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ON THE CUSP FORMS OF CONGRUENCE SUBGROUPS OF AN ALMOST SIMPLE LIE GROUP

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Abstract. In this paper we address the issue of existence of cusp forms for almost simple Lie groups using the approach of the second author combined with local information on supercuspidal representations for p-adic groups known by the first author. We pay special attention to the case of $SL_M(\mathbb{R})$ where we prove various existence results for principal congruence subgroups.

1. Introduction

Existence and construction of cusp forms is a fundamental problem in the modern theory of automorphic forms ([1], [16], [11], [15], [4], [5]). In this paper we address the issue of existence of cusp forms for almost simple Lie groups using the approach of [8] combined with some local information on supercuspidal representations for p-adic groups ([6], [7]). In view of recent development in the analytic number theory ([4], [5]) we pay special attention to the case of SL_M.

Suppose G is a simply connected, absolutely almost simple algebraic group defined over $\mathbb{Q}$, and $G_\infty := G(\mathbb{R})$ is not compact. Let $\mathbb{A}$ and $\mathbb{A}_f$ denote respectively the ring of adeles and finite adeles of $\mathbb{Q}$. For each prime p, let $\mathbb{Z}_p$ denote the p-adic integers inside $\mathbb{Q}_p$. Recall that for almost all primes p, the group G is unramified over $\mathbb{Q}_p$. Thus, G is a group scheme over $\mathbb{Z}_p$, and $G(\mathbb{Z}_p)$ is a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$ ([17], 3.9.1). The group $G(\mathbb{A}_f)$ has a basis of neighborhoods of the identity consisting of open-compact subgroups. Suppose $L \subset G(\mathbb{A}_f)$ is an open–compact subgroup. Set

$$\Gamma_L \ := \ G(\mathbb{Q}) \cap L \subset G(\mathbb{A}_f) \subset G(\mathbb{A}), \tag{1-1}$$

where we identify $G(\mathbb{Q})$ with its image under the diagonal embedding into $G(\mathbb{A}_f)$ and $G(\mathbb{A})$. Now, the projection of $\Gamma_L \subset G(\mathbb{A})$ to G_∞ is a discrete subgroup. We continue to denote this discrete subgroup by Γ_L. It is called a congruence subgroup of $G(\mathbb{Q})$ ([2]). We write $\mathcal{A}_{cusp}(\Gamma_L\backslash G_\infty)$ and $L^2_{cusp}(\Gamma_L\backslash G_\infty)$ for the spaces of cusp forms and its L^2-closure.

Recall the assumption that G is simply connected, absolutely almost simple, and G_∞ is non-compact means it satisfies the strong approximation property ([12], [2] §4.7), i.e., $G(\mathbb{Q})$

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is dense in $G(\mathbb{A}_f)$, and so for any open compact subgroup $L \subset G(\mathbb{A}_f)$:

$$G(\mathbb{A}_f) = G(\mathbb{Q})L \ .$$

We consider a finite family of open compact subgroups

$$\mathcal{F} = \{L\} \tag{1-2}$$

satisfying the following assumptions:

Assumptions 1-3.

(i) *Under the partial ordering of inclusion there exists a subgroup $L_{min} \in \mathcal{F}$ that is a subgroup of all the others.*

(ii) *The groups $L \in \mathcal{F}$ are factorizable, i.e., $L = \prod_p L_p$, and for all but finitely many p's, the group L_p is the maximal compact subgroup $K_p := G(\mathbb{Z}_p)$.*

(iii) *There exists a non-empty finite set of primes T such that for $p \in T$ the group G has a Borel subgroup $B = AU$ and maximal torus A defined over $\mathbb{Z}_p$, and there exists a supercuspidal representation π_p of $G(\mathbb{Q}_p)$ such that $\pi_p^{L_{min,p}} \neq 0$, and for $L \neq L_{min}$ there exists $p \in T$ such that $\pi_p^{L_p} = 0$.*

We note that a simply connected split almost simple group, e.g., SL_M, and Sp_{2M}, defined over $\mathbb{Z}$ satisfies the above assumptions.

Theorem 1-4. *Suppose G is a simply connected, absolutely almost simple algebraic group defined over $\mathbb{Q}$, so that G_∞ is non-compact and $\mathcal{F} = \{L\}$ is a finite set of open compact subgroups of $G(\mathbb{A}_f)$ satisfying assumptions* (1-3). *Then, the orthogonal complement of*

$$\sum_{\substack{L \in \mathcal{F} \\ L_{min} \subsetneq L}} L^2_{cusp}(\Gamma_L \backslash G_\infty)$$

in $L^2_{cusp}(\Gamma_{L_{min}} \backslash G_\infty)$ is a direct sum of infinitely many irreducible unitary representations of G_∞.

Theorem 1-4 is proved in Section 2. For general G, in Sections 3 and 4 we give examples of families of $\mathcal{F}$ using Moy–Prasad filtration subgroups ([6], [7]).

In the case $G = SL_M$, and the congruence subgroups are the principal congruence subgroups $\Gamma(m)$ (see (3-1)) the main theorem has the following form:

Corollary 1-5. *Let $G = SL_M$. Let $n \geq 2$ be an integer. Then, the orthogonal complement of*

$$\sum_{\substack{m \mid n \\ m < n}} L^2_{cusp}(\Gamma(m) \backslash G_\infty)$$

in $L^2_{cusp}(\Gamma(n) \backslash G_\infty)$ is a direct sum of infinitely many irreducible unitary representations of G_∞.

Proof. This follows directly from the examples in Section 4. □

In Section 5, we refine Corollary 1-5 (see Theorem 5-8). As a result, we obtain a generalization of the compact quotient case (obtained in [10]). The corresponding results are contained in Corollaries 5-9, 5-10, and 5-12. For example, in Corollary 5-9, we prove for sufficiently large n that we can take infinitely many spherical representations. Corollary 5-12 improves ([9], Theorem 0-2).

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2. Proof of Theorem 1-4

We recall some results from [8]. For $f \in C_c^\infty(G(\mathbb{A}))$, the adelic compactly supported Poincaré series $P(f)$ is defined as:

$$P(f)(g) \ = \ \sum_{\gamma \in G(\mathbb{Q})} f(\gamma \cdot g). \tag{2-1}$$

Write $g \in G(\mathbb{A}) = G_\infty \times G(\mathbb{A}_f)$ as $g = (g_\infty, g_f)$. We have the following:

$$P(f)(g_\infty, 1) = \sum_{\gamma \in G(k)} f_\infty(\gamma \cdot g_\infty, \gamma). \tag{2-2}$$

The next lemma ([8], Proposition 3.2) describes the restriction of the Poincaré series (2-1) to G_∞.

