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ON THE ESSENTIAL SPECTRUM OF CERTAIN NON-COMMUTATIVE OSCILLATORS

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Abstract. We show here that the spectrum of the family of non-commutative harmonic oscillators $Q^{\mathrm{w}}_{(\alpha,\beta)}(x,D)$ for $\alpha,\beta\in\mathbb{R}_+$ in the range $\alpha\beta=1$, is $[0,+\infty)$ and is entirely essential spectrum. The previous existing results concern the case $\alpha\beta>1$ (case in which $Q^{\mathrm{w}}_{(\alpha,\beta)}(x,D)$ is globally elliptic with a discrete spectrum whose qualitative properties are being extensively studied), and ours therefore extends the picture to the range of parameters $\alpha\beta\geq 1$.

1. Introduction

In this paper, we consider the following system

$$(1.1)\quad Q^{\mathrm{w}}(x,D)=Q^{\mathrm{w}}_{(\alpha,\beta)}(x,D)=A\left(-\frac{\partial_x^2}{2}+\frac{x^2}{2}\right)+J\left(x\partial_x+\frac{1}{2}\right),\quad x\in\mathbb{R},$$

where

$$A=\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix},\quad J=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},\ \alpha,\beta\in\mathbb{R}_+=(0,+\infty).$$

Put $p_0(x,\xi)=(x^2+\xi^2)/2$. System (1.1) is then the Weyl quantization of the matrix-valued quadratic form on $T^*\mathbb{R}=\mathbb{R}\times\mathbb{R}$

$$Q(x,\xi)=Q_{(\alpha,\beta)}(x,\xi)=Ap_0(x,\xi)+iJx\xi.$$

Clearly, one has $Q(x,\xi)^*=Q(x,\xi)$ for all $(x,\xi)\in\mathbb{R}\times\mathbb{R}$.

The operator $Q^{\mathrm{w}}(x,D)$ shall be throughout realized as an unbounded operator Q in $L^2(\mathbb{R};\mathbb{C}^2)$ with maximal domain

$$\mathsf{D}=\{u\in L^2(\mathbb{R};\mathbb{C}^2);\ Q^{\mathrm{w}}(x,D)u\in L^2(\mathbb{R};\mathbb{C}^2)\ \text{(in the }\mathscr{S}'\text{-sense)}\},$$

i.e. Q is the *maximal* realization of $Q^{\mathrm{w}}(x,D)$.

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When $\alpha\beta > 1$ one has $\det Q(x,\xi) \approx (x^2+\xi^2)^2$, whence it follows that Q^{w} is a (**classical**) **globally elliptic self-adjoint** operator in $L^2(\mathbb{R};\mathbb{C}^2)$ (see [20]), whence its realization Q has domain

$$B^2(\mathbb{R};\mathbb{C}^2) = \{u \in L^2(\mathbb{R};\mathbb{C}^2);\ \sum_{j,k\geq 0, j+k\leq 2} \|x^j\partial_x^k u\|_{L^2(\mathbb{R};\mathbb{C}^2)} < +\infty\},$$

and is self-adjoint. Since $B^2(\mathbb{R};\mathbb{C}^2)$ is compactly embedded into $L^2(\mathbb{R};\mathbb{C}^2)$, we have that the spectrum of Q is discrete, made of a diverging (to $+\infty$) sequence of real eigenvalues with finite multiplicities, and it turns out (see [14]) that its lowest eigenvalue is positive. Hence (with *repetitions according to the multiplicity*)

$$\mathrm{Spec}(\mathsf{Q}) = \{0 < \lambda_1 \leq \lambda_2 \leq \ldots \to +\infty\}.$$

System (1.1) is called *non-commutative harmonic oscillator* (NCHO for short), in the terminology introduced by Wakayama and the first author in [13] and [14], and (1.1) is actually a *normal form* of the class introduced there (see [14]).

Wakayama and the first author gave in [13,14], a qualitative description of the spectrum of Q^{w} when $\alpha\beta > 1$, by using $\mathfrak{sl}_2(\mathbb{R})$-symmetries to construct suitable creation-annihilation operators and a basis B made of "twisted" vector-valued Hermite functions.

The case $\alpha = \beta$ is completely understood: the system is unitarily equivalent (through automorphisms of $\mathscr{S}(\mathbb{R};\mathbb{C}^2)$ and $\mathscr{S}'(\mathbb{R};\mathbb{C}^2)$), according to the cases $\alpha = \beta > 1$, $\alpha = \beta = 1$ and $0 < \alpha = \beta < 1$, respectively, to the **scalar** harmonic oscillator $\sqrt{\alpha^2-1}(-\partial_x^2+x^2)/2$, to the scalar $-\partial_x^2/2$, to the scalar $\sqrt{1-\alpha^2}(-\partial_x^2-x^2)/2$, respectively (see Corollary 4.1 of [14]), and one has the following result.

