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Abstract. We use explicit results on modular forms [8] via uniformization theory to obtain embeddings of modular curves in projective spaces.

1. Introduction

Let X be the upper half–plane. Then the group $SL_2(\mathbb{R})$ acts on X as follows:

$$g.z = \frac{az+b}{cz+d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).$$

We let $\mu(g,z) = cz+d$. Next, the $SL_2(\mathbb{R})$–invariant measure on X is defined by $dxdy/y^2$, where the coordinates on X are written in a usual way $z = x + \sqrt{-1}y$, $y > 0$. A discrete subgroup $\Gamma \subset SL_2(\mathbb{R})$ is called a Fuchsian group of the first kind if its fundamental domain $\mathcal{F}_\Gamma$ in X has a finite volume. Then, adding a finite number of points in $\mathbb{R} \cup \{\infty\}$ called cusps, $\mathcal{F}_\Gamma$ can be compactified. In this way we obtain a compact Riemann surface $\mathfrak{R}_\Gamma$.

Let $m \geq 3$ be an integer. We consider the space $S_m(\Gamma)$ of all modular forms of weight m which are cuspidal i.e., this is a space of all holomorphic functions $f : X \to \mathbb{C}$ such that $f(\gamma.z) = \mu(\gamma,z)^m f(z)$ $(z \in X,\ \gamma \in \Gamma)$ which are holomorphic and vanish at every cusp for Γ. The space $S_m(\Gamma)$ is a finite–dimensional Hilbert space under the Petersson inner product:

$$\langle f_1,\ f_2 \rangle = \int_{\Gamma\backslash X} y^m f_1(z)\overline{f_2(z)}\ \frac{dxdy}{y^2}.$$

Let $\xi \in X$ be a fixed point. Then we define the cuspidal modular forms $\Delta_{k,m,\xi}$ in the following way:

$$\langle f,\ \Delta_{k,m,\xi,\chi} \rangle = \frac{d^k f}{dz^k}\Big|_{z=\xi}, \quad f \in S_m(\Gamma), \quad k \geq 0.$$

In [8] we give the following explicit construction of the modular forms $\Delta_{k,m,\xi}$:

$$\Delta_{k,m,\xi}(z) = \epsilon_\Gamma^{-1} 2^{m-2} \pi^{-1} (\sqrt{-1})^m \prod_{i=0}^{k} (m-1+i) \sum_{\gamma \in \Gamma} \frac{1}{\left(\gamma.z - \overline{\xi}\right)^{k+m}} \mu(\gamma,z)^{-m},$$

where $\epsilon_\Gamma = \#(\{\pm 1\} \cap \Gamma)$.

As it is explained in ([8], Section 2), above construction is related to the work of Petersson and refines the construction of Petersson ([13], [14]). We give two rather different proofs of above expansion for $\Delta_{k,m,\xi}$. The one given ([8], Section 2) uses ideas of Selberg, and the one given in the context of representation theory is given in [9] using the methods developed

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in [6]. It is applied in [10] to obtain rather precise estimates on classical Poincaré series which together with a non–vanishing criterion [7] was applied to study non–vanishing of L–functions attached to modular forms. In this paper we give another application of modular forms constructed in [8].

By means of the uniformization theory, we may realize every smooth complex complete curve of genus $g \geq 2$ in the form $\mathfrak{R}_\Gamma$, where Γ has neither elliptic points nor cusps. Then, a well–known result about the Riemann surfaces says that one can use holomorphic differentials of degree $m \geq 3$ to construct the embeddings of $\mathfrak{R}_\Gamma$ into projective spaces. Instead of working with holomorphic differentials of degree m, we may work with the space of cuspidal modular forms $S_{2m}(\Gamma)$ (the isomorphism between the two can be extracted from pages 51, 52 in [4], and it is also well–known). In fact, the idea of using modular forms (via Poincaré series) to construct holomorphic maps on curves is very old one [3]. The advantage working with $S_m(\Gamma)$ is that we can allow Γ which might have elliptic points and cusps which is the case with usual congruence subgroups $\Gamma_0(N)$, and of course we do not have to limit ourselves to the case m is even. So, let Γ be any Fuchsian group of the first kind. We let $t_m = \dim S_m(\Gamma)$. The dimension t_m is explicitly computed in ([4], Theorems 2.5.2, 2.5.3) (see Lemma 2-2 (v) in this paper).

Let $m \geq 3$ such that $t_m \geq 2$. Then selecting a basis $f_0, \ldots, f_{t_m-1}$ of $S_m(\Gamma)$ one may construct a holomorphic map $\mathfrak{R}_\Gamma \longrightarrow \mathbb{P}^{t_m-1}$ given by $z \mapsto (f_0(z) : \cdots : f_{t_m-1}(z))$. In Section 3 in Theorem 3-3, we prove that for $t_m \geq g+1$, where g is the genus of $\mathfrak{R}_\Gamma$, this map is associated to the complete linear system $|\mathfrak{c}_f|$. Here, f is an arbitrary non–zero modular form in $S_m(\Gamma)$, and $\mathfrak{c}_f$ is an (integral) divisor $\mathfrak{c}_f$ of degree $t_m + g - 1$ (see Lemma 2-2 (vi)). Essentially, $\mathfrak{c}_f$ is the divisor of zeroes of f when we subtract necessary contribution of elliptic points and cusps common to all non–zero modular forms. The idea of considering such divisors goes back to Petersson [14]. In Section 3 we give some other information related to obtained projective curves: we compute their degrees, field of rational functions and show that the divisors $\mathfrak{c}_f$ are nothing else than hyperplane intersection divisors. Section 3 is a preliminary section of the paper and the results are based on the standard theory of algebraic curves [5], and as some results obtained in ([8], Section 4) which are refined and generalized in Section 2.

Section 4 is the central section of the paper. In this section we use above mentioned modular forms $\Delta_{k,m,\xi}$ to construct explicit embeddings into projective spaces

$$\mathfrak{R}_\Gamma \hookrightarrow \mathbb{P}^k$$

where k ranges between $g+1$ and $t_m - 1$. When $k = t_m - 1$ we obtain the explicit version of Theorem 3-3. In fact, since our modular forms depend on the parameter $\xi \in X$, we do this a little more generally. Our families of embeddings are C^∞ in ξ when ξ ranges over the complement of the set of all (t_m-1)–order Weierstrass points [14]. The distribution of such points was studied in [12], [11]. It resembles the distribution of zeroes of eigenforms for $\Gamma = SL_2(\mathbb{Z})$ [16], [2]. In Section 4, we determine (t_m-1)–order Weierstrass points in terms of our modular forms $\Delta_{k,m,\xi}$.

In Section 5 (see Proposition 5-1) we study the complete linear system attached to above embedding for $k = t_m - 1$ as a function on ξ on the complement of the set of all $(t_m - 1)$–order Weierstrass points. We use the methods analogous to that in the usual Weierstrass preparation theorem.

2. Preliminaries

In this section we recall and refine some results from [8]. We start by recalling some results from ([4], 2.3).

