|\mathbf{P}_{\phi_{v'}} F_{U,v'}(u)\right\|_{L^1_{du}} \cdot |\, F_V(v')\,|\right](v)\,dv \\
&= \int_V \|\mathbf{P}_{\phi_v} F_{U,v}(u)\|_{L^1_{du}} \cdot |\, F_V(v)\,|\; dv.
\end{aligned}$$

□

If Θ is a skew product on $U\times V$, $\Theta(u,v) = (\phi_v(u),\theta(v))$, we shall write $\Theta_V := \theta$ for the obvious *autonomous factor* of Θ.

5. Factors of interacting maps and densities of bounded variation

Returning to the interacting system we shall usually skip the subscript ϵ indicating the dependence on the interaction strength. So T_ϵ is simply denoted by T.

Autonomous and nonautonomous factors of T. For convex $\Lambda \subset G$ we let $\Lambda^>$ be the set of vertices accessible from Λ and $\Lambda^< := \Lambda^c \setminus \Lambda^>$ those members of $G\setminus\Lambda$ not accessible from Λ. Then $\Lambda^+ := \Lambda \cup \Lambda^<$ is not accessible from its complement (and hence convex in $\mathcal{G}$). Consequently, the action of T on Ω_{Λ^+} is an *autonomous factor of* T, i.e. there is some $T_{\Lambda^+} : \Omega_{\Lambda^+} \to \Omega_{\Lambda^+}$ such that $\pi_{\Lambda^+}\circ T = T_{\Lambda^+}\circ \pi_{\Lambda^+}$.

Observe that any autonomous factor T_{Λ^+} of T is an interacting system on Ω_{Λ^+} with the same single site maps and the obvious factor interaction $\Phi_{\epsilon,\Lambda^+}$ directed according to the restriction $\mathcal{G}_{\Lambda^+}$ of $\mathcal{G}$ to Λ^+, which satisfies all the original assumptions on T (with the same constants in the Lasota-Yorke inequality etc.). Therefore, any statement we prove to hold for all such systems T automatically applies to each autonomous factor of T, so that we may assume that $\Lambda^> \setminus \Lambda$ is empty, i.e. $\Lambda^+ = G$ and $T_{\Lambda^+} = T$. We shall frequently make use of this below without further mention.

Let us therefore consider some convex $\Lambda \subseteq G$ with $\Lambda^> \setminus \Lambda = \emptyset$ so that $\Lambda^c = (\Lambda^c)^+$. Then T is a skew product on $\Omega_\Lambda \times \Omega_{\Lambda^c}$, $T(x,\omega) = (\Theta_\omega x, T_{\Lambda^c}\omega)$. If we fix $\omega \in \Omega_{\Lambda^c}$ and iterate T, the temporally inhomogeneous factor action of T on Ω_Λ is given by the successive application of the maps $\Theta_{T^k_{\Lambda^c}\omega}$, $k \ge 0$. Writing $\Theta^{(i,j)} = \Theta^{(i,j)}_\omega := \Theta_{T^j_{\Lambda^c}\omega}\circ \Theta_{T^{j-1}_{\Lambda^c}\omega}\circ\cdots\circ\Theta_{T^i_{\Lambda^c}\omega} = \pi_\Lambda(T^{j-i}(.,T^i_{\Lambda^c}\omega))$, $0 \le i \le j$, we see that $\Theta^{(i,j)}(x)$ is the Λ-coordinate of the configuration at time j if x was the Λ-coordinate of the configuration at time i and ω was the Λ^c-coordinate of the configuration at time 0. The collection $\{\Theta^{(i,j)}\}$ describing the conditional dynamics of T on Ω_Λ given $\omega \in \Omega_{\Lambda^c}$ will be called a $|\Lambda|$-*dimensional nonautonomous factor of* T. More generally, we shall use the same term for any p-dimensional nonautonomous factor of some autonomous factor T_{Λ^+} of T. The collection of all these p-dimensional nonautonomous factors of T will be denoted by $\mathcal{S}_p(T)$, $p \geq 1$. The transfer operator of $\Theta^{(i,j)}$ will be denoted by $\mathbf{P}_\Theta^{(i,j)}$.

Interacting maps and densities of bounded variation. Next we collect some facts from [18, 20] on Perron-Frobenius operators for interacting systems that act on densities of bounded variation. General references for functions of bounded variation are e.g. [22, 9].

Let $U = U_p := [0,1]^p \subset \mathbb{R}^p$. For $f \in L^1_{du}(U)$ the variation of f is defined as

$$\operatorname{var}(f) := \sup\left\{\int 1_U \cdot f \cdot \operatorname{div}(\varphi)\, du : \varphi \in \mathcal{C}^1(\mathbb{R}^p, \mathbb{R}^p), |\varphi(u)|_\infty \leq 1 \text{ for all } u \in \mathbb{R}^p\right\}$$

where integration extends over all of $\mathbb{R}^p$. Those f, for which this supremum is finite, are called of *bounded variation*. The space of these functions will be denoted by $BV(U_p)$. For $f \in \mathcal{C}^1(U, \mathbb{R})$ a simple calculation yields

$$\operatorname{var}(f) = \int 1_U \cdot |\operatorname{grad} f|_\infty\, du + \int_{\partial U} |f|\, d\sigma,$$

so $\operatorname{var}(f)$ is the sum of the variation of f inside U and the variation caused by the jump to zero on ∂U. Later we shall also make use of an alternative characterization: For $f : U_p \to \mathbb{R}$ we have

$$\operatorname{var}(f) = \sum_{j=1}^{p} \operatorname{var}^{(j)}(f) = \sum_{j=1}^{p} \int_{U_{p-1}} \operatorname{var}_{[0,1]}\left[x_j \mapsto f(x_1, \ldots, x_p)\right]\, dx_1 \ldots dx_{j-1}\, dx_{j+1} \ldots dx_p, \tag{6}$$

where $\operatorname{var}^{(j)}(f)$ denotes the average one-dimensional variation of f in the jth component (recall footnote 2). We let $\mathcal{V}_p$ denote the set of probability densities from $BV(U_p)$, and for $C > 0$ write $\mathcal{V}_p(C) := \{f \in \mathcal{V}_p : \operatorname{var}(f) \leq p \cdot C\}$. $\mathcal{V}_p(C)$ is a compact subset of $L^1_{du}(U_p)$ and the linear span of $\bigcup_{C>0} \mathcal{V}_p(C)$ is dense in $L^1_{du}(U_p)$. In what follows we shall not distinguish functions F defined on some Ω_Λ, $|\Lambda| = p$ and functions defined on U_p.

For later use we note the following: Let the map τ satisfy stably a Lasota-Yorke inequality (see Definition 2). Suppose that $(\Phi_\epsilon)_{0 \leq \epsilon < \epsilon_0}$ is a family of one-sided interactions on the p-dimensional unit cube U_p whose $\mathcal{C}^2$-distance to Id_{U_p} is at most $\kappa\epsilon$ for some $\kappa > 0$ (see Definition 4). Denote by $\mathbf{P}_{U_p,\epsilon}$ the Perron-Frobenius operator of the finite dimensional interacting system $T_{U_p,\epsilon} : U_p \to U_p$. Then there are $\epsilon_1 > 0$, $\gamma \in (\frac{2}{\alpha_k}, 1)$ and $\tilde{C} > 0$ (independent of p) such that for all $\epsilon \in [0, \epsilon_1)$ and all $f \in \mathcal{V}_p$,

$$\operatorname{var}(\mathbf{P}^k_{U_p,\epsilon} f) \leq \gamma \operatorname{var}(f) + p\tilde{C}\ . \tag{7}$$

For $k = 1$ (i.e. when the slope of τ is uniformly bigger than 2 and hence $\alpha_1 > 2$) this was proved in [18], the general case $k > 1$ was treated in the thesis [20]. [3]

Returning to the infinite interacting system, if $\{\Theta^{(i,j)}\}_{0 \leq i \leq j}$ is a p-dimensional nonautonomous factor of T, then inequality (7) carries over to $\mathbf{P}_\Theta^{(i,i+1)}$ immediately in case $k = 1$. For $k > 1$ one has to check that the proof in [20] does not really use that the successively applied interaction maps are identical but that it only requires that the bounds in (5) are uniform for the interactions involved. [4] Therefore (7) extends to

$$\operatorname{var}(\mathbf{P}_\Theta^{(i,i+k)} f) \leq \gamma \operatorname{var}(f) + p\tilde{C}\ . \tag{8}$$

[3]Formally the setup in references [18, 20] was that of Λ being a finite subinterval of $\mathbb{Z}$. The proofs do not make use of this particular structure, however, but apply directly to arbitrary finite index sets Λ.

[4]These are the essential properties of the more special interactions studied in [20]. They are listed in Proposition 2.15 of that thesis, and all further proofs use only those properties.

Iterating this estimate yields $\limsup_{r\to\infty} \mathrm{var}(\mathbf{P}_\Theta^{(i,i+rk)}f) \le p\frac{\tilde{C}}{1-\gamma} =: pC_0$ for all $f \in \mathcal{V}_p$. Finally, since there is some $c > 0$ such that $\mathrm{var}(\mathbf{P}_\Theta^{(i,i+1)}f) \le c\cdot\mathrm{var}(f) + p\tilde{C}$, it follows that for some $C' > 0$,

$$\limsup_{n\to\infty} \mathrm{var}(\mathbf{P}_\Theta^{(i,i+n)}f) \le pC' \quad \text{for all } f \in \mathcal{V}_p. \tag{9}$$

Similarly one shows that

$$\mathbf{P}_\Theta^{(i,i+k)}(\mathcal{V}_p(C_0)) \subseteq \mathcal{V}_p(C_0) \quad \text{and} \quad \mathbf{P}_\Theta^{(i,i+n)}(\mathcal{V}_p(C_0)) \subseteq \mathcal{V}_p(C_1) \quad \text{for all } i,n \ge 0, \tag{10}$$

for some suitable C_1. Enlarging C_0 such that $C_0 > 2$, if necessary, we make sure that the constant density 1_{U_p} belongs to $\mathcal{V}_p(C_0)$ for all $p \ge 0$.

