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# On the Genericity of Cuspidal Automorphic Forms of $\mathrm{SO}_{2n+1}$

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# On the Genericity of Cuspidal Automorphic Forms of $\mathrm{SO}_{2n+1}$

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**Abstract**

We study the irreducible generic cuspidal support up to near equivalence for certain cuspidal automorphic forms of $\mathrm{SO}_{2n+1}$ (Theorem 3.2 and Theorem 4.1), by establishing refined arguments in the theory of local and global Howe duality and theta correspondences ([JngS03], [F95]) and in the theory of Langlands functoriality ([CKPSS01], [JngS03], [GRS01]). The results support a global analogy and generalization of a conjecture of Shahidi on the genericity of tempered local $L$-packets (Conjecture 1.1). The methods are expected to work for other classical groups.

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# 1 Introduction

An automorphic form is called *generic* if it has a nonzero Whittaker-Fourier coefficient. A precise definition will be given in §2. Generic automorphic forms have played distinguished roles in the modern theory of automorphic forms and $L$-functions. On one hand, it seems that generic automorphic forms are more accessible to analytic methods. On the other hand, it seems that non-generic automorphic forms are more sensitive to problems arising from arithmetic and number theory. It seems essential at least to the authors to study various relations between generic automorphic forms and non-generic ones. One such relation is the near equivalence relation between generic cuspidal automorphic representations and non-generic ones. Two irreducible automorphic representations are *nearly equivalent* if at almost all local places, their corresponding local components are equivalent.

The study of the near equivalence relation of automorphic forms originates from the theory of Hecke operators and Hecke eigenforms in the classical theory of modular forms. In the theory of automorphic representations, the notion of near equivalence classes attracts more attention because of the strong multiplicity one theorem for automorphic representations of $\mathrm{GL}(n)$ by Jacquet and Shalika ([JS81]), and because of the pioneer work of Piatetski-Shapiro on the Saito-Kurokawa lift, where the notion of CAP automorphic representations was introduced ([PS83]).

An irreducible cuspidal automorphic representation is called CAP if it is nearly equivalent to an irreducible constituent of an induced representation from an irreducible automorphic representation on a proper parabolic subgroup. CAP automorphic representations are cuspidal ones whose local components are expected to be non-tempered at almost all local places and hence should be counter-examples to the generalized Ramanujan conjecture. It turns out that CAP automorphic representations exist in general, although Jacquet and Shalika showed that when $G = \mathrm{GL}(n)$, no such cuspidal automorphic representations exist ([JS81]). From Arthur's conjecture ([Ar89]), CAP representations form global Arthur packets with degenerate Arthur parameters. The existence of CAP representations makes the theory for general reductive groups more delicate than the theory for general linear groups.

Up to near equivalence, the following conjecture describes a basic structure for the discrete spectrum of square integrable automorphic representations.

**Conjecture 1.1 (CAP Conjecture).** *Let $k$ be a number field and $\mathbb{A}$ be the ring of adeles of $k$. Let $G$ be a reductive algebraic group, which is $k$-quasisplit. For any given irreducible cuspidal automorphic representation $\pi$ of $G(\mathbb{A})$, there exists a standard parabolic subgroup $P = MN$ of $G$ ($P$ may equal $G$) and an irreducible generic cuspidal automorphic representation $\sigma$ of $M(\mathbb{A})$ such that $\pi$ is nearly equivalent to an irreducible constituent of the (normalized or unitarily) induced representation*

$$I(\sigma) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\sigma).$$

We remark first that a version of this conjecture for $G = \mathrm{Sp}(4)$ was stated in the work of I. Piatetski-Shapiro and D. Soudry ([Sdr90] and [PSS87]). Conjecture 1.1 is the natural extension of their conjecture. The point here is that the CAP conjecture above requires the cuspidal datum to be generic, which is clear for the case of $\mathrm{Sp}_4(\mathbb{A})$.

