

On a New Functional Equation for Local Integrals

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1. Introduction and statement of main result

Global integrals of Rankin-Selberg type are ubiquitous in the theory of automorphic forms and their L -functions and have been studied extensively in the literature. The prototype is the original construction for $GL_2 \times GL_2$ [**Ran39**, **Sel40**]. Its extension to $GL_m \times GL_n$ was the subject matter of intensive work by Jacquet, Piatetski-Shapiro and Shalika around 1980 [**JPSS79a**, **JPSS79b**, **JS81b**, **JS81a**] (with some parts completed only in recent years by Jacquet [**Jac04**, **Jac09**]).

Along with the global integrals there is a rich local theory (see [**JPSS83**, **JS90**] and the references above). The local zeta integrals admit functional equations which give rise to the local L and ϵ factors. The purpose of this paper is to derive a variant of the local functional

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equations of Rankin-Selberg integrals. The precise statement is Theorem 1.2 below. On a formal level, one can view these functional equations as a consequence of the ordinary ones. However, they differ in that they don't have mysterious proportionality constants. At any rate we will give a direct proof which is independent of the detailed local theory of Rankin-Selberg integrals. In the supercuspidal case this was proved in [LM]. Here we prove the general case which is complicated by analytic difficulties.

The new functional equations admit a non-split analogue, which is also covered in Theorem 1.2. They are proved similarly, except that they depend on a certain growth condition which we do not address in this paper, but is probably within reach. Thus, our results in the non-split case are conditional.

The new functional equation is one of the ingredients of the paper [LM]. In that paper we formulate a conjecture about Whittaker coefficients of classical groups and the metaplectic double cover of the symplectic group. For the latter, we reduce the global conjecture to a local one, and prove the local conjecture in the supercuspidal case. In a subsequent work we will use the results of this paper (among other things) to extend the argument of [LM] to the non supercuspidal case.

Rankin-Selberg integrals have been a favorite topic of Piatetski-Shapiro. We hope that this note serves as a tribute to Piatetski-Shapiro's legacy.

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1.1. General notation. Let F be a local field of characteristic 0 (although this is probably not essential) with normalized absolute value $|\cdot|$. In the non-archimedean case let q be the size of the residue field.

In general we denote by bold face letters algebraic varieties over F and by plain letters the set of F -points. Thus, $X = \mathbf{X}(F)$.

Let E/F be a quadratic étale algebra over F (either split or inert). Let \mathbf{G} be the restriction of scalars of GL_n from E to F . Thus $G = \mathrm{GL}_n(E)$. Let $\tilde{\mathbf{G}} = \mathrm{GL}_n$ considered as a subgroup of \mathbf{G} . For any subgroup \mathbf{X} of \mathbf{G} we write $\tilde{\mathbf{X}} := \mathbf{X} \cap \tilde{\mathbf{G}}$ (and correspondingly, $\tilde{X} = X \cap \tilde{G}$).

Let K be the standard maximal compact subgroup of G . We denote by I_m the identity matrix in GL_m and by g^t the transpose of the matrix g .

Let \mathbf{N} be the subgroup of upper unitriangular matrices in \mathbf{G} . Fix a unitary character $\psi : F \backslash E \rightarrow \mathbb{C}^*$. Let ψ_N be the character of N , which is trivial on \tilde{N} , given by

$$\psi_N(u) = \psi(u_{1,2} + \dots + u_{n-1,n}), \quad u \in N.$$

When there is no source of confusion we will often write ψ instead of ψ_N .

By a representation of G we will always mean either a smooth complex representation of finite length in the p -adic case or a smooth Fréchet representation of finite length and moderate growth in the archimedean case.

Let $\Pi(G)$ be the set of irreducible representations of G , $\Pi_{\text{gen}}(G)$ the subset of generic representations, and $\Pi_{\text{temp}}(G)$ the subset of tempered representations. For any $\pi \in \Pi_{\text{gen}}(G)$ we let $\mathcal{W}(\pi, \psi)$ be its Whittaker model: every $W \in \mathcal{W}(\pi, \psi)$ is a function satisfying

$$W(ug) = \psi(u)W(g), \quad u \in N, g \in G$$

which is of moderate growth on $N \backslash G$.

Recall that $\pi \in \Pi(G)$ is called \tilde{G} -distinguished if it admits a non-trivial continuous linear form L on π such that $L(\pi(g)v) = L(v)$ for all $v \in \pi$ and all $g \in \tilde{G}$. Recall that such L , if exists, is unique up to scalar. (This is clear in the split case; in the inert case see [Fl91, AG09].)

For $i = 0, \dots, n$ let $H_i \cong \text{GL}_{n-i}(E)$ be the subgroup of G consisting of $\begin{pmatrix} h & \\ & I_i \end{pmatrix}$ where $h \in \text{GL}_{n-i}(E)$ and let N_i be the subgroup of upper unitriangular matrices in H_i . We also let R_i be the unipotent subgroup of G consisting of the matrices of the form $\begin{pmatrix} I_{n-i} & \\ X & I_i \end{pmatrix}$ where the last row of X is zero.

Let w_m be the permutation matrix corresponding to the longest Weyl element in GL_m . Also for $i = 0, \dots, n$ let $w_{n,i} = \begin{pmatrix} w_{n-i} & \\ & I_i \end{pmatrix}$. For $g \in G$, let $g^\sigma = (g^t)^{-1}$ – the Cartan involution, and $g^* = w_n g^\sigma w_n$ – the principal involution. For a subgroup H of G we denote by H^* its image under $*$.

1.2. Let $\pi \in \Pi_{\text{gen}}(G)$. For $W \in \mathcal{W}(\pi, \psi)$, $r = 1, \dots, n-1$, we consider the integrals

$$(1) \quad A_r(W, s) = \int_{\tilde{N}_r \backslash \tilde{H}_r} W(h) |\det h|^{s+1-r} dh,$$

$$(2) \quad A'_r(W, s) = \int_{R_r} \int_{\tilde{N}_r \backslash \tilde{H}_r} W(hx) |\det h|^{s+1-r} dh dx,$$

and

$$(3) \quad B_r(W, s) = \int_{R_r^*} \int_{(\tilde{N}_r)^* \setminus (\tilde{H}_r)^*} W(hxw_{n,r}) |\det h|^{-(s+1-r)} dh dx.$$

Since $W^\sharp(g) := W(w_n g^\sigma) = W(g^* w_n)$ is a Whittaker function (with respect to ψ^{-1}) in the space of the contragredient $\hat{\pi}$ of π , we can relate A'_r to B_r as follows:

$$(4) \quad B_r(W^\sharp, s) = A'_r(\pi(w_{n,r}^* w_n)W, s).$$

The following Lemma is standard. For completeness we will prove it in §2.2 below.

LEMMA 1.1. *Let $\pi \in \Pi_{\text{gen}}(G)$ and $r = 1, \dots, n-1$. Then the integrals defining $A_r(W, s)$, $A'_r(W, s)$ and $B_r(W, s)$ converge absolutely for $\text{Re } s \gg 0$ and these functions admit meromorphic continuation to \mathbb{C} . In the p -adic case they are rational functions in q^s . Moreover, if π is tempered (resp., unitarizable) then $A_r(W, s)$ and $B_r(W, s)$, $A'_r(W, s)$ are absolutely convergent for $\text{Re } s > -1$ (resp. $\text{Re } s \geq 0$).*

We remark that for π unitarizable, the convergence of A_1 was proved in [JS81b, §1,3] for the split case and in [Fli88, §4] for the non-archimedean inert case.

For simplicity we write $A_r(W) = A_r(W, 0)$, and similarly for $B_r(W)$. Our main result is

THEOREM 1.2. *Let $\pi \in \Pi_{\text{gen}}(G)$ be unitarizable and \tilde{G} -distinguished and let $r = 1, \dots, n-1$. Then the identity*

$$A_r(W) = B_r(W), \quad W \in \mathcal{W}(\pi, \psi)$$

holds provided that either

- (1) E/F is split,
- (2) E/F is inert non-archimedean and Assumption 2.8 below holds,
or,
- (3) $E/F = \mathbb{C}/\mathbb{R}$ and Assumptions 2.8 and 2.10 below hold.

Note that the statement of Theorem 1.2 depends implicitly on the choice of Haar measures. If for any subgroup X of G the Haar measure on X^* is compatible with that of X then we only need to specify the Haar measure on R_r . To that end we will take the self-dual Haar measure on E with respect to ψ and use the identification $R_i \simeq E^{(r-1)(n-r)}$ (by coordinates).

REMARK 1.3. It is known that for any $\pi \in \Pi_{\text{gen}}(G)$, $A_1(W, s)$ is holomorphic at $s = 0$ (cf. [FLO12, appendix A]). Thus, $A_1(W)$ is defined. It is likely that the same is true for $A_r(W, s)$ (and $B_r(W, s)$) for

$r > 1$ and that Theorem 1.2 holds for any \tilde{G} -distinguished $\pi \in \Pi_{\text{gen}}(G)$. However, we will not discuss this issue here.

1.3. Let us give a **purely formal** argument for Theorem 1.2 in the split case (ignoring convergence issues). Write $\psi(a, b) = \psi_1(a - b)$ where ψ_1 is a non-trivial character of F . Also write $\pi = \pi_1 \otimes \pi_2$ with $\pi_2 = \hat{\pi}_1$. We can view $A_r(W_1 \otimes W_2)$ as the inner product in $X = N_r(F) \backslash \text{GL}_{n-r}(F)$ of $W_1 |\det|^{(1-r)/2}$ and $W_2 |\det|^{(1-r)/2}$. Expanding it spectrally we get

$$\int_{\Pi_{\text{gen}}(\text{GL}_{n-r}(F))} \sum_{W'} \int_X W_1(({}^g I_r)) W'(g) |\det g|^{(1-r)/2} dg \times \\ \int_X W_2(({}^g I_r)) \overline{W'(g)} |\det g|^{(1-r)/2} dg d_\mu \sigma$$

for an appropriate spectral measure d_μ where W' range over an orthonormal basis in the ψ_1^{-1} -Whittaker space of σ . The usual functional equation in Rankin-Selberg theory (e.g. [JPSS83]) gives a relation (in the sense of analytic continuation)

$$\int_X \int_{R_r^*} W_1(({}^{I_r} g) x w_{n,r}) W'(g) |\det g|^{r/2+s-1} dx dg \\ = \pm \gamma(s, \pi_1 \times \sigma, \psi_1) \int_X W_1(({}^g I_r)) W'(g) |\det g|^{s-r/2} dg.$$

In particular, for $s = \frac{1}{2}$ we get

$$\int_X \int_{R_r^*} W_1(({}^{I_r} g) x w_{n,r}) W'(g) |\det g|^{(r-1)/2} dx dg \\ = \pm \gamma\left(\frac{1}{2}, \pi_1 \times \sigma, \psi_1\right) \int_X W_1(({}^g I_r)) W'(g) |\det g|^{(1-r)/2} dg.$$

Similarly,

$$\int_X \int_{R_r^*} W_2(({}^{I_r} g) x w_{n,r}) \overline{W'(g)} |\det g|^{(r-1)/2} dx dg \\ = \pm \gamma\left(\frac{1}{2}, \pi_2 \times \hat{\sigma}, \psi_1^{-1}\right) \int_X W_2(({}^g I_r)) \overline{W'(g)} |\det g|^{(1-r)/2} dg.$$

Since

$$\gamma\left(\frac{1}{2}, \pi_1 \times \sigma, \psi_1\right) \gamma\left(\frac{1}{2}, \pi_2 \times \hat{\sigma}, \psi_1^{-1}\right) = 1,$$

$A_r(W_1 \otimes W_2)$ equals

$$\begin{aligned} & \int_{\Pi_{\text{gen}}(\text{GL}_{n-r}(F))} \sum_{W'} \int_X \int_{R_r^*} W_1\left(\begin{pmatrix} I_r & \\ & g \end{pmatrix} xw_{n,r}\right) W'(g) |\det g|^{(r-1)/2} dx dg \\ & \quad \times \int_X \int_{R_r^*} W_2\left(\begin{pmatrix} I_r & \\ & g \end{pmatrix} xw_{n,r}\right) \overline{W'(g)} |\det g|^{(r-1)/2} dx dg d_\mu \sigma. \end{aligned}$$

Once again, this is the spectral expansion of $B_r(W_1 \otimes W_2)$.