Now let f be an arbitrary probability density on U_p. For each $\eta > 0$ there is a probability density f_η of bounded variation such that $\|f - f_\eta\|_1 < \eta$. Then also $\|\mathbf{P}_\Theta^{(i,i+n)}f - \mathbf{P}_\Theta^{(i,i+n)}f_\eta\|_1 < \eta$ for all $i, n \ge 0$, and it follows from (9) that

$$\lim_{n\to\infty} dist_{L^1_{du}(U_p)}(\mathbf{P}_\Theta^{(i,i+n)}f, \mathcal{V}_p(C_0)) = 0\ . \tag{11}$$

Since $\mathcal{V}_p(C_0)$ is compact in $L^1_{du}(U_p)$, the sequence $(\mathbf{P}_\Theta^{(i,i+n)}f)_{n\ge 0}$ is relatively compact and all its limit points belong to $\mathcal{V}_p(C_0)$.

Proposition 2. *If the interaction strength ϵ is sufficiently small, $\epsilon \in [0, \epsilon_1)$, we have the following:*

a) For $\nu \in ACC_\Omega$ the conditional distributions of $T^n\nu$ on Ω_{Λ_p} given $\mathcal{F}_{\Lambda_p^c}$ have densities $h^{(n)}_{\Lambda_p,\omega}$, $\omega \in \Omega_{\Lambda_p^c}$, satisfying

$$\lim_{n\to\infty} E_{\pi_{\Lambda_p^c}(T^n\nu)}\left[dist_{L^1_{du}(U_p)}(h^{(n)}_{\Lambda_p,\boldsymbol{\cdot}}, \mathcal{V}_p(C_0))\right] = 0\ .$$

Here and in the sequel the dot indicates the dependence of the respective quantities on ω.

b) $ACC_\Omega(T) \neq \emptyset$.

c) If $\nu \in ACC_\Omega(T)$, then $h^{(0)}_{\Lambda_p,\omega} \in \mathcal{V}_p(C_0)$ for all $p \ge 0$ and ν-a.a. $\omega \in \Omega_{\Lambda_p^c}$, and for each measurable bounded $\chi : \Omega \to \mathbb{R}$ we have

$$\int_\Omega \chi\, d\nu = \int_{\Omega_{\Lambda_p^c}} \int_{\Omega_{\Lambda_p}} \chi(x, T^n_{\Lambda_p^c}\omega) \cdot \mathbf{P}^{(0,n)}_{\Lambda_p,\omega} h^{(0)}_{\Lambda_p,\omega}(x)\, dx\ d\pi_{\Lambda_p^c}\nu(\omega).$$

Here $\mathbf{P}^{(0,n)}_{\Lambda_p,\omega}$ is used as a more suggestive notation for $\mathbf{P}_\Theta^{(0,n)}$ where $\{\Theta^{(i,j)}\}$ is the nonautonomous factor of T on Ω_{Λ_p} given $\omega \in \Omega_{\Lambda_p^c}$.

Proof:

Consider T as a skew product on $\Omega = \Omega_{\Lambda_p} \times \Omega_{\Lambda_p^c}$, $T(x,\omega) = (\Theta_\omega x, T_{\Lambda_p^c}\omega)$, $p \ge 1$ fixed. Take some $\nu \in ACC_\Omega$ with conditional distribution $\nu_{\Lambda_p,\omega}$ on Ω_{Λ_p} given $\mathcal{F}_{\Lambda_p^c}$ which have densities $h_{\Lambda_p,\omega}$.

Fix $n \ge 0$. We claim that $T^n\nu$ has conditional probability distributions $\nu^{(n)}_{\Lambda_p,\omega}$ on Ω_{Λ_p} given $\mathcal{F}_{\Lambda_p^c}$ with densities $h^{(n)}_{\Lambda_p,\omega}$ characterized by

$$h^{(n)}_{\Lambda_p, T^n_{\Lambda_p^c}\omega}(x) = E_{\pi_{\Lambda_p^c}\nu}[\mathbf{P}^{(0,n)}_{\Lambda_p,\boldsymbol{\cdot}} h^{(0)}_{\Lambda_p,\boldsymbol{\cdot}}(x) | T^{-n}_{\Lambda_p^c}\mathcal{F}_{\Lambda_p^c}](\omega)\ . \tag{12}$$

To see this, fix any continuous $\varphi : \Omega_{\Lambda_p} \to \mathbb{R}$. There exists a $\mathcal{F}_{\Lambda_p^c}$-measurable function $g_{\varphi,n}$ on $\Omega_{\Lambda_p^c}$ such that

$$E_{\pi_{\Lambda_p^c}(\nu)}\left[\left(\Theta^{(0,n)}_{\boldsymbol{\cdot}}\nu_{\Lambda_p,\boldsymbol{\cdot}}\right)(\varphi) | T^{-n}_{\Lambda_p^c}\mathcal{F}_{\Lambda_p^c}\right](\omega) = g_{\varphi,n} \circ T^n_{\Lambda_p^c}(\omega). \tag{13}$$

It follows that for each $B \in \mathcal{F}_{\Lambda_p^c}$,

$$\begin{aligned}
E_{T^n\nu}\left[\left(g_{\varphi,n}\circ\pi_{\Lambda_p^c}\right)\cdot\left(1_B\circ\pi_{\Lambda_p^c}\right)\right] &= E_{T^n_{\Lambda_p^c}(\pi_{\Lambda_p^c}\nu)}\left[g_{\varphi,n}\cdot 1_B\right]\\
&= E_{\pi_{\Lambda_p^c}(\nu)}\left[\left(g_{\varphi,n}\circ T^n_{\Lambda_p^c}\right)\cdot\left(1_B\circ T^n_{\Lambda_p^c}\right)\right]\\
&= E_{\pi_{\Lambda_p^c}(\nu)}\left[E_{\pi_{\Lambda_p^c}(\nu)}\left[\Theta_{\bullet}^{(0,n)}\nu_{\Lambda_p,\bullet}(\varphi)\big|T^{-n}_{\Lambda_p^c}\mathcal{F}_{\Lambda_p^c}\right]\cdot 1_{T^{-n}_{\Lambda_p^c}B}\right]\\
&= E_{\pi_{\Lambda_p^c}(\nu)}\left[(\Theta_{\bullet}^{(0,n)}\nu_{\Lambda_p,\bullet})(\varphi)\cdot 1_B\circ T^n_{\Lambda_p^c}\right]\\
&= E_{\pi_{\Lambda_p^c}(\nu)}\left[\nu_{\Lambda_p,\bullet}(\varphi\circ\Theta_{\bullet}^{(0,n)})\cdot 1_B\circ T^n_{\Lambda_p^c}\right]\\
&= E_{\pi_{\Lambda_p^c}(\nu)}\left[\nu_{\Lambda_p,\bullet}\left(\varphi\circ\Theta_{\bullet}^{(0,n)}\cdot 1_B\circ T^n_{\Lambda_p^c}(\bullet)\right)\right]\\
&= E_{\nu}\left[\left(\varphi\circ\pi_{\Lambda_p}\circ T^n\right)\cdot\left(1_B\circ\pi_{\Lambda_p^c}\circ T^n\right)\right]\\
&= E_{T^n\nu}\left[\left(\varphi\circ\pi_{\Lambda_p}\right)\cdot\left(1_B\circ\pi_{\Lambda_p^c}\right)\right]
\end{aligned}$$

Hence, for every $\varphi \in \mathcal{C}(\Omega_{\Lambda_p},\mathbb{R})$ and all $x \in \Omega_{\Lambda_p}$,

$$g_{\varphi,n}(\omega) = \left(g_{\varphi,n}\circ\pi_{\Lambda_p^c}\right)(x,\omega) = E_{T^n\nu}\left[\varphi\circ\pi_{\Lambda_p}\big|\pi^{-1}_{\Lambda_p^c}\mathcal{F}_{\Lambda_p^c}\right](x,\omega) = \nu^{(n)}_{\Lambda_p,\omega}(\varphi) \qquad a.e.\ \omega\ .$$

On the other hand, as $\nu_{\Lambda_p,\omega}$ has density $h_{\Lambda_p,\omega} = h^{(0)}_{\Lambda_p,\omega}$, we have in view of (13)

$$\begin{aligned}
\nu^{(n)}_{\Lambda_p,T_{\Lambda_p^c}\omega}(\varphi) = g_{\varphi,n}(T_{\Lambda_p^c}\omega) &= E_{\pi_{\Lambda_p^c}(\nu)}\left[\int\left(\mathbf{P}^{(0,n)}_{\Lambda_p,\bullet}h^{(0)}_{\Lambda_p,\bullet}\right)(x)\cdot\varphi(x)\,dx\big|T^{-n}_{\Lambda_p^c}\mathcal{F}_{\Lambda_p^c}\right](\omega)\\
&= \int\varphi(x)\cdot E_{\pi_{\Lambda_p^c}(\nu)}\left[\left(\mathbf{P}^{(0,n)}_{\Lambda_p,\bullet}h^{(0)}_{\Lambda_p,\bullet}\right)(x)\big|T^{-n}_{\Lambda_p^c}\mathcal{F}_{\Lambda_p^c}\right](\omega)\,dx \qquad a.e.\,\omega\ .
\end{aligned}$$

Since $C(\Omega_{\Lambda_p},\mathbb{R})$ is separable, equation (12) follows at once.