Secondly, if the parabolic subgroup $P$ in the conjecture is proper, the representation $\pi$ is a CAP automorphic representation or simply a CAP with respect to the cuspidal data $(P; \sigma)$. If the parabolic subgroup $P$ equals $G$, the representation $\pi$ is nearly equivalent to an irreducible generic (having nonzero Whittaker-Fourier coefficients) cuspidal automorphic representation of $G(\mathbb{A})$.

The main interest for us to introduce Conjecture 1.1 is its relation to Arthur's conjecture on the structure of the discrete spectrum of automorphic forms. According to Arthur's conjecture ([Ar89]), for each irreducible cuspidal automorphic representation $\pi$ of $G(\mathbb{A})$, there exists a parabolic subgroup $P = MN$ of $G$ and an irreducible cuspidal automorphic representation $\tau$ of $M(\mathbb{A})$, which is of Ramanujan type, i.e. each of the local components of $\tau$ is tempered. Following a conjecture of Shahidi ([Sh90]), each tempered local $L$-packet should contain a generic member, which is one of the basic assumptions in Arthur's conjecture, $\tau$ must be locally generic at almost all local places. Up to near equivalence, we may take $\tau$ to be locally generic at all local places. Hence the main difference between what comes from Arthur's conjecture and what comes from our CAP conjecture (Conjecture 1.1) is that this irreducible cuspidal automorphic representation $\tau$ of $M(\mathbb{A})$ should be taken to have a nonzero Whittaker-Fourier coefficient. In other words, we expect that an irreducible cuspidal automorphic representation $\pi$ of $G(\mathbb{A})$, which is locally generic at all local places, is nearly equivalent to an irreducible cuspidal automorphic representation of $G(\mathbb{A})$, which has a nonzero

Whittaker-Fourier coefficient. For this moment, we do not know how to prove this statement in general. However, Part (4) of Theorem 4.1 (or 1.3) of this paper shows that for an irreducible cuspidal automorphic representation $\pi$ of $\mathrm{SO}_{2n+1}(\mathbb{A})$ (the $k$-split odd special orthogonal group) with a Bessel model of special type (see §2.2 for definition), if one of the unramifield local components of $\pi$ is generic, then $\pi$ is nearly equivalent to an irreducible cuspidal automorphic representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$ which has a nonzero Whittaker-Fourier coefficient. Theorem 8.1 of [KRS92] also supports our expectation. More examples from the theory of automorphic forms will be provided in our future work.

One of the good features of the CAP conjecture (Conjecture 1.1) is that the irreducible generic cuspidal datum $(P, \sigma)$ attached to the irreducible cuspidal automorphic representation $\pi$ of $G(\mathbb{A})$ is expected to be unique up to association class. Hence it is an invariant attached to $\pi$ for classification up to nearly equivalence. See Theorem 3.2 (or 1.2) for the case of $G = \mathrm{SO}_{2n+1}$.

Finally, the CAP conjecture (Conjecture 1.1) may be viewed as a global analogy and generalization of the Shahidi conjecture, which asserts that every tempered local $L$-packet has a generic member. See our previous work on $p$-adic $\mathrm{SO}_{2n+1}$ ([JngS03] and [JngS04]) for the local generalization of Shahidi's conjecture. Over $\mathbb{R}$, Shahidi's conjecture was proved by D. Vogan ([V78]). The more general version of the CAP conjecture (Conjecture 1.1) can be stated to take care of irreducible cuspidal automorphic representations of $G'(\mathbb{A})$ where $G'$ is an inner form of the $k$-quasisplit $G$. We will not offer any further discussion here for this generalization of the CAP conjecture (Conjecture 1.1), but mention that some evidence in this aspect may be found in [PSS87], [W91], and [P06].

In this paper, we investigate Conjecture 1.1 for $G_n := \mathrm{SO}_{2n+1}$, which is the $k$-split odd special orthogonal group. We prove in §3 that the generic cuspidal data $(P; \sigma)$ in the CAP conjecture is an invariant for irreducible cuspidal automorphic representations up to near equivalence, which is the natural extension of Theorem 5.3 in [JngS03] and Part 2 of Theorem B in [GRS01]. More precisely, we prove in §3.