Throughout this paper, we write

$$t_m = \dim S_m(\Gamma). \tag{2-1}$$

Let $m \geq 2$ be an even integer and $f \in S_m(\Gamma) - \{0\}$. Then $\nu_{z-\xi}(f)$ the order of the holomorphic function f at ξ. For each $\gamma \in \Gamma$, the functional equation $f(\gamma.z) = \mu(\gamma, z)^m f(z)$ shows that $\nu_{z-\xi}(f) = \nu_{z-\xi'}(f)$, where $\xi' = \gamma.\xi$. Also, if we let

$$e_\xi = \#\left(\Gamma_\xi / \Gamma \cap \{\pm 1\}\right),$$

then $e_\xi = e_{\xi'}$. The point ξ is elliptic if $e_\xi > 1$. Next, following ([4], 2.3), we define

$$\nu_\xi(f) = \nu_{z-\xi}(f)/e_\xi.$$

Clearly, $\nu_\xi = \nu_{\xi'}$, and we may let

$$\nu_{\mathfrak{a}_\xi}(f) = \nu_\xi(f),$$

where

$$\mathfrak{a}_\xi \in \mathfrak{R}_\Gamma \text{ is a projection of } \xi \text{ to } \mathfrak{R}_\Gamma,$$

a notation we use throughout this paper.

If $x \in \mathbb{R} \cup \{\infty\}$ is a cups for Γ, then we define $\nu_x(f)$ as follows. Let $\sigma \in SL_2(\mathbb{R})$ such that $\sigma.x = \infty$. We write

$$\{\pm 1\}\sigma\Gamma_x\sigma^{-1} = \{\pm 1\}\left\{\begin{pmatrix} 1 & lh' \\ 0 & 1 \end{pmatrix};\ \ l \in \mathbb{Z}\right\},$$

where $h' > 0$. Then we write the Fourier expansion of f at x as follows:

$$(f|_m\sigma^{-1})(\sigma.z) = \sum_{n=1}^{\infty} a_n e^{2\pi\sqrt{-1}n\sigma.z/h'}.$$

(We remind the reader that m is even.) We let

$$\nu_x(f) = N \geq 0,$$

where N is defined by $a_0 = a_1 = \cdots = a_{N-1} = 0$, $a_N \neq 0$. One easily see that this definition does not depend on σ. Also, if $x' = \gamma.x$, then $\nu_x(f) = \nu_x(f)$. Hence, if $\mathfrak{b}_x \in \mathfrak{R}_\Gamma$ is a cusp corresponding to x, then we may define

$$\nu_{\mathfrak{b}_x} = \nu_x(f).$$

Put

$$\mathrm{div}(f) = \sum_{\mathfrak{a} \in \mathfrak{R}_\Gamma} \nu_{\mathfrak{a}}(f)\mathfrak{a} \in \quad \mathbb{Q} \otimes \mathrm{Div}(\mathfrak{R}_\Gamma),$$

where $\mathrm{Div}(\mathfrak{R}_\Gamma)$ is the group of (integral) divisors on $\mathfrak{R}_\Gamma$.

Using ([4], 2.3), this sum is finite i.e., $\nu_{\mathfrak{a}}(f) \neq 0$ for only a finitely many points. We let

$$\deg(\mathrm{div}(f)) = \sum_{\mathfrak{a} \in \mathfrak{R}_\Gamma} \nu_{\mathfrak{a}}(f).$$

Assume now that $-1 \notin \Gamma$ and $m \geq 1$ is odd. Let $f \in S_m(\Gamma)$, $f \neq 0$. Then $f^2 \in S_{2m}(\Gamma)$. We define $\nu_{\mathfrak{a}}(f) = \nu_{\mathfrak{a}}(f^2)/2$, and define $\mathrm{div}(f)$ and $\deg(\mathrm{div}(f))$ as before. (See [4], page 52.)

Let $\mathfrak{d}_i \in \mathbb{Q} \otimes \mathrm{Div}(\mathfrak{R}_\Gamma)$, $i = 1, 2$. Then we say that $\mathfrak{d}_1 \geq \mathfrak{d}_2$ if their difference $\mathfrak{d}_1 - \mathfrak{d}_2$ belongs to $\mathrm{Div}(\mathfrak{R}_\Gamma)$ and is non–negative in the usual sense.

Lemma 2-2. *Assume that $m \geq 1$ is an integer and $-1 \notin \Gamma$ if m is odd. Assume that $f \in S_m(\Gamma)$, $f \neq 0$. Let t be the number of inequivalent cusps and u (resp., v) the number of inequivalent regular (resp., irregular) cusps for Γ. Then we have the following:*

(i) *For $\mathfrak{a} \in \mathfrak{R}_\Gamma$, we have $\nu_{\mathfrak{a}}(f) \geq 0$.*
(ii) *Let $\mathfrak{a} \in \mathfrak{R}_\Gamma$ is a cusp. If m is even or $\mathfrak{a}$ is regular, then $\nu_{\mathfrak{a}}(f) \geq 1$ is an integer. If $\mathfrak{a}$ is not regular, then $\nu_{\mathfrak{a}}(f) \geq 1/2$ is in $1/2 + \mathbb{Z}$.*
(iii) *If $\mathfrak{a} \in \mathfrak{R}_\Gamma$ is not an elliptic point or a cusp, then $\nu_{\mathfrak{a}}(f) \geq 0$ is an integer. If $\mathfrak{a} \in \mathfrak{R}_\Gamma$ is an elliptic point, then $\nu_{\mathfrak{a}}(f) - \frac{m}{2}(1 - 1/e_{\mathfrak{a}})$ is an integer.*
(iv) *Let g be the genus of $\mathfrak{R}_\Gamma$. Then*

$$\mathit{deg}(\mathit{div}(f)) = m(g-1) + \frac{m}{2}\left(t + \sum_{\mathfrak{a} \in \mathfrak{R}_\Gamma, \ \mathit{elliptic}} (1 - 1/e_{\mathfrak{a}})\right).$$

(v) *Let $[x]$ denote the largest integer $\leq x$ for $x \in \mathbb{R}$. Then,*

$$t_m = \begin{cases} (m-1)(g-1) + (\frac{m}{2} - 1)t + \sum_{\substack{\mathfrak{a} \in \mathfrak{R}_\Gamma, \\ \mathit{elliptic}}} \left[\frac{m}{2}(1 - 1/e_{\mathfrak{a}})\right], & m > 2, \ \mathit{even} \\ g, \ m = 2, \end{cases}$$

and

$$\begin{aligned} t_m =& (m-1)(g-1) + (\frac{m}{2} - 1)u + (\frac{m}{2} - \frac{1}{2})v + \\ &+ \sum_{\substack{\mathfrak{a} \in \mathfrak{R}_\Gamma, \\ \mathit{elliptic}}} \left[\frac{m}{2}(1 - 1/e_{\mathfrak{a}})\right], \quad m \geq 3, \ \mathit{odd}. \end{aligned}$$