Observe next that the distances $dist_{L^1_{du}(U_p)}(\mathbf{P}^{(0,n)}_{\Lambda_p,\omega}h^{(0)}_{\Lambda_p,\omega},\mathcal{V}_p(C_0))$ are uniformly bounded by 2 and tend to zero pointwise as $n\to\infty$ because of (11). Therefore the relation $\pi_{\Lambda_p^c}\circ T^n = T^n_{\Lambda_p^c}\circ\pi_{\Lambda_p^c}$ and a version of Jensen's inequality for random variables taking values in a compact convex subset of a Banach space imply

$$\begin{aligned}
&E_{\pi_{\Lambda_p^c}(T^n\nu)}\left[dist_{L^1_{du}(U_p)}(h^{(n)}_{\Lambda_p,\bullet},\mathcal{V}_p(C_0))\right]\\
&= E_{\pi_{\Lambda_p^c}(\nu)}\left[dist_{L^1_{du}(U_p)}(h^{(n)}_{\Lambda_p,T^n_{\Lambda_p^c}\bullet},\mathcal{V}_p(C_0))\right]\\
&\le E_{\pi_{\Lambda_p^c}(\nu)}\left[E_{\pi_{\Lambda_p^c}\nu}\left[dist_{L^1_{du}(U_p)}(\mathbf{P}^{(0,n)}_{\Lambda_p,\bullet}h^{(0)}_{\Lambda_p,\bullet},\mathcal{V}_p(C_0))\big|T^{-n}_{\Lambda_p^c}\mathcal{F}_{\Lambda_p^c}\right]\right]\\
&= E_{\pi_{\Lambda_p^c}(\nu)}\left[dist_{L^1_{du}(U_p)}(\mathbf{P}^{(0,n)}_{\Lambda_p,\bullet}h^{(0)}_{\Lambda_p,\bullet},\mathcal{V}_p(C_0))\right]\\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned}\tag{14}$$

This proves part a) of the proposition.

Suppose now that ν is the infinite product Lebesgue measure. Then $h^{(0)}_{\Lambda_p,\omega} = 1_{U_p} \in \mathcal{V}_p(C_0)$ for all ω, and it follows from (10) and from the convexity of $\mathcal{V}_p(C_1)$ that $h^{(n)}_{\Lambda_p,\omega} \in \mathcal{V}_p(C_1)$ for all n and ω. Denote by $h^{(n)}_{\Lambda_p} := E_{\pi_{\Lambda_p^c}\nu}[h^{(n)}_{\Lambda_p,\bullet}]$ the marginal densities of $T^n\nu$. Since the $\mathcal{V}_p(C_1)$ are convex, also all $h^{(n)}_{\Lambda_p} \in \mathcal{V}_p(C_1)$ and the same is true, *a fortiori*, for the Césaro averages $H^{(n)}_{\Lambda_p} := \frac{1}{n}\sum_{j=0}^{n-1}h^{(j)}_{\Lambda_p}$. As all $\mathcal{V}_p(C)$ are compact, there exists a subsequence $(n_i)_{i\ge 0}$ such that $(H^{(n_i)}_{\Lambda_p})_{i\ge 0}$ converges for each $p\ge 1$ to some probability density H_{Λ_p} in $\mathcal{V}_p(C_1)$. Passing to a subsequence once more, if necessary, one can make sure that the sequence of measures $\nu_{n_i} = \frac{1}{n_i}\sum_{j=0}^{n_i-1}T^j\nu$ converges to a probability measure μ in the weak topology of measures on Ω. We show that μ has marginal densities H_{Λ_p}: Let $f \in C(\Omega,\mathbb{R})$ be $\mathcal{F}_{\Lambda_p}$-measurable. Then

$$\int_\Omega f\,d\mu = \lim_{i\to\infty}\int_\Omega f\,d\nu_{n_i} = \lim_{i\to\infty}\int_{\Omega_{\Lambda_p}} f\,d(\pi_{\Lambda_p}\nu_{n_i})$$
$$= \lim_{i\to\infty}\int_{\Omega_{\Lambda_p}} f(v)H_{\Lambda_p}^{(n_i)}(v)\,dv = \int_{\Omega_{\Lambda_p}} f(v)H_{\Lambda_p}(v)\,dv$$

For the T-invariance of μ we want to argue along the usual lines: Let $f \in C(\Omega,\mathbb{R})$ be Λ_p-measurable. We must show that

$$\int_\Omega f\circ T\,d\mu = \lim_{i\to\infty}\int_\Omega f\circ T\,d\nu_{n_i} = \lim_{i\to\infty}\int_\Omega f\,d\nu_{n_i} = \int_\Omega f\,d\mu\ .$$

Only the first identity needs additional justification because $T=\Phi_\epsilon\circ T_0$ is not necessarily continuous. It is continuous if τ is; otherwise the discontinuities of T are contained in the countable union of hyperplanes $\bigcup_{g\in G}\bigcup_{j=0}^N \pi_{\{g\}}^{-1}(\{a_j\})$ which has μ-measure zero, because, as we just proved, all finite-dimensional marginals of μ are absolutely continuous with respect to Lebesgue measure.

To finish the proof of part b) of the proposition, we still must show that $\mu \in ACC_\Omega(T)$. To this end fix some $\Lambda = \Lambda_{p_0}$ and consider $p \geq p_0$. Denote points in Ω by (u,v,ω) with $u\in\Omega_\Lambda$, $v\in\Omega_{\Lambda_p\setminus\Lambda}$ and $\omega\in\Omega_{\Lambda_p^c}$. Then

$$h_{\Lambda_p,\omega}^{(n)}(u,v) = h_{\Lambda,(v,\omega)}(u)\cdot\int h_{\Lambda_p,\omega}(\tilde u,v)\,d\tilde u\ .$$

Hence, with the notation $\mathrm{var}^{(\Lambda)}(f) := \sum_{j\in\Lambda}\mathrm{var}^{(j)}(f)$ for $f: U_p\to\mathbb{R}$, we have

$$\mathrm{var}^{(\Lambda)}(h_{\Lambda_p,\omega}^{(n)}) = \int\left\{\int h_{\Lambda_p,\omega}^{(n)}(\tilde u,v)\,d\tilde u\cdot\mathrm{var}(h_{\Lambda,(v,\omega)})\right\}dv \leq C_1\,p_0\cdot\int\int h_{\Lambda_p,\omega}^{(n)}(\tilde u,v)\,d\tilde u dv = C_1\,p_0$$

because $h_{\Lambda,(v,\omega)}\in\mathcal{V}_{p_0}(C_1)$. Just as above it follows that $\mathrm{var}^{(\Lambda)}(h_{\Lambda_p}^{(n)})\leq C_1\,p_0$, $\mathrm{var}^{(\Lambda)}(H_{\Lambda_p}^{(n)})\leq C_1\,p_0$, and finally $\mathrm{var}^{(\Lambda)}(H_{\Lambda_p})\leq C_1\,p_0$ uniformly for all $p\geq p_0$. This allows to apply Lemma 4 from [16] which asserts that $\mu\in ACC_\Omega(T)$.

If, from the beginning, $\nu\in ACC_\Omega(T)$, then $h_{\Lambda_p,\omega} = h_{\Lambda_p,\omega}^{(n)}$ for all n and p. Therefore (14) implies that $dist_{L^1_{du}(\Lambda_p)}(h_{\Lambda_p,\omega},\mathcal{V}_p(C_0)) = 0$ for ν-a.e. $\omega\in\Omega_{\Lambda_p^c}$. Since $\mathcal{V}_p(C_0)$ is compact this proves the first statement of part c). For the final assertion we observe that $\int_\Omega\chi\,d\nu$ equals

$$\int_\Omega \chi\circ T^n d\nu = E_{\pi_{\Lambda_p^c}\nu}\left[\int_{\Omega_{\Lambda_p}}\chi(\Theta_{\bullet}^{(0,n)}x, T_{\Lambda_p^c}^n(\bullet))\,h_{\Lambda_p,\bullet}(x)\,dx\right]$$
$$= E_{\pi_{\Lambda_p^c}\nu}\left[\int_{\Omega_{\Lambda_p}}\chi(x, T_{\Lambda_p^c}^n(\bullet))\,\mathbf{P}_{\Lambda_p,\bullet}^{(0,n)}\,h_{\Lambda_p,\bullet}(x)\,dx\right].$$

□

6. A probabilistic coupling procedure for nonautonomous factors of the interacting system

Throughout this section, $\{\Theta^{(i,j)}\}_{0\leq i\leq j}$ denotes some fixed $(p+1)$-dimensional nonautonomous factor of the interacting system, on $U\times V = \Omega_{\{h\}}\times\Omega_\Lambda$, with $\Lambda\subset G$ inaccessible from $h\in G$, $|\Lambda| = p$, and $\Lambda\cup\{h\}$ convex in $\mathcal{G}$. Writing $\Theta^{(i,j)}(u,v) =: (\phi_v^{(i,j)}(u),\theta^{(i,j)}(v))$, $\{\theta^{(i,j)}\}$ is a p-dimensional nonautonomous factor of T, while the $\phi_v^{(i,j)}$ belong to one-dimensional nonautonomous factors. Observe that $\phi_v^{(i,k)} = \phi_{\theta^{(i,j)}(v)}^{(j,k)}\circ\phi_v^{(i,j)}$ whenever $0\leq i\leq j\leq k$.