Lemma 2-3. *Let $f \in C_c^\infty(G(\mathbb{A}))$. Assume that L is an open compact subgroup of $G(\mathbb{A}_f)$ such that f is right–invariant under L. Define the congruence subgroup Γ_L of G_∞ as in (1-1). Then:*

- (i) *The function in (2-2) is a compactly supported Poincaré series on G_∞ for Γ_L.*
- (ii) *If $P(f)$ is cuspidal, then the function in (2-2) is cuspidal for Γ_L.*

Let S be a finite set of places, containing ∞, and large enough so that G is defined over $\mathbb{Z}_p$ for $p \notin S$. We use the decomposition of $G(\mathbb{A})$ given by:

$$G(\mathbb{A}) = G_S \times G^S \ , \text{ where } G_S := \prod_{p \in S} G(\mathbb{Q}_p) \ , \text{ and } G^S = \prod_{p \notin S}{}' G(\mathbb{Q}_p) \ . \tag{2-4}$$

Set $G^S(\mathbb{Z}_p) := \prod_{p \notin S} G(\mathbb{Z}_p)$, and

$$\Gamma(S) \ := \ G^S(\mathbb{Z}_p) \cap G(\mathbb{Q}) \ \text{ (the intersection is taken in } G^S). \tag{2-5}$$

We view $\Gamma(S) \subset G(\mathbb{Q}) \subset G(\mathbb{A})$. Set

$$\Gamma_S \;=\; \text{image of } \Gamma(S) \text{ under the projection map } G(\mathbb{A}) = G_S \times G^S \to G_S\,.$$

Since $G(\mathbb{Q})$ is a discrete subgroup of $G(\mathbb{A})$, it follows that Γ_S is a discrete subgroup of G_S.

For each $p \in S - \{\infty\}$, we choose an open–compact subgroup L_p, and we set

$$\begin{aligned} L \;&=\; \Big(\prod_{p\in S-\{\infty\}} L_p\Big) \;\times\; G^S(\mathbb{Z}_p) \\ \Gamma_L \;&=\; L \cap G(\mathbb{Q}) \;=\; \Big(\prod_{p\in S-\{\infty\}} L_p\Big) \;\cap\; \Gamma_S\,. \end{aligned} \tag{2-6}$$

The group Γ_L is a discrete subgroup of G_∞.

Let $\mathfrak{g}_\infty$ be the (real) Lie algebra of G_∞, and K_∞ a maximal compact subgroup. We have the following non-vanishing criterion ([8], Theorem 4.2):

Lemma 2-7. *Assume that for each prime p we have a function $f_p \in C_c^\infty(G(\mathbb{Q}_p))$ so that $f_p(1) \neq 0$, and $f_p = 1_{G(\mathbb{Z}_p)}$ is the characteristic function of $G(\mathbb{Z}_p)$ for all $p \notin S$. Assume further that for $p \in S - \{\infty\}$, L_p is an open-compact subgroup so that f_p is right-invariant under L_p. Note. Since the set $\big(K_\infty \times \prod_{p\in S-\{\infty\}} \operatorname{supp}(f_p)\big)$ is compact, the intersection $\Gamma_S \cap \big(K_\infty \times \prod_{p\in S-\{\infty\}} \operatorname{supp}(f_p)\big)$ is a finite set. It can be written as follows:*

$$\bigcup_{j=1}^{l} \gamma_j \cdot (K_\infty \cap \Gamma_L). \tag{2-8}$$

Set

$$c_j = \prod_{p\in S-\{\infty\}} f_p(\gamma_j).$$

Then, the K_∞-invariant map $C^\infty(K_\infty) \longrightarrow C^\infty(K_\infty \cap \Gamma_L \setminus K_\infty)$ given by

$$\alpha \;\mapsto\; \hat{\alpha}(k) \;:=\; k \;\mapsto\; \sum_{j=1}^{l} \sum_{\gamma\in K_\infty\cap\Gamma} c_j \cdot \alpha(\gamma_j \gamma \cdot k) \tag{2-9}$$

is non–trivial, and, for every $\delta \in \hat{K}_\infty$, contributing in the decomposition of the closure of the image of (2-9) in $L^2(K_\infty \cap \Gamma_L \setminus K_\infty)$, we can find a non-trivial $f_\infty \in C_c^\infty(G_\infty)$ so that the following hold:

(i) *$E_\delta(f_\infty) = f_\infty$.*
(ii) *The Poincaré series $P(f)$ and its restriction to G_∞ (which is a Poincaré series for Γ_L) are non–trivial, where $f \overset{def}{=} f_\infty \otimes_p f_p \in C_c^\infty(G(\mathbb{A}))$.*
(iii) *$E_\delta(P(f)) = P(f)$ and $P(f)$ is right–invariant under L.*
(iv) *The support of $P(f)|_{G_\infty}$ is contained in the set of the form $\Gamma_L \cdot C$, where C is a compact set which is right–invariant under K_∞, and $\Gamma_L \cdot C$ is not whole G_∞.*

We begin the proof of Theorem 1-4. We apply the above considerations to $L_{\min}$. By hypothesis, this group is factorizable. Take S sufficiently large so that it contains T, and if $p \notin S$ then the group G is unramified over $\mathbb{Q}_p$ so that it is defined over $\mathbb{Z}_p$, and $L_{\min,p} = G(\mathbb{Z}_p)$. Thus, (2-6) holds for $L = L_{\min}$. We apply Lemma 2-7. To do this, we construct functions $f_p \in C_c^\infty(G(\mathbb{Q}_p))$ such that $f_p(1) \neq 0$, f_p is L_p-invariant on the right, and $f_p = 1_{G(\mathbb{Z}_p)}$ for all $p \notin S$. We need to define f_p for $p \in S - \{\infty\}$. We let $f_p = 1_{L_{\min,p}}$ for $p \in S - \{\infty\} - T$. For $p \in T$, we use our assumption there exists a supercuspidal representation π_p such that $\pi_p^{L_{\min,p}} \neq 0$. We let f_p be a matrix coefficient of π_p such that $f_p(1) \neq 0$ and f_p is $L_{\min,p}$-invariant on the right.

Having completed the construction of the functions f_p for all finite p, we see that we meet all assumptions of Lemma 2-7. By that lemma, we can select $\delta \in \hat{K}_\infty$ and $f_\infty \in C_c^\infty(G_\infty)$ such that (i)—(iv) of Lemma 2-7 hold.

We can decompose into closed irreducible $G(\mathbb{A})$ invariant subspaces

(2-10) $$L^2_{cusp}(G(k) \setminus G(\mathbb{A})) = \oplus_j \ \mathcal{H}_j.$$

Now, we prove the following lemma:

Lemma 2-11. *We maintain above assumptions. Then,*

$$P(f) \in L^2_{cusp}(G(\mathbb{Q}) \setminus G(\mathbb{A}))^{L_{min}},$$

and we can decompose according to (2-10)

(2-12) $$P(f) = \sum_j \psi_j, \quad \psi_j \in \mathfrak{H}^j.$$

Then we have the following:

- (i) *For all j, $\psi_j \in \mathcal{A}_{cusp}(G(\mathbb{Q}) \setminus G(\mathbb{A}))$ is right–invariant under L_{min}, and transforms according to δ i.e., $E_\delta(\psi_j) = \psi_j$.*
- (ii) *Assume $\psi_j \neq 0$. Then $\pi_p^j \simeq \pi_p$ for all $p \in T$.*
- (iii) *The number of indices j in (2-12) such that $\psi_j \neq 0$ is infinite.*
- (iv) *The closure of G_∞–invariant subspace in $L^2_{cusp}(\Gamma_{L_{min}} \setminus G_\infty)$ generated by $P(f)|_{G_\infty}$ is an orthogonal direct sum of infinitely many non–equivalent irreducible unitary representations of G_∞ which contain δ.*

Proof. This follows from ([8], Proposition 5.3 and Theorem 7.2). We remark that the formulation of (iv) follows from the the proof of ([8], Theorem 7.2). □