Theorem 1.1 ([14]). *When $\alpha = \beta > 1$ one has*

$$\mathrm{Spec}(\mathsf{Q}) = \left\{\sqrt{\alpha^2-1}\left(N+\frac{1}{2}\right);\ N \in \mathbb{Z}_+\right\} \tag{1.2}$$

(where $\mathbb{Z}_+ = \{0,1,\ldots\}$), with eigenvalues of multiplicity 2.

When $\alpha = \beta = 1$ one has

$$\mathrm{Spec}(\mathsf{Q}) = \mathrm{Spec}_{\mathrm{ess}}(\mathsf{Q}) = [0,+\infty). \tag{1.3}$$

When $\alpha = \beta < 1$ one has

$$\mathrm{Spec}(\mathsf{Q}) = \mathrm{Spec}_{\mathrm{ess}}(\mathsf{Q}) = \mathbb{R}. \tag{1.4}$$

Here $\mathrm{Spec}_{\mathrm{ess}}$ *denotes the essential spectrum (the complement in $\mathbb{C}$ of the discrete spectrum).*

It is interesting to notice the appearance of the "symplectic" parameter $\sqrt{\alpha^2-1}$ (denoted by $\ell = \sqrt{\alpha\beta-1}$ in [13,14]).

When $\alpha \neq \beta$ things are highly nontrivial. In this case, when $\alpha\beta > 1$, in [14] (see also [13]) to understand the spectrum two kinds of sets, $\Sigma_0^{\pm}$ and $\Sigma_\infty^{\pm}$, were introduced. (The $\pm$ stands in this case for the **parity**: the system preserves parity, whence it follows that one can study, separately, the even case, +, and the odd one, −, respectively.) The sets $\Sigma_0^\pm$ are described as the sets of those eigenvalues that are roots of particular polynomials, whereas the sets $\Sigma_\infty^\pm$ are described as the sets of those eigenvalues that are zeroes of particular meromorphic functions (defined through continued fractions). These polynomials and meromorphic functions are related to certain three-term recurrence systems. Corresponding to eigenvalues belonging to $\Sigma_0 = \Sigma_0^+ \cup \Sigma_0^-$ one has eigenfunctions which are written as a linear combination of *finitely* many elements of the basis B, whereas corresponding to eigenvalues belonging to $\Sigma_\infty = \Sigma_\infty^+ \cup \Sigma_\infty^-$ one has eigenfunctions which are written as a linear combination of *infinitely* many elements of the basis B. While one **always** has $\Sigma_0^+ \cap \Sigma_0^- = \emptyset$, the intersection $\Sigma_\infty^+ \cap \Sigma_\infty^-$ has a very difficult description and is yet to be understood. Upon defining

$$E_\pm(\lambda) = \{u \in L^2(\mathbb{R};\mathbb{C}^2);\ Q^{\rm w}(x,D)u = \lambda u,\ u\ \textbf{even/odd}\},$$

one has the following theorem.

Theorem 1.2 ([14]; see also [13])**.** *When $\alpha \neq \beta$, one has*

$$\Sigma_0^+ \subset \Sigma_\infty^+,\ \Sigma_0^- \subset \Sigma_\infty^-,$$
$$\mathrm{Spec}(Q^{\rm w}(x,D)) = \Sigma_\infty^+ \cup \Sigma_\infty^-,$$

and

$$\dim E_\pm(\lambda) = \begin{cases} 2, & \textit{whenever } \lambda \in \Sigma_0^\pm \\ \\ 1, & \textit{whenever } \lambda \in \Sigma_\infty^\pm \setminus \Sigma_0^\pm \end{cases},\ \pm\textit{-respectively.}$$

Notice that the theorem says nothing about whether $\Sigma_0^\pm \cap (\Sigma_\infty^\mp \setminus \Sigma_0^\mp) \neq \emptyset$, case that would yield an eigenvalue of multiplicity 3.

The sets $\Sigma_0^\pm$ and $\Sigma_\infty^\pm$, although explicitly described, are complicated (see [13, 14] and also [15, 17, 18]), and it would be desirable to have also other (hopefully simpler) descriptions.

It is a remarkable fact, proved by Ochiai in [11] (see also [12]), that the spectral problem for $Q^{\rm w}$ is equivalent to a family of third-order Fuchsian differential equations with four regular singularities in the complex unit disk. Furthermore, important work on the spectral zeta-function $\zeta_Q(s)$, and its special values, associated with $Q^{\rm w}(x,D)$, defined by

$$\zeta_{\mathsf{Q}}(s) = \sum_{\lambda\in\mathrm{Spec}(\mathsf{Q})} \frac{1}{\lambda^s},\quad s\in\mathbb{C},\ \mathrm{Re}\, s > 1,$$

has been started by Ichinose and Wakayama (see [6, 7]; see also [18] and [8]). It is also worth mentioning that numerical study of the spectrum $Q^{\rm w}(x,D)$ has been carried out by Nagatou, Nakao and Wakayama in [10], and that one can study the spectrum by Rellich's perturbation theory in the limit $\alpha\beta \to +\infty$ with α/β a fixed constant $\neq 1$ (see [15]). Furthermore, the study of Poisson-type relations for the spectral distribution, and clustering theorems of the spectrum were proved in Parmeggiani [16, 17] (see also [18]).