Given *nonnegative* functions F, G of bounded variation on $U\times V$ with $\int F = \int G$, our aim is to estimate the difference $\mathbf{P}_\Theta^{(0,n)}F - \mathbf{P}_\Theta^{(0,n)}G$ in L^1_{dudv} using contraction properties of the transfer operators $\mathbf{P}_\theta^{(i,j)}$ and $\mathbf{P}_{\phi_v}^{(i,j)}$ corresponding to the lower dimensional transformations $\theta^{(i,j)}$ and $\phi_v^{(i,j)}$ on V respectively U.

Recall from Lemma 1 that for sufficiently small interaction strength $\epsilon < \epsilon^*$, there are $K > 0$ and $\gamma \in (0,1)$ such that the transfer operator $\mathbf{P}_\phi^{(i,i+n)}$ of any one-dimensional nonautonomous factor satisfies the Lasota-Yorke inequality

$$\operatorname{var}\left(\mathbf{P}_\phi^{(i,i+n)} f\right) \le K\gamma^n \operatorname{var}(f) + C_0 \|f\|_1 \qquad \text{for } f \in BV([0,1]),\ n \ge 0, \tag{15}$$

and is exponentially contracting in that

$$\sup_{\substack{0 \le f,g \in BV[0,1], \\ \int f\,du = \int g\,du}} \ \sup_{i \ge 0} \left\{ \frac{1}{2} \|\mathbf{P}_\phi^{(i,i+n)}(f-g)\|_{L^1_{du}} \right\} \le K\,\gamma^n \cdot (\operatorname{var}(f) + \operatorname{var}(g)) \ . \tag{16}$$

Henceforth we assume (w.l.o.g.) that $K \ge 2C_0$.

The coupling procedure: We let $F^{(0)} := F$, $G^{(0)} := G$. Given $F^{(n)}$ and $G^{(n)}$ we define $F^{(n)} =: F_{U,\cdot}^{(n)} \cdot F_V^{(n)}$ and $G^{(n)} =: G_{U,\cdot}^{(n)} \cdot G_V^{(n)}$ via standard factorization, see section 4. Let

$$H_V^{(n)} := \min\left\{F_V^{(n)}, G_V^{(n)}\right\}, \quad \widehat{F}_V^{(n)} := F_V^{(n)} - H_V^{(n)}, \quad \widehat{G}_V^{(n)} := G_V^{(n)} - H_V^{(n)},$$

and inductively define

$$F^{(n+1)} := \mathbf{P}_\Theta^{(n,n+1)}\left(F_{U,\cdot}^{(n)} \cdot \widehat{F}_V^{(n)}\right), \quad G^{(n+1)} := \mathbf{P}_\Theta^{(n,n+1)}\left(G_{U,\cdot}^{(n)} \cdot \widehat{G}_V^{(n)}\right),$$

so that $F_V^{(n+1)} = \mathbf{P}_\theta^{(n,n+1)} \widehat{F}_V^{(n)}$.

Lemma 5. *The coupling procedure preserves BV-spaces: If $0 \le F, G \in BV([0,1]^{p+1})$, then $0 \le F_V, G_V \in BV([0,1]^p)$ and for a.e. $v \in V$ we have $0 \le F_{U,v}^{(n)}, G_{U,v}^{(n)} \in BV([0,1])$ for all $n \ge 0$.*

Proof: Evidently, $F_V \in BV([0,1]^p)$ and by (6), we have $F(.,v) \in BV([0,1])$, and hence also $F_{U,v}^{(0)} \in BV([0,1])$ for a.e. $v \in V$. But each $F_{U,v}^{(n)}$ is a finite convex combination of densities $\mathbf{P}_{\phi_{v'}}^{(i,i+n)} F_{U,v'}^{(0)}$, cf. Remark 8. □

The following shows how to use this scheme to obtain bounds for the convergence rate on $U \times V$ from (16) and from bounds for that on U.

Lemma 6. *Assume that $\Psi_p(n) > 0$, $n \ge 1$ are constants such that for all $n \ge 1$,*

$$\sup_{\substack{0 \le f,g \in BV([0,1]^p), \\ \int f\,dv = \int g\,dv}} \ \sup_{l \ge 0} \left\{ \frac{1}{2} \|\mathbf{P}_\theta^{(l,l+n)}(f-g)\|_{L^1_{dv}} \right\} \le \Psi_p(n) \cdot (\operatorname{var}(f) + \operatorname{var}(g)).$$

Then, defining $\Psi_p(0) := 1$ for all p and

$$\Psi_{p+1}(n) := \frac{2K^3}{\gamma} \cdot \sum_{k=0}^{n} \gamma^{n-k} \left(\Psi_p(k) + \gamma^k\right) \ ,$$

we have, for $n \ge 1$,

$$\sup_{\substack{0 \le F,G \in BV([0,1]^{p+1}), \\ \int\int F\,du\,dv = \int\int G\,du\,dv}} \ \sup_{l \ge 0} \left\{ \frac{1}{2} \|\mathbf{P}_\Theta^{(l,l+n)}(F-G)\|_{L^1_{du\,dv}} \right\} \le \Psi_{p+1}(n) \ \cdot (\operatorname{var}(F) + \operatorname{var}(G)) \ .$$

Proof: First of all it is sufficient to consider the case $l = 0$, since the corresponding estimate for $l > 0$ then easily follows by applying the same argument to the system given by $\widehat{\Theta}^{(i,j)} := \Theta^{(l+i,l+j)}$, $0 \le i \le j$, for which the condition on $\Psi_p(n)$ is again satisfied.

It is clear that all functions involved in the construction are nonnegative and that $\{\widehat{F}_V^{(n)} > 0\} \cap \{\widehat{G}_V^{(n)} > 0\} = \emptyset$. These two functions give the portions of mass on the factor V which have not been cancelled up to step n, since

$$\mathbf{P}_\theta^{(0,n)}(F_V - G_V) = F_V^{(n)} - G_V^{(n)} = \widehat{F}_V^{(n)} - \widehat{G}_V^{(n)}.$$

We also have $\int \widehat{F}_V^{(n)} dv = \int \widehat{G}_V^{(n)} dv$, so that

$$\|\widehat{F}_V^{(n)}\|_{L^1_{dv}},\ \|\widehat{G}_V^{(n)}\|_{L^1_{dv}} = \frac{1}{2}\|\mathbf{P}_\theta^{(0,n)}(F_V - G_V)\|_{L^1_{dv}} \leq \Psi_p(n)\cdot(\mathrm{var}(F_V)+\mathrm{var}(G_V)).$$

Moreover, as $\|F^{(n)}\|_{L^1_{dudv}} = \|F_V^{(n)}\|_{L^1_{dv}} = \|\mathbf{P}_\theta^{(n-1,n)}\widehat{F}_V^{(n-1)}\|_{L^1_{dv}} = \|\widehat{F}_V^{(n-1)}\|_{L^1_{dv}}$, and analogously for $G^{(n)}$, we get

$$\|F^{(n)}\|_{L^1_{dudv}} + \|G^{(n)}\|_{L^1_{dudv}} \leq 2\Psi_p(n-1)\cdot(\mathrm{var}(F_V)+\mathrm{var}(G_V))$$

and

$$\|H_V^{(l+1)}\|_{L^1_{dv}} \leq \|F_V^{(l+1)}\|_{L^1_{dv}} \leq \|\widehat{F}_V^{(l)}\|_{L^1_{dv}} \leq \Psi_p(l)\cdot(\mathrm{var}(F_V)+\mathrm{var}(G_V)) \quad \text{for } l \geq 0.$$

For later use we introduce sequences of auxiliary functions as follows. We can choose non-negative measurable $f_{V,k}$, $k \geq 0$, on V such that $0 \leq \ldots \leq f_{V,k} \leq \ldots \leq f_{V,0} = F_V^{(0)} = F_V$ and $\mathbf{P}_\theta^{(0,k)} f_{V,k} = F_V^{(k)}$. This is easily seen as $\mathbf{P}_\theta^{(0,k)}$ is a finite positive linear combination of the transfer operators of the single branches of $\theta^{(0,k)}$, each of which is a simple invertible positive operator. We then let $f_k(u,v) := F_{U,v}^{(0)}(u)\cdot f_{V,k}(v)$, so that $\mathbf{P}_\theta^{(0,k)} f_k = F^{(k)}$.

For the transfer operator of Θ a simple induction shows that

$$\mathbf{P}_\Theta^{(0,n)} F = \sum_{k=0}^{n-1} \mathbf{P}_\Theta^{(k,n)}\left(F_{U,\cdot}^{(k)} H_V^{(k)}\right) + F^{(n)},$$

so that

$$\|\mathbf{P}_\Theta^{(0,n)}(F-G)\|_{L^1_{dudv}} \leq \sum_{k=0}^{n-1} \|\mathbf{P}_\Theta^{(k,n)}\left((F_{U,\cdot}^{(k)} - G_{U,\cdot}^{(k)})\, H_V^{(k)}\right)\|_{L^1_{dudv}} + \|F^{(n)}\|_{L^1_{dudv}} + \|G^{(n)}\|_{L^1_{dudv}}. \tag{17}$$