**Theorem 1.2 (Theorem 3.2).** *Let $(P; \sigma)$ and $(Q; \tau)$ be two pairs of generic cuspidal data of $G_n(\mathbb{A})$. If the two induced representations $\mathrm{Ind}_{P(\mathbb{A})}^{G_n(\mathbb{A})}(\sigma)$ and $\mathrm{Ind}_{Q(\mathbb{A})}^{G_n(\mathbb{A})}(\tau)$ share the same irreducible unramified local constituent at almost all local places, then the two induced representations are equivalent as (global) representations of $G_n(\mathbb{A})$.*

With this theorem at hand, it makes sense to determine the generic cuspidal datum attached to an irreducible cuspidal automorphic representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$ up to near equivalence, namely to determine the explicit structure of CAP automorphic representations. It should be noted that various cases of CAP automorphic representations with respect to certain proper parabolic subgroups have been constructed and characterized by many authors. Some remarks and references can be found in [GJR02]. These examples of CAP automorphic representations confirm the CAP conjecture (Conjecture 1.1) positively. However, it seems that not much progress has been made for the characterization of an irreducible non-generic cuspidal automorphic representation to be nearly equivalent to an irreducible generic cuspidal automorphic representation, that is, for the case when $P = G$. It is clear that such cuspidal automorphic representations are hard to construct. Their existence may follow from the Weyl law in the spectral theory and the stabilization of the Arthur-Selberg trace formula.

We introduce in §2 the Fourier coefficients and the Bessel models of automorphic forms attached to nilpotent orbits $\mathcal{O}_{[2r+1,1^{2(n-r)}]}$ given by partitions of the form $[2r+1, 1^{2(n-r)}]$ for $0 \leq r \leq n$. We consider the $k$-stable orbits. In the case of the subregular nilpotent orbit, the $k$-stable orbits are parameterized by the square classes of $k^\times$. From Proposition 2.1 and Proposition 2.2, when the $k$-stable class in the nilpotent orbit $\mathcal{O}_{[2r+1,1^{2(n-r)}]}$ is parameterized by the discriminant of the quadratic form defining $\mathrm{SO}_{2n+1}$, the non-vanishing of one such Fourier coefficient of an irreducible cuspidal automorphic representation $\pi$ of $\mathrm{SO}_{2n+1}(\mathbb{A})$ implies that $\pi$ has a nonzero Whittaker Fourier coefficient, which is the Fourier coefficient attached to the regular nilpotent orbit. In general, for an irreducible cuspidal automorphic representation $\pi$ of $\mathrm{SO}_{2n+1}(\mathbb{A})$, there should be a maximal index $r \in \{1, 2, \cdots, n-1\}$ such that $\pi$ has a nonzero Fourier coefficient attached to a $k$-rational orbit in $\mathcal{O}_{[2r+1,1^{2(n-r)}]}$. Such an index is expected to be related to the first occurrence of $\pi$ in the theory of theta correspondence and to the existence of a certain pole on the positive real axis of the standard $L$-function associated to $\pi$. Since our goal in this paper is to detect a family of irreducible cuspidal automorphic representations which are nearly equivalent to an irreducible generic cuspidal automorphic representation, we investigate how the Fourier coefficient attached to the subregular nilpotent orbit plays the role in our theory. The result can be stated as follows.

**Theorem 1.3 (Theorem 4.1).** *Let $\pi$ be an irreducible cuspidal automorphic* *representation of $G_n(\mathbb{A})$ with a Bessel model of special type $(D_\lambda, 1, \psi_{n,n-1;\lambda})$, where $\lambda \not\equiv d \mod (k^\times)^2$ (see (2.11) for the definition). Then $\pi$ enjoys the following four properties.*