The rigorous proof of Theorem 1.2 will be given in the following sections. The idea of the proof is to express A_r as a Fourier coefficient of A_1 with respect to the unipotent radical U_{r+1} of the standard parabolic subgroup of type $(n-r+1, 1, \dots, 1)$. (We call this procedure “model transition”.) Similarly, we express A'_r as a Fourier coefficient of $A'_1 = A_1$ with respect to a conjugate of U_{r+1}^* . This is done in §3 and §4 respectively. Finally, we use the relation (14) below to get the required identity. The definitions of the Fourier coefficients above require some care since they are not given by absolutely convergent integrals. The analysis will be facilitated by the estimates of §2.

1.4. Let us introduce some more notation which will be used for the rest of the paper.

Let \mathbf{T} be the group of diagonal matrices in \mathbf{G} . We write an element $t \in T$ as $\text{diag}(t_1, \dots, t_n)$. Let \mathbf{N}° be the derived group of \mathbf{N} . For any subgroup \mathbf{X} of \mathbf{N} we write $\mathbf{X}^\circ = \mathbf{X} \cap \mathbf{N}^\circ$ which is a normal subgroup of \mathbf{X} containing (possibly strictly) the derived group of \mathbf{X} ; let $\mathbf{X}^{\text{ab}} = \mathbf{X}^\circ \backslash \mathbf{X}$. Let \mathcal{P} be the mirabolic subgroup of G consisting of elements whose last row is $(0, \dots, 0, 1)$.

For $i = 1, \dots, n-1$, we introduce some more algebraic subgroups of G .

- Let $G_i \cong \text{GL}_i(E)$ be the subgroup of G consisting of $\begin{pmatrix} I_{n-i} & \\ & h \end{pmatrix}$ where $h \in \text{GL}_i(E)$.
- Let V_i be the unipotent radical of the standard parabolic subgroup of G of type $(n-i, i)$.
- Let U_i be the unipotent radical of the standard parabolic subgroup of G of type $(n-i, 1, \dots, 1)$.
- Let $C_i = U_i \cap H_{i-1}$, consisting of elements $u = I_n + v$ in N where v is zero outside the $(n-i+1)$ -th column. Then $U_i = C_i \times U_{i-1}$ (with $U_0 = 1$).
- For $k = 1, \dots, i$ let $C_{i,k} = C_k \cap V_i$ and ${}_k C_i = C_k \cap G_i$.
- Let $D_i = C_{i+1}^t$, $D_{i,k} = D_k \cap V_i^t$ and ${}_k D_i = D_k \cap G_i$. Thus $R_i = \prod_{k=1}^{i-1} D_{i,k}$.

- Let T_i be the i -dimensional torus

$$T_i = T \cap G_{i+1} \cap \mathcal{P} = \{\text{diag}(t_1, \dots, t_n) : t_1 = \dots = t_{n-i-1} = t_n = 1\}.$$

- For $m = 1, \dots, r$ let $R_{r,m} = R_r \cap H_m = \prod_{k=m}^{r-1} D_{r,k}$. In particular, $R_{r,1} = R_r$ and $R_{r,r} = 1$.
- Let α_i be the simple root $\alpha_i(t) = \frac{t_i}{t_{i+1}}$ of T and let N_{α_i} denote the corresponding one parameter root subgroup of N . It is the image of the homomorphism $\lambda_i : E \rightarrow N$ given by $\lambda_i(x) = I_n + x e_{i,i+1}$, $x \in E$ where $e_{i,j}$ is the matrix whose (i,j) -th entry is 1 and all other entries are zero.

For a group \mathbf{X} defined over F , we denote the quotient $\mathbf{X}(F) \backslash \mathbf{X}(E)$ by $[X]$. For example $[U_i^{\text{ab}}] = (\tilde{U}_i^\circ \backslash \tilde{U}_i) \backslash (U_i^\circ \backslash U_i)$ is isomorphic to $\prod_{k=n-1}^{n-i} [N_{\alpha_k}]$ through the embedding of N_{α_k} in U_i . If $X \subset N$, we can consider ψ as a character on $[X^{\text{ab}}]$.

If f is a function on a group Y and $X = Y \rtimes Z$, then we often extend f to a function on X via the quotient map $X \rightarrow Y$.

For any group X we write δ_X for the modulus function of X . We also write δ_0 for the modulus function of the upper triangular Borel subgroup $T \rtimes N$. Similarly for $\tilde{\delta}_0$.

If X is a locally compact Abelian group then we denote by \hat{X} its Pontryagin dual.

Note that $[C_i^\circ, D_i] = N_{\alpha_{n-i}}$ where as usual we denote by $[x, y]$ the commutator $xyx^{-1}y^{-1}$. This gives rise to a non-degenerate pairing between C_i° and D_i given by $\langle c, d \rangle = \langle c, d \rangle_\psi = \psi([c, d])^{-1}$. Moreover, $\langle \cdot, \cdot \rangle$ induces non-degenerate pairings between $[C_i^\circ]$ and \tilde{D}_i , between $C_{m,i}$ and $D_{m,i}$, and between ${}_i C_m^\circ$ and ${}_i D_m$ for $i < m < n$.

For any Whittaker function W on G with respect to ψ we have the following basic property:

$$(5) \quad W(hzxy) = \psi(z) \langle y, x \rangle W(hx), \quad h \in H_{i+1}, x \in D_i, y \in C_i^\circ, z \in C_i.$$

The convention of Haar measures will be as follows. Recall that we fixed the Haar measure on E to be the self-dual Haar measure with respect to ψ . We also fix once and for all an arbitrary Haar measure on F . Note that ψ gives rise to an isomorphism between $\widehat{E/F}$ and F . Under this isomorphism, the Haar measure on $\widehat{E/F}$ dual to the quotient measure on E/F coincides with the Haar measure on F .

In the sequel we will encounter Haar measures on two types of algebraic subgroups of G (or quotients thereof): subgroups of \tilde{G} and unipotent subgroups of G . In the former case, we will take the unnormalized Tamagawa measure with respect to the ‘‘obvious’’ gauge form

and the above measure on F . For instance for \tilde{G} we take the gauge form $\frac{\wedge dg_{i,j}}{\det g^n}$; for unipotent groups we take the product of the differentials of the coordinates; for diagonal groups we take the logarithmic differentials of the coordinates. In the second case, all unipotent subgroups of G in hand will have natural coordinates and we will take the product of the Haar measures of the coordinates (using the above measure on E).

We endow all compact groups with Haar measures of total volume one.

We will use the following convention for iterated and double integrals: we write

$$\int_A \left(\int_B \int_C f(a, b, c) dc db \right) da$$

to mean that the double integral over B and C converges and the integral over A (as an iterated integral) converges.

2. Basic estimates

2.1. For any $\pi \in \Pi_{\text{gen}}(\tilde{G})$ and $s \in \mathbb{R}$ we write $\pi[s] = \pi \otimes |\det \cdot|^s$.

Let $\pi \in \Pi_{\text{gen}}(\tilde{G})$. Denoting by \times parabolic induction. We write

$$\pi = \pi_1[s_1] \times \cdots \times \pi_k[s_k]$$

with π_i square-integrable and $s_1, \dots, s_k \in \mathbb{R}$. This presentation is unique up to permutation. By an abuse of language we call the s_i 's the exponents of π . For instance, if π is tempered then the exponents are 0. If π is unitarizable then the exponents are in the open interval $(-\frac{1}{2}, \frac{1}{2})$.

We will need the following variant of the estimates of [JPSS79a].

LEMMA 2.1. *Suppose that $\pi \in \Pi_{\text{gen}}(\tilde{G})$ with unitary central character and all its exponents are bigger than λ . Then for any W in the Whittaker space of π there exists a Schwartz function $\phi \in \mathcal{S}(F^{n-1})$ such that*

$$(6) \quad |W(tk)| \leq \tilde{\delta}_0^{\frac{1}{2}}(t) |\det t|^\lambda |t_n|^{-n\lambda} \phi(\alpha_1(t), \dots, \alpha_{n-1}(t))$$

for all $t \in \tilde{T}$, $k \in \tilde{K}$.

PROOF. Indeed, in the non-archimedean case, it is easy to see that for any standard parabolic $\tilde{P} = \tilde{M}\tilde{U}$, any cuspidal exponent of π along \tilde{P} (viewed as a $\mathbb{R}_{>0}$ -valued quasi-character of the center $Z(\tilde{M})$ of \tilde{M}) has the form

$$t \mapsto |\det t|^\lambda |t_n|^{-n\lambda} \prod_{i=1}^{n-1} |\alpha_i(t)|^{x_i}, \quad t = \text{diag}(t_1, \dots, t_n) \in Z(\tilde{M})$$

with $x_i \geq 0$ for all i . The estimate (6) follows from [LM09, Theorem 3.1]. In the Archimedean case the result follows from Lemma 15.2.3 and statement (1) on the bottom of p. 369 of [Wal92]. We note that the Λ in [ibid.] is of the form $\rho + n\lambda\varpi + \sum y_i\alpha_i$ where $y_i \geq 0$, ρ is the half-sum of positive roots and ϖ is the fundamental weight corresponding to α_{n-1} . \square

Suppose that E/F is inert. Then we define exponents for $\pi \in \Pi_{\text{gen}}(G)$ in a similar way (with $|\cdot|_E$ instead of $|\cdot|$). If E/F splits then the exponents of $\pi = \pi_1 \otimes \pi_2 \in \Pi_{\text{gen}}(G)$ are by definition the union of the exponents of π_i , $i = 1, 2$.

COROLLARY 2.2. *Suppose that $\pi \in \Pi_{\text{gen}}(G)$ with unitary central character and all its exponents are bigger than λ . Then for any $W \in \mathcal{W}(\pi, \psi)$ there exists a Schwartz function $\phi \in \mathcal{S}(F^{n-1})$ such that*

$$(7) \quad |W(tk)| \leq \delta_0^{\frac{1}{2}}(t) |\det t|^{2\lambda} |t_n|^{-2n\lambda} \phi(\alpha_1(t), \dots, \alpha_{n-1}(t))$$

for all $t \in \tilde{T}$, $k \in \tilde{K}$.