As the right-hand terms have been estimated before, we turn to the summands on the left, obtaining

$$\begin{aligned}\left\|\mathbf{P}_\Theta^{(k,n)}\left((F_{U,\cdot}^{(k)} - G_{U,\cdot}^{(k)})\, H_V^{(k)}\right)\right\|_{L^1_{dudv}} &\leq \textstyle\int_V \|\, \mathbf{P}_{\phi_v}^{(k,n)}(F_{U,v}^{(k)} - G_{U,v}^{(k)})\ \|_{L^1_{du}} \cdot H_V^{(k)}(v)\, dv \\ &\leq 2K\gamma^{n-k}\cdot\left[\textstyle\int_V \mathrm{var}_U(F_{U,v}^{(k)})\cdot H_V^{(k)}(v)\, dv + \int_V \mathrm{var}_U(G_{U,v}^{(k)})\cdot H_V^{(k)}(v)\, dv\right]\end{aligned}$$

by Lemma 4 and (16), as $F_{U,v}^{(k)}, G_{U,v}^{(k)} \in BV[0,1]$. Now, by Lemma 3,

$$\begin{aligned}\mathrm{var}_U(F_{U,v}^{(k)}) = \mathrm{var}_U\left(\frac{F^{(k)}(.,v)}{F_V^{(k)}(v)}\right) &= \frac{1}{F_V^{(k)}(v)}\cdot\mathrm{var}_U\left(\mathbf{P}_\Theta^{(0,k)} f_k\right) \\ &= \frac{1}{F_V^{(k)}(v)}\cdot\mathrm{var}_U\left[\mathbf{P}_\theta^{(0,k)}\left[v' \mapsto \mathbf{P}_{\phi_{v'}}^{(0,k)} F_{U,v'}^{(0)}\cdot f_{V,k}(v')\right](v)\right].\end{aligned}$$

Recalling Remark 8 this is at most

$$\frac{1}{F_V^{(k)}(v)}\cdot\mathbf{P}_\theta^{(0,k)}\left[v' \mapsto \mathrm{var}_U\left(\mathbf{P}_{\phi_{v'}}^{(0,k)} F_{U,v'}^{(0)}\right)\cdot f_{V,k}(v')\right](v)\ ,$$

which by the Lasota-Yorke inequality (15) is bounded above by

$$\begin{aligned}&\frac{1}{F_V^{(k)}(v)}\cdot\mathbf{P}_\theta^{(0,k)}\left[v' \mapsto \left(K\gamma^k\,\mathrm{var}_U\left(F_{U,v'}^{(0)}\right) + C_0\right)\cdot f_{V,k}(v')\right](v) \\ =\ &\frac{K\gamma^k}{F_V^{(k)}(v)}\cdot\ \mathbf{P}_\theta^{(0,k)}\left[v' \mapsto \mathrm{var}_U\left(F_{U,v'}^{(0)}\right)\cdot f_{V,k}(v')\right](v) +\ C_0 \\ =\ &\frac{K\gamma^k}{F_V^{(k)}(v)}\cdot\ \mathbf{P}_\theta^{(0,k)}\left[v' \mapsto \mathrm{var}_U\left(F(.,v')\right)\cdot\frac{f_{V,k}(v')}{F_V(v')}\right](v) +\ C_0 \\ \leq\ &\frac{K\gamma^k}{F_V^{(k)}(v)}\cdot\ \mathbf{P}_\theta^{(0,k)}\left[v' \mapsto \mathrm{var}_U\left(F(.,v')\right)\right](v) + C_0\ .\end{aligned}$$

Using this estimate for $\mathrm{var}_U(F_{U,v}^{(k)})$, $k \geq 0$, and letting $\Psi_p(-1) := \Psi_p(0) := 1$ we find that

$$\begin{aligned}
&\int_V \mathrm{var}_U(F_{U,v}^{(k)}) \cdot H_V^{(k)}(v)\,dv \\
&\leq C_0 \int_V H_V^{(k)}(v)\,dv + K\gamma^k \int_V \mathbf{P}_\theta^{(0,k)}\left[v' \mapsto \mathrm{var}_U\left(F(.,v')\right)\right](v) \cdot \frac{H_V^{(k)}(v)}{F_V^{(k)}(v)}\,dv \\
&\leq C_0 \Psi_p(k-1)(\mathrm{var}(F_V) + \mathrm{var}(G_V)) + K\gamma^k \int_V \mathbf{P}_\theta^{(0,k)}\left[v' \mapsto \mathrm{var}_U\left(F(.,v')\right)\right](v)\,dv \\
&= C_0 \Psi_p(k-1)(\mathrm{var}(F_V) + \mathrm{var}(G_V)) + K\,\gamma^k \int_V \mathrm{var}_U\left(F(.,v)\right)\,dv.
\end{aligned}$$

Of course, the same is true with F replaced by G. Therefore,

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{P}_\Theta^{(k,n)}\left((F_{U,.}^{(k)} - G_{U,.}^{(k)})\,H_V^{(k)}\right)\|_{L^1_{du\,dv}} \leq \\
&\qquad K\gamma^{n-k} \cdot \left[2C_0\Psi_p(k-1)(\mathrm{var}(F_V) + \mathrm{var}(G_V)) + K\,\gamma^k\left(\mathrm{var}^{(1)}(F) + \mathrm{var}^{(1)}(G)\right)\right].
\end{aligned}$$

Finally, since $\mathrm{var}(F_V) + \mathrm{var}^{(1)}(F) \leq \mathrm{var}(F)$, cf. (6), we obtain returning to (17)

$$\begin{aligned}
\frac{1}{2}\left\|\mathbf{P}_\Theta^{(0,n)}(F-G)\right\|_{L^1_{du\,dv}} &\leq K^2\left[\mathrm{var}(F) + \mathrm{var}(G)\right]\sum_{k=0}^n \gamma^{n-k}(\Psi_p(k-1) + \gamma^k) \\
&\leq \frac{2K^2}{\gamma}\left[\mathrm{var}(F) + \mathrm{var}(G)\right]\sum_{k=0}^n \gamma^{n-k}\left(\Psi_p(k) + \gamma^k\right).
\end{aligned}$$

□

Starting from (15), i.e. taking $\Psi_1(n) := K\gamma^n$, $n \geq 1$, we define the sequences Ψ_p, $p \geq 2$, by the recursion formula of the lemma. We then obtain

Proposition 3. *If $\epsilon < \epsilon^*$, and Θ is any p-dimensional nonautonomous factor of T, then*

$$\sup_{\substack{0 \leq f,g \in BV([0,1]^p),\\ \int f\,dv = \int g\,dv}} \sup_{l \geq 0}\left\{\frac{1}{2}\|\mathbf{P}_\Theta^{(l,l+n)}(f-g)\|_{L^1_{dv}}\right\} \leq \Psi_p(n) \cdot (\mathrm{var}(f) + \mathrm{var}(g))$$

for any $p \geq 1$, $n \geq 0$. Letting $\eta := 4K^3\gamma^{-1}$, we have

$$\Psi_p(n) \leq t^{-n}\left(\frac{\eta}{1-\gamma t}\right)^p \qquad \text{for any } t \in (0, \gamma^{-1}).$$

In particular, for any $q > \gamma$ there is some $\alpha_0 \in (0,1)$ such that $\Psi_p(n) \leq q^n$ whenever $p \leq \alpha_0 n$.

Proof: Let $\Psi_1^*(n) := \gamma^n$, $n \geq 0$, and Ψ_p^* be the p-th convolution power of Ψ_1^*, i.e. $\Psi_{p+1}^*(n) := \sum_{k=0}^n \gamma^{n-k}\Psi_p^*(k)$. A straightforward induction shows that $\Psi_p(n) \leq \eta^p \cdot \Psi_p^*(n)$. The generating function $\varphi_p(t) := \sum_{n\geq 0}\Psi_p^*(n)\,t^n$ equals $(\varphi_1(t))^p = (1-\gamma t)^{-p}$, $t \in (0,\gamma^{-1})$. For such t we therefore have

$$\Psi_p^*(n) \leq t^{-n}\varphi_p(t) = t^{-n}(1-\gamma t)^{-p}.$$

For the second assertion let $q \in (\gamma, 1)$ and take $t \in (q^{-1}, \gamma^{-1})$ so large that $\eta^{-1}(1-\gamma t) \leq 1$. Choose $\alpha_0 > 0$ so small that also $[t\eta^{-\alpha_0}(1-\gamma t)^{\alpha_0}] > q^{-1}$. Then, if $\alpha := \frac{n}{p} \leq \alpha_0$, we find that

$$\Psi_p(n) \leq \left[t\left(\frac{(1-\gamma t)}{\eta}\right)^\alpha\right]^{-n} \leq q^n.$$

□

7. Proof of the theorems

Again we write $T := T_\epsilon$ thus suppressing ϵ in most notations.

Proof of Theorem 1: The assertion $ACC_\Omega(T) \neq \emptyset$ was already shown in Proposition 2. Fix any $\mathcal{G}$-exhausting sequence $(\Lambda_p)_{p\geq 1}$ and let $\mu, \tilde{\mu} \in ACC_\Omega(T)$ with $\mu_{|\mathcal{F}_{tail}} = \tilde{\mu}_{|\mathcal{F}_{tail}}$ and conditional densities $\{h_{\Lambda,\omega}\}_{\omega\in\Omega_{\Lambda^c}}$ and $\{\tilde{h}_{\Lambda,\omega}\}_{\omega\in\Omega_{\Lambda^c}}$ respectively. We have to show that $\mu = \tilde{\mu}$.

For $p \geq 1$ define $\lambda_p := \pi_{\Lambda_p^c}\mu \wedge \pi_{\Lambda_p^c}\tilde{\mu}$ [5], $\mu_p := \pi_{\Lambda_p^c}\mu - \lambda_p$ and $\tilde{\mu}_p := \pi_{\Lambda_p^c}\tilde{\mu} - \lambda_p$. These are $T_{\Lambda_p^c}$-invariant measures on $(\Omega_{\Lambda_p^c}, \mathcal{F}_{\Lambda_p^c})$. Let $P := \frac{1}{2}(\mu + \tilde{\mu})$ and denote $f := \frac{d\mu}{dP}$, $\tilde{f} := \frac{d\tilde{\mu}}{dP}$. Let $g : \Omega \to \mathbb{R}$ be a bounded measurable function that depends only on the coordinates in Λ_{p_0} for some $p_0 \geq 1$. g will serve as a test function, i.e. we shall show that $E_\mu[g] = E_{\tilde{\mu}}[g]$. As p_0 and g are arbitrary, this will prove $\mu = \tilde{\mu}$.