*(1) The first occurrence of $\pi$ in the Witt tower $\widetilde{\mathrm{Sp}}_{2m}(\mathbb{A})$ for $m \geq 1$ satisfies*

$$n - 1 \leq m_{0,\psi}(\pi) \leq n.$$

*(2) Assume that $m_{0,\psi}(\pi) = n$, i.e. the $\psi$-theta lift $\Theta_{n,n}(\pi, \psi)$ of $\pi$ to $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ is cuspidal. Then either $\pi$ is nearly equivalent to an irreducible, generic, cuspidal, automorphic representation of $G_n(\mathbb{A})$, or $\pi$ is CAP with respect to the cuspidal data*

$$(P_1; (\alpha_\lambda \cdot |\cdot|^{\frac{1}{2}}) \otimes \sigma),$$

*where $P_1$ is the standard parabolic subgroup whose Levi part is isomorphic to $\mathrm{GL}_1 \times \mathrm{SO}_{2n-1}$, and $\sigma$ is a cuspidal representation of $\mathrm{SO}_{2n-1}(\mathbb{A})$, such that $\sigma \otimes \chi_\lambda$ is in the image of the $\psi^{-1}$-theta correspondence from a certain $\psi$-generic, cuspidal representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$.*

*(3) Assume that $\pi$ is generic at one finite place. Then $m_{0,\psi}(\pi) = n$.*

*(4) Assume that $\pi$ is generic at a finite odd place $\nu$, where $\pi_\nu$ is unramified and $\lambda$ is a unit. Then $\pi$ is nearly equivalent to a cuspidal generic representation. In particular, $\pi$ is locally generic almost everywhere.*

As a consequence, we obtain the rigidity of the local genericity of irreducible cuspidal automorphic representations, which have nonzero Bessel model of special type (§2.2).

**Corollary 1.4.** *Let $\pi$ be an irreducible cuspidal automorphic representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$ with Bessel models of special type. Then $\pi$ is either nearly equivalent to an irreducible generic cuspidal automorphic representation $\pi'$ of $\mathrm{SO}_{2n+1}(\mathbb{A})$, or has no generic local components at almost all finite local place.*

For this paper, the rigidity of local genericity of irreducible cuspidal automorphic representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$ is still conditional, however, general results of this type are expected, since it is compatible with Arthur's conjecture on the $\mathrm{SL}_2(\mathbb{C})$-types of the discrete automorphic representations ([Ar89],

[Ar04], and [Cl]). It is not hard to see that this corollary is a reformulation and generalization of the main result of [PSS87], which is for $\mathrm{SO}_5$. The proof is based on the ideas of [PSS87] and results from [F95]. Basically, it is a refined argument using local and global Howe duality and theta correspondences. To the authors' knowledge, this is the first result proved for higher rank groups in this aspect. The main idea here is to filter irreducible cuspidal automorphic representations by a family of Bessel models or Fourier-Jacobi coefficients, which has been used in many occasions in the theory automorphic forms and $L$-functions. In §4, we try to explain the idea by filtering cuspidal automorphic representations through the Bessel models attached to the regular and the subregular nilpotent orbits. Some interesting examples of irreducible cuspidal automorphic representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$ with the first occurrence being $n-1$ are constructed at the end of §4. As explained in Remark 4.7, the discussion in §4 does not quite complete the proof of Conjecture 1.1 for irreducible cuspidal automorphic representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$ with Bessel model of special type. However, this will be treated in our forthcoming work by establishing Moeglin's work ([M97a] and [M97b]) for the reductive dual pairs $(\mathrm{SO}_{2n+1}, \widetilde{\mathrm{Sp}}_{2m})$.

It is expected from a certain generalization of the automorphic descent ([GRS01]) that there exist irreducible cuspidal automorphic representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$ which have no non-trivial Bessel models attached to $\mathcal{O}_{[2r+1,1^{2(n-r)}]}$ for $r = n-1$ or $n$. The corresponding local theory has been suggested through the work of Moeglin and Waldspurger for $p$-adic fields ([M96] and [MW87]) and the orbit method for real or complex reductive groups ([V97]). Some interesting global results for $\mathrm{Sp}_{2m}$ have been worked out in [GRS03] and in [GG].