2.2. Next, we will prove the following slightly stronger version of Lemma 1.1. The argument is standard.

LEMMA 2.3. *Let $\pi \in \Pi_{\text{gen}}(G)$, $W \in \mathcal{W}(\pi, \psi)$ and $r = 1, \dots, n-1$. Suppose that the exponents of π are $\geq \lambda$. Then $A_r(W, s)$, $A'_r(W, s)$ and $B_r(W, s)$ are given by absolutely convergent integrals for $\text{Re } s > -2\lambda - 1$ which extend to meromorphic functions in $s \in \mathbb{C}$. In the p -adic case they are rational functions in q^s . Moreover for any s in the range of convergence there exists N and C such that*

$$|A_r(\pi(g)W, s)|, |A'_r(\pi(g)W, s)| \leq C \|g\|^N.$$

PROOF. We start with A_r . Let \tilde{K}_r be the standard maximal compact subgroup of \tilde{H}_r , \tilde{T}'_r the diagonal torus in H_r and $\tilde{\delta}_r$ the modulus function of the upper triangular Borel subgroup $\tilde{T}'_r \times \tilde{N}_r$ of \tilde{H}_r . We rewrite A_r using the Iwasawa decomposition as

$$(8) \quad A_r(W, s) = c_r \int_{\tilde{K}_r} \int_{\tilde{T}'_r} W(tk) |\det t|^{s+1-r} \tilde{\delta}_r^{-1}(t) dt dk$$

for a suitable constant $c_r > 0$. Note that $\delta_0^{\frac{1}{2}}(t) \tilde{\delta}_r^{-1}(t) = |\det t|^r$. Therefore by (7) there exists $\phi \in \mathcal{S}(F^{n-1})$ such that

$$|W(tk)| |\det t|^{\text{Re } s+1-r} \tilde{\delta}_r^{-1}(t) \leq |\det t|^{\text{Re } s+1+2\lambda} \phi(\alpha_1(t), \dots, \alpha_{n-1}(t))$$

for all $t \in \tilde{T}'_r$, $k \in \tilde{K}_r$. Thus the integral (8) converges provided that $\text{Re } s > -2\lambda - 1$ as claimed.

The meromorphic continuation of $A_r(W, s)$ also follows from (8) by using the asymptotic behavior of Whittaker functions (cf. [JPSS79a] for the p -adic case and [Jac04] for the archimedean case).

We turn to A'_r . Again, we use Iwasawa decomposition to write it as

$$c_r \int_{R_r} \int_{\tilde{K}_r} \int_{\tilde{T}'_r} W(txk) |\det t|^{s+1-r} \tilde{\delta}_r^{-1}(t) dt dk dx.$$

We note that the Iwasawa decomposition of x is given by uak' where $a = \text{diag}(a_1, \dots, a_n)$ with $a_n = 1$ and $|a_{n-r+1}|_E \geq \max|x_{i,j}|_E$ in the inert case and $|a_{n-r+1}^{(l)}| \geq \max|x_{i,j}^{(l)}|$, $l = 1, 2$ in the split case where we write $a_i = (a_i^{(1)}, a_i^{(2)})$ and similarly for the $x_{i,j}$'s. The convergence follows as in the case of A_r . (It will also follow from the argument below.)

The moderate growth property of A_r and A'_r is also clear.

Let us prove meromorphic continuation of A'_r . Recall that for $m = 1, \dots, r$ we write $R_{r,m} = R_r \cap H_m$. Thus, $R_{r,1} = R_r$ and $R_{r,r} = 1$. Recall that $C_{r,m}$ and $D_{r,m}$ are in duality via the commutator. Observe that for $m < r$, by (5) (applied with $i = m$, $h = hy$ and $z = e$) for any $\phi \in \mathcal{S}(C_{r,m})$ we have

$$(9) \quad \pi(\phi)W(hyx) = \hat{\phi}(x)W(hyx)$$

for all $h \in H_r$, $x \in D_{r,m}$ and $y \in R_{r,m+1}$.

We introduce auxiliary integrals

$$A_{r,m}(W) = \int_{R_{r,m}} \int_{\tilde{N}_r \backslash \tilde{H}_r} W(hx) |\det h|^{s+1-r} dh dx.$$

Using the Dixmier-Malliavin theorem and (9), one proves the meromorphic continuation of $A_{r,m}$ by descending induction on m . Since $A'_r = A_{r,1}$ the analytic properties of A'_r follow.

Finally, the case of B_r follows from the relation (4). \square

REMARK 2.4. Note that the use of the Dixmier-Malliavin Theorem is not essential. It suffice to write W as a convolution by a sufficiently smooth test function (which is elementary) in order to get the meromorphic continuation strip by strip. The same remark applies to all future applications of the Dixmier-Malliavin Theorem.

2.3. Next, we discuss the split case in more detail. For any $\lambda \in \mathbb{C}^n$ we write

$$|t|^\lambda = \prod_{i=1}^n |t_i|^{\lambda_i}, \quad t = \text{diag}(t_1, \dots, t_n) \in \tilde{T}.$$

For any $\lambda \in \mathbb{C}^n$ let φ_λ be the function on \tilde{G} given by

$$\varphi_\lambda(ntk) = \tilde{\delta}_0^{\frac{1}{2}}(t)|t|^\lambda, \quad n \in \tilde{N}, t \in \tilde{T}, k \in \tilde{K}.$$

Let f_0 be the function on \tilde{N}^{ab} given by

$$(10) \quad f_0(v) = \int_{\tilde{N}^\circ} \varphi_0(w_n uv) du.$$

For convenience let ψ' be a non-degenerate character of \tilde{N} . Recall the Jacquet integral

$$J(g, \lambda) = \int_{\tilde{N}} \varphi_\lambda(w_n ug) \psi'(u)^{-1} du.$$

The integral converges absolutely for $\text{Re } \lambda_1 > \cdots > \text{Re } \lambda_n$ and admits holomorphic continuation to $\lambda \in \mathbb{C}^n$. Moreover, $J(g, \lambda)$ is a Whittaker function for the induced representation $I(\lambda)$. In particular, for any $\lambda \in \mathbb{C}^n$ such that $\text{Re } \lambda$ is dominant there exists $\phi \in \mathcal{S}(F^{n-1})$ such that

$$(11) \quad |J(t, \lambda)| \leq \phi(\alpha_1(t), \dots, \alpha_{n-1}(t)) \tilde{\delta}_0^{\frac{1}{2}}(t) |t|^{\text{Re } w_n \lambda}$$

for all $t \in \tilde{T}$.

LEMMA 2.5. f_0 is defined (i.e., the integral (10) converges) and $f_0 \in L^2(\tilde{N}^{\text{ab}})$. Moreover, its Fourier transform at $\psi'_t = \psi'(t \cdot t^{-1})^{-1}$, $t \in \tilde{T}$ is given by $\tilde{\delta}_0^{-\frac{1}{2}}(t) J(t, 0)$.

PROOF. We follow the argument of [SV, Lemma 6.3.1]. For $\text{Re } \lambda$ dominant, the function

$$f_\lambda(v) = \int_{\tilde{N}^\circ} \varphi_\lambda(w_n uv) du, \quad v \in \tilde{N}^{\text{ab}}$$

is smooth and belongs to $L^1(\tilde{N}^{\text{ab}})$. Its Fourier transform at ψ'_t is $\tilde{\delta}_0^{-\frac{1}{2}}(t) |t|^{-w_n \lambda} J(t, \lambda)$. Note that $\{\psi'_t : t \in \tilde{T}\}$ is a co-null subset of $\widehat{\tilde{N}^{\text{ab}}}$ and that the push forward of the measure $|t_1| dt$ on \tilde{T} via $t \mapsto \psi'_t$ is the restriction of the Haar measure on $\widehat{\tilde{N}^{\text{ab}}}$. Therefore, by Fourier inversion we have

$$(12) \quad f_\lambda(I_n) = \int_{\tilde{T} \cap \tilde{P}} J(t, \lambda) |t|^{-w_n \lambda} \tilde{\delta}_0^{-\frac{1}{2}}(t) |t_1| dt.$$

(The integral on the right-hand side converges absolutely by (11).) Fix a dominant $\lambda_0 \in \mathbb{R}^n$. We claim that for any $\epsilon > 0$ there exists $\phi \in \mathcal{S}(F^{n-1})$ such that

$$(13) \quad |J(t, \lambda)| \tilde{\delta}_0^{-\frac{1}{2}}(t) \leq |t|^{\epsilon w_n \lambda_0} \phi(\alpha_1(t), \dots, \alpha_{n-1}(t))$$

for all $t \in \tilde{T} \cap \tilde{\mathcal{P}}$ and $\lambda = s\lambda_0$ with $|s| \leq \epsilon$. Indeed, by the maximum modulus principle it is enough to prove this inequality for $|s| = \epsilon$. In the p -adic case this follows immediately from the fact that on the set of regular λ we have

$$J(t, \lambda) \tilde{\delta}_0^{-\frac{1}{2}}(t) = \mathbf{1}_{I^{n-1}}(\alpha_1(t), \dots, \alpha_{n-1}(t)) \sum_{w \in S_n} c_w(\lambda) |t|^{w\lambda}$$

where I is the largest fractional ideal on which ψ' is trivial, S_n is the Weyl group and c_w are holomorphic functions on the set of regular λ (which can be explicated by Shintani's formula [Shi76]). The Archimedean case follows from [Jac04, Theorem 2].

Take $\lambda = s\lambda_0$, $s \in (0, 1)$. It follows from (13) and the dominated convergence theorem that the right-hand side of (12) is a continuous function at $s = 0$. On the other hand $\lambda \mapsto \varphi_\lambda(w_n u)$ is monotone decreasing in s . Therefore by Lebesgue's monotone convergence theorem (or Fatou's lemma) we conclude the convergence of (10) for $v = I_n$. The same argument works for any $v \in \tilde{N}^{\text{ab}}$, or alternatively we can use the fact that $\varphi_0(gv) \leq \max_{k \in \tilde{K}} \varphi_0(kv) \varphi_0(g)$ for all $g \in \tilde{G}$.

Moreover, for $\text{Re } \lambda$ dominant we have

$$\int_{\tilde{N}^{\text{ab}}} |f_\lambda(v)|^2 dv = \int_{\tilde{T} \cap \tilde{\mathcal{P}}} |J(t, \lambda)|^2 |t|^{-2w_n \text{Re } \lambda} \tilde{\delta}_0^{-1}(t) |t_1| dt.$$

Once again, taking $\lambda = s\lambda_0$, the right-hand side is continuous at $s = 0$, which implies that $f_0 \in L^2(\tilde{N}^{\text{ab}})$. In addition, it follows once again from (13) that \widehat{f}_λ converges in $L^2(\widehat{\tilde{N}^{\text{ab}}})$ as $s \rightarrow 0$. Hence, f_λ converges (necessarily to f_0) in $L^2(\tilde{N}^{\text{ab}})$ as $s \rightarrow 0$. We deduce that

$$\widehat{f}_0 = \lim_{s \rightarrow 0} \widehat{f}_\lambda$$

in $L^2(\widehat{\tilde{N}^{\text{ab}}})$ and therefore $\widehat{f}_0(\psi'_t) = \tilde{\delta}_0^{-\frac{1}{2}}(t) J(t, 0)$ for all $t \in \tilde{T}$ as required. \square

Let Ξ be the standard zonal spherical function of \tilde{G} , i.e.

$$\Xi(g) = \int_{\tilde{K}} \varphi_0(kg) dk = c \int_{\tilde{N}} \varphi_0(w_n u g) \varphi_0(w_n u) du, \quad g \in \tilde{G}$$

for some scalar $c > 0$.

