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HADAMARD'S AND CALABI-YAU'S CONJECTURES ON NEGATIVELY CURVED AND MINIMAL SURFACES

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1. Introduction

In this paper we consider two related problems. Let (M, s) be a Riemann surface with a complete Riemannian metric s on M and let

$$\Psi : (M, s) \to B_1 \subset \mathbb{R}^3$$

be an isometrical immersion, and let B_1 be the unit ball.

Problem 1 (Hadamard's conjecture, [Ha], cf. [R2]). Is it possible that the metric s has a negative Gaussian curvature?

If the Gaussian curvature K of s is a negative constant then such an immersion is impossible even into the whole space $\mathbb{R}^3$ (Hilbert, [Hi]). Hilbert's theorem is valid for $K \leq const < 0$ (Efimov, [E]). On the other hand there exists a complete bounded surface in $\mathbb{R}^3$ with nonpositive Gaussian curvature (Rosendorn, [R1], [R2]).

Problem 2 (Calabi-Yau problem, [Y]). Is it possible that an immersion Ψ is minimal?

Jorge and Xavier, [J-X] , proved the existence of a complete minimally immersed surface between two planes. On the other hand there are many non-existence results under certain extra conditions on the surface, see e.g. [H], [X].

The aim of this paper is to show that to both problems the answer is YES. And even more, the following theorem holds.

Theorem. *There exists a complete surface of negative Gaussian curvature minimally immersed in $\mathbb{R}^3$ which is a subset of the unit ball.*

Our example of a minimal surface is somewhat similar to the example of Jorge and Xavier: we also use the Weierstrass representation of minimal surfaces and the Runge approximation theorem.

2. Preliminaries

Let $\Omega \subset \mathbb{C}$ be a domain and $\varphi : \Omega \to \mathbb{C}^3$ be a conformal map $\varphi = (\varphi_1, \varphi_2, \varphi_3)$, satisfying, $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 \equiv 0$. Then

$$X(z) = Re \int_{z_0}^{z} \varphi \tag{1}$$

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is a minimal surface in $\mathbb{R}^3$. Also every minimal surface $X : \Omega \to \mathbb{R}^3$ can be locally represented in the form (1) and if Ω is simply connected then X is globally represented by (1), (see [C], [O]). In order for Ω to be immersed in $\mathbb{R}^3$ one requires

$$\sum_{i=1}^{3} |\varphi_i(z)| \neq 0$$

for all $z \in \Omega$.

Let us assume that $\varphi_1 - i\varphi_2 \not\equiv 0$ and set

$$f = \varphi_1 - i\varphi_2,$$
$$g = \varphi_3/(\varphi_1 - i\varphi_2)$$

then f is a holomorphic and g is a meromorphic function on Ω. The surface (1) can be obtained by

$$X(z) = Re \int_{z_0}^{z} (\frac{1}{2}f(1-g^2), \frac{i}{2}f(1+g^2), fg). \tag{2}$$

This is called the Weierstrass representation of a minimal surface. The induced metric s_X on Ω is given by:

$$s_X = (\frac{1}{2}|f|(1+|g|^2)|dz|)^2. \tag{3}$$

The poles of g are the zeros of f and a pole of order k of g corresponds to a zero of order $2k$ of f. The curvature K_X of (M, s) is given by:

$$K_X = -\left(\frac{4|g'|}{|f|(1+|g|^2)^2}\right)^2.$$

The meromorphic map g has an important geometrical meaning: it is the composition of the Gauss map of $X(m)$ with the stereographic projection of the unit sphere to the equatorial plane, from the north pole. Let the minimal immersion $X : \Omega \to \mathbb{R}^3$ be given by (2) and let h be a holomorphic function on $\Omega, h \neq 0$ in Ω. Set $\tilde{f} = fh, \tilde{g} = g/h$, and

$$\tilde{X}(z) = Re \int_{z_0}^{z} (\frac{1}{2}\tilde{f}(1-\tilde{g}^2), \frac{i}{2}\tilde{f}(1+\tilde{g}^2), fg). \tag{4}$$

Then $\tilde{X} : \Omega \to \mathbb{R}^3$ is a minimal immersion.

Notation. Let D_r be a disk on $\mathbb{C}$: $|z| < r$, $S_r := \partial D_r$, and let $B_r \subset \mathbb{R}^3$ be a ball $|x| < r$. Let $E \subset \mathbb{C}$ be a set, $\varepsilon > 0$. By $U[\varepsilon](E)$ we denote an ε-neighbourhood of the set E.