Lemma 2-13. *Let $L \in \mathcal{F}$. Then, the restriction map gives an isomorphism of unitary G_∞ representations*

$$L^2_{cusp}(G(\mathbb{Q}) \setminus G(\mathbb{A}))^L \simeq L^2_{cusp}(\Gamma_L \setminus G_\infty),$$

which is norm preserving up to a scalar $vol_{G(\mathbb{A}_f)}(L)$ i.e.,

$$\int_{G(\mathbb{Q})\setminus G(\mathbb{A})} |\psi(g)|^2 dg = vol_{G(\mathbb{A}_f)}(L) \int_{\Gamma_L \setminus G_\infty} |\psi(g_\infty)|^2 dg_\infty.$$

Proof. Since we assume that G is simply connected, absolutely almost simple over $\mathbb{Q}$ and G_∞ is not compact, satisfies the strong approximation property: we have $G(\mathbb{A}_f) = G(\mathbb{Q})L$. Now, the claim is a particular case of the computations contained in the proof of ([8], Theorem 7-2, see (7-6) there). □

Lemma 2-14. *Let $L \in \mathcal{F}$. Then, the natural embedding is an embedding of unitary G_∞–representations:*

$$L^2_{cusp}(\Gamma_L \setminus G_\infty) \hookrightarrow L^2_{cusp}(\Gamma_{L_{min}} \setminus G_\infty).$$

This means it is norm preserving up to a scalar $[\Gamma_L : \Gamma_{L_{min}}]$ i.e.,

$$\int_{\Gamma_{L_{min}} \setminus G_\infty} |\psi(g_\infty)|^2 dg_\infty = [\Gamma_L : \Gamma_{L_{min}}] \int_{\Gamma_L \setminus G_\infty} |\psi(g_\infty)|^2 dg_\infty,$$

for $\psi \in L^2_{cusp}(\Gamma_L \setminus G_\infty)$.

Proof. This is ([9], Lemma 1-9). □

Lemma 2-15. *Let $L \in \mathcal{F}$. Then, we have*

$$[\Gamma_L : \Gamma_{L_{min}}] = vol_{G(\mathbb{A}_f)}(L)/vol_{G(\mathbb{A}_f)}(L_{min}).$$

Proof. Obviously, we have

$$[L : L_{\min}] = vol_{G(\mathbb{A}_f)}(L)/vol_{G(\mathbb{A}_f)}(L_{\min}).$$

But

$$L/L_{\min} = (L \cap G(\mathbb{Q})) \,/\, (L_{\min} \cap G(\mathbb{Q})) = \Gamma_L/\Gamma_{L_{\min}},$$

since

$$G(\mathbb{A}_f) = G(\mathbb{Q})L_{\min}.$$

□

Lemma 2-16. *We have the following commutative diagram of unitary G_∞–representations:*

$$\begin{array}{ccc} L^2_{cusp}(G(\mathbb{Q}) \setminus G(\mathbb{A}))^L & \longrightarrow & L^2_{cusp}(G(\mathbb{Q}) \setminus G(\mathbb{A}))^{L_{min}} \\ \simeq\big\downarrow & & \simeq\big\downarrow \\ L^2_{cusp}(\Gamma_L \setminus G_\infty) & \longrightarrow & L^2_{cusp}(\Gamma_{L_{min}} \setminus G_\infty), \end{array}$$

where the horizontal maps are inclusions, the vertical maps are isomorphisms, and the inner products are normalized as follows: (i) on the spaces in the first row, we take usual Petersson inner product $\int_{G(\mathbb{Q})\setminus G(\mathbb{A})} \psi(g)\overline{\varphi(g)}dg$, and (ii) on $L^2_{cusp}(\Gamma_U \setminus G_\infty)$, the inner product is the normalized integral $vol_{G(\mathbb{A}_f)}(U) \int_{\Gamma_U \setminus G_\infty} \psi(g_\infty)\overline{\varphi(g_\infty)}dg$, where $U \in \{L_{min}, L\}$.

Proof. The lemma follows immediately from Lemmas 2-13, 2-14, and 2-15. □

Lemma 2-17. *Let $L \in \mathcal{F}$, $L \neq L_{min}$. Then, $P(f)|_{G_\infty}$ is orthogonal to $L^2_{cusp}(\Gamma_L \setminus G_\infty)$ in $L^2_{cusp}(\Gamma_{L_{min}} \setminus G_\infty)$ if and only if $P(f)$ is orthogonal to $L^2_{cusp}(G(\mathbb{Q}) \setminus G(\mathbb{A}))^L$.*

Proof. This follows from Lemma 2-16. $\square$

By Lemma 2-11 (iv), the closure $\mathcal{U}$ of G_∞-invariant subspace in $L^2_{cusp}(\Gamma_{L_{\min}}\backslash G_\infty)$ generated by $P(f)|_{G_\infty}$ is an orthogonal direct sum of infinitely many non-equivalent irreducible unitary representations of G_∞. By Lemma 2-17, $\mathcal{U}$ is orthogonal to

$$\sum_{\substack{L\in\mathcal{F}\\ L\neq L_{\min}}} L^2_{cusp}(\Gamma_L \setminus G_\infty)$$

if and only if $P(f)$ is orthogonal to $L^2_{cusp}(G(\mathbb{Q}) \setminus G(\mathbb{A}))^L$ for all $L \in \mathcal{F}$, $L \neq L_{\min}$. The following lemma completes the proof of Theorem 1-4.

Lemma 2-18. *Let $L \in \mathcal{F}$, $L \neq L_{min}$. Then, $P(f)$ is orthogonal to $L^2_{cusp}(G(\mathbb{Q}) \setminus G(\mathbb{A}))^L$.*

Proof. Here we use the very last assumption that $\pi^{L_p} = 0$ for all $L \in \mathcal{F}$ such that $L \neq L_{\min}$.

We remind the reader that f_p is matrix coefficient of π_p for $p \in T$. Since $L_{\min} \subset L$, we have $L_{\min,p} \subset L_p$ for all primes p (the groups are factorizable), we obtain that

$$\int_L f(gl)dl = 0, \quad \text{for all } g \in G(\mathbb{A}).$$

Hence

$$\int_L P(f)(gl)dl = 0, \quad \text{for all } g \in G(\mathbb{A}).$$

Finally, let $\varphi \in L^2_{cusp}(G(\mathbb{Q}) \setminus G(\mathbb{A}))^L$. Then we compute

$$vol_{G(\mathbb{A}_f)}(L) \cdot \int_{G(\mathbb{Q})\setminus G(\mathbb{A})} P(f)(g)\overline{\varphi(g)}dg =$$
$$\int_{G(\mathbb{Q})\setminus G(\mathbb{A})} \int_L P(f)(gl)\overline{\varphi(gl)}dldg =$$
$$\int_{G(\mathbb{Q})\setminus G(\mathbb{A})} \left(\int_L P(f)(gl)dl\right) \overline{\varphi(g)}dg = 0.$$

This proves the lemma. $\square$

3. Open compact subgroups of $G(\mathbb{Q}_p)$ and congruence subgroups in G_∞

Fix a positive integer M, and consider the algebraic reductive group SL_M. In $\mathrm{SL}_M(\mathbb{Z})$, we consider an alternative to the usual principal congruence subgroup

$$\Gamma(n) \ := \ \{ g = (g_{i,j}) \in \mathrm{SL}_M(\mathbb{Z}) \mid g_{i,j} \equiv \delta_{i,j} \bmod n \} \ . \tag{3-1}$$

For a prime power p^k ($k > 0$), we set

$$\Gamma_1(p^k) \ := \ \{ g = (g_{i,j}) \in \mathrm{SL}_M(\mathbb{Z}) \mid p^{(k-1)} \mid g_{i,j} \text{ for } i < j \quad p^k \mid (g_{i,j} - \delta_{i,j}) \text{ for } i \geq j \ \}. \tag{3-2}$$