(vi) *There exists an integral divisor $\mathfrak{c}_f \geq 0$ of degree $t_m + g - 1$ such that*

$$\begin{aligned} \mathit{div}(f) =& \mathfrak{c}_f + \sum_{\mathfrak{a} \in \mathfrak{R}_\Gamma, \ \mathit{elliptic}} \left(\frac{m}{2}(1 - 1/e_{\mathfrak{a}}) - \left[\frac{m}{2}(1 - 1/e_{\mathfrak{a}})\right]\right) \mathfrak{a} + \\ &+ \begin{cases} \sum_{\substack{\mathfrak{b} \in \mathfrak{R}_\Gamma, \\ \mathit{cusp}}} \mathfrak{b} & m \geq 2 \ \mathit{is\ even} \\ \sum_{\substack{\mathfrak{b} \in \mathfrak{R}_\Gamma, \\ \mathit{regular\ cusp}}} \mathfrak{b} + \sum_{\substack{\mathfrak{b} \in \mathfrak{R}_\Gamma, \\ \mathit{irregular\ cusp}}} \frac{1}{2}\mathfrak{b}, & m \geq 1 \ \mathit{is\ odd} \end{cases} \end{aligned}$$

Proof. This is ([8], Lemma 4-1). □

Definition 2-3. *Let $\xi \in X$ or let ξ be a cusp for Γ. Let $m \geq 1$ be an integer such that $S_m(\Gamma) \neq 0$. We define (see Lemma 2-2 (iv))*

$$k_{\xi,m} \overset{def}{=} \sup_{\substack{f \in S_m(\Gamma)\\ f \neq 0}} \mathfrak{c}_f(\mathfrak{a}_\xi).$$

Let $\xi \in X$ be a non–elliptic point. Then $k_{\xi,m} = \sup_{\substack{f \in S_m(\Gamma)\\ f\neq 0}} \nu_{z-\xi}(f)$ which agrees with the definition given in ([8], Lemma 4-2). Going back to the general set–up of Definition 2-3, we observe that there exists a unique to a scalar $f_{\xi,m} \in S_m(\Gamma) - \{0\}$ such that $c_{f_{\xi,m}}(\mathfrak{a}_\xi) = k_{\xi,m}$. (See [8], Lemma 4-2 for the case ξ non–elliptic; the general case has similar proof.) By Lemma 2-2 (vi), we may write

$$\text{div}(f_{\xi,m}) = \mathfrak{d}_{\xi,m} + k_{\xi,m}\mathfrak{a}_\xi + \sum_{\mathfrak{a} \in \mathfrak{R}_\Gamma,\ \ elliptic} \left(\frac{m}{2}(1-1/e_{\mathfrak{a}}) - \left[\frac{m}{2}(1-1/e_{\mathfrak{a}})\right]\right)\mathfrak{a} +$$

$$(2\text{-}4) \qquad + \begin{cases} \sum_{\substack{\mathfrak{b} \in \mathfrak{R}_\Gamma,\\ cusp}} \mathfrak{b} & m \geq 2 \text{ is even} \\ \sum_{\substack{\mathfrak{b} \in \mathfrak{R}_\Gamma,\\ regular\ cusp}} \mathfrak{b} + \sum_{\substack{\mathfrak{b} \in \mathfrak{R}_\Gamma,\\ irregular\ cusp}} \frac{1}{2}\mathfrak{b}, & m \geq 1 \text{ is odd,} \end{cases}$$

where $\mathfrak{d}_{\xi,m}$ is a non–negative integral divisor of degree (see Lemma 2-2 (vi))

$$(2\text{-}5) \qquad \deg(\mathfrak{d}_{\xi,m}) = g - 1 + t_m - k_{\xi,m} \leq g, \quad \text{for } m \geq 3.$$

We may define the usual spaces of meromorphic functions

$$L(k\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}) = \{F \in \mathbb{C}(\mathfrak{R}_\Gamma);\ \ div(F) + k\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m} \geq 0\}, \quad k = 0, 1, 2, \ldots,$$

where $\mathbb{C}(\mathfrak{R}_\Gamma)$ is the field of rational functions on $\mathfrak{R}_\Gamma$.

We have the inclusions

$$L(0 \cdot \mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}) \subset L(1 \cdot \mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}) \subset \cdots;$$

the dimension of the spaces increase for not more than one. We collect some results that has the proofs similar to those given by ([8], Section 4).

Lemma 2-6. *Let $\xi \in X$ or let ξ be a cusp for Γ. Let $m \geq 3$ be an integer such that $S_m(\Gamma) \neq 0$. Then*

$$t_m - 1 \leq k_{\xi,m} \leq t_m + g - 1.$$

Proof. This is ([8], Lemma 4-4) when ξ is not elliptic. In other two cases the upper bound has the same proof. For the lower bound, we need to adjust the proof of ([8], Lemma 4-2(iv)). Instead of working in $S_m(\Gamma)$, we work in the space of meromorphic functions $\{f/f_{\xi,m};\ \ f \in S_m(\Gamma)\}$, where $f_{\xi,m}$ is fixed but the proof is essentially the same. In more details, we restrict to a neighborhood of $\mathfrak{a}_\xi$ and observe that the functions from that space of functions have poles from order 0 up to at most $k_{\xi,m}$, choosing a local chart we can switch to the vector space of functions holomorphic at $\mathfrak{a}_\xi$ having a zero of order at most $k_{\xi,m}$. Then, one easily adjust the proof ([8], Lemma 4-2(iv)). ☐

We include the following lemma which just the standard Riemann–Roch:

Lemma 2-7. *Let $\xi \in X$ or let ξ be a cusp for Γ. Let $m \geq 3$ be an integer such that $S_m(\Gamma) \neq 0$. Then, for $k \geq g - t_m + k_{\xi,m}$, we have the following*[1]*:*

$$\dim L(k\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}) = t_m + k - k_{\xi,m}.$$

Proof. The Riemann–Roch theorem says

$$\dim L(k\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}) = \deg(k\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}) + \dim L(K - k\mathfrak{a}_\xi - \mathfrak{d}_{\xi,m}) + 1 - g.$$

Here K is the canonical divisor. The claim is obvious if $g = 0$. Indeed, Lemma 2-6 implies that

$$g - t_m + k_{\xi,m} = 1.$$

Since, also $K = 0$, the Riemann–Roch theorem and (2-5) implies that

$$\dim L(k\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}) = t_m + k - k_{\xi,m}.$$

So, we assume that $g \geq 1$. Then, $K \neq 0$ and we have

$$\deg(K) = 2g - 2.$$

Thus, as usual, we see that if

$$\deg(K - k\mathfrak{a}_\xi - \mathfrak{d}_{\xi,m}) = 2g - 2 - k - \deg(\mathfrak{d}_{\xi,m}) < 0$$

i.e.,

$$k \geq 2g - 1 - \deg(\mathfrak{d}_{\xi,m}) = g - t_m + k_{\xi,m},$$

then $\dim L(K - k\mathfrak{a}_\xi - \mathfrak{d}_{\xi,m}) = 0$ and with the aid of (2-5) the lemma follows. □

Lemma 2-8. *Let $\xi \in X$ or let ξ be a cusp for Γ. Let $m \geq 3$ be an integer such that $S_m(\Gamma) \neq 0$. Then, for all $0 \leq i \leq k_{\xi,m}$, we have*

$$L((k_{\xi,m} - i)\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}) = \left\{f/f_{\xi,m};\quad f = 0 \text{ or } \mathfrak{c}_f(\mathfrak{a}_\xi) \geq i\right\}.$$

Moreover, $\dim L(\mathfrak{d}_{\xi,m}) = 1$ i.e., it consists of constants.