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# 2 Nilpotent Orbits and Bessel Models

Let $k$ be a field of characteristic zero. Let $(V_{2n+1}, (\cdot,\cdot))$ be a nondegenerate quadratic vector space over $k$ of dimension $2n+1$ with Witt index $n$. The symmetric bilinear form is given by

$$J_{2n+1}(d) = \begin{pmatrix} & & J_n \\ & d & \\ J_n & & \end{pmatrix} \tag{2.1}$$

where we define $J_n := \begin{pmatrix} & & 1 \\ & J_{n-2} & \\ 1 & & \end{pmatrix}$ inductively, and $d \in k^{\times}$. We may choose a basis

$$\{e_1, \cdots, e_n, e_0, e_{-n}, \cdots, e_{-1}\} \tag{2.2}$$

such that

$$(e_i, e_j) = \begin{cases} 1 & \text{if } j = -i, \\ 0 & \text{if } j \neq -i, \\ d & \text{if } j = i = 0. \end{cases}$$

For each $r \in \{1, 2, \cdots, n\}$, we have the following partial polarization

$$V_{2n+1} = X_r \oplus V_{2(n-r)+1} \oplus X_r^* \tag{2.3}$$

where $X_r$ is a totally isotropic subspace of dimension $r$ and $X_r^*$ is the dual of $X_r$ with respect to the non-degenerate bilinear form $(\cdot, \cdot)$, and the subspace $V_{2(n-r)+1}$ is the orthogonal complement of $X_r \oplus X_r^*$. Without loss of generality, we may assume that $X_r$ is generated by $e_1, \cdots, e_r$ and $X_r^*$ is generated by $e_{-r}, \cdots, e_{-1}$.

We first consider a family of special nilpotent orbits of the odd special orthogonal Lie algebra $\mathfrak{so}_{2n+1}(\bar{k})$, $\bar{k}$ is the algebraic closure of $k$, and then consider the $k$-rational orbit structures in case $k$ is a number field. Attached to such $k$-rational orbits, we define for an automorphic form, Fourier coefficients and Bessel models.

## 2.1 A family of special nilpotent orbits.

We first consider the absolute theory over the algebraic closure $\bar{k}$ of $k$. Without loss of generality, we may consider the theory over $\mathbb{C}$, the field of complex numbers.

We recall basic facts on nilpotent orbits of complex classical Lie algebras from [CM93]. For $\mathfrak{so}_{2n+1}(\mathbb{C})$, the nilpotent orbits or nilpotent conjugacy classes are in one to one correspondence with the partitions of $2n+1$ with even parts occurring with even multiplicity. Using the notation from [CM93], the set of such partitions is denoted by $\mathcal{P}_1(2n+1)$. We call such partitions *odd orthogonal partitions*.

For $r \in \{0, 1, 2, \cdots, n\}$, the partition $[2r+1, 1^{2(n-r)}]$ belongs to $\mathcal{P}_1(2n+1)$. By Theorem 6.3.7, [CM93], the partition $[2r+1, 1^{2(n-r)}]$ is special for every $r \in \{0, 1, 2, \cdots, n\}$ and also the involution of partitions stabilizes this family of special partitions. The nilpotent orbit corresponding to $[2r+1, 1^{2(n-r)}]$ is denoted by

$$\mathcal{O}_{n,r} = \mathcal{O}_{[2r+1, 1^{2(n-r)}]}.$$

The orbit corresponding to a special partition is called a special nilpotent orbit. It is easy to see that this family of special nilpotent orbits is linearly ordered. When $r = n$, $\mathcal{O}_{n,n} = \mathcal{O}_{[2n+1]}$ is the regular nilpotent orbit, and when $r = 0$, $\mathcal{O}_{n,0} = \mathcal{O}_{[1^{2n+1}]}$ is the zero nilpotent orbit.