It follows from (12) that

$$h_{\Lambda_p, T^n_{\Lambda_p^c}\omega}(x) = h^{(n)}_{\Lambda_p, T^n_{\Lambda_p^c}\omega}(x) = E_{\pi_{\Lambda_p^c}\mu}[(\mathbf{P}^{(0,n)}_{\Lambda_p,\cdot} h_{\Lambda_p,\cdot})(x) | T^{-n}_{\Lambda_p^c}\mathcal{F}_{\Lambda_p^c}](\omega)$$

Therefore, for $p \geq p_0$,

$$\begin{aligned}
E_\mu[g] &= E_{\pi_{\Lambda_p^c}\mu}\left[\int h_{\Lambda_p,\cdot}(x)\cdot g(x)\,dx\right] = E_{\pi_{\Lambda_p^c}\mu}\left[\int h_{\Lambda_p, T^n_{\Lambda_p^c}(\cdot)}(x)\cdot g(x)\,dx\right] \\
&= E_{\pi_{\Lambda_p^c}\mu}\left[\int E_{\pi_{\Lambda_p^c}\mu}\left[\left(\mathbf{P}^{(0,n)}_{\Lambda_p,\cdot} h_{\Lambda_p,\cdot}\right)(x)\,\middle|\, T^{-n}_{\Lambda_p^c}\mathcal{F}_{\Lambda_p^c}\right]\cdot g(x)\,dx\right] \\
&= E_{\pi_{\Lambda_p^c}\mu}\left[E_{\pi_{\Lambda_p^c}\mu}\left[\int \left(\mathbf{P}^{(0,n)}_{\Lambda_p,\cdot} h_{\Lambda_p,\cdot}\right)(x)\cdot g(x)\,dx \,\middle|\, T^{-n}_{\Lambda_p^c}\mathcal{F}_{\Lambda_p^c}\right]\right] \\
&= E_{\pi_{\Lambda_p^c}\mu}\left[\int \left(\mathbf{P}^{(0,n)}_{\Lambda_p,\cdot} h_{\Lambda_p,\cdot}\right)(x)\cdot g(x)\,dx\right] \\
&= E_P\left[f(\cdot)\cdot\int\left(\mathbf{P}^{(0,n)}_{\Lambda_p^c,\pi_{\Lambda_p^c}(\cdot)} h_{\Lambda_p,\pi_{\Lambda_p^c}(\cdot)}\right)(x)\cdot g(x)\,dx\right]
\end{aligned}$$

and an analogous identity holds for $E_{\tilde{\mu}}[g]$. Hence

$$\begin{aligned}
\left|E_\mu[g] - E_{\tilde{\mu}}[g]\right| \leq& \left|E_P\left[(f-\tilde{f})(\cdot)\cdot\int\left(\mathbf{P}^{(0,n)}_{\Lambda_p,\pi_{\Lambda_p^c}(\cdot)} h_{\Lambda_p,\pi_{\Lambda_p^c}(\cdot)}\right)(x)\cdot g(x)\,dx\right]\right| \\
&+ \left|E_P\left[\tilde{f}(\cdot)\cdot\int\left(\mathbf{P}^{(0,n)}_{\Lambda_p,\pi_{\Lambda_p^c}(\cdot)}\left(h_{\Lambda_p,\pi_{\Lambda_p^c}(\cdot)} - \tilde{h}_{\Lambda_p,\pi_{\Lambda_p^c}(\cdot)}\right)\right)(x)\cdot g(x)\,dx\right]\right| \\
\leq& E_P\left[\left|E_P\left[f-\tilde{f}|\pi^{-1}_{\Lambda_p^c}\mathcal{F}_{\Lambda_p^c}\right](\cdot)\right|\cdot\left|\int\left(\mathbf{P}^{(0,n)}_{\Lambda_p,\pi_{\Lambda_p^c}(\cdot)} h_{\Lambda_p,\pi_{\Lambda_p^c}(\cdot)}\right)(x)\cdot g(x)\,dx\right|\right] \\
&+ \|g\|_\infty\cdot E_{\tilde{\mu}}\left[\int\left|\left(\mathbf{P}^{(0,n)}_{\Lambda_p,\pi_{\Lambda_p^c}(\cdot)}\left(h_{\Lambda_p,\pi_{\Lambda_p^c}(\cdot)} - \tilde{h}_{\Lambda_p,\pi_{\Lambda_p^c}(\cdot)}\right)\right)(x)\right|\,dx\right] \\
\leq& \|g\|_\infty\cdot E_P\left[\left|E_P\left[f-\tilde{f}|\pi^{-1}_{\Lambda_p}\mathcal{F}_{\Lambda_p^c}\right]\right|\right] + 4C_0\|g\|_\infty\cdot p\,\Psi_p(n)
\end{aligned}$$

by Proposition 3 and part c) of Proposition 2. Since $\lim_{n\to\infty}\Psi_p(n) = 0$ for all $p \geq 1$, we find that

$$\begin{aligned}
\left|E_\mu[g] - E_{\tilde{\mu}}[g]\right| &\leq \|g\|_\infty\cdot\liminf_{p\to\infty} E_P\left[\left|E_P\left[f-\tilde{f}|\pi^{-1}_{\Lambda_p}\mathcal{F}_{\Lambda_p^c}\right]\right|\right] \\
&= \|g\|_\infty\cdot E_P\left[\left|E_P\left[f-\tilde{f}|\mathcal{F}_{tail}\right]\right|\right] = \|g\|_\infty\cdot E_P[0] = 0
\end{aligned}$$

since $\mu_{|\mathcal{F}_{tail}} = \tilde{\mu}_{|\mathcal{F}_{tail}}$. □

From this point on we specialize to $\mathcal{G}_1 = (\mathbb{N}, \geq)$. In particular, $\Lambda_p = \{0, 1, \ldots, p-1\}$, $p \geq 1$.

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[5]For two finite measures P_1 and P_2 on the same measurable space, $P_1 \wedge P_2$ denotes the finite measure defined by $(P_1 \wedge P_2)(A) := \sup\{Q(A) :\ Q \text{ a finite measure}, Q \leq P_1, Q \leq P_2\}$.

Proof of Theorem 2: To prove uniqueness of the invariant measure we fix $\mu, \mu^* \in ACC_\Omega(T)$ with conditional densities $\{h^{(*)}_{\Lambda,\omega}\}$ and show that $\int_\Omega \chi \, d\mu = \int_\Omega \chi \, d\mu^*$ for every $\chi \in Lip_f(\Omega)$. This class of test functions is rich enough to let us conclude that $\mu = \mu^*$.

Let χ be a function of, say, the first M coordinates only, so that we can regard it as a function on any Ω_{Λ_p}, $p \geq M$. Take some $q \in (\gamma, 1)$ and choose α_0 as in Proposition 3. For any $n \geq \alpha_0^{-1}$ we let $p = p(n) := [\alpha_0 \, n]$, so that $\Psi_p(n) \leq q^n$. Consequently,

$$\left| \int_{\Omega_{\Lambda_p}} \chi \cdot \mathbf{P}^{(0,n)}_{\Lambda_p,\omega} \left(h_{\Lambda_p,\omega} - 1_{U_p} \right) \, dx \right| \leq 4 \, \|\chi\|_\infty \, C_0 \, \alpha_0 \cdot n \, q^n \tag{18}$$

for such n and any $\omega \in \Omega_{\Lambda_p^c}$, since $h_{\Lambda_p,\omega}, 1_{U_p} \in \mathcal{V}_p(C_0)$.

Now take some $\alpha_1 \in (0, \alpha_0)$ and choose $\epsilon_1 = \epsilon_1(\alpha_1, q)$ as in Proposition 1. For $\epsilon \in (0, \epsilon_1)$ we then have

$$\sup_{x \in \Omega_{\Lambda_p}, \omega, \omega' \in \Omega_{\Lambda_p^c}} \left| (\chi \circ T^n_{\Lambda_p,\omega})(x) - (\chi \circ T^n_{\Lambda_p,\omega'})(x) \right| \leq M \ \mathrm{Lip}(\chi) \cdot q^n \tag{19}$$

provided $p = [\alpha_0 \, n] > M + \alpha_1 \, n$ which holds for $n \geq n_0(\alpha_0, \alpha_1, M)$. For such n and any $\omega, \omega' \in \Omega_{\Lambda_p^c}$ therefore

$$\left| \int_{\Omega_{\Lambda_p}} \chi \cdot \mathbf{P}^{(0,n)}_{\Lambda_p,\omega} 1_{U_p} \, dx - \int_{\Omega_{\Lambda_p}} \chi \circ T^n_{\Lambda_p,\omega'} \, dx \right| \leq M \ \mathrm{Lip}(\chi) \cdot q^n .$$

Combined with (18) this yields

$$\left| \int_{\Omega_{\Lambda_p}} \chi \cdot \mathbf{P}^{(0,n)}_{\Lambda_p,\omega} h_{\Lambda_p,\omega} \, dx - \int_{\Omega_{\Lambda_p}} \chi \circ T^n_{\Lambda_p,\omega'} \, dx \right| \leq 4 \, C_0 \, \alpha_0 \ \mathrm{Lip}_M(\chi) \cdot n \, q^n$$

for $n \geq n_0$. Since $\int_\Omega \chi \, d\mu = \int_{\Omega_{\Lambda_p^c}} \int_{\Omega_{\Lambda_p}} \chi \cdot \mathbf{P}^{(0,n)}_{\Lambda_p,\omega} h_{\Lambda_p,\omega} \, dx \, d\pi_{\Lambda_p^c}\mu(\omega)$ by part c) of Proposition 2, we end up with

$$\left| \int_\Omega \chi \, d\mu - \int_{\Omega_{\Lambda_p}} \chi \circ T^n_{\Lambda_p,\omega'} \, dx \right| \leq 4 \, C_0 \, \alpha_0 \ \mathrm{Lip}_M(\chi) \cdot n \, q^n$$

for $n \geq n_0$, and analogously for μ^*. As the second term does not depend on the invariant measure chosen, and the right-hand bound goes to zero as $n \to \infty$, we are done. □

As the arguments leading to exponential mixing in time and space respectively are quite different, we prefer to first give separate proofs of these two aspects. Combining theorems 4 and 5 immediately gives Theorem 3.