Proof. This is proved in the same way as ([8], Lemma 4-13). □

Lemma 2-9. *Let $\xi \in X$ or let ξ be a cusp for Γ. Let $m \geq 3$ be an integer such that $S_m(\Gamma) \neq 0$. Then, for all $0 \leq i \leq t_m - g - 1$, there exists $f \in S_m(\Gamma)$ such that $\mathfrak{c}_f(\mathfrak{a}_\xi) = i$. In particular, if $t_m \geq g + 1$, then there exists $f \in S_m(\Gamma)$ such that $\mathfrak{c}_f(\mathfrak{a}_\xi) = 0$.*

Proof. Indeed, by Lemma 2-7, we have

$$L((k_{\xi,m} - i)\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}) \neq L((k_{\xi,m} - i - 1)\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m})$$

if $k_{\xi,m} - i - 1 \geq g - t_m + k_{\xi,m}$. Now, we apply Lemma 2-8. □

The following proposition is the main result of the present section. We use notion of linear systems of divisors. We refer to ([5], page 147) for this notion.

[1]Lemma 2-2 (v) and the assumption $S_m(\Gamma) \neq 0$ imply that $t_m \geq g$. Hence, we can take $k = k_{\xi,m}$.

Proposition 2-10. *Let $m \geq 3$ be an integer such that $S_m(\Gamma) \neq 0$. Then, for any non–zero $f \in S_m(\Gamma)$, we have the following:*

$$L(\mathfrak{c}_f) = \{g/f; \quad g \in S_m(\Gamma)\}.$$

Furthermore, if $t_m \geq g+1$ the complete linear system $|\mathfrak{c}_f|$ has no base points.

Proof. By definition, we have the following:

$$L(\mathfrak{c}_f) = \{F \in \mathbb{C}(\mathfrak{R}_\Gamma); \quad \mathrm{div}(F) + \mathfrak{c}_f \geq 0.\}$$

Next, if $g \in S(\Gamma)$, $g \neq 0$, then by Lemma 2-2 (vi) we obtain

$$\text{(2-11)} \qquad \mathrm{div}\left(\frac{g}{f}\right) + \mathfrak{c}_f = \mathrm{div}(g) - \mathrm{div}(f) + \mathfrak{c}_f = \mathfrak{c}_g - \mathfrak{c}_f + \mathfrak{c}_f = \mathfrak{c}_g \geq 0.$$

This shows that $g/f \in L(\mathfrak{c}_f)$.

Let $\xi \in X$ or let ξ be a cusp for Γ. By Lemma 2-8, we have the following:

$$\dim L(k_{\xi,m}\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}) = t_m.$$

Again, by Lemma 2-2 (vi), we have

$$\mathrm{div}(f_{\xi,m}/f) = k_{\xi,m}\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m} - \mathfrak{c}_f.$$

Thus, the divisors $\mathfrak{c}_f$ and $k_{\xi,m}\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}$ are linearly equivalent. This implies that

$$\dim L(\mathfrak{c}_f) = \dim L(k_{\xi,m}\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}) = t_m.$$

This proves the first claim of the proposition.

The assumption $t_m \geq g+1$ implies that for every $\zeta \in X$ or a cusp, there exists $g \in S_m(\Gamma)$ such that $\mathfrak{c}_g(\mathfrak{a}_\zeta) = 0$ (see Lemma 2-9). This means that $\mathfrak{a}_\zeta \notin \mathrm{supp}(\mathfrak{c}_g)$ for some $g \in S_m(\Gamma) - \{0\}$. Then, by (2-11), we have that $g/f \notin L(\mathfrak{c}_f - \mathfrak{a}_\zeta)$. This proves the last claim of the proposition. □

3. Maps to Projective Spaces

Assume that $m \geq 3$ and $t_m \geq 2$. We have a holomorphic map

$$\mathfrak{R}_\Gamma \longrightarrow \mathbb{P}^{t_m - 1}$$

defined by

$$\text{(3-1)} \qquad \mathfrak{a}_z \mapsto (f_0(z) : \cdots : f_{t_m-1}(z))$$

where $f_0, \ldots, f_{t_m-1}$ is a basis of $S_m(\Gamma)$. For any non–zero $f \in S_m(\Gamma)$, this map can be written as follows:

$$\text{(3-2)} \qquad \mathfrak{a}_z \mapsto (f_0(z)/f(z) : \cdots : f_{t_m-1}(z)/f(z)).$$

As with any holomorphic map, this one has attached divisor $\mathfrak{d}$. The divisor $\mathfrak{d}$ is defined by

$$-\mathfrak{d} = \sum_{\mathfrak{a} \in \mathfrak{R}_\Gamma} \min\left(\mathrm{div}\left(\frac{f_i}{f}\right)(\mathfrak{a}),\ 0 \leq i \leq t_m - 1\right)\mathfrak{a}.$$

When $t_m \geq g+1$, it is very easy to compute $\mathfrak{d}$. By Lemma 2-2 (vi), we have the following:

$$\begin{aligned}
-\mathfrak{d} &= \sum_{\mathfrak{a}\in\mathfrak{R}_\Gamma} \min\left(\operatorname{div}\left(\frac{f_i}{f}\right)(\mathfrak{a}),\ 0\leq i\leq t_m-1\right)\mathfrak{a} \\
&= \sum_{\mathfrak{a}\in\mathfrak{R}_\Gamma} \min\left(\mathfrak{c}_{f_i}(\mathfrak{a})-\mathfrak{c}_f(\mathfrak{a}),\ 0\leq i\leq t_m-1\right)\mathfrak{a} \\
&= -\mathfrak{c}_f + \sum_{\mathfrak{a}\in\mathfrak{R}_\Gamma} \min\left(\mathfrak{c}_{f_i}(\mathfrak{a}),\ 0\leq i\leq t_m-1\right)\mathfrak{a} \\
&= -\mathfrak{c}_f,
\end{aligned}$$

where in the last equality we use Lemma 2-9 (and the assumption $t_m \geq g+1$).

Now, we prove the main result of this section. By Lemma 2-2 (vi), for non–zero $f, g \in S_m(\Gamma)$, we have that

$$\operatorname{div}\left(\frac{f}{g}\right) = \operatorname{div}(f) - \operatorname{div}(g) = \mathfrak{c}_f - \mathfrak{c}_g.$$

Thus, the divisors $\mathfrak{c}_f$ and $\mathfrak{c}_g$ are equivalent. Hence, the corresponding complete linear systems are equal

$$|\mathfrak{c}_f| = |\mathfrak{c}_g|.$$

Theorem 3-3. *Let $m \geq 3$ such that $t_m \geq g+1, 2$. Then, the holomorphic map (3-1) is attached to the (base point free) complete linear system $|\mathfrak{c}_f|$ where $f \in S_m(\Gamma)$ is any non–zero modular form. Moreover, if $t_m \geq g+2$, then the divisor $\mathfrak{c}_f$ is very ample; in particular, the holomorphic map (3-1) is an embedding.*

Proof. By above computation, the divisor attached to the map (3-2) is $\mathfrak{c}_f$. The linear system of the holomorphic map (3-2) is by definition equal to

$$\left\{\operatorname{div}\left(\sum_{i=0}^{t_m-1} a_i f_i/f\right) + \mathfrak{c}_f;\quad a_0,\ldots,a_{t_m-1}\in\mathbb{C}\right\}.$$

By Proposition 2-10, it is equal to the complete linear system $|\mathfrak{c}_f|$. This proves the first claim.