$\Gamma_1(p^k)$ is the set of elements in $\Gamma(p^{(k-1)})$ which modulo p^k are unipotent upper triangular. If n is a positive integer, with prime factorization $n = {p_1}^{e_1} \cdots {p_s}^{e_s}$, set

$$\Gamma_1(n) \ := \ \bigcap_i \Gamma_1({p_i}^{e_i}) \ . \tag{3-3}$$

We note that if n is divisible by m, then $\Gamma_1(n)$ is a subgroup of $\Gamma_1(m)$. However, we also note $\Gamma_1(n)$ is not a normal subgroup of $\Gamma(1) = \mathrm{SL}_M(\mathbb{Z})$, and that $\Gamma_1(n)$ is not necessarily a normal subgroup of $\Gamma_1(m)$ when $m|n$. To rectify this situation, we define a stronger notion of divisibility of integers. Define n to be a strong multiple of m (or m divides n strongly) if n is a multiple of m and every prime p that occurs in the factorization of n also occurs in the factorization of m. We use the notation $m|_s n$. The following is elementary:

Proposition 3-4. (i) *For $k \geq 1$, the group $\Gamma_1(p^k)$ is a normal subgroup of $\Gamma_1(p)$.*
(ii) *If $m|_s n$, then $\Gamma_1(n)$ is a normal subgroup of $\Gamma_1(m)$.*

For k a positive integer, define open compacts of $\mathrm{SL}_M(\mathbb{Q}_p)$:

$$\begin{aligned} \mathcal{K}_{p,k} \ &:= \ \{ \, (g_{i,j}) \in \mathrm{SL}_M(\mathbb{Z}_p) \ \big| \ p^k \,|\, (g_{i,j} - \delta_{i,j}) \ \forall \, i,j \ \} \ , \quad \text{and} \\ \mathcal{J}_{p,k^+} \ &:= \ \{ \, (g_{i,j}) \in \mathrm{SL}_M(\mathbb{Z}_p) \ \big| \ p^{(k-1)} \,|\, g_{i,j} \text{ for } i < j, \text{ and } p^k \,|\, (g_{i,j} - \delta_{i,j}) \text{ for } i \geq j \ \}. \end{aligned} \tag{3-5}$$

The group $\mathcal{K}_{p,k}$ is the well-known k-th congruence subgroup of $\mathrm{SL}_M(\mathbb{Z}_p)$. Set

$$\mathcal{K}_n \ := \ \prod_{p|n} \mathcal{K}_{p,e_i} \ \prod_{p\nmid n} G(\mathbb{Z}_p) \ , \quad \text{and} \quad \mathcal{J}_n \ := \ \prod_{p|n} \mathcal{J}_{p,(e_i)^+} \ \prod_{p\nmid n} G(\mathbb{Z}_p) \ . \tag{3-6}$$

A consequence of the Chinese Remainder Theorem is the following:

Proposition 3-7.

$$\Gamma(n) \ = \ \mathcal{K}_n \ \cap \ G(\mathbb{Q}) \, , \quad \text{and} \quad \Gamma_1(n) \ = \ \mathcal{J}_n \ \cap \ G(\mathbb{Q})$$

The groups $\mathcal{J}_{k^+}$ are the same as certain Moy-Prasad filtration subgroups. Let $\mathcal{J}$ denote the subgroup consisting of elements in $\mathrm{SL}_M(\mathbb{Z}_p)$ which are upper triangular modulo p. It is an Iwahori subgroup of $\mathrm{SL}_M(\mathbb{Q}_p)$. Let $\mathcal{B}(\mathrm{SL}_M(\mathbb{Q}_p))$ be the Bruhat-Tits building of $\mathrm{SL}_M(\mathbb{Q}_p)$. Let C be the alcove in $\mathcal{B}(\mathrm{SL}_M(\mathbb{Q}_p))$ fixed by $\mathcal{J} \cap \mathrm{SL}_M(\mathbb{Q}_p)$. Then, for any $x \in C$, and $k \in \mathbb{N}$, we have $\mathcal{J}_{k^+} \cap \mathrm{SL}_M(\mathbb{Q}_p) = \big(\mathrm{SL}_M(\mathbb{Q}_p)\big)_{x,k^+}$.

Note that the k-th congruence subgroups $\mathcal{K}_{p,k}$ are normal in $\mathrm{SL}_M(\mathbb{Z}_p)$. For $k > 0$, we now formulate and prove a result on cusp forms associated to certain characters of the group $\mathcal{K}_{p,k}/\mathcal{K}_{p,(k+1)}$. To simplify notation (since p will be fixed), we shorten $\mathcal{K}_{p,k}$ to $\mathcal{K}_k$. Set $\mathcal{L} = \mathfrak{sl}_M(\mathbb{Z}_p)$, and

(3-8) $$\mathcal{L}_k \;:=\; \{\ (x_{i,j}) \in \mathcal{L} \ \big|\ p^k \mid x_{i,j}\ \} \;=\; p^k\mathcal{L}\ .$$

The quotient $\mathcal{L}/\mathcal{L}_1$ is naturally the Lie algebra $\mathfrak{sl}_M(\mathbb{F}_p)$. The map $X \to p^kX$ gives a natural isomorphism τ_k of $\mathcal{L}/\mathcal{L}_1$ with $\mathcal{L}_k/\mathcal{L}_{(k+1)}$ Recall there is an isomorphism

(3-9) $$\theta\ :\ \mathcal{L}_k/\mathcal{L}_{(k+1)}\ \to\ \mathcal{K}_k/\mathcal{K}_{(k+1)}$$

so that for $x \in \mathcal{L}_k$, that

$$\theta(x) \;=\; 1 + x \bmod \mathcal{K}_{(k+1)} \ \text{ in }\ \mathrm{GL}_M(\mathbb{Z}_p)\ .$$

Therefore, there is a natural isomorphism of $\mathcal{K}_k/\mathcal{K}_{(k+1)}$ with $\mathfrak{sl}_M(\mathbb{F}_p)$. Any choice of a non-trivial additive character ψ of $\mathbb{F}_p$ give an identification of the Pontryagin dual of $\mathfrak{sl}_M(\mathbb{F}_p)$ with $\mathfrak{gl}_M(\mathbb{F}_p)/\mathbb{F}_p$, via the composition of the pairing

(3-10) $$\begin{array}{ccc} \mathfrak{sl}_M(\mathbb{F}_p)\ \times\ \mathfrak{gl}_M(\mathbb{F}_p) & \longrightarrow & \mathbb{F}_p \\ (\,X\,,\,Y\,) & \longrightarrow & \mathbf{tr}(\,XY\,) \end{array}$$

with ψ. Write

(3-11) $$\begin{array}{c} \psi_Y\ :\ \mathfrak{sl}_M(\mathbb{F}_p)\ \longrightarrow \mathbb{C}^\times \\ X \xrightarrow{\psi_Y} \psi(\,\mathbf{tr}(\,XY\,)\,)\ . \end{array}$$

Take $Y \in \mathfrak{gl}_M(\mathbb{F}_p)$ to be an element whose characteristic polynomial is irreducible. Such elements exists since there is a finite extension of $\mathbb{F}_p$ of degree M. The following proposition is elementary.