3. Proof of the Theorem

Lemma. *Let* $X \in C^\infty(\bar{D}_1; \mathbb{R}^3)$ *and*

$$X : D_1 \to B_r \subset \mathbb{R}^3, \tag{5}$$

$r > 0$, *be a minimal immersion,* $X(0) = 0, K_X < 0$. *Assume that* (D_1, s_X) *is a geodesic disk of radius* ρ *centred in 0. Then for every* $\varepsilon, \rho > 0$ *there exists a minimal immersion*

$$Y : D_1 \to B_R \subset \mathbb{R}^3,$$

$R = \sqrt{r^2 + s^2} + \varepsilon$, *such that* (D_1, s_Y) *is a geodesic disk of radius* $\rho + s, K_Y < 0$ *and*

$$|X - Y| < \varepsilon \text{ on } D_{1-\varepsilon}$$

This Lemma will be proved in Section 4. We now show that the Theorem is a consequence of the Lemma.

We define a sequence of minimal immersions

$$X_n : D_1 \to \mathbb{R}^3$$

by induction over $n = 1, 2, \ldots$. Let $X_1 : (D_1, |dz|) \to \mathbb{R}^3$ be any minimal immersion such that $K_{X_1} < 0, X_1(0) = 0$ and let (D_1, s_{X_1}) be a disk of radius 1. Let $\varepsilon_n, n = 1, 2, \ldots$, be a sequence of positive numbers which will be specified later. Assume that a minimal immersion $X_{n-1} = X$ is already defined. Set $\varepsilon = \varepsilon_n, s = 1/n$ and let the minimal immersion Y be defined by the lemma. Define $X_n = Y$. If the ε_k tend sufficiently fast to zero as $k \to \infty$ then the following holds:

(a) $X_k \to \Psi$ as $k \to \infty$ in the open disk D_1 and

$$\Psi : D_1 \to \mathbb{R}^3$$

is a minimal immersion and $K_\Psi < 0$;

(b)

$$X_k : D_1 \to B_{r_k} \subset \mathbb{R}^3$$

where $r_k \leq r_{k-1} + 1/k^2$ and hence $r_k \leq 2$ for all k;

(c) since (D_1, s_{X_k}) is a geodesic disk of radius ρ_k, where ρ_k is given by

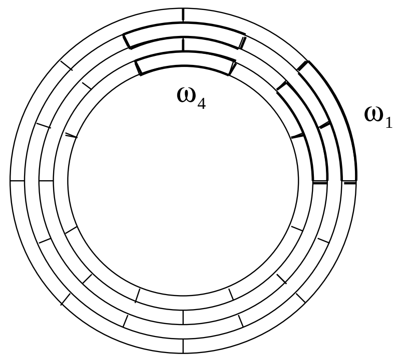
$$\rho_k = \sum_{j=1}^{k} 1/j$$

the metric (D_1, s_X) is complete provided that the ε_k tend to zero sufficiently fast.

The theorem is proved.

4. Proof of the Lemma

(4.1) Consider the following labyrinth, see the diagram below.



It's not difficult to find a way from the inner circle to the outside but any such way is fairly long although the Euclidean distance is short.

Now we give a formal description of a partition of the unit disk, which is illustrated by the diagram above.

N-partition of a disk. Let $N \in \mathbb{N}$. Denote $r_i = 1 - i/N^3$, $i = 0, \dots, 2N^2 + 1$,
$\mathfrak{A} = D_1 \backslash D_{r_{2N^2+1}}$,
$A_i = D_{r_{2i}} \backslash D_{r_{2i+1}}$, $\tilde{A}_i = D_{r_{2i-1}} \backslash D_{r_{2i}}$,

$$A = \cup_{i=0}^{N^2} A_i,$$
$$\tilde{A} = \cup_{i=1}^{N^2} \tilde{A}_i,$$
$$S = \cup_{i=0}^{2N^2} S_{r_i}.$$

Denote by l_θ a ray in $\mathbb{C}$, $l_\theta = \alpha e^{i\theta}$, $\alpha > 0$,

$$l = \cup_{i=0}^{N} l_{i2\pi/N},$$
$$\tilde{l} = \cup_{i=0}^{N} l_{(2i+1)\pi/N}.$$

$L = l \cap A$, $\tilde{L} = \tilde{l} \cap \tilde{A}$, $H = S \cup L \cup \tilde{L}$, $P = U[1/4N^3](H)$, $\Omega = \mathfrak{A} \backslash P$, $s_i = l_{i\pi/N} \cap \mathfrak{A}$.