As for the multiplicity of the lowest eigenvalue, one has results in Parmeggiani [15], in Hiroshima and Sasaki [3], and in the more recent paper by Wakayama [21]; however, our knowledge of the lowest eigenvalue is still incomplete. It is finally worth mentioning the recent study of Dicke-type crossings among the eigenvalues of certain families of NCHOs carried out by Hirokawa in [2], which is related to the study of self-adjoint operators with non-commutative coefficients such as the Rabi model or the Jaynes-Cumming model, describing the interaction between a one-mode photon and a two-level atom.

In this paper, we extend the knowledge of the spectrum of $Q^{\mathrm{w}}_{(\alpha,\beta)}(x,D)$, thought of as its maximal realization Q, to the case $\alpha\beta = 1$, with $\alpha, \beta > 0$, case in which the NCHO is no longer globally elliptic, proving that when $\alpha\beta = 1$ one has

$$\mathrm{Spec}(\mathsf{Q}) = \mathrm{Spec}_{\mathrm{ess}}(\mathsf{Q}) = [0, +\infty).$$

The proof is based on a metaplectic factorization (in $\mathscr{S}'(\mathbb{R};\mathbb{C}^2)$) of $Q^{\mathrm{w}}(x,D)$, with local metaplectic operators (i.e. none of the metaplectic operators involved is the Fourier transform).

2. A fundamental lemma and a few consequences of it

We prove in this section a fundamental lemma, following the approach of [18] in his proof of Theorem 8.2.1. Although the lemma will be used only in the case $\alpha\beta = 1$, we shall prove it in the general case (i.e. $\alpha, \beta > 0$ and no restriction on $\alpha\beta$) because it will be useful also in a subsequent study.

We have to introduce some notation. For $\alpha, \beta \in \mathbb{R}_+$, let

$$0 < \delta := \sqrt{\alpha\beta},\ \mathsf{s}(\delta) := \begin{cases} \mathrm{sgn}(\alpha\beta - 1),\ \delta \neq 1, \\ \\ 0,\ \alpha\beta = 1. \end{cases},\ \epsilon := \sqrt{|\alpha\beta - 1|},$$

and $\omega_\pm := \dfrac{\alpha \pm \beta}{2}$. Let $v_\pm := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \mp i \end{bmatrix}$ be the orthonormal eigenvectors of J belonging to $\pm i$, respectively. Hence $Jv_\pm = \pm i v_\pm$ and, furthermore, $KJv_\pm = v_\mp$, where $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Let $W_0 := [v_+|v_-]$. Let us consider the (global, in the sense of Shubin, see [20], or [9], or [18]) symbols

$$p_0(x,\xi) = \frac{\xi^2 + x^2}{2},\ e(x,\xi) = x\xi,$$

$$p_\delta(x,\xi) = \frac{\xi^2 + \mathsf{s}(\delta)x^2}{2},\ L_\delta(x,\xi) = \frac{\xi^2 + (\delta^2 - 1)x^2}{2},$$

and the linear symplectomorphisms $\chi_\delta, \chi_\pm \colon \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}\times\mathbb{R}$, where

$$\chi_\delta \colon (x,\xi) \longmapsto (\delta^{1/2}x, \delta^{-1/2}\xi),\quad \chi_\pm \colon (x,\xi) \longmapsto (x, \xi \pm x),$$

which are associated with the metaplectic operators

$$U_\delta \colon f(x) \longmapsto (U_\delta f)(x) = \delta^{-1/4} f(x/\delta^{1/2}),$$

$$U_\pm \colon f(x) \longmapsto (U_\pm f)(x) = e^{\pm i x^2/2} f(x),$$

respectively. Notice that $U_-^* = U_-^{-1} = U_+$, and that $U_\delta^* = U_\delta^{-1} = U_{1/\delta}$.

Lemma 2.1. *There exist metaplectic operators* U_0 *and* U_ϵ*, isometries of* $L^2(\mathbb{R};\mathbb{C}^2)$ *and automorphisms of* $\mathscr{S}'(\mathbb{R};\mathbb{C}^2)$ *and of* $\mathscr{S}(\mathbb{R};\mathbb{C}^2)$*, such that for all* $\alpha,\beta \in \mathbb{R}_+$ *and for all* $\lambda \in \mathbb{C}$*, on* $\mathscr{S}'(\mathbb{R};\mathbb{C}^2)$ *the operator* $Q^{\mathrm{w}}(x,D) - \lambda$ *can be factored as (recall that* $D_x = -i\partial_x$*)*

$$Q^{\mathrm{w}}(x,D) - \lambda = A^{1/2}\mathsf{U}_0^*\Big(\frac{1}{2}D_x^2 - \lambda V(x)\Big)\mathsf{U}_0 A^{1/2}, \quad \textit{when} \quad \epsilon = 0, \tag{2.1}$$