Theorem 4. (Exponential spatial mixing for $\mathcal{G}_1 = (\mathbb{N}, \geq)$) *In the situation of Theorem 2, if $\epsilon \in (0.\epsilon^*)$, there are $q \in (0,1)$ and $K > 0$ such that for any $\varphi, \psi \in Lip_f(\Omega)$ which depend only on the first M coordinates we have*

$$\left| \int (\varphi \circ T^l) \cdot (\psi \circ \sigma^m) \, d\mu_\epsilon - \int \varphi \, d\mu_\epsilon \cdot \int \psi \circ \sigma^m \, d\mu_\epsilon \right| \leq K \, Q^M \, L_M(\varphi, \psi) \cdot q^{l+m}$$

for $m > M$, where $L_M(\varphi, \psi) := (\|\varphi\|_\infty \|\psi\|_\infty + \|\varphi\|_\infty M \, \mathrm{Lip}(\psi) + \|\psi\|_\infty M \, \mathrm{Lip}(\varphi))$ and σ denotes the spatial left shift on Ω, provided $l < \alpha_0^{-1}(m + M)$.

Proof: We are going to extend the argument used in the proof of Theorem 2. To this end we apply Proposition 3 with $p := m$. Take some $q \in (\gamma, 1)$ and α_0 as in that Proposition. Choose $n = n(m)$ such that $m = p = [\alpha_0 n/2]$. We then have $\Psi_{2m}(n) \leq q^n$.

Choose $\alpha := \alpha_0/8$ so that $l < \alpha_0^{-1}(m + M)$ implies $\alpha(n + l) \leq [\alpha_0 n/2] = m$ for all $m > M$. Henceforth we assume that $l < \alpha_0^{-1}(m + M)$ and write $\tilde{\varphi} := \varphi \circ T^l$ and $\tilde{\psi} := \psi \circ \sigma^m$.

Considering the splitting $\Omega = \Omega_{\Lambda_m} \times \Omega_{\Lambda_{2m} \setminus \Lambda_m} \times \Omega_{\Lambda_m^c}$, Proposition 1 in view of our choice $\alpha(n + l) \geq m$ shows that there are $\epsilon_1 = \epsilon_1(\alpha, q), Q = Q(\alpha, q) > 0$ such that in case $\epsilon \in (0, \epsilon_1)$ we have

$$\sup_{\substack{x\in\Omega_{\Lambda_{2m}},\ v\in\Omega_{\Lambda_{2m}\setminus\Lambda_m},\\ \omega,\omega^*\in\Omega_{\Lambda_{2m}^c}}} \left|(\tilde\varphi\circ T^n)(x,\omega)-(\tilde\varphi\circ T^n)(\pi_{\Lambda_m}(x),v,\omega^*)\right| \le M\,Q^M\ \mathrm{Lip}(\varphi)\cdot q^n \tag{20}$$

(compare (19)) and,

$$\sup_{x\in\Omega_{\Lambda_{2m}},\ \omega,\omega'\in\Omega_{\Lambda_{2m}^c}} \left|\left(\tilde\psi\circ T^n\right)(x,\omega)-\left(\tilde\psi\circ T^n\right)(x,\omega')\right| \le M\,Q^M\ \mathrm{Lip}(\psi)\cdot q^n. \tag{21}$$

We are going to show that for arbitrary $\omega,\omega',\omega''\in\Omega_{\Lambda_{2m}^c}$ and

$$A := \int_{\Omega_{\Lambda_{2m}}} \tilde\varphi\,\tilde\psi\cdot\mathbf{P}^{(0,n)}_{\Lambda_{2m},\omega} h_{\Lambda_{2m},\omega}\,dx\,,$$

$$A^* := \left(\int_{\Omega_{\Lambda_{2m}}} \tilde\varphi\cdot\mathbf{P}^{(0,n)}_{\Lambda_{2m},\omega''} h_{\Lambda_{2m},\omega''}\,dx\right)\left(\int_{\Omega_{\Lambda_{2m}}} \tilde\psi\cdot\mathbf{P}^{(0,n)}_{\Lambda_{2m},\omega'} h_{\Lambda_{2m},\omega'}\,dx\right)$$

we have

$$|A-A^*| \le 12\,C_1\,M\,Q^M\ (\|\varphi\|_\infty+\mathrm{Lip}(\varphi))\ (\|\psi\|_\infty+\mathrm{Lip}(\psi))\cdot q^n. \tag{22}$$

Integrating this estimate with respect to $d\pi_{\Lambda_{2m}^c}\mu(\omega)$, $d\pi_{\Lambda_{2m}^c}\mu(\omega')$, and $d\pi_{\Lambda_{2m}^c}\mu(\omega'')$ and using the T-invariance of μ immediately yields

$$\left|\int_\Omega \tilde\varphi\,\tilde\psi\,d\mu-\int_\Omega\varphi\,d\mu\cdot\int_\Omega\psi\circ\sigma^m\,d\mu\right| \le 12\,C_1\,M\,Q^M\ \cdot(\|\varphi\|_\infty+\mathrm{Lip}(\varphi))\cdot(\|\psi\|_\infty+\mathrm{Lip}(\psi))\cdot q^n.$$

Since $q^n\le q^m$, this implies the assertion of the theorem. To begin with let

$$\begin{aligned} A_1 &:= \int_{\Omega_{\Lambda_{2m}}} \tilde\varphi\,\tilde\psi\cdot\mathbf{P}^{(0,n)}_{\Lambda_{2m},\omega}1_{\Lambda_{2m}}\,dx = \int_{\Omega_{\Lambda_{2m}}}\left(\tilde\varphi\circ T^{(0,n)}_{\Lambda_{2m},\omega}\right)(x)\cdot\left(\tilde\psi\circ T^{(0,n)}_{\Lambda_{2m},\omega}\right)(x)\,dx\\ &= \int_{\Omega_{\Lambda_{2m}}}\left(\tilde\varphi\circ T^{(0,n)}\right)(x,\omega)\cdot\left(\tilde\psi\circ T^{(0,n)}\right)(x,\omega)\,dx, \end{aligned}$$

and observe that

$$|A-A_1|\le 4C_1\|\varphi\|_\infty\|\psi\|_\infty\cdot\Psi_{2m}(n).$$

Take any $v^*\in\Omega_{\Lambda_{2m}\setminus\Lambda_m}$, $\omega',\omega^*\in\Omega_{\Lambda_{2m}^c}$ and let

$$\begin{aligned} A_2 &:= \int_{\Omega_{\Lambda_{2m}}}\left(\tilde\varphi\circ T^{(0,n)}\right)(\pi_{\Lambda_m}(x),v^*,\omega^*)\cdot\left(\tilde\psi\circ T^{(0,n)}\right)(x,\omega')\,dx\\ &= \left(\int_{\Omega_{\Lambda_{2m}}}\left(\tilde\varphi\circ T^{(0,n)}\right)(\pi_{\Lambda_m}(x),v^*,\omega^*)\,dx\right)\cdot\left(\int_{\Omega_{\Lambda_{2m}}}\left(\tilde\psi\circ T^{(0,n)}\right)(x,\omega')\,dx\right). \end{aligned}$$

(Observe that the two factors of the integrand as a function of x only depend on the coordinates in Λ_m and Λ_m^c respectively.) According to (20) and (21) we obtain

$$|A_1-A_2|\le M\,Q^M\ (\|\psi\|_\infty\,\mathrm{Lip}(\varphi)+\|\varphi\|_\infty\,\mathrm{Lip}(\psi))\cdot q^n.$$

For arbitrary $\omega''\in\Omega_{\Lambda_{2m}^c}$ and

$$\begin{aligned} A_3 &:= \left(\int_{\Omega_{\Lambda_{2m}}}\left(\tilde\varphi\circ T^{(0,n)}\right)(x,\omega'')\,dx\right)\cdot\left(\int_{\Omega_{\Lambda_{2m}}}\left(\tilde\psi\circ T^{(0,n)}\right)(x,\omega')\,dx\right)\\ &= \left(\int_{\Omega_{\Lambda_{2m}}}\left(\tilde\varphi\circ T^{(0,n)}_{\Lambda_{2m},\omega''}\right)(x)\,dx\right)\cdot\left(\int_{\Omega_{\Lambda_{2m}}}\left(\tilde\psi\circ T^{(0,n)}_{\Lambda_{2m},\omega'}\right)(x)\,dx\right)\\ &= \left(\int_{\Omega_{\Lambda_{2m}}}\tilde\varphi\cdot\mathbf{P}^{(0,n)}_{\Lambda_{2m},\omega''}1_{\Lambda_{2m}}\,dx\right)\cdot\left(\int_{\Omega_{\Lambda_{2m}}}\tilde\psi\cdot\mathbf{P}^{(0,n)}_{\Lambda_{2m},\omega'}1_{\Lambda_{2m}}\,dx\right), \end{aligned}$$

estimate (20) shows that

$$|A_2-A_3|\le M\,Q^M\,\|\psi\|_\infty\,\mathrm{Lip}(\varphi)\cdot q^n.$$

Finally, recalling the last of the expressions for A_3 above, we find that

$$|A_3 - A^*| \leq 8C_1\|\varphi\|_\infty\|\psi\|_\infty\Psi_{2m}(n).$$

Combining the estimates for these intermediate steps leads to (22). □

Theorem 5. (Exponential mixing in time for $\mathcal{G}_1 = (\mathbb{N}, \geq)$) *In the situation of Theorem 2, if $\epsilon \in (0, \epsilon^*)$, there are $q \in (0,1)$ and $K, Q > 0$ such that for any $\varphi, \psi \in Lip_f(\Omega)$ which depend only on the first M coordinates we have*

$$\left|\int (\varphi \circ T^l) \cdot (\psi \circ \sigma^m)\, d\mu_\epsilon - \int \varphi\, d\mu_\epsilon \cdot \int \psi \circ \sigma^m\, d\mu_\epsilon\right| \leq K\, Q^M\, L_M(\varphi, \psi) \cdot q^{l+m}$$

if $l \geq (\alpha_0)^{-1}(m + M)$ where $L_M(\varphi, \psi) := (\|\varphi\|_\infty\|\psi\|_\infty + \|\varphi\|_\infty M \operatorname{Lip}(\psi) + \|\psi\|_\infty M \operatorname{Lip}(\varphi))$ and α_0 is the constant from Proposition 3.