LEMMA 2.6. *The function $v \mapsto \int_{\tilde{N}^\circ} \Xi(uv) du$, $v \in \tilde{N}^{\text{ab}}$ is well-defined and belongs to $L^2(\tilde{N}^{\text{ab}})$.*

PROOF. For any $v \in \tilde{N}^{\text{ab}}$ we have

$$\begin{aligned}
 c^{-1} \int_{\tilde{N}^\circ} \Xi(uv) du &= \int_{\tilde{N}^\circ} \int_{\tilde{N}} \varphi_0(w_n v' uv) \varphi_0(w_n v') dv' du \\
 &= \int_{\tilde{N}^\circ} \int_{\tilde{N}} \varphi_0(w_n uv' v) \varphi_0(w_n v') dv' du \\
 &= \int_{\tilde{N}^{\text{ab}}} \int_{\tilde{N}^\circ} \int_{\tilde{N}^\circ} \varphi_0(w_n uv_1 v_2 v) \varphi_0(w_n v_1 v_2) dv_1 du dv_2 \\
 &= \int_{\tilde{N}^{\text{ab}}} \int_{\tilde{N}^\circ} \int_{\tilde{N}^\circ} \varphi_0(w_n uv_2 v) \varphi_0(w_n v_1 v_2) dv_1 du dv_2 \\
 &= \int_{\tilde{N}^{\text{ab}}} f_0(v'v) f_0(v') dv' < \infty
 \end{aligned}$$

by the previous lemma. Thus, also by the previous lemma, $v \mapsto \int_{\tilde{N}^\circ} \Xi(uv) du$ is the Fourier transform of $c|J(t, 0)|^2 \tilde{\delta}_0^{-1}(t)$. Hence $\int_{\tilde{N}^\circ} \Xi(u \cdot) du \in L^2(\tilde{N}^{\text{ab}})$. \square

From Lemma 2.6 and the fact that smooth matrix coefficients of a tempered representation are bounded by Ξ ([CHH88, Sun09]) we get

COROLLARY 2.7. *Suppose that E/F is split and $\pi \in \Pi_{\text{temp}}(G)$ is \tilde{G} -distinguished. Then for any $W \in \mathcal{W}(\pi, \psi)$*

$$\int_{[N^\circ]} |A_1(\pi(v)W)| dv < \infty,$$

and

$$\int_{[N^{\text{ab}}]} \left(\int_{[N^\circ]} |A_1(\pi(vu)W)| dv \right)^2 du < \infty.$$

In the non-split case we will make the following assumption.

ASSUMPTION 2.8. *Corollary 2.7 holds in the non-split case as well.*

In the p -adic case, it is likely that this can be deduced from the results of [Lag08, KT08], but we don't address this issue here.

2.4. Observe that

$$A_1(W) = \int_{\tilde{N} \backslash \tilde{\mathcal{P}}} W(g) dg$$

since $\tilde{H}_1 \tilde{N} = \tilde{\mathcal{P}}$. Therefore A_1 is a $\tilde{\mathcal{P}}$ -invariant functional.

Suppose that E/F splits and $\pi \in \Pi_{\text{gen}}(G)$ is unitarizable and \tilde{G} -distinguished. Then $\pi = \pi_1 \otimes \hat{\pi}_1$ where π_1 is an irreducible unitarizable representations of $\text{GL}_n(F)$. It is known that A_1 is \tilde{G} -invariant [Ber84, Bar03].

We can write $\psi(a, b) = \psi_1(a - b)$ for some character ψ_1 of F and take W of the form $W(x, y) = W_1(x)\hat{W}_1(y)$ with $W_1 \in \mathcal{W}(\pi_1, \psi_1)$ and $\hat{W}_1 \in \mathcal{W}(\hat{\pi}, \psi_1^{-1})$. Then $W^\sharp(x, y)$ has the form $W_2(y)\hat{W}_2(x)$ with $W_2 \in \mathcal{W}(\pi_1, \psi_1)$ and $\hat{W}_2 \in \mathcal{W}(\hat{\pi}, \psi_1^{-1})$. By uniqueness we have $A_1(W^\sharp) = cA_1(W)$ for some positive constant c . Since $A_1(W) = A_1((W^\sharp)^\sharp) = cA_1(W^\sharp) = c^2A_1(W)$ we get that $c^2 = 1$ and hence $c = 1$. Thus,

$$(14) \quad A_1(W^\sharp) = A_1(W).$$

In the non-archimedean inert case, again when π is unitarizable and \tilde{G} -distinguished, it is still true by [Off11, Theorem 3.1] that A_1 is \tilde{G} -invariant and that (14) holds [Off11, Corollary 7.2].

We also record the following result of Matringe.

THEOREM 2.9 ([Mat11]). *Suppose that E/F is a quadratic extension of p -adic fields. Let τ be the Galois involution. Then the \tilde{G} -distinguished representations $\pi \in \Pi_{\text{gen}}(G)$ are precisely the irreducible representations of the form*

$$\pi = \pi_1 \times \cdots \times \pi_k \times \hat{\sigma} \times \sigma^\tau$$

where $\pi_i \in \Pi(\text{GL}_{n_i}(E))$, $i = 1, \dots, k$ is square-integrable and $\text{GL}_{n_i}(F)$ -distinguished and $\sigma \in \Pi_{\text{gen}}(\text{GL}_m(E))$, where $n = n_1 + \cdots + n_k + 2m$.

ASSUMPTION 2.10. *Both Theorem 2.9 and the relation (14) hold for $E/F = \mathbb{C}/\mathbb{R}$.*

Note that the characterization of generic irreducible representations of $\text{GL}_n(\mathbb{C})$ which are distinguished by a unitary group was given in [FLO12]. It is likely that the methods of [loc. cit.] apply for the analogue of Theorem 2.9 in the archimedean case. So Assumption 2.10 is certainly within reach.

3. Model transition – first case

Throughout this section let $\pi \in \Pi_{\text{temp}}(G)$ be \tilde{G} -distinguished. In the non-split case we will work under Assumption 2.8. Our goal is to express the A_r 's in terms of A_1 .

3.1.

LEMMA 3.1. *Let $W \in \mathcal{W}(\pi, \psi)$ and $i = 1, \dots, n - 1$. Then*

- (1) *The function $u \mapsto A_1(\pi(u)W)$ lies in $L^1([U_i^\circ])$.*
- (2) *Define the function $\mathcal{F}_{W,i}$ on U_i^{ab} by*

$$(15) \quad \mathcal{F}_{W,i}(u) = \int_{[U_i^\circ]} A_1(\pi(vu)W) dv.$$

Then $\mathcal{F}_{W,i} \in L^2([U_i^{\text{ab}}])$.

PROOF. It is clear that $\mathcal{F}_{W,i}$, once defined, is left invariant under U_i° and thus defines a function on U_i^{ab} , and in fact on $[U_i^{\text{ab}}]$ since A_1 is $\tilde{\mathcal{P}}$ -invariant. We will show that for $i < n$ we have

$$(16) \quad \int_{[U_i^\circ]} |A_1(\pi(v)W)| \, dv < \infty,$$

and

$$(17) \quad \int_{[U_i^{\text{ab}}]} \left(\int_{[U_i^\circ]} |A_1(\pi(vu)W)| \, dv \right)^2 \, du < \infty.$$

In the case $i = n - 1$ ($U_{n-1} = N$) this is precisely Corollary 2.7 and Assumption 2.8.

The p -adic case follows trivially from this. In the archimedean case we will prove (16) and (17) by descending induction on i .

Assume that (16) and (17) hold for $i = k$ and consider the case $i = k - 1$. Recall that $U_k^\circ = C_k^\circ \times U_{k-1}^\circ$. By Dixmier-Malliavin, we can assume that $W = \pi(f_1)W'$ where $f_1 \in C_c^\infty(C_k^\circ)$. Set $\tilde{f}_1(u) = \int_{\tilde{C}_k^\circ} |f_1(vu)| \, dv \in C_c([C_k^\circ])$. Then for $u \in U_{k-1}^{\text{ab}}$:

$$\begin{aligned} \int_{[U_{k-1}^\circ]} |A_1(\pi(vu)W)| \, dv &= \int_{[U_{k-1}^\circ]} \left| \int_{C_k^\circ} f_1(v') A_1(\pi(vuv')W') \, dv' \right| \, dv \\ &\leq \int_{C_k^\circ} \int_{[U_{k-1}^\circ]} |f_1(v') A_1(\pi(vuv')W')| \, dv \, dv'. \end{aligned}$$

Since v' normalizes U_{k-1}° and commutes with U_{k-1} modulo U_{k-1}° we can write this as

$$\begin{aligned} &\int_{C_k^\circ} \int_{[U_{k-1}^\circ]} |f_1(v') A_1(\pi(vv'u)W')| \, dv \, dv' = \\ &= \int_{[C_k^\circ]} \int_{\tilde{C}_k^\circ} \int_{[U_{k-1}^\circ]} |f_1(v'_1 v'_2) A_1(\pi(vv'_1 v'_2 u)W')| \, dv \, dv'_1 \, dv'_2. \end{aligned}$$

Since v'_1 normalizes \tilde{U}_{k-1}° we get

$$\begin{aligned} &\int_{[C_k^\circ]} \int_{[U_{k-1}^\circ]} \tilde{f}_1(v') |A_1(\pi(vv'u)W')| \, dv \, dv' \\ &\leq \sup \tilde{f}_1 \cdot \int_{[C_k^\circ]} \int_{[U_{k-1}^\circ]} |A_1(\pi(vv'u)W')| \, dv \, dv' \\ &= \sup \tilde{f}_1 \cdot \int_{[U_k^\circ]} |A_1(\pi(vu)W')| \, dv. \end{aligned}$$

This proves (16) for $i = k - 1$. Moreover to prove (17) for $i = k - 1$, we only need to show that

$$\int_{[U_{k-1}^{\text{ab}}]} \left(\int_{[U_k^\circ]} |A_1(\pi(vu)W)| dv \right)^2 du < \infty.$$

Assume that $W = \pi(f_2)W''$ where $f_2 \in C_c^\infty(N_{\alpha_{n-k}})$. Let

$$\tilde{f}_2(u) = \int_{\tilde{N}_{\alpha_{n-k}}} |f_2(vu)| dv \in C_c([N_{\alpha_{n-k}}]).$$

Then since $N_{\alpha_{n-k}}$ normalizes U_{k-1} and U_{k-1}° while $\tilde{N}_{\alpha_{n-k}}$ normalizes U_k° and \tilde{U}_k° we have

$$\begin{aligned} & \int_{[U_{k-1}^{\text{ab}}]} \left(\int_{[U_k^\circ]} |A_1(\pi(vu)W)| dv \right)^2 du \\ & \leq \int_{[U_{k-1}^{\text{ab}}]} \left(\int_{[U_k^\circ]} \int_{N_{\alpha_{n-k}}} |A_1(\pi(vuu')W'') f_2(u')| du' dv \right)^2 du \\ & = \int_{[U_{k-1}^{\text{ab}}]} \left(\int_{N_{\alpha_{n-k}}} \int_{[U_k^\circ]} |A_1(\pi(vu'u)W'')| dv |f_2(u')| du' \right)^2 du \\ & = \int_{[U_{k-1}^{\text{ab}}]} \left(\int_{[N_{\alpha_{n-k}}]} \int_{[U_k^\circ]} |A_1(\pi(vu'u)W'')| dv \tilde{f}_2(u') du' \right)^2 du \\ & \leq \int_{[N_{\alpha_{n-k}}]} \tilde{f}_2(u')^2 du' \cdot \int_{[U_{k-1}^{\text{ab}}]} \int_{[N_{\alpha_{n-k}}]} \left(\int_{[U_k^\circ]} |A_1(\pi(vu'u)W'')| dv \right)^2 du' du \\ & = \int_{[N_{\alpha_{n-k}}]} \tilde{f}_2(u')^2 du' \cdot \int_{[U_k^{\text{ab}}]} \left(\int_{[U_k^\circ]} |A_1(\pi(vu)W'')| dv \right)^2 du < \infty \end{aligned}$$

where we used the induction hypothesis, the Cauchy-Schwarz inequality and the fact that $[U_k^{\text{ab}}] = [U_{k-1}^{\text{ab}}] \times [N_{\alpha_{n-k}}]$. This completes the induction step. \square

3.2. Consider now the Fourier transform of $\mathcal{F}_{W,i}$ as an L^2 -function of $[\widehat{U_i^{\text{ab}}}]$, the group of characters of $[U_i^{\text{ab}}]$.