and as

$$Q^{\mathrm{w}}(x,D) - \lambda = \frac{1}{\delta}A^{1/2}\mathsf{U}_\epsilon^*\Big(\epsilon p_\delta^{\mathrm{w}}(x,D) - \frac{\lambda}{\delta}V_\epsilon(x)\Big)\mathsf{U}_\epsilon A^{1/2}, \quad \textit{when} \quad \epsilon > 0, \tag{2.2}$$

where

$$V(x) = \begin{bmatrix} \omega_+ & -\omega_- e^{-ix^2} \\ -\omega_- e^{ix^2} & \omega_+ \end{bmatrix}, \quad V_\epsilon(x) = V(x/\epsilon^{1/2}),$$

and

$$\mathsf{U}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} U_- U_\delta^* & 0 \\ 0 & U_+ U_\delta^* \end{bmatrix}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}, \quad \mathsf{U}_\epsilon = (U_{1/\epsilon}^* \otimes I_2)\mathsf{U}_0.$$

The metaplectic operators U_0 *and* U_ϵ *are* local *(i.e. leave* $C_0^\infty(\mathbb{R};\mathbb{C}^2)$ *invariant).*

Proof. One writes

$$Q^{\mathrm{w}}(x,D) - \lambda = \frac{1}{\delta}A^{1/2}\Big(\delta p_0^{\mathrm{w}}(x,D) + iJe^{\mathrm{w}}(x,D) - \lambda\delta A^{-1}\Big)A^{1/2}.$$

Hence, thinking of $A^{1/2}$ as an automorphism of $\mathscr{S}'(\mathbb{R};\mathbb{C}^2)$, we have that $(Q^{\mathrm{w}}(x,D) - \lambda)u = 0$ in $\mathscr{S}'(\mathbb{R};\mathbb{C}^2)$ is equivalent to

$$(\delta p_0^{\mathrm{w}}(x,D) + iJe^{\mathrm{w}}(x,D) - \lambda\delta A^{-1})v = 0, \quad v := A^{1/2}u \in \mathscr{S}'(\mathbb{R};\mathbb{C}^2).$$

Since

$$\delta p_0(x,\xi) \mp x\xi = (L_\delta \circ \chi_\mp \circ \chi_\delta^{-1})(x,\xi),$$

we have

$$\delta p_0^{\mathrm{w}}(x,D) \mp e^{\mathrm{w}}(x,D) = U_\delta U_\mp^{-1} L_\delta^{\mathrm{w}}(x,D) U_\mp U_\delta^{-1}.$$

As $A^{-1} = (\omega_+ I - \omega_- KJ)/\delta^2$, we may write, using the diagonalizer W_0 of J,

$$W_0^*\Big(\delta p_0^{\mathrm{w}}(x,D) + iJe^{\mathrm{w}}(x,D) - \lambda\delta A^{-1}\Big)W_0 =$$

$$= \begin{bmatrix} \delta p_0^{\mathrm{w}}(x,D) - e^{\mathrm{w}}(x,D) - \frac{\lambda}{\delta}\omega_+ & \frac{\lambda}{\delta}\omega_- \\ \frac{\lambda}{\delta}\omega_- & \delta p_0^{\mathrm{w}}(x,D) + e^{\mathrm{w}}(x,D) - \frac{\lambda}{\delta}\omega_+ \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} U_\delta U_+ & 0 \\ 0 & U_\delta U_- \end{bmatrix}}_{=:\mathsf{U}^*} L_\delta^{\mathrm{w}}(x,D) \underbrace{\begin{bmatrix} U_- U_\delta^* & 0 \\ 0 & U_+ U_\delta^* \end{bmatrix}}_{=:\mathsf{U}} - \frac{\lambda}{\delta}\begin{bmatrix} \omega_+ & -\omega_- \\ -\omega_- & \omega_+ \end{bmatrix} =$$

$$= \mathsf{U}^*\Big(L_\delta^{\mathrm{w}}(x,D) - \frac{\lambda}{\delta}\mathsf{U}\begin{bmatrix} \omega_+ & -\omega_- \\ -\omega_- & \omega_+ \end{bmatrix}\mathsf{U}^*\Big)\mathsf{U} =$$

(since $U_+U_\delta^*(U_-U_\delta^*)^*\colon f \longmapsto e^{ix^2}f$)

$$= \mathsf{U}^*\Big(L_\delta^{\mathrm{w}}(x,D) - \frac{\lambda}{\delta}V(x)\Big)\mathsf{U},$$

where

$$V(x) := \begin{bmatrix} \omega_+ & -\omega_- e^{-ix^2} \\ -\omega_- e^{ix^2} & \omega_+ \end{bmatrix}.$$

Hence, at this point we have obtained the factorization

$$Q^{\mathrm{w}}(x,D) - \lambda = \frac{1}{\delta}A^{1/2}W_0\mathsf{U}^*\Big(L_\delta^{\mathrm{w}}(x,D) - \frac{\lambda}{\delta}V(x)\Big)\mathsf{U}W_0^*A^{1/2}. \tag{2.3}$$