Let us denote by $\omega_j, j = 1, \dots, 2N$, the union of the segment s_j and those components of the set Ω which have nonempty intersection with s_j.

Let the curve $\sum \subset D_1 \backslash \Omega$ connect the point 0 and S_1. Then we have

$$\text{length } (\Sigma) > 10N.$$

Let h be a continuous function in $D_1, h \geq 1$ on D_1, $h \geq N^4$ on Ω. Let a smooth curve σ connect 0 and S_1 in D_1. Then

$$\int_\sigma h ds > N \tag{6}$$

where ds is the arc length parameter on σ.

(4.2) Let $G : D_1 \to S^2$ be the Gauss map of the minimal surface (5). Since X is smooth in $\bar{D}_1$ the map G is continuous in $\bar{D}_1$. Hence, for any $\delta > 0$ there exists $N = N(\delta)$ such that for every domain ω_i of the N-partition of D_1 the following inequality holds

$$diam G(\omega_i) < \delta.$$

(4.3) *Proposition.* Let $E_1, E_2 \subset D_1 \subset \mathbb{C}$ be compact such that each complement $\complement E_i$ is connected, $i = 1, 2$, and $E_1 \cap E_2 = \emptyset$. Let g be a meromorphic function on $D_{1+\epsilon_0}, \epsilon_0 > 0$ and $g' \neq 0$ on D_1. Let $T > 1$. Then there exists a holomorphic function $h(z)$ on D_1,

$$h(z) = h[T, E_1, E_2, g](z)$$

such that

$$|1 - h| < 1/T \text{ on } E_1$$

$$|h - T| < 1/T \text{ on } E_2.$$

$$(g/h)' \neq 0 \text{ on } D_1$$

Proof. There exist Jordan domains $E_1', E_2' \subset D_1$ such that $E_1 \subset\subset E_1', E_2 \subset\subset E_2'$ and $E_1' \cap E_2' = \emptyset$. By Runge's theorem for any $\epsilon_1 > 0$ there exists a holomorphic function w on $\mathbb{C}$ such that

$$|w| < \epsilon_1 \text{ on } E_1',$$

$$|w - \ln T| < \epsilon_1 \text{ on } E_2'.$$

¿From the above inequalities it follows that

$$w' \to 0 \text{ on } E_1 \cup E_2 \text{ as } \epsilon \to 0.$$

Since g'/g is a nonvanishing meromorphic function on D_1 we can choose $\epsilon > 0$ so small such that

$$d := w' - g'/g \neq 0 \text{ on } E_1 \cup E_2.$$

Since the set of zeros and the set of poles of d are discrete there exists a Jordan domain $E \subset D_1$ such that $E_1 \subset E, E_2 \subset E$,

$$d \neq 0 \text{ on } E$$

and

(7) $$1/d \neq 0 \text{ on } \partial E.$$

Denote $q := g/g'$, then q is a holomorphic function on $D_{1+\epsilon_0}$. Denote by $z_1, ..., z_n$ the zeros of q in D_1 and by $k_1, ..., k_n$ their orders. Since $1/d$ is a holomorphic function on E we obtain by the theorem of Walsh, [W], that for any $\delta > 0$ there exists a holomorphic function $s_\delta(z)$ on D_1 such that

$$|s_\delta - 1/d| < \delta \text{ on } E$$

and for all $i = 1, ..., n$

$$|s_\delta(z) + q(z)| = o(|z - z_i|^{k_i}) \text{ as } z \to z_i.$$

Denote $y = 1/s_\delta + 1/q$. Then y be a holomorphic function on D_1 and

$$y - g'/g \neq 0 \text{ on } D_1. \tag{8}$$

¿From (7) it follows that

$$|1/s_\delta - d| \to 0 \text{ on } \partial E \text{ as } \delta \to 0$$

$$|y - w'| \to 0 \text{ on } E \text{ as } \delta \to 0. \tag{9}$$

Let $z_0 \in E_1$. Set

$$w_1(z) = \int_{z_0}^{z} y(s)ds.$$

¿From (9) it follows that for sufficientlly small $\delta > 0$ we have

$$|w_1| < 2\epsilon \text{ on } E_1,$$

$$|w_1 - \ln T| < 2\epsilon \text{ on } E_2.$$

Let us denote

$$h = e^{w_1(z)}.$$

Then

$$(g/h)' = (g' - w_1' g)h$$

and from (8) it follows that

$$(g/h)' \neq 0 \text{ on } D_1.$$

For sufficiently small $\epsilon > 0$ we evidently have $|1-h| < 1/T$ on E_1 and $|T-h| < 1/T$ on E_2 as required.