We denote by $[\widehat{U_i^{\text{ab}}}]_{\text{reg}}$ the set of non-degenerate characters on $[U_i^{\text{ab}}]$, which is an open subset of $[\widehat{U_i^{\text{ab}}}]$. The group \tilde{T}_i acts on $[U_i^{\text{ab}}]$ by conjugation. Therefore it acts on $[\widehat{U_i^{\text{ab}}}]$ and it acts simply transitively on $[\widehat{U_i^{\text{ab}}}]_{\text{reg}}$. Let

$$\kappa = \kappa^i : \tilde{T}_i \rightarrow [\widehat{U_i^{\text{ab}}}]$$

be the open embedding

$$\kappa(t) = \psi(t \cdot t^{-1})^{-1}.$$

The image of κ is $\widehat{[U_i^{\text{ab}}]}_{\text{reg}}$.

For any function f on $\widehat{[U_i^{\text{ab}}]}$ we write $\kappa^* f = f \circ \kappa$ (a function on \tilde{T}_i) and for any function g on \tilde{T}_i we write $\kappa_* g$ to be the extension of g by 0 to $\widehat{[U_i^{\text{ab}}]}$. Thus,

$$\kappa_* : C_c^\infty(\tilde{T}_i) \rightarrow C_c^\infty(\widehat{[U_i^{\text{ab}}]}).$$

Let $\phi \in \mathcal{S}([U_i^{\text{ab}}])$, considered also as a function on $[U_i]$. Define

$$(18) \quad \mathcal{J}_i(W, \phi) = \int_{[U_i^{\text{ab}}]} \phi(u) \mathcal{F}_{W,i}(u) \, du = \int_{[U_i]} \phi(u) A_1(\pi(u)W) \, du.$$

Set $S_i = \tilde{T}_i \times [U_i^\circ]$. Note that for $t \in \tilde{T}_i$,

$$(19) \quad \delta_{S_i}(t) = |t_{n-i}|^{i-1} \prod_{j=n-i+1}^{n-1} |t_j|^{n-2j+1}.$$

LEMMA 3.2. *Let $f \in C_c^\infty(\tilde{T}_i)$ and $W' = \pi(f\delta_{S_i}^{-1})W$. Then $\kappa^* \widehat{\mathcal{F}_{W',i}}$ is given by $t \mapsto |t_{n-i}^{-1}| \mathcal{J}_i(\pi(t)W, \widehat{\kappa_* f})$. In particular, it is a smooth function on \tilde{T}_i .*

PROOF. We have

$$\begin{aligned} \mathcal{F}_{W',i}(u) &= \int_{[U_i^\circ]} \int_{\tilde{T}_i} f(t') \delta_{S_i}^{-1}(t') A_1(\pi(vut')W) \, dt' \, dv \\ &= \int_{[U_i^\circ]} \int_{\tilde{T}_i} f(t') A_1(\pi(vt'^{-1}ut')W) \, dt' \, dv = \int_{\tilde{T}_i} f(t') \mathcal{F}_{W,i}(t'^{-1}ut') \, dt'. \end{aligned}$$

It is easy to see that the value of the Fourier transform of $\mathcal{F}_{W,i}(t'^{-1} \cdot t')$ at $\kappa(t)$ is $|t_{n-i}^{-1}| \kappa^* \widehat{\mathcal{F}_{\pi(t)W,i}}(t')$. Note that the push-forward of the measure $d^a t' = |t_{n-i}^{-1}| \, dt'$ on \tilde{T}_i via κ is the dual measure to du on $[U_i^{\text{ab}}]$. Thus, by Parseval's identity, the value of the Fourier transform of $\mathcal{F}_{W',i}$ at $\kappa(t)$ is

$$\begin{aligned} |t_{n-i}^{-1}| \int_{\tilde{T}_i} f(t') \kappa^* \widehat{\mathcal{F}_{\pi(t)W,i}}(t') \, d^a t' &= \\ |t_{n-i}^{-1}| \int_{[U_i^{\text{ab}}]} \widehat{\kappa_* f}(u) \mathcal{F}_{\pi(t)W,i}(u) \, du &= |t_{n-i}^{-1}| \mathcal{J}_i(\pi(t)W, \widehat{\kappa_* f}) \end{aligned}$$

as claimed. \square

Define $\mathcal{E}_i(W)$ to be the value of the Fourier transform of $\mathcal{F}_{W,i}$ at ψ^{-1} . Thus $\mathcal{E}_i(W)$ is a regularization of the integral

$$\mathcal{E}_i(W) = \int_{[U_i]} A_1(\pi(u)W) \psi(u)^{-1} \, du$$

which is in general not absolutely convergent. Note that $\mathcal{E}_0(W) = A_1(W)$.

REMARK 3.3. We expect that in the non-archimedean case, the integral

$$\int_{[U_i]} A_1(\pi(u)W)\psi(u)^{-1} du$$

makes sense as a stable integral, i.e. stabilizes for large enough compact open subgroups of $[U_i]$. However, we will not address this issue here.

REMARK 3.4. One can define $\mathcal{E}_i(W)$ (in the tempered case) without knowing that $\mathcal{F}_{W,i} \in L^2([U_i^{\text{ab}}])$. Namely, one can show that the restriction of the Fourier transform of $\mathcal{F}_{W,i}$ (as a distribution) to $[\widehat{U_i^{\text{ab}}}]_{\text{reg}}$ is smooth. Therefore, in the non-split case we would only need to know the first part of Corollary 2.7.

By Lemma 3.2 we have

$$(20) \quad \mathcal{E}_i(\pi(f\delta_{S_i}^{-1})W) = \mathcal{J}_i(W, \widehat{\kappa_* f}).$$

The rest of this section is devoted to the proof of the following result.

PROPOSITION 3.5. *For $i = 1, \dots, n$ we have*

$$A_i(W) = \mathcal{E}_{i-1}(W)$$

for any $W \in \mathcal{W}(\pi, \psi)$.

Fix $i = 0, \dots, n-1$. By (20) it suffices to prove that for all $f \in C_c^\infty(\tilde{T}_i)$ we have

$$(21) \quad A_{i+1}(\pi(f\delta_{S_i}^{-1})W) = \mathcal{J}_i(W, \widehat{\kappa_* f}).$$

REMARK 3.6. Since $A_{i+1}(\pi(\cdot)W)$ is a distribution on $C_c^\infty(\tilde{T}_i)$ and $\mathcal{J}_i(W, \cdot)$ is a tempered distribution on $\mathcal{S}([U_i^{\text{ab}}])$ it suffices to prove (21) for functions of the form $f(t) = \prod_{k=1}^i \phi_k(\frac{t_{n-k}}{t_{n-k+1}})$ when $t \in \tilde{T}_i$, with $\phi_k \in C_c^\infty(F^*)$.

3.3. To prove (21), we use induction on i . The case $i = 0$ is trivial ($T_0 = 1$).

By Remark 3.6, we can assume $f(t) = f_1(t)f_2(t)$, where $f_2 \in C_c^\infty(\tilde{T}_{i-1})$ considered as a function on \tilde{T}_i via the obvious projection $\tilde{T}_i \rightarrow \tilde{T}_{i-1}$ and $f_1(t) = \phi(\alpha_{n-i}(t)) = \phi(\frac{t_{n-i}}{t_{n-i+1}})$ with $\phi \in C_c^\infty(F^*)$.

Let $j : \tilde{T}_i \rightarrow [\widehat{N_{\alpha_{n-i}}}]$ be the map $t \mapsto \psi(t \cdot t^{-1})^{-1}$, which factors through α_{n-i} . For any function g on \tilde{T}_i which depends only on $\alpha_{n-i}(t)$

let j_*g be the function on $\widehat{[N_{\alpha_{n-i}}]}$ which is 0 on the trivial character and given by $g(t)$ on $j(t)$.

Then $\widehat{\kappa_*^i f} = \widehat{j_* f_1 \kappa_*^{i-1} f_2}$ where we consider $\widehat{\kappa_*^{i-1} f_2}$ as function on $[U_i^{\text{ab}}]$ that factors through $[U_{i-1}^{\text{ab}}]$, and $\widehat{j_* f_1}$ factors through $[N_{\alpha_{n-i}}]$. More precisely, $\widehat{j_* f_1}(\lambda_{n-i}(x)) = \hat{\phi}(x)$ for $x \in F \setminus E$. We will consider $\widehat{j_* f_1}$ as a function on $[C_i]$ as $C_i = N_{\alpha_{n-i}} \times C_i^\circ$.

Recall that $U_i = C_i \times U_{i-1}$. By induction hypothesis we have

$$\begin{aligned} \mathcal{J}_i(W, \widehat{\kappa_*^i f}) &= \int_{[C_i]} \widehat{j_* f_1}(u) \mathcal{J}_{i-1}(\pi(u)W, \widehat{\kappa_*^{i-1} f_2}) du \\ &= \int_{[C_i]} \widehat{j_* f_1}(u) A_i(\pi(f_2 \delta_{S_{i-1}}^{-1})\pi(u)W) du \\ &= \int_{[C_i]} \widehat{j_* f_1}(u) \left(\int_{\tilde{T}_{i-1}} f_2(t) \delta_{S_{i-1}}(t)^{-1} A_i(\pi(tu)W) dt \right) du. \end{aligned}$$

We will express A_i in terms of A_{i+1} . Let $T_i^0 \subset T_i$ be the one-dimensional subtorus consisting of $\text{diag}(t_1, \dots, t_n)$ with $t_k = 1$ if $k \neq n-i$. Observe that the map

$$(\tilde{N}_{H_{i+1}} \setminus \tilde{H}_{i+1} \times \tilde{T}_i^0) \times \tilde{D}_i \rightarrow \tilde{N}_{H_i} \setminus \tilde{H}_i$$

is an open embedding with dense image. Thus,

$$(22) \quad A_i(W) = \int_{\tilde{T}_i^0} \int_{\tilde{D}_i} A_{i+1}(\pi(\xi t)W) |\det t|^{1-i} d\xi dt.$$

Therefore, we can write $\mathcal{J}_i(W, \widehat{\kappa_*^i f})$ as

$$\int_{[C_i]} \left(\int_{\tilde{T}_{i-1}} \int_{\tilde{T}_i^0} \int_{\tilde{D}_i} \widehat{j_* f_1}(u) f_2(t) \delta_{S_{i-1}}^{-1}(t) A_{i+1}(\pi(\xi t' tu)W) |t'_{n-i}|^{1-i} d\xi dt' dt \right) du.$$

From (19) we get for $t'' = tt'$ with $t \in \tilde{T}_{i-1}$ and $t' \in \tilde{T}_i^0$:

$$\delta_{S_{i-1}}(t)^{-1} |t'_{n-i}|^{1-i} |t_{n-i+1}|^{n-i-1} = \delta_{S_i}(t'')^{-1}.$$

As $\tilde{T}_i = \tilde{T}_{i-1} \times \tilde{T}_i^0$, $\mathcal{J}_i(W, \widehat{\kappa_*^i f})$ equals

$$\int_{[C_i]} \left(\int_{\tilde{T}_i} \int_{\tilde{D}_i} \widehat{j_* f_1}(u) f_2(t) \delta_{S_i}^{-1}(t) A_{i+1}(\pi(\xi tu)W) |t_{n-i+1}|^{1+i-n} d\xi dt \right) du.$$