Now, when $\epsilon = 0$, that is $\alpha\beta = 1 = \delta$, we then have $L_1^{\mathrm{w}}(x,D) = D_x^2/2$, so that

$$Q^{\mathrm{w}}(x,D) - \lambda = A^{1/2}\mathsf{U}_0^*\Big(\frac{1}{2}D_x^2 - \lambda V(x)\Big)\mathsf{U}_0A^{1/2},$$

where

$$\mathsf{U}_0 = \mathsf{U}W_0^* = \frac{1}{\sqrt{2}}\begin{bmatrix} U_-U_\delta^* & 0 \\ 0 & U_+U_\delta^* \end{bmatrix}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}, \tag{2.4}$$

which is the sought factorization, as claimed.
Next, when $\epsilon > 0$, consider $\chi_{1/\epsilon}\colon (x,\xi) \longmapsto (\epsilon^{-1/2}x, \epsilon^{1/2}\xi)$, and the associated metaplectic operator $U_{1/\epsilon}\colon f \longmapsto \epsilon^{1/4}f(\epsilon^{1/2}\,\cdot)$. Since

$$U_{1/\epsilon}^*L_\delta^{\mathrm{w}}(x,D)U_{1/\epsilon} = (L_\delta \circ \chi_{1/\epsilon})^{\mathrm{w}}(x,D) = \frac{\epsilon}{2}(D_x^2 + \mathsf{s}(\delta)x^2) = \epsilon p_\delta^{\mathrm{w}}(x,D),$$

we get from (2.3)

$$Q^{\mathrm{w}}(x,D) - \lambda = \frac{1}{\delta}A^{1/2}\mathsf{U}_\epsilon^*\Big(\epsilon p_\delta^{\mathrm{w}}(x,D) - \frac{\lambda}{\delta}V_\epsilon(x)\Big)\mathsf{U}_\epsilon A^{1/2},$$

where this time

$$\mathsf{U}_\epsilon = (U_{1/\epsilon}^* \otimes I_2)\mathsf{U}W_0^*, \tag{2.5}$$

and

$$V_\epsilon(x) = U_{1/\epsilon}^*V(x)U_{1/\epsilon} = V(x/\epsilon^{1/2})$$

(of course, as a multiplication operator). This concludes the proof of the lemma. □

Therefore we have in particular that

$$Q^{\mathrm{w}}(x,D) = \frac{\epsilon}{\delta}A^{1/2}\mathsf{U}_\epsilon^*\,p_\delta^{\mathrm{w}}(x,D)\mathsf{U}_\epsilon A^{1/2}, \text{ when } \epsilon > 0, \tag{2.6}$$

and

$$Q^{\mathrm{w}}(x,D) = \frac{1}{2}A^{1/2}\mathsf{U}_0^*\,D_x^2\,\mathsf{U}_0A^{1/2}, \text{ when } \epsilon = 0, \tag{2.7}$$

where the metaplectic operators U_0 and U_ϵ are given in (2.4) and (2.5), respectively.

Notice that neither U_0 nor U_ϵ commutes with A.

Recalling that $Q^{\mathrm{w}}(x,D)$ is realized as the maximal operator Q, with domain

$$\mathsf{D} = \mathsf{D}_{\alpha,\beta} = \{u \in L^2(\mathbb{R};\mathbb{C}^2);\ Q^{\mathrm{w}}(x,D)u \in L^2(\mathbb{R};\mathbb{C}^2)\},$$

we have that, when $\alpha\beta = 1$, D can be described as

$$\mathsf{D} = \{u \in L^2(\mathbb{R};\mathbb{C}^2);\ D_x^2 \mathsf{U}_0 A^{1/2} u \in L^2(\mathbb{R};\mathbb{C}^2)\} = \tag{2.8}$$
$$= \{u \in L^2(\mathbb{R};\mathbb{C}^2);\ \mathsf{U}_0 A^{1/2} u \in H^2(\mathbb{R};\mathbb{C}^2)\},$$

whereas, when $0 < \alpha\beta < 1$, it can be described as

$$\mathsf{D} = \{u \in L^2(\mathbb{R};\mathbb{C}^2);\ (D_x^2 - x^2)\mathsf{U}_\epsilon A^{1/2} u \in L^2(\mathbb{R};\mathbb{C}^2)\}.$$

Moreover, when $\alpha\beta \geq 1$ we have $Q^{\mathrm{w}}(x,D) \geq 0$ on $\mathscr{S}(\mathbb{R};\mathbb{C}^2)$. This is already well-known when $\alpha\beta > 1$ (see [14], or [18]), and when $\alpha,\beta \in \mathbb{R}_+$ with $\alpha\beta = 1$ it follows from Lemma 2.1, for we have

$$\big(Q^{\mathrm{w}}(x,D)u,u\big) = \frac{1}{2}\big(D_x^2 \mathsf{U}_0 A^{1/2}u, \mathsf{U}_0 A^{1/2} u\big) =$$
$$= \frac{1}{2}\|\partial_x \mathsf{U}_0 A^{1/2} u\|_{L^2}^2 \geq 0, \quad \forall u \in \mathscr{S}(\mathbb{R};\mathbb{C}^2).$$

As another immediate consequence of Lemma 2.1, we may establish the self-adjointness of Q, also for the range of values $0 < \alpha\beta \leq 1$.
This also follows from the arguments of Hörmander [5], whose extension to our system presents no problem: using the Weyl-Hörmander pseudodifferential calculus in the “global” setting (see, e.g., [20], or [9], or [18]) one sees that Q is the closure of its restriction of $\mathscr{S}(\mathbb{R};\mathbb{C}^2)$ on which it is symmetric.