By definition $|\mathfrak{c}_f|$ is very ample if it is base–point–free and it defines an embedding. By Proposition 2-10, the complete linear system $|\mathfrak{c}_f|$ is base–point–free for $t_m \geq g+1$. By Lemma 2-2 (vi), we obtain we obtain

$$\deg(\mathfrak{c}_f) = g-1+t_m \geq (g-1)+(g+2) = 2g+1.$$

By the standard theory of algebraic curves, any divisor of degree $\geq 2g+1$ is very ample. In fact, we did not have to check that $|\mathfrak{c}_f|$ is base–point free in advance since this holds for any any divisor of degree $\geq 2g+1$. □

Corollary 3-4. *Assume that $m \geq 3$ is given such that $t_m \geq g+2$. Then, if $f_0, \ldots, f_{t_m-1}$ is a basis of $S_m(\Gamma)$, then we denote by*

$$\mathcal{C} = \mathcal{C}(f_0, \ldots, f_{t_m-1}). \tag{3-5}$$

the image of the map (3-1). Then, $\mathcal{C}$ is a irreducible smooth projective curve in $\mathbb{P}^{t_m-1}$ which has a degree $t_m + g - 1$.

Proof. The first claim follows by Chow's theorem. By the standard theory, the degree is given by

$$\deg(\mathcal{C}) = \deg(\mathfrak{c}_f) = t_m + g - 1,$$

by Lemma 2-2 (vi). □

Corollary 3-6. *Assume that $m \geq 3$ is given such that $t_m \geq g+2$. Then, for the curve $\mathcal{C}$ given by (3-5), the hyperplane intersection divisor for $x_0 = 0$ is $\mathfrak{c}_{f_0}$.*

Proof. Indeed, the corresponding map we can write has follows:

$$\mathfrak{a}_z \longmapsto (f_0(z) : \cdots : f_{t_m-1}(z)) = (1 : f_1(z)/f_0(z) \cdots : f_{t_m-1}(z)/f_0(z)).$$

Next, by Lemma 2-2 (vi), we note that

$$\mathrm{div}(f_i/f_0) = \mathrm{div}(f_i) - \mathrm{div}(f_0) = \mathfrak{c}_{f_i} - \mathfrak{c}_{f_0},$$

where $\mathfrak{c}_{f_i}, \mathfrak{c}_{f_0} \geq 0$. Finally, Lemma 2-9, shows that if $\mathfrak{c}_{f_0}(\mathfrak{a}_\zeta) > 0$ for some $\mathfrak{a}_\zeta \in \mathfrak{R}_\Gamma$, then there exists $i \geq 1$ such that $\mathfrak{c}_{f_i}(\mathfrak{a}_\zeta) = 0$. Now, the proof is immediate. □

Since f_0 could be any non–zero modular form in $S_m(\Gamma)$, we see that the divisors $\mathfrak{c}_f$ are just the hyperplane intersection divisors. Hence, their geometric interpretation.

Corollary 3-7. *Assume that $m \geq 3$ is given such that $t_m \geq g+2$. Then, the field of rational functions $\mathbb{C}(\mathfrak{R}_\Gamma)$ is generated over $\mathbb{C}$ by the rational functions f_i/f_0, $1 \leq i \leq t_m - 1$.*

Proof. We denote the homogeneous coordinates on $\mathbb{P}^{t_m-1}$ by $(x_0 : \cdots : x_{t_m-1})$. We remark that $\mathcal{C}$ does not lie in any hyperplane, and, in particular, not in $x_0 = 0$. Thus, the field of rational functions on $\mathcal{C}$ is generated by x_i/x_0, $1 \leq i \leq t_m - 1$. Now, apply the map (3-1). □

4. Wronskians and explicit version of Theorem 3-3

Using, some further results from [8], we can make Theorem 3-3 more explicit. This is the goal of the present section. We start with the following lemma:

Lemma 4-1. *Let $m \geq 1$. Then, for any sequence $f_1, \ldots, f_k \in S_m(\Gamma)$, the Wronskian*

$$W(f_1, \ldots, f_k)(z) \overset{def}{=} \begin{vmatrix} f_1(z) & \cdots & f_k(z) \\ \frac{df_1(z)}{dz} & \cdots & \frac{df_k(z)}{dz} \\ & \cdots & \\ \frac{d^{k-1}f_1(z)}{dz^{k-1}} & \cdots & \frac{d^{k-1}f_k(z)}{dz^{k-1}} \end{vmatrix}$$

is a cuspidal modular form in $S_{k(m+k-1)}(\Gamma)$.

Proof. This is a standard fact (see [15], ([1], page 162), ([8], Theorem 3-8 (iii)). We leave details to the reader. □

We recall the following construction of elements of $S_m(\Gamma)$ (see [8], Proposition 2.1). We assume that $m \geq 3$. Put $\epsilon_\Gamma = \#(\{\pm 1\} \cap \Gamma)$. Let $\xi \in X$. Then, $k \geq 0$, the series

$$\Delta_{k,m,\xi}(z) \overset{def}{=} \frac{(m-1)m\cdots(m+k-1)(2\sqrt{-1})^m}{4\epsilon_\Gamma \pi} \sum_{\gamma\in\Gamma} \left(\gamma.z - \overline{\xi}\right)^{-k-m} \mu(\gamma, z)^{-m}, \tag{4-2}$$

converges absolutely and uniformly on compact sets to an element of $S_m(\Gamma)$ which satisfies

$$\langle f,\ \Delta_{k,m,\xi}\rangle = \frac{d^k f(z)}{dz^k}\Big|_{z=\xi},\quad f \in S_m(\Gamma),\ k \geq 0, \tag{4-3}$$

where the Petersson inner product on $S_m(\Gamma)$ is defined by

$$\langle f_1,\ f_2\rangle = \int_{\Gamma\backslash X} y^m f_1(z)\overline{f_2(z)}\ \frac{dxdy}{y^2}.$$

We put

$$W_k(z,\xi) \overset{def}{=} \begin{vmatrix} \Delta_{0,m,\xi}(z) & \cdots & \Delta_{k,m,\xi}(z) \\ \frac{d\Delta_{0,m,\xi}(z)}{dz} & \cdots & \frac{d\Delta_{k,m,\xi}(z)}{dz} \\ & \cdots & \\ \frac{d^k\Delta_{0,m,\xi}(z)}{dz^k} & \cdots & \frac{d^k\Delta_{k,m,\xi}(z)}{dz^k} \end{vmatrix}. \tag{4-4}$$

The next lemma collects some properties of those functions.