Write $u \in C_i$ as $u_1 u_2$ with $u_1 \in C_i^\circ$ and $u_2 \in N_{\alpha_{n-i}}$. Then the above integration can be written as

$$\int_{[C_i^\circ]} \int_{[N_{\alpha_{n-i}}]} \left(\int_{\tilde{T}_i} \int_{\tilde{D}_i} \right) = \int_{[C_i^\circ]} \left(\int_{[N_{\alpha_{n-i}}]} \int_{\tilde{T}_i} \int_{\tilde{D}_i} \right)$$

as the inner triple integral is absolutely convergent. It is clear from (5) that:

$$(23) \quad A_{i+1}(\pi(u)W) = \psi(u)A_{i+1}(W), \quad u \in U_i.$$

Thus $A_{i+1}(\pi(\xi tu_1 u_2)W) = A_{i+1}(\pi(\xi t)W) \langle tu_1 t^{-1}, \xi \rangle \psi(tu_2 t^{-1})$. We get

$$\begin{aligned} \mathcal{J}_i(W, \widehat{\kappa_* f}) &= \int_{[C_i^\circ]} \left(\int_{[N_{\alpha_{n-1}}]} \int_{\tilde{T}_i} \int_{\tilde{D}_i} \widehat{j_* f_1}(u_2) f_2(t) \delta_{S_i}^{-1}(t) \right. \\ &\quad \left. A_{i+1}(\pi(\xi t)W) |t_{n-i+1}|^{1+i-n} \langle tu_1 t^{-1}, \xi \rangle \psi(tu_2 t^{-1}) d\xi dt du_2 \right) du_1. \end{aligned}$$

Integrating first over u_2 and using Fourier inversion, $\mathcal{J}_i(W, \widehat{\kappa_* f})$ equals

$$\int_{[C_i^\circ]} \left(\int_{\tilde{T}_i} \int_{\tilde{D}_i} f(t) \delta_{S_i}^{-1}(t) A_{i+1}(\pi(\xi t)W) |t_{n-i+1}|^{1+i-n} \langle tu_1 t^{-1}, \xi \rangle d\xi dt \right) du_1.$$

It will follow from Lemma 3.8 below that we can switch the order of integration, and integrate over $[C_i^\circ]$ before the integration over \tilde{T}_i . Changing u_1 to $t^{-1}u_1 t$ we get:

$$\mathcal{J}_i(W, \widehat{\kappa_* f}) = \int_{\tilde{T}_i} \int_{[C_i^\circ]} \left(\int_{\tilde{D}_i} f(t) \delta_{S_i}^{-1}(t) A_{i+1}(\pi(\xi t)W) \langle u_1, \xi \rangle d\xi \right) du_1 dt.$$

By Lemma 3.7 below $A_{i+1}(\pi(\xi)W) \in \mathcal{S}(\tilde{D}_i)$. Hence by Fourier inversion the above is:

$$\mathcal{J}_i(W, \widehat{\kappa_* f}) = \int_{\tilde{T}_i} f(t) \delta_{S_i}^{-1}(t) A_{i+1}(\pi(t)W) dt = A_{i+1}(\pi(f \delta_{S_i}^{-1}))W.$$

This completes the induction step and thus the proof of Proposition 3.5.

It remains to show the following two Lemmas.

LEMMA 3.7. *The function $\xi \mapsto A_{i+1}(\pi(\xi)W)$ on D_i lies in $\mathcal{S}(D_i)$.*

LEMMA 3.8. *For any $f \in C_c^\infty(\tilde{T}_i)$,*

$$\int_{[C_i^\circ]} \int_{\tilde{T}_i} \left(\int_{\tilde{D}_i} f(t) A_{i+1}(\pi(\xi t)W) \langle tu_1 t^{-1}, \xi \rangle d\xi \right) dt du_1$$

is absolutely convergent.

PROOF OF LEMMA 3.7. A priori $A_{i+1}(\pi(\cdot)W)$ is a tempered function on D_i . We can assume $W = \pi(\phi')W'$ where $\phi' \in C_c^\infty(C_i^\circ)$. It follows from (5) that

$$\begin{aligned} A_{i+1}(\pi(\xi)W) &= \int_{C_i^\circ} A_{i+1}(\pi(\xi u)W') \phi'(u) du \\ &= \int_{C_i^\circ} A_{i+1}(\pi(\xi)W') \phi'(u) \langle u, \xi \rangle du = A_{i+1}(\pi(\xi)W') \hat{\phi}'(\xi). \end{aligned}$$

The Lemma follows. \square

PROOF OF LEMMA 3.8. We can assume $W = \pi(f')W'$ where $f' \in C_c^\infty(\tilde{D}_i)$. Then we need to consider

$$\int_{[C_i^\circ]} \int_{\tilde{T}_i} \left| \left(\int_{\tilde{D}_i} \int_{\tilde{D}_i} f(t)f'(\xi')A_{i+1}(\pi(\xi t\xi')W') \langle tu_1t^{-1}, \xi \rangle d\xi' d\xi \right) \right| dt du_1.$$

By changing $\xi \mapsto \xi t(\xi')^{-1}t^{-1}$ and using (23), the inner double integral inside the absolute value is

$$\int_{\tilde{D}_i} \int_{\tilde{D}_i} f(t)f'(\xi')A_{i+1}(\pi(\xi tu_1)W') \langle tu_1t^{-1}, t(\xi')^{-1}t^{-1} \rangle d\xi' d\xi.$$

Integrating over ξ' we can express the above as

$$\int_{[C_i^\circ] \times \tilde{T}_i} \left| \left(\int_{\tilde{D}_i} f''(g)A_{i+1}(\pi(\xi g)W') d\xi \right) \right| dg$$

for some Schwartz function f'' on $[C_i^\circ] \times \tilde{T}_i$. The Lemma now follows from the previous one. \square

4. Model transition – second case

4.1. Throughout this section let $\pi \in \Pi_{\text{temp}}(G)$ be \tilde{G} -distinguished. Our goal is to express $A'_r(W)$ (defined in (2)) in terms of $A'_1(W) = A_1(W)$. We keep the proviso of Assumptions 2.8 and 2.10.

Let $N_{G_r} = N \cap G_r$. Define the function $\mathcal{F}'_{W,r}$ on $[N_{G_r}^{\text{ab}}]$ by

$$\mathcal{F}'_{W,r}(u) = \int_{[R_r \times N_{G_r}^\circ]} A_1(\pi(vu)W) dv.$$

LEMMA 4.1. Let $\hat{w}_r := \begin{pmatrix} I_{n-r} & \\ & w_r \end{pmatrix} = w_{n,r}^* w_n$, and let $u \mapsto \hat{u}$ be the automorphism of N_{G_r} given by $\hat{u} = \hat{w}_r u^\sigma \hat{w}_r^{-1}$. Then for $W \in \mathcal{W}(\pi, \psi)$ we have the identity $\mathcal{F}'_{W^\sharp, r-1}(u) = \mathcal{F}'_{\pi(\hat{w}_r)W, r}(\hat{u})$, $u \in N_{G_r}$. In particular,

- (1) The integral defining $\mathcal{F}'_{W,r}$ is absolutely convergent.
- (2) $\mathcal{F}'_{W,r} \in L^2([N_{G_r}^{\text{ab}}])$.
- (3) The Fourier transform of $\mathcal{F}'_{W,r}$ is smooth at the non-degenerate elements of $\widehat{[N_{G_r}^{\text{ab}}]}$. Let $\mathcal{E}'_r(W) = \widehat{\mathcal{F}'_{W,r}}(\psi^{-1})$. Then $\mathcal{E}'_r(\pi(\hat{w}_r)W) = \mathcal{E}'_{r-1}(W^\sharp)$.

PROOF. Recall that $W^\sharp \in \mathcal{W}(\hat{\pi}, \psi^{-1})$. Thus by (15)

$$\begin{aligned} \mathcal{F}'_{W^\sharp, r-1}(u) &= \int_{[U_{r-1}^\circ]} A_1(\hat{\pi}(vu)W^\sharp) dv \\ &= \int_{[U_{r-1}^\circ]} A_1([\pi(v^\sigma u^\sigma)W]^\sharp) dv = \int_{[U_{r-1}^\circ]} A_1(\pi(v^\sigma u^\sigma)W) dv, \end{aligned}$$

where the last equation follows from the relation (14). Since $\hat{w}_r(U_{r-1}^\circ)^\sigma \hat{w}_r^{-1} = R_r \rtimes N_{G_r}^\circ$, we deduce that the above is

$$\int_{[R_r \rtimes N_{G_r}^\circ]} A_1(\pi(\hat{w}_r v \hat{w}_r)W) dv = \mathcal{F}'_{\pi(\hat{w}_r)W, r}(\hat{u}).$$

The two claims on convergence follow. The last claim $\mathcal{E}'_r(\pi(\hat{w}_r)W) = \mathcal{E}_{r-1}(W^\#)$ follows from the fact that $\psi^{-1}(\hat{u}) = \psi(u)$. (Note that $\mathcal{E}_r(W^\#)$ is the Fourier transform of $\mathcal{F}_{W^\#, r}$ at ψ because $W^\# \in \mathcal{W}(\hat{\pi}, \psi^{-1})$.) \square

The goal of this section is to prove:

PROPOSITION 4.2. *For $r = 1, \dots, n-1$ we have*

$$A'_r(W) = \mathcal{E}'_r(W).$$

4.2. In order to prove the proposition we will prove a more general auxiliary statement (Proposition 4.4 below). From now on we fix a positive integer $r < n$.

Let $1 \leq i \leq r-1$. Let ${}_i U_r = U_i \cap G_r$, then ${}_i U_r^{\text{ab}} = {}_i U_r^\circ \setminus {}_i U_r$ is an i -dimensional vector space. Let

$$\mathcal{F}'_{W, r, i}(u) = \int_{[R_r \rtimes {}_i U_r^\circ]} A_1(\pi(vu)W) dv, \quad u \in [{}_i U_r^{\text{ab}}].$$

We have:

- LEMMA 4.3. (1) *The integral defining $\mathcal{F}'_{W, r, i}$ is absolutely convergent.*
 (2) $\mathcal{F}'_{W, r, i} \in L^2([{}_i U_r^{\text{ab}}])$.
 (3) *The Fourier transform of $\mathcal{F}'_{W, r, i}$ is smooth at the non-degenerate elements of $\widehat{[{}_i U_r^{\text{ab}}]}$.*

PROOF. From the argument in Lemma 4.1 and the bounds (16) and (17), we get:

$$\int_{[R_r \rtimes N_{G_r}^\circ]} |A_1(\pi(vu)W)| dv < \infty$$

and

$$\int_{[N_{G_r}^{\text{ab}}]} \left(\int_{[R_r \rtimes N_{G_r}^\circ]} |A_1(\pi(vu)W)| dv \right)^2 du < \infty.$$

Since $N_{G_r} = {}_{r-1}U_r$, we get the claim of convergence for $i = r-1$. The same induction argument as in the proof of Lemma 3.1 gives the claims of convergence. The proof of the last smoothness claim is also identical to the proof of Lemma 3.2. \square

Define $\mathcal{E}'_{r,i}(W) = \widehat{\mathcal{F}'_{W,r,i}(\psi^{-1})}$. Clearly, $\mathcal{E}'_r = \mathcal{E}'_{r,r-1}$.
 Define the integral

$$A'_{r,i}(W) = \int_{\tilde{R}_{r,i+1} \setminus R_r} A_{i+1}(\pi(x)W) dx.$$

It is clear that $A'_r = A'_{r,r-1}$ (since $R_{r,r} = 1$). Thus, Proposition 4.2 is a special case of the following result which will be proved in the rest of the section.