However, we shall here prove directly the self-adjointness of Q for the sake of having a self-contained approach.

Recall also that the operator $D_x^2 - x^2$, realized as an unbounded operator in L^2 defined on its maximal domain, is self-adjoint, with (essential) spectrum $(-\infty,+\infty)$.

Corollary 2.2. *The operator* Q *is self-adjoint also for* $0 < \alpha\beta \leq 1$.

Proof. We give a proof in the case $\alpha\beta = 1$, the other case $0 < \alpha\beta < 1$ being completely similar. For simplicity we write D_* for $\mathscr{D}(\mathsf{Q}^*)$. Put $\mathsf{F} = \mathsf{U}_0 A^{1/2}$. Since

$$\mathsf{D}_* = \{v \in L^2;\ \mathsf{D} = \mathsf{F}^{-1}(H^2) \ni u \longmapsto (Qu,v) = \frac{1}{2}(D_x^2 \mathsf{F}u, \mathsf{F}v) \text{ is bndd}\},$$

using (2.7) we may consider

$$\mathsf{F}(\mathsf{D}_*) = \{\mathsf{F}v \in L^2;\ H^2 \ni \mathsf{F}u \longmapsto (D_x^2 \mathsf{F}u, \mathsf{F}v) \text{ is bndd}\}.$$

As H^2 is the maximal domain of D_x^2, we thus conclude that $\mathsf{F}(\mathsf{D}_*) = H^2(\mathbb{R};\mathbb{C}^2)$, whence $\mathsf{D}_* = \mathsf{D}$. □

Therefore, in particular, for all $\alpha,\beta \in \mathbb{R}_+$ with $\alpha\beta = 1$, the operator Q is *self-adjoint*, with spectrum *contained* in $[0,+\infty)$. We know that when $\alpha = \beta = 1$, our NCHO $Q^{\mathrm{w}}(x,D)$ is thus isometrically equivalent to $(D_x^2/2)I_2$, whence the spectrum of Q is indeed the whole half-ray $[0,+\infty)$. In the next section, we will see that this is indeed the case also for all $\alpha, \beta \in \mathbb{R}_+$ with $\alpha\beta = 1$.

We close the section by stating the following classical characterization of the bottom of the essential spectrum of a self-adjoint operator (Persson's Theorem; see [1]), that we state already in the case of a self-adjoint system of the kind we shall have to consider in what follows (the generalization to these systems presents no problem).

Proposition 2.3. *Let* $F = F^* \in L^\infty(\mathbb{R}; \mathsf{M}_2(\mathbb{C})) \cap C^\infty(\mathbb{R}; \mathsf{M}_2(\mathbb{C}))$*, and let* $\mathsf{P} = \mathsf{P}^*$ *be the realization of* $D_x^2/2 + F(x)$ *with domain* $H^2(\mathbb{R}; \mathbb{C}^2)$ *(i.e. the maximal realization). Hence* P *is semibounded from below. We have*

$$\inf \mathrm{Spec}_{\mathrm{ess}}(\mathsf{P}) = \sup_{K \subset\subset \mathbb{R}} \inf \Big\{ \frac{(\mathsf{P}\varphi, \varphi)}{\|\varphi\|^2};\ \varphi \in C_0^\infty(\mathbb{R} \setminus K; \mathbb{C}^2),\ \varphi \neq 0 \Big\}. \tag{2.9}$$

3. Proof of the main result

In this section we prove our result, that we recall next.

Theorem 3.1. *Let* $\alpha, \beta \in \mathbb{R}_+$ *with* $\alpha\beta = 1$*. For* $Q^{\mathrm{w}}(x, D)$*, realized as the maximal extension* Q *with domain* D *(see (2.8)), we have*

$$\mathrm{Spec}(\mathsf{Q}) = \mathrm{Spec}_{\mathrm{ess}}(\mathsf{Q}) = [0, +\infty).$$