Proposition 3-12. *Suppose $Y \in \mathfrak{gl}_M(\mathbb{F}_p)$ has irreducible characteristic polynomial:*

(i) *If $\mathfrak{p}(\mathbb{F}_p) \subsetneq \mathfrak{gl}_M(\mathbb{F}_p)$ is any parabolic subalgebra of $\mathfrak{gl}_M(\mathbb{F}_p)$, then $Y \notin \mathfrak{p}(\mathbb{F}_p)$.*

(ii) *The character ψ_Y of $\mathfrak{sl}_M(\mathbb{F}_p)$ is a cusp form.*

(iii) *The inflation of ψ_Y to $\mathcal{K}_k$ via* (3-9)*, when extended to $SL_M(\mathbb{Q}_p)$ by setting it zero off $\mathcal{K}_k$ is a cusp form.*

(iv) *For each positive integer k, there exists an irreducible supercuspidal representation (ρ, W_ρ) which has a non-zero $\mathcal{K}_{k+1}$ fixed vector, but no non-zero $\mathcal{K}_k$-fixed vector.*

Proof. To prove part (i), suppose $\mathfrak{p}(\mathbb{F}_p) \subsetneq \mathfrak{gl}_M(\mathbb{F}_p)$ is any parabolic subalgebra and $\mathfrak{p}(\mathbb{F}_p) = \mathfrak{m}(\mathbb{F}_p) + \mathfrak{u}(\mathbb{F}_p)$ is a Levi decomposition. For any $Z \in \mathfrak{p}(\mathbb{F}_p)$, let $Z_{\mathfrak{m}(\mathbb{F}_p)}$ be the projection of Z to $\mathfrak{m}(\mathbb{F}_p)$. Then Z and $Z_{\mathfrak{m}(\mathbb{F}_p)}$ have the same characteristic polynomial, and the latter characteristic polynomial is clearly not irreducible. Thus, if $Y \in \mathfrak{gl}_M(\mathbb{F}_p)$, it cannot lie in any $\mathfrak{p}(\mathbb{F}_p) \subsetneq \mathfrak{gl}_M(\mathbb{F}_p)$.

To prove (ii), suppose $x \in \mathfrak{sl}_M(\mathbb{F}_p)$, and $\mathfrak{p}(\mathbb{F}_p) = \mathfrak{m}(\mathbb{F}_p) + \mathfrak{u}(\mathbb{F}_p)$ is a proper parabolic subalgebra. Then

$$\int_{\mathfrak{u}(\mathbb{F}_p)} \psi_Y(x+n)\,dn \;=\; \psi_Y(x)\ \int_{\mathfrak{u}(\mathbb{F}_p)} \psi_Y(n)\,dn$$

Since Y is not contained in any parabolic subalgebra of $\mathfrak{gl}_M(\mathbb{F}_p)$, the integrand of the integral on the right side is a non-trivial character of $\mathfrak{u}(\mathbb{F}_p)$ and therefore the integral is zero. Whence, ψ_Y is a cusp form.

To prove (iii), denote the inflation of ψ_Y by $\psi_Y \circ \theta^{-1}$. Suppose $P \subset \mathrm{SL}_M(\mathbb{Q}_p)$ is a parabolic subgroup. Then P is conjugate to a standard 'block upper triangular' parabolic subgroup $Q = M_Q N_Q$ of $\mathrm{SL}_M(\mathbb{Q}_p)$, i.e.,

$$P \ = \ g^{-1}Qg \ = \ (\,g^{-1}M_Q g\,)\,(\,g^{-1}N_Q g\,) \quad \text{and} \quad U_P = g^{-1}N_Q g.$$

Since $\mathrm{SL}_M(\mathbb{Q}_p) = Q\mathcal{K}$, express g as $g = v_g k_g$ with $v_g \in Q$, and $k_g \in \mathcal{K}$. Then,

$$\begin{aligned}
\int_{U_P} \psi_Y \circ \theta^{-1}(xn)\ dn \ &= \ \int_{U_Q} \psi_Y \circ \theta^{-1}(xg^{-1}ug)\ du \\
&= \ \int_{U_Q} \psi_Y \circ \theta^{-1}(xk_g^{-1}v_g^{-1}uv_gk_g)\ du \\
&= \ c \int_{U_Q} \psi_Y \circ \theta^{-1}(xk_g^{-1}uk_g)\ du \quad \text{(suitable constant } c) \\
&= \ c \int_{k_g^{-1}U_Q k_g} \psi_Y \circ \theta^{-1}(xu)\ du
\end{aligned}$$

From the last line, since $\psi_Y \circ \theta^{-1}$ has support in $\mathcal{K}_k$, to prove the integral vanishes, it suffices to do so when $x \in \mathcal{K}_k$. In this situation the integral vanishes by part (ii). Thus $\psi_Y \circ \theta^{-1}$ is a cusp form on $\mathrm{SL}_M(\mathbb{Q}_p)$.

In regards to part (iv), let $V_{\psi_Y \circ \theta^{-1}}$ be the representation of $\mathrm{SL}_M(\mathbb{Q}_p)$ generated by the left translates of the cusp form ${}_{\psi_Y \circ \theta^{-1}}$. It is a finite direct sum of supercuspidal representations, and by Frobenius reciprocity, any irreducible subrepresentation σ of $V_{\psi_Y \circ \theta^{-1}}$ contains the character $\psi_Y \circ \theta^{-1}$. This means σ contains a non-zero $\mathcal{K}_{(k+1)}$-fixed vector, but no $\mathcal{K}_k$-fixed vector. □

Suppose G is split simple algebraic group defined over $\mathbb{Z}_p$. Let B be a Borel subgroup of G, T a maximal torus of B, and $A \subset T$ the maximal split torus in T. Set

(3-13) $$\mathcal{G} := G(\mathbb{Q}_p) \ \text{ and } \ \mathcal{K} := G(\mathbb{Z}_p) \text{ a maximal compact subgroup of } \mathcal{G}.$$

The choice of B determines an Iwahori subgroup $\mathcal{I} \subset \mathcal{K}$.

Let $\mathcal{B}(\mathcal{G})$ be the Bruhat-Tits building of $\mathcal{G}$. Let $C = \mathcal{B}(\mathcal{G})^{\mathcal{I}}$ be the fixed points of of the Iwahori subgroup $\mathcal{I}$. It is an alcove in $\mathcal{B}(\mathcal{G})$. Take x_0 to be the barycenter of C, and let ℓ be the rank of G. Set

(3-14) $$k' \ := \ k + (\frac{1}{\ell+1}) \quad \text{and} \quad k'' \ := \ k + (\frac{2}{\ell+1})$$

Then, in terms of the Moy-Prasad filtration subgroups, we have

(3-15) $$\mathcal{I}_k \ = \ \mathcal{G}_{x_0,k} \quad \text{and} \quad \mathcal{I}_{k^+} \ = \ \mathcal{G}_{x_0,k'}\ .$$

Let Δ and Δ^{aff} be the simple roots and simple affine roots respectively with respect to the Borel and Iwahori subgroups B and $\mathcal{I}$ respectively. We recall that every $\alpha \in \Delta$ is the gradient part of a unique root $\psi \in \Delta^{\mathrm{aff}}$. In this way, we view Δ as a subset of Δ^{aff}. We recall

$$\text{(3-16)} \qquad \text{the quotient} \quad \mathcal{G}_{x_0,k'}/\mathcal{G}_{x_0,k''} \quad \text{is canonically} \prod_{\psi\in\Delta^{\mathrm{aff}}} U_{(\psi+k)}/U_{(\psi+k+1)} \ .$$

We further recall that a character χ of $\mathcal{G}_{x_0,k'}/\mathcal{G}_{x_0,k''}$ is non-degenerate if the restriction of χ to any $U_{(\psi+k)}$ is non-trivial. In particular, it is clear there exists a non-degenerate character χ of $\mathcal{G}_{x_0,k'}/\mathcal{G}_{x_0,k''}$ for any integer $k \geq 0$. For convenience, we identify a function on $\mathcal{G}_{x_0,k'}/\mathcal{G}_{x_0,k''}$ with its inflation to the group $\mathcal{G}_{x_0,k'}$.