PROPOSITION 4.4. *For any $i = 0, \dots, r-1$ the integrals defining $A'_{r,i}(W)$ are absolutely convergent, and*

$$A'_{r,i}(W) = \mathcal{E}'_{r,i}(W).$$

4.3. As in §3.2, we can write $\mathcal{E}'_{r,i}(W)$ using an absolutely convergent integral. For $\phi \in \mathcal{S}([{}_iU_r^{\text{ab}}])$ regarded as a function of $[R_r \times {}_iU_r]$, define

$$\mathcal{J}'_i(W, \phi) = \int_{[{}_iU_r^{\text{ab}}]} \phi(u) \mathcal{F}'_{W,r,i}(u) du = \int_{[R_r \times {}_iU_r]} \phi(u) A_1(\pi(u)W) du.$$

Then for $f \in C_c^\infty(\tilde{T}_i)$, by the same argument as in §3.2 we have

$$\mathcal{E}'_{r,i}(\pi(f\delta_{S'_i}^{-1})W) = \mathcal{J}'_i(W, \widehat{\kappa_* f})$$

where $S'_i = \tilde{T}_i \times [{}_iU_r^\circ \times R_r]$. Note that the embedding ${}_iU_r \hookrightarrow U_i$ induces an isomorphism ${}_iU_r^{\text{ab}} \simeq U_i^{\text{ab}}$. We can therefore view κ as a map $\kappa : \tilde{T}_i \rightarrow [{}_iU_r^{\text{ab}}]$. Note that for $t \in \tilde{T}_i$:

$$(24) \quad \delta_{S'_i}(t) = |\det t|^{n-r} |t_{n-i}|^{i-1} \prod_{j=1}^{i-1} |t_{n-j}|^{-r+1+2j}.$$

Thus, it suffices to show that for any $i = 0, \dots, r-1$ and $f \in C_c^\infty(\tilde{T}_i)$, the integrals

$$A'_{r,i}(\pi(f\delta_{S'_i}^{-1})W) = \int_{\tilde{R}_{r,i+1} \setminus R_r} \int_{\tilde{T}_i} A_{i+1}(\pi(xt)W) f(t) \delta_{S'_i}^{-1}(t) dt dx$$

are absolutely convergent and equal $\mathcal{J}'_i(W, \widehat{\kappa_* f})$.

4.4. We use induction on i , the case $i = 0$ is a tautology. By Remark 3.6 we may assume that $f(t) = f_1(t)f_2(t)$ with $f_2 \in C_c^\infty(\tilde{T}_{i-1})$ and $f_1(t) = \phi'(\alpha_{n-i}(t))$ where $\alpha_{n-i}(t) = \frac{t_{n-i}}{t_{n-i+1}}$ and $\phi' \in C_c^\infty(F^*)$.

Note that ${}_iU_r = {}_{i-1}U_r \times {}_iC_r$, thus ${}_iU_r^{\text{ab}} = {}_{i-1}U_r^{\text{ab}} \times {}_iC_r^{\text{ab}}$; $\widehat{\kappa_* f} = \widehat{j_* f_1 \kappa_*^{i-1} f_2}$ where $\widehat{\kappa_*^{i-1} f_2}$ is a function on $[{}_iU_r^{\text{ab}}]$ that factors through $[{}_{i-1}U_r^{\text{ab}}]$ and $\widehat{j_* f_1}$ factors through $[N_{\alpha_{n-i}}]$. More precisely, $\widehat{j_* f_1}(\lambda_{n-i}(x)) =$

$\widehat{\phi}'(x)$ for $x \in F \setminus E$. We will consider $\widehat{j_* f_1}$ as a function on $[_i C_r]$ via $_i C_r = N_{\alpha_{n-i}} \times _i C_r^\circ$.

For the induction step we write

$$\begin{aligned} \mathcal{J}'_i(W, \widehat{\kappa_* f}) &= \int_{[R_r \rtimes_i U_r]} \widehat{\kappa_* f}(u) A_1(\pi(u)W) du \\ &= \int_{[_i C_r]} \int_{[R_r \rtimes_{i-1} U_r]} \widehat{j_* f_1}(u) \widehat{\kappa_*^{i-1} f_2}(v) A_1(\pi(vu)W) dv du. \end{aligned}$$

By the induction hypothesis this is equal to

$$(25) \quad \int_{[_i C_r]} \widehat{j_* f_1}(u) \left(\int_{\tilde{R}_{r,i} \setminus R_r} \int_{\tilde{T}_{i-1}} A_i(\pi(xtu)W) f_2(t) \delta_{S'_{i-1}}^{-1}(t) dt dx \right) du = \int_{[_i C_r^\circ]} \Lambda(\pi(v)W) dv$$

where $\Lambda(W)$ is the following absolutely convergent triple integral:

$$(26) \quad \int_{[N_{\alpha_{n-i}}]} \widehat{j_* f_1}(u_1) \int_{\tilde{R}_{r,i} \setminus R_r} \int_{\tilde{T}_{i-1}} A_i(\pi(xtu_1)W) f_2(t) \delta_{S'_{i-1}}^{-1}(t) dt dx du_1.$$

4.5.

LEMMA 4.5. *We have*

$$\Lambda(W) = \int_{(D_{r,i} \tilde{R}_{r,i}) \setminus R_r} \left(\int_{\tilde{T}_i} \int_{D_{r,i}} \int_{_i \tilde{D}_r} A_{i+1}(\pi(\xi y x t)W) f(t) \delta_{S'_i}^{-1}(t) |t_{n-i+1}|^{1+i-r} d\xi dy dt \right) dx.$$

PROOF. Note that $N_{\alpha_{n-i}}$ and $R_{r,i}$ commute. Making a change of variable $x \mapsto (tu_1 t^{-1})x(tu_1^{-1} t^{-1})$ in (26) we get that $\Lambda(W)$ equals

$$\int_{\tilde{R}_{r,i} \setminus R_r} \int_{\tilde{T}_{i-1}} \int_{[N_{\alpha_{n-i}}]} \widehat{j_* f_1}(u_1) A_i(\pi(tu_1 t^{-1} x t)W) f_2(t) \delta_{S'_{i-1}}^{-1}(t) du_1 dt dx.$$

By (22) this is

$$\int_{\tilde{R}_{r,i} \setminus R_r} \int_{\tilde{T}_{i-1}} \int_{[N_{\alpha_{n-i}}]} \widehat{j_* f_1}(u_1) \left(\int_{\tilde{T}_i^0} \int_{\tilde{D}_i} A_{i+1}(\pi(\xi t' tu_1 t^{-1} x t)W) f_2(t) \delta_{S'_{i-1}}^{-1}(t) |t'_{n-i}|^{1-i} d\xi dt' \right) du_1 dt dx.$$

Note that ξ and $t' tu_1 (t't)^{-1}$ commute. Applying (23) we get

$$A_{i+1}(\pi(\xi t' tu_1 t^{-1} x t)W) = A_{i+1}(\pi(\xi t' x t)W) \psi(t' tu_1 (t't)^{-1}).$$

It is now clear that the quadruple integral over t', t, ξ, u_1 is absolutely convergent, i.e.

$$\int_{\tilde{T}_{i-1}} \int_{[N\alpha_{n-i}]} \int_{\tilde{T}_i^0} \int_{\tilde{D}_i} |A_{i+1}(\pi(\xi t' x t) W) \widehat{j_* f_1}(u_1) f_2(t) \delta_{S'_{i-1}}^{-1}(t) |t'_{n-i}|^{1-i}| d\xi dt' du_1 dt < \infty.$$

We can integrate over u_1 first. By Fourier inversion we get:

$$\Lambda(W) = \int_{\tilde{R}_{r,i} \setminus R_r} \left(\int_{\tilde{T}_{i-1}} \int_{\tilde{T}_i^0} \int_{\tilde{D}_i} A_{i+1}(\pi(\xi t' x t) W) f(tt') \delta_{S'_{i-1}}^{-1}(t) |t'_{n-i}|^{1-i} d\xi dt' dt \right) dx.$$

Next, we write $x \in \tilde{R}_{r,i} \setminus R_r$ as $x'y'$ with $y' \in [D_{r,i}]$ and $x' \in (D_{r,i} \tilde{R}_{r,i}) \setminus R_r$. It follows from Lemma 3.7 that

$$\Lambda(W) = \int_{(D_{r,i} \tilde{R}_{r,i}) \setminus R_r} \left(\int_{[D_{r,i}]} \int_{\tilde{T}_{i-1}} \int_{\tilde{T}_i^0} \int_{\tilde{D}_i} A_{i+1}(\pi(\xi t' y' x' t) W) f(tt') \delta_{S'_{i-1}}^{-1}(t) |t'_{n-i}|^{1-i} d\xi dt' dt dy' \right) dx'$$

with the inner integrals being absolutely convergent. Make a change of variable $y' \mapsto (t')^{-1} y' t'$ and notice that T_i^0 acts trivially by conjugation on $(D_{r,i} \tilde{R}_{r,i}) \setminus R_r$. We get

$$\Lambda(W) = \int_{(D_{r,i} \tilde{R}_{r,i}) \setminus R_r} \left(\int_{[D_{r,i}]} \int_{\tilde{T}_{i-1}} \int_{\tilde{T}_i^0} \int_{\tilde{D}_i} A_{i+1}(\pi(\xi y' x' t' t) W) f(tt') \delta_{S'_{i-1}}^{-1}(t) |t'_{n-i}|^{1-i+r-n} d\xi dt' dt dy' \right) dx'.$$

Finally, note that $\tilde{D}_i = \tilde{D}_{r,i} \times {}_i \tilde{D}_r$, and from (24),

$$\delta_{S'_{i-1}}^{-1}(t) |t'_{n-i}|^{r+1-i-n} |t_{n-i+1}|^{r-i-1} = \delta_{S'_i}^{-1}(tt').$$

Unwinding the integrals we obtain the Lemma. \square

REMARK 4.6. Since $f(t)$ is an arbitrary function in $C_c^\infty(\tilde{T}_i)$, from the Lemma we get the convergence of

$$\int_{(D_{r,i} \tilde{R}_{r,i}) \setminus R_r} \left(\int_{D_{r,i}} \int_{{}_i \tilde{D}_r} A_{i+1}(\pi(\xi y x) W) d\xi dy \right) dx.$$

In particular we get:

$$(27) \quad \Lambda(W) = \int_{\tilde{T}_i} \int_{(D_{r,i} \tilde{R}_{r,i}) \setminus R_r} \left(\int_{D_{r,i}} \int_{{}_i \tilde{D}_r} A_{i+1}(\pi(\xi y x t) W) f(t) \delta_{S'_i}^{-1}(t) |t_{n-i+1}|^{1+i-r} d\xi dy \right) dx dt.$$

4.6. From (25) and (27), we get:

$$(28) \quad \mathcal{J}'_i(W, \widehat{\kappa_* f}) = \int_{[i, C_r^{\circ}]} \left(\int_{\tilde{T}_i} \int_{(D_{r,i} \tilde{R}_{r,i}) \setminus R_r} \left(\int_{D_{r,i}} \int_{i \tilde{D}_r} A_{i+1}(\pi(\xi y x t v) W) f(t) \delta_{S'_i}^{-1}(t) |t_{n-i+1}|^{1+i-r} d\xi dy \right) dx dt \right) dv.$$

Note that ${}_i C_r^{\circ}$ and $D_{r,i} \tilde{R}_{r,i}$ commute. Making a change of variable $x \mapsto (tvt^{-1})x(tv^{-1}t^{-1})$, the above is:

$$\mathcal{J}'_i(W, \widehat{\kappa_* f}) = \int_{[i, C_r^{\circ}]} \left(\int_{\tilde{T}_i} \int_{(D_{r,i} \tilde{R}_{r,i}) \setminus R_r} \left(\int_{D_{r,i}} \int_{i \tilde{D}_r} A_{i+1}(\pi(\xi t v t^{-1} y x t) W) f(t) \delta_{S'_i}^{-1}(t) |t_{n-i+1}|^{1+i-r} d\xi dy \right) dx dt \right) dv.$$

From (23), we get

$$A_{i+1}(\pi(\xi t v t^{-1} y x t) W) = A_{i+1}(\pi(\xi y x t) W) \langle t v t^{-1}, \xi \rangle.$$