Lemma 4-5. *Let $m \geq 3$ and $k \geq 0$. Let $\xi \in X$. Then, we have the following:*

(i) $W_k(\cdot, \xi) \in S_{(k+1)(m+k)}(\Gamma)$.
(ii) *The function $W_k(\xi) \overset{def}{=} W_k(\xi, \xi)$ is real–analytic and non–negative function on X.*
(iii) *We have $W_k(\gamma.\xi) = |\mu(\gamma, \xi)|^{2(k+1)m+2k(k+1)}\, W_k(\xi)$, for all $\gamma \in \Gamma$ and $\xi \in X$.*
(iv) *The sequence $\Delta_{0,m,\xi}, \ldots, \Delta_{k,m,\xi}$ is linearly independent if and only if $W_k(\xi) = 0$.*
(v) *If $k \geq t_m$, then W_k is identically zero.*
(vi) *The set of zeroes of W_k is Γ–invariant. If W_k is not identically zero, then the zeroes of W_k belongs to finitely many Γ–orbits.*
(vii) *For any non–elliptic point ξ which satisfies $W_{t_m-1}(\xi) \neq 0$, we have $k_{\xi,m} = t_m - 1$.*
(viii) *Let ξ be an elliptic point. Then $W_k(\xi) = 0$ for all $k \geq 0$ if the order of Γ_ξ does not divide m.*
(ix) *Let ξ be an elliptic point. Then $W_k(\xi) = 0$ for all $k \geq 1$.*

Proof. (i) follows from Lemma 4-1. Next, using (4-3), we can write (4-4) in the following form:

$$W_k(z,\xi) = \begin{vmatrix} \langle \Delta_{0,m,\xi}, \Delta_{0,m,z}\rangle & \cdots & \langle \Delta_{k,m,\xi}, \Delta_{0,m,z}\rangle \\ \langle \Delta_{0,m,\xi}, \Delta_{1,m,z}\rangle & \cdots & \langle \Delta_{k,m,\xi}, \Delta_{1,m,z}\rangle \\ & \cdots & \\ \langle \Delta_{0,m,\xi}, \Delta_{k,m,z}\rangle & \cdots & \langle \Delta_{k,m,\xi}, \Delta_{k,m,z}\rangle \end{vmatrix}. \tag{4-6}$$

This shows that the Gramm determinant of the sequence of modular forms $\Delta_{0,m,\xi}, \ldots \Delta_{k,m,\xi}$ is $W_k(\xi)$. This proves (iv). Also, one of the basic property of the Gramm determinant implies $W_k(\xi) \geq 0$ for all $\xi \in X$. This proves that the function in (ii) is non–negative. It is real analytic by ([8], Proposition 2-7 (i)). This completes the proof of (iii). Next, (iv) is a consequence of Lemma 4-1 and the following obvious consequence of (4-6) $W_k(z,\xi) = \overline{W_k(\xi, z)}$. (v) follows directly from (iv). The first part of (vi) follows from (iii). The last part of (vi) follows from the following observation. By (iv), the claim is sufficient to establish for $k = t_m - 1$. But, by the proof of ([8], Theorem 3-5 (iii)), those orbits are determined as the zeroes of the cuspidal modular form given by the Wronskian $W(f_1, \ldots, f_{t_m})$, attached to a basis $f_1, \ldots, f_{t_m}$ of $S_m(\Gamma)$. Finally, (vii) follows from (vi) and ([8], Theorem 3-5 (iii)). We prove (viii). We observe that $\gamma \longrightarrow \mu(\gamma,\xi)$ is a character $\Gamma_\xi \longrightarrow \mathbb{C}^\times$ by the cocycle relation

$$\mu(\gamma_1\gamma_2, \xi) = \mu(\gamma_1, \gamma_2.\xi)\mu(\gamma_2,\xi) = \mu(\gamma_1,\xi)\mu(\gamma_2,\xi).$$

Its kernel is trivial. Indeed, if

$$\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma_\xi$$

satisfies

$$c_\gamma \xi + d_\gamma = \mu(\gamma, \xi) = 1,$$

then taking the imaginary part we find $c_\gamma = 0$ and $d_\gamma = 1$. Also

$$a_\gamma \xi + b_\gamma = \frac{a_\gamma \xi + b_\gamma}{c_\gamma \xi + d_\gamma} = \gamma.\xi = \xi,$$

implies $a_\gamma = 1$ and $b_\gamma = 0$. Hence, the group Γ_ξ can be considered as a subgroup of $\mathbb{C}^\times$. In particular, it is cyclic. So, if the order of Γ_ξ does not divide m, we can find $\gamma \in \Gamma_\xi$ such that $\mu(\gamma,\xi)^m \neq 1$. Hence, for $f \in S_m(\Gamma)$, $f(\xi) = f(\gamma.\xi) = \mu(\gamma,\xi)^m f(\xi)$ implies $f(\xi) = 0$. In particular,

$$\langle \Delta_{0,m,\xi}, \Delta_{0,m,\xi}\rangle = \Delta_{0,m,\xi}(\xi) = 0.$$

Hence, $\Delta_{0,m,\xi} = 0$. Since $W_k(\xi)$ can be computed using (4-6) with $z = \xi$, the first row there consists of 0's in our case. Thus, we have proved that if m is not divisible by the order of Γ_ξ, then $W_k(\xi) = 0$ for all $k \geq 0$. This proves (viii). Finally, we prove (ix). By (viii) we may assume that the order of Γ_ξ divides m. Then, since the order of Γ_ξ is greater than 2 (this fact is elementary [2]), then $W_1(\xi) = 0$ by (i). Now, (iv) implies that $\Delta_{0,m,\xi}$ and $\Delta_{1,m,\xi}$ are linearly dependent. Thus, the sequence $\Delta_{0,m,\xi}, \ldots, \Delta_{k,m,\xi}$ is linearly dependent for $k \geq 1$. Hence, $W_k(\xi) = 0$ by (iv). □

[2]If not, then by the definition of an elliptic point (see the begining of Section 2), we find that the order of Γ_ξ is equal to 2. But in $SL_2(\mathbb{R})$ elements of order ≤ 2 are ± 1. Then, we would get $e_\xi = 1$ which is a contradiction with the definition of an elliptic point.

Let $\xi \in X$. Then, we define for all $k \geq 0$

$$\Xi_{k,m,\xi}(z) = \begin{vmatrix} \langle \Delta_{0,m,\xi}, \Delta_{0,m,\xi} \rangle & \cdots & \langle \Delta_{0,m,\xi}, \Delta_{k-1,m,\xi} \rangle & \Delta_{0,m,\xi} \\ \langle \Delta_{1,m,\xi}, \Delta_{0,m,\xi} \rangle & \cdots & \langle \Delta_{1,m,\xi}, \Delta_{k-1,m,\xi} \rangle & \Delta_{1,m,\xi} \\ & \cdots & \cdots & \cdots \\ \langle \Delta_{k,m,\xi}, \Delta_{0,m,\xi} \rangle & \cdots & \langle \Delta_{k,m,\xi}, \Delta_{k-1,m,\xi} \rangle & \Delta_{k,m,\xi} \end{vmatrix}$$

(4-7)

$$= \begin{vmatrix} \Delta_{0,m,\xi}(\xi) & \cdots & \frac{d^{k-1}\Delta_{0,m,\xi}}{dz^{k-1}}\big|_{z=\xi} & \Delta_{0,m,\xi} \\ \Delta_{1,m,\xi}(\xi) & \cdots & \frac{d^{k-1}\Delta_{1,m,\xi}}{dz^{k-1}}\big|_{z=\xi} & \Delta_{1,m,\xi} \\ & \cdots & \cdots & \cdots \\ \Delta_{k,m,\xi}(\xi) & \cdots & \frac{d^{k-1}\Delta_{k,m,\xi}}{dz^{k-1}}\big|_{z=\xi} & \Delta_{k,m,\xi} \end{vmatrix}.$$