Lemma 3-17. *Let p be a prime such that G is unramified over $\mathbb{Q}_p$. Let $\mathcal{I}_{k^+}$ $(k \geq 0)$ denote the subgroup in* (3-15)*, and let χ be a non-degenerate character of $\mathcal{G}_{x_0,k'}/\mathcal{G}_{x_0,k''}$. Then,*

(i) *The inflation of χ to $\mathcal{G}_{x_0,k'}$, when extended to $\mathcal{G}$ by zero outside $\mathcal{G}_{x_0,k'}$, is a cusp form of $\mathcal{G}$.*
(ii) *For each $k \geq 0$, there exists an irreducible supercuspidal representation (ρ_p, W_p) which has a non-zero $\mathcal{I}_{(k+1)^+}$–invariant vector but no non-zero $\mathcal{I}_{(k)^+}$–invariant vector.*

Proof. To prove part (i), suppose $x \in \mathcal{G} = G(\mathbb{Q}_p)$, and $\mathcal{U} = U(\mathbb{Q}_p)$ is the unipotent radical of a proper parabolic subgroup $\mathcal{P} = P(\mathbb{Q}_p)$ of $\mathcal{G}$. We need to show

$$\text{(3-18)} \qquad \int_{\mathcal{U}} \chi(xu)\; du \;=\; 0\,.$$

Take $Q \subset \mathcal{G}$ to be a $\mathbb{Z}_p$-defined parabolic subgroup so that $\mathcal{Q} = Q(\mathbb{Q}_p)$ is $\mathcal{G}$ conjugate to $\mathcal{P}$, i.e., $P = gQg^{-1}$, with $g \in \mathcal{G}$. Let V and $\mathcal{V}$ denote the unipotent radical of Q, and its group of k_v-rational points. We use the Iwasawa decomposition $\mathcal{G} = \mathcal{K}\mathcal{Q}$ to write g as $g = kh$. Then,

$$\text{(3-19)} \qquad \begin{aligned} \int_{\mathcal{U}} \psi_Y(xu)\; du \;&=\; \int_{\mathcal{V}} \psi_Y(\,x\,gvg^{-1}\,)\; dn \quad (u = gvg^{-1}) \\ &=\; \int_{\mathcal{V}} \psi_Y(\,x\,khvh^{-1}k\,)\; du \\ &=\; c\int_{\mathcal{V}} \psi_Y(\,x\,kvk^{-1}\,)\; dv \quad \text{(for a suitable constant } c\text{)} \\ &=\; c\int_{k\mathcal{V}k^{-1}} \psi_Y(\,x\,n\,)\; dn\ . \end{aligned}$$

In particular, we can reduce to the case where the parabolic P is a $\mathbb{Z}_p$-defined subgroup of $\mathcal{G}$. But, then P is $\mathcal{K}$-conjugate to a standard parabolic subgroup of $\mathcal{G}$ with respect to the maximal split torus A. So, we can and do assume P is a standard parabolic.

Observe that since $\mathrm{supp}(\chi) = \mathcal{G}_{x_0,k'}$, to show (3-18), it suffices to take $x \in \mathcal{G}_{x_0,k'}$. Then, $xu \in \mathcal{G}_{x_0,k'}$ if and only if $u \in \mathcal{G}_{x_0,k'} \cap \mathcal{U}$, so

$$\int_{\mathcal{U}} \chi(xu)\ du \ = \ \int_{\mathcal{G}_{x_0,k'}\ \cap\ \mathcal{U}} \chi(xu)\ du \tag{3-20}$$

The intersection $\mathcal{G}_{x_0,k'} \cap \mathcal{U}$ is a product of affine root subgroups. Combining this with the fact that χ is a character, we see that the integral over $\mathcal{G}_{x_0,k'} \ \cap\ \mathcal{U}$ is a product of integrals over the affine root subgroups. Since U is the radical of a proper standard parabolic subgroup, at least one of the A-roots in U is the gradient of an affine root $\psi \in \Delta^{\text{aff}}$. But then

$$\int_{\mathcal{G}_{x_0,k'}\ \cap\ \mathcal{U}_\psi} \chi(xu)\ du \ = \ 0 \tag{3-21}$$

since χ is a non-trivial character of $\mathcal{U}_{(\psi+k)} = \mathcal{G}_{x_0,k'} \ \cap\ \mathcal{U}_\psi$. Thus, χ is a cusp form. This completes the proof of part (i).

To prove part (ii), let V_χ denote the vector space spanned by left translations of χ. That χ is a cusp form of $\mathcal{G}$ means V_χ, as a representation of $\mathcal{G}$, is a direct sum of finitely many irreducible cuspidal representations, and by Frobenius reciprocity each irreducible cuspidal representation σ which appears when restricted to $\mathcal{G}_{x_0,k'}$ contains the character χ. In particular, σ contains a non-zero $\mathcal{G}_{x_0,k''}$ vector, whence a non-zero $\mathcal{I}_{(k+1)^+}$-fixed vector. The fact that σ contains the non-degenerate character χ and σ is assumed to be irreducible means it cannot have a $\mathcal{G}_{x_0,k'} = \mathcal{I}_{k^+}$-fixed vector. So (ii) holds.

□

4. Examples of open compact subgroups $\mathcal{F}$ satisfying assumptions (1-3)

We produce examples of finite sets $\mathcal{F}$ of open compact subgroups of $G(\mathbb{A}_f)$ satisfying the assumptions (1-3).

Suppose $G = \mathrm{SL}_M$.

- Fix a positive integer D. For each positive divisor d of D, set $\mathcal{K}_d$ as in (3-6). Then, as a consequence of Proposition (3-12), the finite family
$$\mathcal{F} \ = \ \{\, \mathcal{K}_d \ \big| \ d \,|\, D\,\}$$
satisfies the assumptions in (1-3). Whence, Theorem (1-4) applies to this family. As already mentioned in Proposition (3-7) $\mathcal{K}_d \cap \mathrm{SL}_M(\mathbb{Q})$ is the principal congruence subgroup $\Gamma(d)$ of (3-1).
- Fix a positive integer D. For each positive divisor d of D, set $\mathcal{J}_d$ as in (3-6). Then, as a consequence of Lemma (3-17), the finite family
$$\mathcal{F} \ = \ \{\, \mathcal{J}_d \ \big| \ d \,|\, D\,\}$$
satisfies the assumptions in (1-3). Whence, Theorem (1-4) applies to this family. Here, $\mathcal{K}_d \cap \mathrm{SL}_M(\mathbb{Q})$ is the subgroup $\Gamma_1(d)$ of (3-2).