Clearly ξ commutes with y and acts trivially by conjugation on $(D_{r,i} \tilde{R}_{r,i}) \setminus R_r$. Thus the above is:

$$\mathcal{J}'_i(W, \widehat{\kappa_* f}) = \int_{[i, C_r^{\circ}]} \left(\int_{(D_{r,i} \tilde{R}_{r,i}) \setminus R_r} \int_{\tilde{T}_i} \left(\int_{D_{r,i}} \int_{i \tilde{D}_r} A_{i+1}(\pi(y x \xi t) W) f(t) \delta_{S'_i}^{-1}(t) |t_{n-i+1}|^{1+i-r} \langle t v t^{-1}, \xi \rangle d\xi dy \right) dt dx \right) dv.$$

The same argument as in Lemma 3.8, using Remark 4.6 in place of Lemma 3.7, shows that the triple integral over x , t and v above is absolutely convergent. Thus we may integrate over v first.

$$\mathcal{J}'_i(W, \widehat{\kappa_* f}) = \int_{(D_{r,i} \tilde{R}_{r,i}) \setminus R_r} \int_{\tilde{T}_i} \int_{[i, C_r^{\circ}]} \left(\int_{D_{r,i}} \int_{i \tilde{D}_r} A_{i+1}(\pi(y x \xi t) W) f(t) \delta_{S'_i}^{-1}(t) |t_{n-i+1}|^{1+i-r} \langle t v t^{-1}, \xi \rangle d\xi dy \right) dv dt dx.$$

Change $v \mapsto t^{-1}vt$:

$$\mathcal{J}'_i(W, \widehat{\kappa_* f}) = \int_{(D_{r,i} \tilde{R}_{r,i}) \setminus R_r} \int_{\tilde{T}_i} \int_{[i, C_r^{\circ}]} \left(\int_{D_{r,i}} \int_{i \tilde{D}_r} A_{i+1}(\pi(y x \xi t) W) f(t) \delta_{S'_i}^{-1}(t) \langle v, \xi \rangle d\xi dy \right) dv dt dx.$$

By Lemma 3.7, $A_{i+1}(\pi(y x \xi t) W)$ is a Schwartz function in ξ and y for fixed x and t . By Fourier inversion we get

$$\mathcal{J}'_i(W, \widehat{\kappa_* f}) = \int_{(D_{r,i} \tilde{R}_{r,i}) \setminus R_r} \int_{\tilde{T}_i} \left(\int_{D_{r,i}} A_{i+1}(\pi(y x t) W) f(t) \delta_{S'_i}^{-1}(t) dy \right) dt dx.$$

If we could unwind this expression we would get $\mathcal{J}'_i(W, \widehat{\kappa_* f}) = A'_{r,i}(\pi(f\delta_{S'_i}^{-1})W)$ and complete the induction step of the proof of Proposition 4.4.

4.7. We are left to show:

LEMMA 4.7. *Fix $i = 0, \dots, r-1$. Assume that the iterated integral*

$$\int_{(D_{r,i}\tilde{R}_{r,i})\backslash R_r} \left(\int_{D_{r,i}} A_{i+1}(\pi(yx)W) dy \right) dx$$

converges for all $W \in \mathcal{W}(\pi, \psi)$. Then the integral

$$\int_{\tilde{R}_{r,i+1}\backslash R_r} A_{i+1}(\pi(x)W) dx = \int_{(D_{r,i}\tilde{R}_{r,i})\backslash R_r} \int_{D_{r,i}} A_{i+1}(\pi(yx)W) dy dx$$

is absolutely convergent for all $W \in \mathcal{W}(\pi, \psi)$.

To prove the Lemma, we need another auxiliary result:

LEMMA 4.8. *Fix $j = 1, \dots, r-1$ and let R' be a subgroup of G which commutes with $D_{r,j}$ and centralizes C_j modulo ${}_jC_r^\circ$. Let R'' be a subgroup of R' and let $A(W)$ be a linear form on $\mathcal{W}(\pi, \psi)$ satisfying $A(\pi(rv)W) = \psi(v)A(W)$ for all $r \in R''$ and $v \in C_j$. Assume that the iterated integral*

$$\int_{R''\backslash R'} \left(\int_{D_{r,j}} A(\pi(yx)W) dy \right) dx$$

converges for all $W \in \mathcal{W}(\pi, \psi)$. Then the integral

$$\int_{R''\backslash R'} A(\pi(x)W) dx$$

is absolutely convergent for all $W \in \mathcal{W}(\pi, \psi)$.

PROOF. Let $W = \pi(\varphi)W'$ where $\varphi \in \mathcal{S}(C_{r,j})$. Then

$$\begin{aligned} \int_{R''\backslash R'} \left(\int_{D_{r,j}} A(\pi(yx)W) dy \right) dx \\ = \int_{R''\backslash R'} \left(\int_{D_{r,j}} \int_{C_{r,j}} A(\pi(yxv)W') \varphi(v) dv dy \right) dx. \end{aligned}$$

Since x centralizes C_j modulo ${}_j C_r^\circ$ and $D_{r,j}$ commutes with ${}_j C_r^\circ$ we get by a change of variable $v \mapsto x^{-1}vx$ that

$$\begin{aligned} & \int_{R'' \setminus R'} \left(\int_{D_{r,j}} \int_{C_{r,j}} A(\pi(yvx)W') \varphi(v) dv dy \right) dx \\ &= \int_{R'' \setminus R'} \left(\int_{D_{r,j}} \int_{C_{r,j}} A(\pi(yx)W') \langle v, y \rangle \varphi(v) dv dy \right) dx \\ &= \int_{R'' \setminus R'} \left(\int_{D_{r,j}} A(\pi(xy)W') \hat{\varphi}(y) dy \right) dx = \int_{R'' \setminus R'} A(\pi(x)\pi(\hat{\varphi})W') dx. \end{aligned}$$

The claim follows from the Dixmier-Malliavin theorem as φ is an arbitrary Schwartz function. \square

PROOF OF LEMMA 4.7. Fix i and r and assume that the conditions of the Lemma hold. Note that

$$(D_{r,i} \tilde{R}_{r,i}) \setminus R_r = [R_{r,i+1}] \times \prod_{j=1}^{i-1} D_{r,j}.$$

We can apply Lemma 4.8 iteratively for $j = 1, \dots, i-1$, where $R' = R_{r,i+1} \times \prod_{k=j+1}^{i-1} D_{r,k}$, $R'' = \tilde{R}_{r,i+1}$ and $A(W) = \int_{D_{r,i}} A_{i+1}(\pi(x)W) dx$. We conclude that the iterated integrals

$$\int_{[R_{r,i+1}]} \left(\int_{D_{r,i}} A_{i+1}(\pi(yx)W) dy \right) dx$$

converge. Apply Lemma 4.8 once again for $j = i$ with $R' = R_{r,i+1}$, $R'' = \tilde{R}_{r,i+1}$ and $A(W) = A_{i+1}(W)$. From (5), the conditions of the Lemma are satisfied. We get that the integral

$$\int_{[R_{r,i+1}]} A_{i+1}(\pi(x)W) dx$$

is absolutely convergent for all $W \in \mathcal{W}(\pi, \psi)$. Note that $\tilde{R}_{r,i+1} \setminus R_r = [R_{r,i+1}] \times \prod_{j=1}^i D_{r,j}$. From the above convergence, we can derive the claim in Lemma 4.7 using the Dixmier-Malliavin Theorem, (9) and the last claim of Lemma 2.3. \square

The proof of Proposition 4.4, and hence of Proposition 4.2, is now complete.

5. The functional equation

5.1. Finally, we prove Theorem 1.2.

We continue to work under the Assumptions 2.8 and 2.10.

Suppose first that $\pi \in \Pi_{\text{temp}}(G)$ is \tilde{G} -distinguished. Then by (4), Proposition 4.2 and Lemma 4.1 we have

$$B_r(W^\sharp) = A'_r(\pi(w_{n,r}^* w_n)W) = \mathcal{E}'_r(\pi(w_{n,r}^* w_n)W) = \mathcal{E}'_r(\pi(\hat{w}_r)W) = \mathcal{E}_{r-1}(W^\sharp).$$

Theorem 1.2 now follows from Propositions 3.5 in this case.

Now let $\pi \in \Pi_{\text{gen}}(G)$ be unitarizable and \tilde{G} -distinguished. Consider the inert case. Let τ be the Galois involution of E/F . By Assumption 2.10 we can write

$$\pi = \pi_1 \times \cdots \times \pi_k \times \sigma_1^\vee[-s_1] \times \cdots \times \sigma_l^\vee[-s_l] \times \sigma_1^\tau[s_1] \times \cdots \times \sigma_l^\tau[s_l]$$

where π_1, \dots, π_k are square-integrable and distinguished, $\sigma_1, \dots, \sigma_l$ are square-integrable and $-\frac{1}{2} < s_1, \dots, s_l < \frac{1}{2}$ (since π is unitarizable). Consider the Jacquet integral $J(\varphi, \underline{s})$, $\underline{s} \in \mathbb{C}^l$ with respect to the inducing representation. We then view the identity

$$A_r(J(\varphi, \underline{s})) = B_r(J(\varphi, \underline{s}))$$

as an identity of holomorphic functions in the strip $-\frac{1}{2} < \text{Re } \underline{s}_i < \frac{1}{2}$. Since it holds for $\underline{s}_i \in i\mathbb{R}$, it holds in the entire strip. Specializing to $\underline{s} = (s_1, \dots, s_l)$ we get $A_r(W) = B_r(W)$. A similar argument works in the split case.

5.2. We can extend Theorem 1.2 (still under Assumptions 2.8 and 2.10) as follows.

THEOREM 5.1. *Let $\pi \in \Pi_{\text{gen}}(G)$ be unitarizable and \tilde{G} -distinguished. Then for $W \in \mathcal{W}(\pi, \psi)$ and $1 \leq i \leq r \leq n-1$ we have*

$$(29) \quad \int_{R_{r,i}} \int_{\tilde{N}_r \backslash \tilde{H}_r} W(hy) |\det h|^{1-r} dh dy \\ = \int_{R_{r,r-i+1}^*} \int_{\tilde{N}_r^* \backslash \tilde{H}_r^*} W(hxw_{n,r}) |\det h|^{r-1} dh dx.$$

PROOF. It follows from Lemma 2.3 and its proof that both sides of the equation are absolutely convergent. Once again, we can reduce to the tempered case by analytic continuation.

The equation for $i = r$ is just Theorem 1.2. Now assume that the equation holds for $i = l+1$, and we prove it for $i = l$. By induction

hypothesis:

$$\begin{aligned}
& \int_{R_{r,l}} \int_{\tilde{N}_r \backslash \tilde{H}_r} W(hy) |\det h|^{1-r} dh dy \\
&= \int_{D_{r,l}} \int_{R_{r,l+1}} \int_{\tilde{N}_r \backslash \tilde{H}_r} W(huv) |\det h|^{1-r} dh du dv \\
&= \int_{D_{r,k}} \left(\int_{R_{r,r-l}^*} \int_{\tilde{N}_r^* \backslash \tilde{H}_r^*} W(hxw_{n,r}v) |\det h|^{r-1} dh dx \right) dv \\
&= \int_{D_{r,l}} \left(\int_{R_{r,r-l}^*} \int_{R_{r,r-l+1}^*} \int_{\tilde{N}_r^* \backslash \tilde{H}_r^*} W(hx\xi w_{n,r}v) |\det h|^{r-1} dh dx d\xi \right) dv.
\end{aligned}$$

As $W(hx\xi w_{n,r}v) = W(hx\xi w_{n,r}) \langle w_{n,r}^{-1} \xi w_{n,r}, v \rangle$, we get by Fourier inversion that the above is just

$$\int_{R_{r,r-l+1}^*} \int_{\tilde{N}_r^* \backslash \tilde{H}_r^*} W(hxw_{n,r}) |\det h|^{r-1} dh dx$$

as required. \square

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