We develop $\Xi_{k,m,\xi}$ into power series centered at ξ:

$$\Xi_{k,m,\xi}(z) = \sum_{j=k}^{\infty} W_{k,j}(\xi)\,(z-\xi)^j\,, \tag{4-8}$$

where we let

$$W_{k,j}(\xi) = \begin{vmatrix} \Delta_{0,m,\xi}(\xi) & \cdots & \frac{d^{k-1}\Delta_{0,m,\xi}}{dz^{k-1}}\big|_{z=\xi} & \frac{d^{j}\Delta_{0,m,\xi}}{dz^{j}}\big|_{z=\xi} \\ \Delta_{1,m,\xi}(\xi) & \cdots & \frac{d^{k-1}\Delta_{1,m,\xi}}{dz^{k-1}}\big|_{z=\xi} & \frac{d^{j}\Delta_{1,m,\xi}}{dz^{j}}\big|_{z=\xi} \\ & \cdots & \cdots & \cdots \\ \Delta_{k,m,\xi}(\xi) & \cdots & \frac{d^{k-1}\Delta_{k,m,\xi}}{dz^{k-1}}\big|_{z=\xi} & \frac{d^{j}\Delta_{k,m,\xi}}{dz^{j}}\big|_{z=\xi} \end{vmatrix}.$$

Note that

$$W_{k,k}(\xi) = W_k(\xi).$$

Thus, $\nu_{z-\xi}(\Xi_{k,m,\xi}) = k$ if and only if $W_k(\xi) \neq 0$.

Lemma 4-9. *Let $m \geq 3$ such that $S_m(\Gamma) \neq 0$. Assume that $\xi \in X$ satisfies $W_{t_m-1}(\xi) \neq 0$. Then, we have the following:*

(i) *The modular forms $\Xi_{0,m,\xi}, \Xi_{1,m,\xi}, \ldots, \Xi_{t_m-1,m,\xi}$ form the basis of $S_m(\Gamma)$.*
(ii) *We have $\nu_{z-\xi}(\Xi_{k,m,\xi}) = k$ for $0 \leq k \leq t_m - 1$.*
(iii) *We may take $f_{\xi,m} = \Xi_{t_m-1,m,\xi}$, up to a scalar $\neq 0$. In particular, $k_{\xi,m} = t_m - 1$.*

Proof. (i) is obvious from (4-7) and (4-8) combined with Lemma 4-5 (iv). (ii) follows from the comments before the statement of the lemma combined with the fact that $W_{t_m-1}(\xi) \neq 0$ implies $W_k(\xi) \neq 0$ for all $0 \leq k \leq t_m - 1$. (See Lemma 4-5 (iv).) Finally, (iii) follows from Lemma 4-5 (vii). □

Now, we prove the main result of the present section. The present theorem is a slight generalization of Theorem 3-3 since we are dealing with specific linear systems.

Theorem 4-10. *Let $m \geq 3$ such that $t_m \geq g+2$. Then, for $1 \leq k \leq t_m - 1$, we define a continuous map $\{W_{t_m-1}(\xi) \neq 0\} \times \mathfrak{R}_\Gamma \longrightarrow \mathbb{P}^k$ by $(\xi,\ \mathfrak{a}_z) \mapsto (\Xi_{t_m-1-k,m,\xi}(z) : \cdots : \Xi_{t_m-1,m,\xi}(z))$ which is real analytic in ξ and holomorphic in z. Moreover, for each ξ it sends $\mathfrak{a}_\xi$ to $(1 : 0 : \cdots : 0)$. Finally, if $k \geq g+1$ (which is satisfied for $k = t_m - 1$), then the map* *is attached to base point free complete linear system $|k\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}|$; $k\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}$ is very ample and it defines an embedding of $\mathfrak{R}_\Gamma$ into $\mathbb{P}^k$. The field of rational functions on $\mathfrak{R}_\Gamma$ is generated over $\mathbb{C}$ by*

$$\frac{\Xi_{t_m-g-2,m,\xi}}{\Xi_{t_m-1,m,\xi}}, \dots, \frac{\Xi_{t_m-2,m,\xi}}{\Xi_{t_m-1,m,\xi}}.$$

Proof. The map is well–defined by Lemma 4-9 (i). The fact that the map is continuous, real analytic in ξ, and holomorphic in z follows easily from ([8], Proposition 2-7) and its definition. It maps $\mathfrak{a}_\xi$ to $(1 : 0 : \cdots : 0)$ by Lemma 4-9 (ii).

Next, the assumption $t_m \geq g + 2$ implies that $t_m - 1 \geq 1$. Hence, Lemma 4-5 (ix) imply that ξ is not elliptic when $W_{t_m-1}(\xi) \neq 0$. Then, Lemma 4-9 (iii) implies that $k_\xi = t_m - 1$. Hence, (2-5) implies that the degree of $k\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}$ is $k + g$. Assuming that $k \geq g + 1$ we obtain the divisor $k\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}$ of degree $k + g \geq 2g + 1$. By the standard theory, this implies that divisor $k\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}$ very ample. In particular, $|k\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}|$ is base–point–free.

By Lemma 4-9 (ii), (iii), Lemma 2-7, and Lemma 2-8, the basis of $L(k\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m})$ is given by

$$\frac{\Xi_{t_m-1-k,m,\xi}}{\Xi_{t_m-1,m,\xi}}, \dots, \frac{\Xi_{t_m-1,m,\xi}}{\Xi_{t_m-1,m,\xi}}.$$

Thus, the map attached to the base point free complete linear system $|k\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}|$ is exactly the map $\mathfrak{R}_\Gamma \longrightarrow \mathbb{P}^k$ by $\mathfrak{a}_z \mapsto (\Xi_{t_m-1-k,m,\xi}(z) : \cdots : \Xi_{t_m-1,m,\xi}(z))$ which is embedding since the divisor $k\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}$ very ample.

The very last claim has the same proof as Corollary 3-7. ☐

We remark that for $k = g + 1$, we constructed a family of explicit embeddings of $\mathfrak{R}_\Gamma$ into $\mathbb{P}^{g+1}$. They are quite different than usual embeddings into $\mathbb{P}^{g+1}$ constructed with the aid of the complete linear system $(2g + 1)\mathfrak{a}$, where $\mathfrak{a} \in \mathfrak{R}_\Gamma$ is any point. For m large enough, we have that $t_m \geq 2g + 2$. Then, among the maps in Theorem 4-10 is the map attached to $|(2g+1)\mathfrak{a}_\xi + \mathfrak{d}_{\xi,m}|$. It defines an embedding into $\mathbb{P}^{2g+1}$. Choosing appropriately the $(g+1)$–dimensional linear subspace E of $\mathbb{P}^{2g+1}$, we may project the curve to get the curve in E which is in fact the one that comes from $|(2g + 1)\mathfrak{a}_\xi|$. One may write down the equations for this in terms of the divisor $\mathfrak{d}_{\xi,m}$.