Proof. We shall follow the approach by Sasaki [19]. Since Q is self-adjoint and nonnegative on D, we shall prove that for any given $\lambda \in [0, +\infty)$, the operator $\mathsf{Q} - \lambda$ has 0 in the essential spectrum (in other words, by the Weyl criterion, see e.g. [4], for each $\lambda \geq 0$ one may construct a corresponding Weyl sequence). Therefore, by Lemma 2.1, this is in turn reduced to proving that for every fixed $\lambda \in [0, +\infty)$, the operator

$$\frac{1}{2}D_x^2 - \lambda V(x) =: \frac{1}{2}D_x^2 - \lambda\omega_+ + \lambda\omega_- \begin{bmatrix} 0 & e^{-ix^2} \\ e^{ix^2} & 0 \end{bmatrix} =: \frac{1}{2}D_x^2 - \mu + F(x)$$

has 0 in the essential spectrum. To prove this, it suffices to prove that $\mathrm{Spec}_{\mathrm{ess}}(\frac{1}{2}D_x^2 + F(x)) = [0, +\infty)$, because one may then construct a Weyl sequence for $\frac{1}{2}D_x^2 + F(x) - \mu$, for all $\mu \in [0, +\infty)$. For short, we keep writing $\frac{1}{2}D_x^2 + F(x)$ also for its maximal realization in $L^2(\mathbb{R}; \mathbb{C}^2)$ with maximal domain $H^2(\mathbb{R}; \mathbb{C}^2)$, on which it is self-adjoint, by virtue of the fact that for the vector-valued potential F we have $F = F^* \in L^\infty(\mathbb{R}; \mathsf{M}_2(\mathbb{C})) \cap C^\infty(\mathbb{R}; \mathsf{M}_2(\mathbb{C}))$.

We have the following basic proposition.

Proposition 3.2. $\mathrm{Spec}_{\mathrm{ess}}(\dfrac{1}{2}D_x^2 + F(x)) = [0, +\infty)$.

Proof of the proposition. Since (as a Riemann generalized integral) one has $\int_0^\infty e^{ix^2} dx = \sqrt{2\pi}(1+i)/4 =: \omega_0$, we note that

$$\lim_{R \to +\infty} \pm \int_0^{\pm R} F(x)dx = \lambda\omega_- \begin{bmatrix} 0 & \bar{\omega}_0 \\ \omega_0 & 0 \end{bmatrix}. \tag{3.10}$$

Define, for $R \in \mathbb{R}$,

$$W_+(x;R) := \int_R^x F(s)ds, \quad W_-(x;R) = \int_x^{-R} F(s)ds.$$

Then

$$W_\pm = W_\pm^*, \quad W_\pm'(x;R) = \pm F(x), \quad \text{and} \quad W_\pm(\pm R;R) = 0 \quad (\pm-\text{respectively}).$$

By virtue of (3.10) we have that there exists $C_0 > 0$ such that for all R

$$\sup_{x\in\mathbb{R}} \|W_\pm(x;R)\|_{\mathbb{C}^2\to\mathbb{C}^2} \le C_0,$$

and that

$$\alpha(R) := \sup_{x\ge R} \|W_+(\cdot,R)\|_{\mathbb{C}^2\to\mathbb{C}^2} + \sup_{x\le -R} \|W_-(\cdot,R)\|_{\mathbb{C}^2\to\mathbb{C}^2} \longrightarrow 0, \text{ as } R\to+\infty.$$

For $u \in C_0^\infty(\mathbb{R};\mathbb{C}^2)$, and χ_R the characteristic function of $[-R,R]$, consider

$$(u,Fu) = \int_{\mathbb{R}} \langle u(x), F(x)u(x)\rangle_{\mathbb{C}^2} dx = (u,\chi_R Fu) + (u,(1-\chi_R)Fu)$$

$$= (u,\chi_R Fu) + \underbrace{\int_{-\infty}^{-R} \langle u(x), F(x)u(x)\rangle_{\mathbb{C}^2} dx}_{I_-(R)} + \underbrace{\int_R^{+\infty} \langle u(x), F(x)u(x)\rangle_{\mathbb{C}^2} dx}_{I_+(R)}.$$

Now,

$$I_-(R) = -\underbrace{\Big[\langle u(x), W_-(x;R)u(x)\rangle_{\mathbb{C}^2}\Big]_{-\infty}^{-R}}_{=0}$$

$$+2\int_{-\infty}^{-R} \mathsf{Re}\,\langle u'(x), W_-(x;R)u(x)\rangle_{\mathbb{C}^2} dx.$$

It therefore follows that for $R > 0$

$$\Big|\int_{-\infty}^{-R} \langle u(x), F(x)u(x)\rangle_{\mathbb{C}^2} dx\Big| \le \sup_{x\le -R} \|W_-(\cdot,R)\|_{\mathbb{C}^2\to\mathbb{C}^2} \int_{-\infty}^0 \big(|u|^2 + |u'|^2\big)dx,$$

and, analogously,

$$\Big|\int_R^{+\infty} \langle u(x), F(x)u(x)\rangle_{\mathbb{C}^2} dx\Big| \le \sup_{x\ge R} \|W_+(\cdot,R)\|_{\mathbb{C}^2\to\mathbb{C}^2} \int_0^{+\infty} \big(|u|^2 + |u'|^2\big)dx.$$