Proof: Let $p = [\alpha_0 l]$ where α_0 is the constant from Proposition 3 and assume that $p \geq m + M$. We abbreviate $\Lambda := \Lambda_p$. For $\omega \in \Omega_{\Lambda^c}$ let

$$g(\omega) := \int_{\Omega_\Lambda} \psi \circ \sigma^m(x) h_{\Lambda,\omega}(x)\, dx$$

and observe that

$$\int_{\Omega_{\Lambda^c}} g(\omega)\, d(\pi_{\Lambda^c}\mu)(\omega) = \int \psi \circ \sigma^m\, d\mu\ .$$

Denote

$$B_1 := \int (\varphi \circ T^l) \cdot (\psi \circ \sigma^m)\, d\mu = \int_{\Omega_{\Lambda^c}} \int_{\Omega_\Lambda} (\varphi \circ T^{(0,l)}_{\Lambda,\omega}(x)) \cdot (\psi \circ \sigma^m(x)) h_{\Lambda,\omega}(x)\, dx\, d(\pi_{\Lambda^c}\mu)(\omega)$$

$$B_2 := \int_{\Omega_{\Lambda^c}} \int_{\Omega_\Lambda} \varphi \circ T^{(0,l)}_{\Lambda,\omega}(x) \cdot (\psi \circ \sigma^m(x) h_{\Lambda,\omega}(x) - g(\omega))\, dx\, d(\pi_{\Lambda^c}\mu)(\omega)$$

$$B_3 := \int_{\Omega_{\Lambda^c}} \left(\int_{\Omega_\Lambda} \varphi \circ T^{(0,l)}_{\Lambda,\omega}(x)(1 - h_{\Lambda,\omega}(x))\, dx\right) g(\omega)\, d(\pi_{\Lambda^c}\mu)(\omega)$$

$$B_4 := \int_{\Omega_{\Lambda^c}} \left(\int_{\Omega_\Lambda} \varphi \circ T^{(0,l)}_{\Lambda,\omega}(x) h_{\Lambda,\omega}(x)\, dx\right) g(\omega)\, d(\pi_{\Lambda^c}\mu)(\omega)$$

(The notation $\psi \circ \sigma^m(x)$ is justified because this function depends only on the first $m + M$ coordinates of (x, ω) and x already represents the first $p \geq m + M$ ones.)

Since $B_1 = B_2 + B_3 + B_4$ we see that

$$\begin{aligned} &\left|\int (\varphi \circ T^l) \cdot (\psi \circ \sigma^m)\, d\mu - \int \varphi\, d\mu \cdot \int \psi \circ \sigma^m\, d\mu\right| \\ \leq &|B_2| + |B_3| + \left|B_4 - \int \varphi\, d\mu \cdot \int \psi \circ \sigma^m\, d\mu\right|\ . \end{aligned} \tag{23}$$

We estimate the three terms on the right-hand side. To this end let

$$\rho_{\Lambda,\omega}(x) := \psi \circ \sigma^m(x)\, h_{\Lambda,\omega}(x) - g(\omega)$$

and denote by $\rho^+_{\Lambda,\omega}$ and $\rho^-_{\Lambda,\omega}$ the positive and negative parts of $\rho_{\Lambda,\omega}$, respectively. Then

$$\begin{aligned} |B_2| &= \left|\int_{\Omega_{\Lambda^c}} \int_{\Omega_\Lambda} \varphi(x) \cdot \mathbf{P}^{(0,l)}_{\Lambda,\omega}(\rho^+_{\Lambda,\omega} - \rho^-_{\Lambda,\omega})(x)\, dx\, d(\pi_{\Lambda^c}\mu)(\omega)\right| \\ &\leq \|\varphi\|_\infty \cdot 2q^l\, (\operatorname{var}(\rho^+_{\Lambda,\omega}) + \operatorname{var}(\rho^-_{\Lambda,\omega})) \quad \text{by Proposition 3} \\ &\leq 4\|\varphi\|_\infty \cdot q^l\ \operatorname{var}(\rho_{\Lambda,\omega}) \\ &\leq 4\|\varphi\|_\infty \cdot q^l \cdot (M\ \operatorname{Lip}(\psi) + \|\psi\|_\infty\, p(C_0 + 2)) \end{aligned} \tag{24}$$

with C_0 as in Proposition 2c.

We turn to the estimate for $|B_3|$. Since

$$\left|\int_{\Omega_\Lambda} \varphi \circ T^{(0,l)}_{\Lambda,\omega}(x)\,(1-h_{\Lambda,\omega}(x))\,dx\right| = \left|\int_{\Omega_\Lambda} \varphi(x)\mathbf{P}^{(0,l)}_{\Lambda,\omega}\,(1_\Lambda - h_{\Lambda,\omega})\,(x)\,dx\right| \le 4\|\varphi\|_\infty \cdot q^l\,pC_0$$

by Proposition 3, we see immediately that

$$|B_3| \le 4\|\varphi\|_\infty\,\|\psi\|_\infty \cdot q^l\,pC_0 \tag{25}$$

In order to estimate the B_4-term in (23), we need to control the dependence on ω of the inner integral in B_4. So let $\omega,\omega' \in \Omega_{\Lambda^c}$.

$$\begin{aligned}
&\left|\int_{\Omega_\Lambda} \varphi \circ T^{(0,l)}_{\Lambda,\omega}(x)\,h_{\Lambda,\omega}(x)\,dx - \int_{\Omega_\Lambda} \varphi \circ T^{(0,l)}_{\Lambda,\omega'}(x)\,h_{\Lambda,\omega'}(x)\,dx\right| \\
&\le \left|\int_{\Omega_\Lambda} \varphi(x)\mathbf{P}^{(0,l)}_{\Lambda,\omega}\,(h_{\Lambda,\omega}(x) - h_{\Lambda,\omega'}(x))\,\,dx\right| + \int_{\Omega_\Lambda} \left|\varphi \circ T^{(0,l)}_{\Lambda,\omega}(x) - \varphi \circ T^{(0,l)}_{\Lambda,\omega'}(x)\right| h_{\Lambda,\omega}(x)\,dx \\
&\le 4\|\varphi\|_\infty \cdot q^l\,pC_0 + M\,Q^M\ \mathrm{Lip}(\varphi)\cdot q^l
\end{aligned}$$

if $\epsilon > 0$ is small enough by Proposition 3 and Proposition 1. This means that this integral, as a function of ω, is nearly constant, and since $\int_{\Omega_{\Lambda^c}} g(\omega)\,d(\pi_{\Lambda^c}\mu)(\omega) = \int \psi\circ\sigma^m\,d\mu$, it follows that

$$\begin{aligned}
&\left|B_3 - \int_\Omega \varphi\,d\mu \cdot \int_\Omega \psi\circ\sigma^m\,d\mu\right| \\
&= \left|\int_{\Omega_{\Lambda^c}} \left(\int_{\Omega_\Lambda} \varphi \circ T^{(0,l)}_{\Lambda,\omega}(x)\,h_{\Lambda,\omega}(x)\,dx\right)\cdot\left(g(\omega) - \int \psi\circ\sigma^m\,d\mu\right)\,d(\pi_{\Lambda^c}\mu)(\omega)\right| \\
&\le \|\psi\|_\infty \cdot \left(4\|\varphi\|_\infty\,pC_0 + M\,Q^M\ \mathrm{Lip}(\varphi)\right)\,q^l\ .
\end{aligned}$$

Combining this with (23) and (24) we arrive at

$$\begin{aligned}
&\left|\int (\varphi\circ T^l)\cdot(\psi\circ\sigma^m)\,d\mu - \int\varphi\,d\mu\cdot\int\psi\circ\sigma^m\,d\mu\right| \\
&\le 4\|\varphi\|_\infty\cdot q^l\cdot(M\ \mathrm{Lip}(\psi) + \|\psi\|_\infty\,pC_0) + \|\psi\|_\infty\cdot\left(4\|\varphi\|_\infty\,pC_0 + M\,Q^M\ \mathrm{Lip}(\varphi)\right)\,q^l \\
&\le K\,Q^M\ \mathrm{Lip}_M(\varphi)\ \mathrm{Lip}_M(\psi)\cdot\tilde q^l
\end{aligned}$$

for any $\tilde q > q$ where $K > 0$ is a suitable constant (observe that $p = [\alpha_0 l]$). □

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