Recall that we have been assuming G is simply connected, absolutely almost simple over $\mathbb{Q}$ and G_∞ is not compact. Let $S_f = \{p_1, \dots, p_r, p_{r+1}, \dots, p_{r+s}\}$ be primes satisfying the following:

(i) The group G is unramified at any prime $v \notin S_f$. For such a prime set $L_v = G(\mathbb{Z}_p)$.
(ii) For $1 \le i \le r$, we are given open compact subgroups $L_{p_i} \subset G(\mathbb{Q}_{p_i})$.
(iii) For $(r+1) \le i \le (r+s)$, the group G is unramified at p_i. For each p_i, take C_i to be an alcove in the Bruhat-Tits building and $x(C_i)$ the barycenter of C_i.

Fix some exponents $e_{r+1}, \dots, e_{r+s}$, and set

$$D \;=\; p_{r+1}^{e_{r+1}} \cdots p_j^{e_j} \cdots p_{r+s}^{e_{r+s}} \;.$$

For $d = p_{r+1}^{\alpha_{r+1}} \cdots p_{r+s}^{\alpha_{r+s}}$ a divisor of D, set

$$\mathcal{L}_d \;:=\; \prod_{i=1}^{r} L_{p_i} \prod_{i=(r+1)}^{(r+s)} G(\mathbb{Q}_{p_i})_{x(C_i),\alpha_i'} \prod_{v\notin S_f} G(\mathbb{Z}_p) \tag{4-1}$$

Then, $\mathcal{F} = \{\ \mathcal{L}_d \ \big|\ d \,|\, D\ \}$ is a family of open compact subgroups of $G(\mathbb{A}_f)$ satisfying the assumption (1-3). Whence, Theorem (1-4) applies to this family.

We note that if we had selected a different choice of alcoves C_i', then the groups $G(\mathbb{Q}_{p_i})_{x(C_i),\alpha_i'}$ and $G(\mathbb{Q}_{p_i})_{x(C_i'),\alpha_i'}$ are conjugate in $G(\mathbb{Q}_{p_i})$, say $g_{p,i}\, G(\mathbb{Q}_{p_i})_{x(C_i),\alpha_i'} g_{p,i}^{-1} = G(\mathbb{Q}_{p_i})_{x(C_i'),\alpha_i'}$. Denote by $\mathcal{L}_d'$, the open compact subgroup of $G(\mathbb{A}_f)$ obtained in (4-1) by replacing $G(\mathbb{Q}_{p_i})_{x(C_i),\alpha_i'}$ with $G(\mathbb{Q}_{p_i})_{x(C_i'),\alpha_i'}$. In $G(\mathbb{A}_f)$ the element

$$g = \prod_{i=1}^{r} 1_{G(\mathbb{Q}_{p_i})} \prod_{i=(r+1)}^{(r+s)} g_{p,i} \prod_{v\notin S_f} 1_{G(\mathbb{Q}_p)}$$

conjugates $\mathcal{L}_d$ to $\mathcal{L}_d'$. Since $G(\mathbb{A}_f)$ satisfies strong approximation, $g = g_{\mathbb{Q}} h_{\mathcal{L}_d}$, with $g_{\mathbb{Q}} \in G(\mathbb{Q})$ and $h_{\mathcal{L}_d} \in \mathcal{L}_d$. It follows $\mathcal{L}_d$ to $\mathcal{L}_d'$ are conjugate by the element $g_{\mathbb{Q}}$. In particular, the intersections

$$\mathcal{L}_d \cap G(\mathbb{Q}) \quad\text{and}\quad \mathcal{L}_d' \cap G(\mathbb{Q})$$

are conjugate by the element $g_{\mathbb{Q}}$ in $G(\mathbb{Q})$.

5. Some Additional Results for SL_M

In this section we let $G = SL_M$. We fix a finite non–emptyset of primes T. Put

$$\mathbb{N}_T = \{n \in \mathbb{N}; \nu_p(n) > 0 \iff p \in T\}.$$

For $n \in \mathbb{N}_T$, we define open–compact subgroup $\mathcal{K}_n$ of $G(\mathbb{A}_f)$ by (3-6). By our assumption $n \in \mathbb{N}_T$, the group

$$\mathcal{K}_{n/\prod_{p\in T} p}$$

is well–defined. We remind the reader (see Proposition 3-7) that the corresponding congruence subgroups are principal congruence subgroups $\Gamma(n)$ and $\Gamma(n/\prod_{p\in T} p)$.

We will use the results and constructions of Section 2. We consider $L = \mathcal{K}_n$. The set of primes T defined above will be the set of primes used in the Assumption 1-3. For each prime number p, we select the $f_p \in C_c^\infty(G(\mathbb{Q}_p))$ as in the paragraph after the statement of Lemma 2-7. But in addition to that we may require a little bit more on f_p for $p \in T$.

Lemma 5-1. *Using the notation introduced in (3-5), we may choose $f_p \in C_c^\infty(G(\mathbb{Q}_p))$, for $p \in T$, such that it is a cuspidal function (a finite sum of matrix coefficients of the supercuspidal representation π_p), right–invariant under $\mathcal{K}_{p,\nu_p(n)}$, $f_p(1) \neq 0$, and, which is new, such that* $\mathrm{supp}(f_p)$ *is contained in $\mathcal{K}_{p,\nu_p(n)-1}$.*

Proof. This follows from Proposition 3-12 (iii). □

Since $K_\infty = SO(M)$, we have

$$\max_{i,j} \max_{k=(k_{ij})\in K_\infty} |k_{ij}| = 1. \tag{5-2}$$

In the following lemmas, we prove some simple properties of the intersection of the principal congruence subgroups with K_∞.

Lemma 5-3. *Let $m \geq 1$. If $g \in \Gamma(m)$ is a diagonal element, then $g_{ii} \in \{\pm 1\}$ for all $i = 1, \ldots, M$.*

Proof. Since $g_{11}g_{22}\cdots g_{MM} = 1$, the claim follows. □

Lemma 5-4. *Let $m \geq 3$. If $g \in \Gamma(m)$ is a diagonal element, then $g = id$.*

Proof. By Lemma 5-3, $g_{ii} \in \{\pm 1\}$ for all $i = 1, \ldots, M$. Since $g_{ii} \equiv 1 \ (mod\ m)$, we obtain $m|(\pm 1 - 1)$. Finally, $m \geq 3$ implies that $g_{ii} = 1$ for all $i = 1, \ldots, M$. □

Lemma 5-5. *Let $m \geq 3$. Then $\Gamma(m) \cap K_\infty = \{id\}$.*

Proof. Let $g = (g_{ij}) \in \Gamma(m) \cap K_\infty$. Then, by definition of $\Gamma(m)$, $m|g_{ij}$ for $i \neq j$. But $|g_{ij}| \leq 1 < m$. Hence $g_{ij} = 0$ for $i \neq j$. Thus, g is a diagonal element of $\Gamma(m)$. Hence, Lemma 5-4 implies the claim. □

Now, thanks to Lemma 5-1, we can improve Lemma 2-7 considerably.