5. On the divisor $\mathfrak{d}_{\xi,m}$

In this section we consider the case when Γ has no cusps and elliptic points. This forces $g \geq 2$. By Lemma 2-2 (v), $t_m = \dim S_m(\Gamma) = (m - 1)(g - 1) \neq 0$ for $m \geq 3$. Then, we may observe that Lemma 4-9 (iii) shows that $f_{\xi,m}$ can be taken to depend continuously on ξ on the set defined by $W_{t_m-1}(\xi) \neq 0$. More precisely, we may let $f_{\xi,m} = \Xi_{t_m-1,m,\xi}$. Hence, we may view $(\xi, z) \mapsto f_{\xi,m}(z)$ as a continuous function which is real analytic in ξ and holomorphic in z. This will help us understand the divisor $\mathfrak{d}_{\xi,m}$. So far, we only know that it does not contain $\mathfrak{a}_\xi$ in its support, it is non–negative, integral, and it has a degree g (see (2-5)).

Proposition 5-1. *Let* $W_{t_m-1}(\xi_0) \neq 0$. *Then, there exists* $\delta > 0$ *and* C^∞*–functions* $b_0, \ldots, b_{g-1}$ *on the set* $W_{t_m-1}(\xi) \neq 0$, $|\xi - \xi_0| < \delta$, *such that if* $r_1(\xi), \ldots, r_g(\xi)$ *are all zeroes of the polynomial*

$$z^g + b_{g-1}(\xi) z^{g-1} + \cdots + b_0(\xi) = 0,$$

then

$$\mathfrak{d}_{\xi,m} = \sum_{i=1}^{g} \mathfrak{a}_{r_i(\xi)}.$$

Proof. We imitate the proof of the Weierstrass preparation theorem. Let $\zeta_1, \ldots, \zeta_l \in X$ be fixed representatives of points from the support of $\mathfrak{d}_{\xi_0,m}$. We select open circles D_i with centers ζ_i, $1 \leq i \leq l$ such that $\gamma.D_i \cap D_j \neq \emptyset$, $\gamma \in \Gamma$, implies $\gamma = \pm 1$. In particular, $f_{\xi_0,m}(\zeta_i) = 0$, but $f_{\xi_0,m}(z) \neq 0$ for $z \in D_i - \{\zeta_i\}$. We select a circle C_i with center ζ_i, say $|z - \zeta_i| = r_i$ which is completely inside D_i. Then, for $z \in C_i$, we have $f_{\xi_0,m}(z) \neq 0$. Thus, there exists $\delta = \delta_z > 0$ such that $|\xi - \xi_0| < \delta_z$ and $|w - z| < \delta_z$, $w \in C_i$, implies $f_\xi(w) \neq 0$. Because of the compactness of C_i, there exists $\delta > 0$ such that $|\xi - \xi_0| < \delta$ implies $f_{\xi,m}(z) \neq 0$ for all $z \in C_i$. Now, we apply the residue theorem to count zeroes of $f_{\xi,m}$ inside each D_i. We have the following:

$$\sum_{\substack{|\zeta-\zeta_i|<r_i \\ f_{\xi,m}(\zeta)=0}} \nu_{z-\zeta}(f_{\xi,m}) = \frac{1}{2\pi\sqrt{-1}} \int_{C_i} \frac{\partial f_{\xi,m}(z)/\partial z}{f_{\xi,m}(z)} dz, \quad \text{for} \quad |\xi - \xi_0| < \delta, \ 1 \leq i \leq l.$$

Obviously, we have that the left–hand side is continuous in ξ. Thus, we have

$$\sum_{\substack{|\zeta-\zeta_i|<r_i \\ f_{\xi,m}(\zeta)=0}} \nu_{z-\zeta}(f_{\xi,m}) = \nu_{z-\zeta_i}(f_{\xi_0,m}), \quad \text{for} \quad |\xi - \xi_0| < \delta, \ 1 \leq i \leq l.$$

Because of the assumption $\gamma.D_i \cap D_i \neq \emptyset$ for $\gamma \neq \pm 1$, we see that the support of the divisor

$$\mathfrak{d}_{\xi,m,i} \overset{def}{=} \sum_{\substack{|\zeta-\zeta_i|<r_i \\ f_{\xi,m}(\zeta)=0}} \nu_{z-\zeta}(f_{\xi,m}) \mathfrak{a}_\zeta$$

consists of Γ–non–equivalent points $\mathfrak{a}_\zeta$ when ζ ranges over $f_{\xi,m}(\zeta) = 0$ in $|\zeta - \zeta_i| < r_i$. As it is also $\gamma.D_i \cap D_j \neq \emptyset$ for $\gamma \neq \pm 1$, we see that for $i \neq j$ the supports of just constructed divisors $\mathfrak{d}_{\xi,m,i}$ and $\mathfrak{d}_{\xi,m,j}$ are disjoint. Finally, because of the same assumptions we have

$$\mathfrak{d}_{\xi_0,m} = \sum_{i=1}^{l} \nu_{z-\zeta_i}(f_{\xi_0,m}) \mathfrak{a}_{\zeta_i}.$$

In particular,

$$\sum_{i=1}^{l} \nu_{z-\zeta_i}(f_{\xi_0,m}) = \deg\left(\mathfrak{d}_{\xi_0,m}\right) = g.$$

By above discussion, we obtain

$$\deg\,(\sum_{i=1}^{l}\mathfrak{d}_{\xi,m,i})=\sum_{i=1}^{l}\sum_{\substack{|\zeta-\zeta_i|<r_i\\ f_{\xi,m}(\zeta)=0}}\nu_{z-\zeta}(f_{\xi,m})=\sum_{i=1}^{l}\nu_{z-\zeta_i}(f_{\xi_0,m})=g.$$

Thus, we finally obtain

$$\mathfrak{d}_{\xi,m}=\sum_{i=1}^{l}\mathfrak{d}_{\xi,m,i},\quad |\xi-\xi_0|<\delta.$$

Again, by the residue theorem, for $m\geq 0$, we have the following:

$$\sum_{\substack{|\zeta-\zeta_i|<r_i\\ f_{\xi,m}(\zeta)=0}}\nu_{z-\zeta}(f_{\xi,m})\zeta^m=\frac{1}{2\pi\sqrt{-1}}\int_{C_i}z^m\frac{\partial f_{\xi,m}(z)/\partial z}{f_{\xi,m}(z)}dz,\quad \text{for}\quad |\xi-\xi_0|<\delta,\ 1\leq i\leq l.$$

This implies that the sum

$$\sum_{i=1}^{l}\sum_{\substack{|\zeta-\zeta_i|<r_i\\ f_{\xi,m}(\zeta)=0}}\nu_{z-\zeta}(f_{\xi,m})\zeta^m$$

is a C^∞–function on $|\xi-\xi_0|<\delta$. In another words, if $r_1(\xi),\cdots,r_g(\xi)$ are all zeroes of f_ξ in the union of the circles $|\zeta-\zeta_i|<r_i$ counted with multiplicity, then

$$r_1^m(\xi)+\cdots+r_g^m(\xi),\quad m\geq 0,$$

is a C^∞–function.

Let $\sigma_1(\xi),\ldots,\sigma_g(\xi)$ be the elementary symmetric polynomials in $r_1(\xi),\cdots,r_g(\xi)$. These are polynomials in the sums of powers, hence C^∞–functions. □

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