(4.4) Let N and T be sufficiently large positive constants which will be specified later. We define a sequence of minimal immersions

$$F_k : D_1 \to \mathbb{R}^3$$

$k = 0, \ldots, 2N^3 = K$ by induction over k. Set $F_0 = X$. Assume that a map F_{i-1} is already defined. Let us pick a point $q_i \in S^1$ such that

$$dist(q_i, G(\omega_i)) = 1/\sqrt{N} \tag{10}$$

We assume that in the orthogonal coordinates x_1, x_2, x_3 in $\mathbb{R}^3$ the vector q_i is directed along x_3. Let (2) be the Weierstrass representation of F_{i-1}. Set

$$h = h[T, D_1 \backslash U[1/4N^3](\omega_i), \omega_i, g],$$

$\tilde{f} = fh, \tilde{g} = g/h$ and let $\tilde{X}$ be defined by (4). Set $F_i := \tilde{X}$. Then

(11) $$\pi(F_i) = \pi(F_{i-1}),$$

where π is the orthogonal projection $\mathbb{R}^3$ to the x_3 axes. If $K_{F_{i-1}} < 0$ then

$$K_{F_i} < 0.$$

Denote $g_{F_i} = a|dz|$. By (3) we have

$$a \geq |f| max(|h|, |g|/|h|).$$

Hence $a \to \infty$ on ω_i as $T \to \infty$, $g_{F_i} \to g_{F_{i-1}}$ on $D_1 \backslash U[1/4N^3](\omega_i)$ as $T \to \infty$. By (7) the following inequality holds on the set $U[1/4N^3](\omega_i) \backslash \omega_i$:

$$a \geq 1/4\sqrt{N}.$$

Thus, by (6) for sufficiently large T the geodesic distance between the points 0 and S_1 in the metric (D_1, g_{F_K}) is no less than $\sqrt{N}/4$.

Let d be a geodesic disk in (D_1, g_{F_K}) of radius $\rho + s$ with centre at 0. Since the Gaussian curvature of g_{F_K} is nonpositive, ∂d is a smooth curve in D_1. For sufficiently large N it follows:

(12) $$D_{1-\varepsilon} \subset d$$

and

(13) $$|X - F_K| < \varepsilon \text{ on } D_{1-\varepsilon}.$$

Let $\eta \in \partial d$. If

$$\eta \in D_1 \setminus \bigcup_{i=1}^{K} U[1/4N](\omega_i)$$

then

(14) $$F_K(\eta) = X(\eta) + o(1) \quad \text{as} \quad T \to \infty.$$

If $\eta \in U[1/4N](\omega_i)$ for some $j, 1 \leq j \leq K$, then from (11) it follows that the vector $F_K(\eta)$ has the form

(15) $$F_K(\eta) = X(\eta) + tp(\eta) + o(1) \quad \text{as} \quad T \to \infty$$

where $t = t(N,T) \in \mathbb{R}, < p(\eta), q_i >= 0$. By (10)

(16) $$\langle X(\eta), q_j \rangle \to 0 \quad \text{as} N \to \infty.$$

¿From (12) and (13) it immediately follows that

$$\overline{\lim}_{\substack{N\to\infty \\ T\to\infty}} t(N,T) \leq s.$$

So as a consequence of (14), (15), (16) and of the inequality $|X(\eta)| \leq r$ we have

$$\overline{\lim}_{\substack{N\to\infty \\ T\to\infty}} |F_K(\eta)| \leq \sqrt{r^2 + s^2}.$$

Let $w : D_1 \to d$ be a biholomorphic map such that $w(0) = 0, w'(0) > 0$. Set $Y = F_K \circ w$. By Carathodory's theorem on the convergence of a sequence of conformal maps, (see [G]), it follows from (9) that for every $\delta > 0$ and for all sufficiently large N the following inequality holds.

$$|X - Y| < \varepsilon \text{ on } D_{1-\varepsilon}$$

The lemma is proved.

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