Lemma 5-6. *Let $n \in \mathbb{N}_T$, $n \geq 3\prod_{p\in T} p$. Then, <u>for any</u> $\delta \in \hat{K}_\infty$, there exists $f_\infty \in C_c^\infty(G_\infty)$ such that the following holds:*

- (i) *$E_\delta(f_\infty) = f_\infty$.*
- (ii) *Select f_p ($p \in T$) as in Lemma 5-1, and let $f_p = char_{G_{\mathbb{Z}_p}}$ for $p \notin T$. Put $f = f_\infty \otimes_p f_p$. Then the Poincaré series $P(f)$ and its restriction to G_∞ (which is a Poincaré series for $\Gamma(n)$) are non–zero.*
- (iii) *$E_\delta(P(f)) = P(f)$, $E_\delta(P(f)|_{G_\infty}) = P(f)|_{G_\infty}$, and $P(f)$ is right–invariant under $\mathcal{K}_n$.*

(iv) *The support of* $P(f)|_{G_\infty}$ *is contained in the set of the form* $\Gamma(n) \cdot C$*, where* C *is a compact set which is right–invariant under* K_∞*, and* $\Gamma(n) \cdot C$ *is not whole* G_∞*.*

(v) $P(f)$ *is cuspidal and* $P(f)|_{G_\infty}$ *is* $\Gamma(n)$*–cuspidal.*

Proof. We use Lemmas 2-3 and 2-7. We also use the notation introduced in the paragraph before and in Lemma 5-1. This meets all assumptions of Lemma 2-7 (with $L = \mathcal{K}_n$). We let $S = T \cup \{\infty\}$.

We need to study the intersection (2-8). In our case it is given by

$$\Gamma_S \cap \left[K_\infty \times \prod_{p \in T} \text{supp } (f_p) \right]. \tag{5-7}$$

Thanks to the Lemma 5-1, this is a subset of

$$\Gamma_S \cap \left[K_\infty \times \prod_{p \in T} \mathcal{K}_{p,\nu_p(n)-1} \right].$$

But projecting down to the first factor, this intersection becomes

$$K_\infty \cap \Gamma(n/\prod_{p \in T} p).$$

By Lemma 5-5, and our assumption $n \geq 3 \prod_{p \in T} p$, it is trivial. Whence, (5-7) consists of identity only. In particular, in (2-8), we have $K_\infty \cap \Gamma = \{1\}$, $l = 1$, $\gamma_1 = 1$, and $c_1 \neq 0$. We remark that $\Gamma = \Gamma(n)$ (see (2-6)).

Next, by Lemma 2-7, we need to study the map (2-9). Thanks to above computations, this map is $\alpha \mapsto c_1 \cdot \alpha$. Hence, it is essentially identity. Now, (i)–(iv) of the lemma follow from (i)–(iv) from Lemma 2-7 for any K_∞–type δ. Finally, (v) follows from Lemma 2-3, and ([8], Proposition 5.3). □

Now, we prove the main result of this section. It is analogous (and it generalizes) the main result of [10] (see [10], Theorem 0-1).

Theorem 5-8. *Let* $n \in \mathbb{N}_T$*,* $n \geq 3 \prod_{p \in T} p$*. Then, for any* $\delta \in \hat{K}_\infty$*, the orthogonal complement of*

$$\sum_{\substack{m|n \\ m<n}} L^2_{cusp}(\Gamma(m) \backslash G_\infty)$$

in $L^2_{cusp}(\Gamma(n) \backslash G_\infty)$ *contains a direct sum of infinitely many non–equivalent irreducible unitary representations of* G_∞ *all containing* δ*.*

Proof. The family of open compact subgroups

$$\mathcal{F} = \{\mathcal{K}_m : \quad 1 \leq m \leq n, \quad m|n\}$$

meet all assumptions of Assumption 1-3 with $\mathcal{K}_n$ contained in all other $\mathcal{K}_m \in \mathcal{F}$. (See Section 4.) Now, the proof is the same as the proof of Theorem 1-4. We leave the details to the reader.

□

Corollary 5-9. *Let $n \in \mathbb{N}_T$, $n \geq 3\prod_{p\in T} p$. Then the orthogonal complement of*

$$\sum_{\substack{m|n\\ m<n}} L^2_{cusp}(\Gamma(m)\backslash G_\infty)$$

in $L^2_{cusp}(\Gamma(n)\backslash G_\infty)$ contains a direct sum of infinitely many non–equivalent irreducible unitary K_∞–spherical representations of G_∞.

Corollary 5-10. *Let $n \in \mathbb{N}_T$, $n \geq 3\prod_{p\in T} p$. Then, for every $\delta \in \hat{K}_\infty$, the orthogonal complement of*

$$\sum_{\substack{m|n\\ m<n}} L^2_{cusp}(\Gamma(m)\backslash G_\infty)$$

in $L^2_{cusp}(\Gamma(n)\backslash G_\infty)$ contains a direct sum of infinitely many non–equivalent irreducible unitary representations of G_∞ all contaning δ which are not in the discrete series or in the limits of discrete series for G_∞.

Proof. As in ([10], Proposition 4.2). □

We warn the reader that $G_\infty = SL_M(\mathbb{R})$ has no discrete series when $M \geq 3$.

Let $P_\infty = M_\infty A_\infty N_\infty$ be the Langlands decomposition of a minimal parabolic subgroup of G_∞. We let $\mathfrak{a}_\infty$ be the real Lie algebra of A_∞ and $\mathfrak{a}^*_\infty$ its complex dual. We use Vogan's theory of minimal K_∞–types ([13], [14]). Any $\epsilon \in \hat{M}_\infty$ is fine ([14], Definition 4.3.8).

Let $\epsilon \in \hat{M}_\infty$. Following ([14], Definition 4.3.15), we let $A(\epsilon)$ be the set of K_∞–types δ such that δ is fine ([14], Definition 4.3.9) and ϵ occurs in $\delta|_{M_\infty}$. Applying ([14], Theorem 4.3.16), we obtain that $A(\epsilon)$ is not empty and for $\delta \in A(\epsilon)$, we have the following:

$$\delta|_M = \oplus_{\epsilon' \in \{w(\epsilon);\ w\in W\}} \epsilon', \tag{5-11}$$

where $W = N_{K_\infty}(A_\infty)/M_\infty$ is the Weyl group of A_∞ in G_∞. Since the restriction map implies $\mathrm{Ind}_{M_\infty A_\infty N_\infty}^{G_\infty}(\epsilon \otimes \exp\nu(\ )) \simeq \mathrm{Ind}_{M_\infty}^{K_\infty}(\epsilon)$ as K_∞–representations, by Frobenius reciprocity and (5-11) we see for every $\nu \in \mathfrak{a}^*_\infty$ there exists a unique irreducible subquotient $J_{\epsilon\otimes\nu}(\delta)$ of $\mathrm{Ind}_{M_\infty A_\infty N_\infty}^{G_\infty}(\epsilon \otimes \exp\nu(\ ))$ containing the K_∞–type δ.

One important example is the case $\epsilon = \mathbf{1}_{M_\infty}$. Then $\mu = \mathbf{1}_{K_\infty} \in A(\mathbf{1}_M)$, and $J_{\epsilon\otimes\nu}(\delta)$ is the unique K_∞–spherical irreducible subquotient of $\mathrm{Ind}_{M_\infty A_\infty N_\infty}^{G_\infty}(\epsilon \otimes \exp\nu(\ ))$.

Corollary 5-12. *Let $n \in \mathbb{N}_T$, $n \geq 3\prod_{p\in T} p$. Let $\epsilon \in \hat{M}_\infty$. Then, for every $\delta \in A(\epsilon)$, there exists infinitely many $\nu \in \mathfrak{a}^*$ such that $J_{\epsilon\otimes\nu}(\delta)$ appears in the orthogonal complement of*

$$\sum_{\substack{m|n \\ m<n}} L^2_{cusp}(\Gamma(m)\backslash G_\infty)$$

in $L^2_{cusp}(\Gamma(n)\backslash G_\infty)$.

Proof. As in ([10], Theorem 4.8). ☐

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