Therefore we get

$$|(u,(1-\chi_R)Fu)| \le 2\alpha(R)\Big(\|u\|_{L^2}^2 + (u, \frac{1}{2}D_x^2 u)\Big),$$

whence, picking $R_0 \gg 1$ so that, say, $\alpha(R) < 1/4$ for all $R \ge R_0$,

$$(1-2\alpha(R))\frac{1}{2}D_x^2 + \chi_R F - 2\alpha(R) \le \frac{1}{2}D_x^2 + F \tag{3.11}$$

$$\le (1+2\alpha(R))\frac{1}{2}D_x^2 + \chi_R F + 2\alpha(R)$$

on $C_0^\infty(\mathbb{R};\mathbb{C}^2)$, and also on $H^2(\mathbb{R};\mathbb{C}^2)$. Since $\chi_R F$ is D_x^2-compact for every $R>0$, we thus conclude by Proposition 2.3 that for all $R \geq R_0$

$$\begin{aligned} -2\alpha(R) &= \inf \mathrm{Spec}_{\mathrm{ess}}\Big((1-2\alpha(R))\frac{1}{2}D_x^2 + \chi_R F - 2\alpha(R)\Big) \\ &\leq \inf \mathrm{Spec}_{\mathrm{ess}}(\frac{1}{2}D_x^2 + F) \\ &\leq \inf \mathrm{Spec}_{\mathrm{ess}}\Big((1+2\alpha(R))\frac{1}{2}D_x^2 + \chi_R F + 2\alpha(R)\Big) = 2\alpha(R). \end{aligned}$$

Hence, by taking the limit as $R \to +\infty$,

$$0 = \inf \mathrm{Spec}_{\mathrm{ess}}(\frac{1}{2}D_x^2 + F) \in \mathrm{Spec}_{\mathrm{ess}}(\frac{1}{2}D_x^2 + F) \subset [0,+\infty).$$

We have now to show that $[0,+\infty) \subset \mathrm{Spec}_{\mathrm{ess}}(\frac{1}{2}D_x^2 + F)$. Since 0 is in the essential spectrum, we may take a corresponding Weyl sequence $\{w_k\}_{k\geq 1} \subset H^2(\mathbb{R};\mathbb{C}^2)$, that is, a sequence such that (w-L^2 stands for "weakly in L^2")

$$\text{(3.12)} \qquad \begin{cases} \|w_k\|_{L^2} = 1, \quad w_k \overset{\text{w}-L^2}{\longrightarrow} 0, \\ \|(\frac{1}{2}D_x^2 + F)w_k\|_{L^2} \longrightarrow 0, \end{cases} \quad \text{as } k \to +\infty.$$

By (3.11) we get constants $C_1 > 0, C_2 \in \mathbb{R}$ such that

$$C_1 D_x^2/2 + C_2 \leq D_x^2/2 + F \ \text{ on } H^2(\mathbb{R};\mathbb{C}^2),$$

so that $\|D_x w_k\| \leq C_3$ for all $k \geq 1$. It therefore follows that we may choose a subsequence $\{w_{k_j}\}_{j\geq 1} \subset \{w_k\}_{k\geq 1}$ with $D_x w_{k_j} \overset{\text{w}-L^2}{\longrightarrow} w_0$ for some $w_0 \in L^2$. On the other hand, $w_k \overset{\text{w}-L^2}{\longrightarrow} 0$ implies that $w_0 = 0$. Next, as $\chi_R F$ is D_x^2-compact, by possibly passing to a subsequence (that we keep denoting by w_{k_j}) we have $(w_{k_j}, \chi_R F w_{k_j}) \to 0$ as $j \to +\infty$, so that from (3.11), (3.12) we obtain

$$(1-2\alpha(R)) \limsup_{j\to+\infty} \|D_x w_{k_j}\|_{L^2}^2 \leq 2\alpha(R) \longrightarrow 0, \ \text{ as } R \to +\infty.$$

Hence $\|D_x w_{k_j}\|_{L^2} \to 0$ as $j \to +\infty$. Now, for each $\gamma \geq 0$, put $u_j := e^{i\sqrt{2\gamma}\,x} w_{k_j}$. Obviously, we have $u_j \in H^2(\mathbb{R};\mathbb{C}^2)$ and

$$(\frac{1}{2}D_x^2 + F)u_j = e^{i\sqrt{2\gamma}\,x}(\frac{1}{2}D_x^2 + F)w_{k_j} + \gamma\, e^{i\sqrt{2\gamma}\,x} w_{k_j} + \sqrt{2\gamma}\, e^{i\sqrt{2\gamma}\,x} D_x w_{k_j}.$$

Therefore $\|(D_x^2/2 + F - \gamma)u_j\|_{L^2} \longrightarrow 0$ as $j \to +\infty$. Since $\|u_j\|_{L^2} = 1$ and $u_j \overset{\text{w}-L^2}{\longrightarrow} 0$ as $j \to +\infty$, we thus have that $\gamma \in \mathrm{Spec}_{\mathrm{ess}}(D_x^2/2 + F)$, for all $\gamma \geq 0$. This concludes the proof of the proposition. □

By Lemma 2.1 and Proposition 3.2 the proof of the theorem is now complete. □

Remark 3.3. *The case $0 < \alpha\beta < 1$ seems to be of a very different nature, and to require another approach. We shall deal with this case in a subsequent paper.*

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