The following discussion is trivial when $r = 0$. Hence we assume that $r \neq 0$. For each $r \in \{1, 2, \cdots, n\}$, there is a standard parabolic Lie subalgebra $\mathfrak{p}_n^r = \mathfrak{m}_n^r \mathfrak{n}_n^r$ (unique up to conjugation) with Levi subalgebra

$$\mathfrak{m}_n^r \cong (\mathfrak{gl}_1)^{\oplus r} \oplus \mathfrak{so}_{2(n-r)+1}$$

and nilpotent radical $\mathfrak{n}_n^r$, whose quotient modulo its derived subalgebra $[\mathfrak{n}_n^r, \mathfrak{n}_n^r]$ is given by

$$\mathfrak{n}_n^r / [\mathfrak{n}_n^r, \mathfrak{n}_n^r] \cong \mathbb{C}^{\oplus(r-1)} \oplus \mathbb{C}^{\oplus(2(n-r)+1)}.$$

Note that if $r = 0$, then $\mathfrak{p}_n^0 = \mathfrak{so}_{2n+1}$. Hence we may assume that the above space is zero when $r = 0$. When $r \neq 0$, the adjoint action of $\mathfrak{m}_n^r$ on $\mathfrak{n}_n^r$ induces an action of $(\mathrm{GL}_1)^r \times \mathrm{SO}_{2(n-r)+1}$ on the affine space $\mathbb{C}^{\oplus(r-1)} \oplus \mathbb{C}^{\oplus(2(n-r)+1)}$ such that for each

$$g = (t_1, t_2, \cdots, t_r; h) \in (\mathrm{GL}_1)^r \times \mathrm{SO}_{2(n-r)+1}$$

and each

$$v = (x_1, x_2, \cdots, x_{r-1}; u) \in \mathbb{C}^{\oplus(r-1)} \oplus \mathbb{C}^{\oplus(2(n-r)+1)},$$

we have

$$g \circ v = (t_1 t_2^{-1} x_1, t_2 t_3^{-1} x_2, \cdots, t_{r-1} t_r^{-1} x_{r-1}; t_r u h^{-1}). \qquad (2.4)$$

Moreover, the nilpotent orbit $\mathcal{O}_{[2r+1, 1^{2(n-r)}]}$ corresponds to the Zariski open orbit of $(\mathrm{GL}_1)^r \times \mathrm{SO}_{2(n-r)+1}$ on the affine space $\mathbb{C}^{\oplus(r-1)} \oplus \mathbb{C}^{\oplus(2(n-r)+1)}$. It is clear from the action given by (2.4) that we may choose the representatives of the Zariski open orbit to have the following form

$$v = (1, 1, \cdots, 1; u) \in \mathbb{C}^{\oplus(r-1)} \oplus \mathbb{C}^{\oplus(2(n-r)+1)},$$

that is, $x_1 = \cdots = x_{r-1} = 1$. Since the set of such vectors is stable under the the subgroup

$$(\mathrm{GL}_1)^\Delta \times \mathrm{SO}_{2(n-r)+1}$$

of $(\mathrm{GL}_1)^r \times \mathrm{SO}_{2(n-r)+1}$, where $(\mathrm{GL}_1)^\Delta$ is $\mathrm{GL}_1$, diagonally embedded inside $(\mathrm{GL}_1)^r$, this induces further the action of $\mathrm{GL}_1 \times \mathrm{SO}_{2(n-r)+1}$ on the affine space $\mathbb{C}^{\oplus(2(n-r)+1)}$ as follows: for each $(t; h) \in \mathrm{GL}_1 \times \mathrm{SO}_{2(n-r)+1}$ and each $u \in \mathbb{C}^{\oplus(2(n-r)+1)}$, we have

$$(t; h) \circ u = tuh^{-1}. \tag{2.5}$$

Consider the partial polarization of $V_{2n+1}$

$$V_{2n+1} = X_r \oplus V_{2(n-r)+1} \oplus X_r^*$$

as in (2.3), with $V_{2(n-r)+1} \cong \mathbb{C}^{\oplus(2(n-r)+1)}$. By Witt's theorem, we know that the affine space $V_{2(n-r)+1}$ with the group action of $\mathrm{GL}_1 \times \mathrm{SO}_{2(n-r)+1}$ is a regular prehomogeneous vector space and decomposes over $\mathbb{C}$ (or $\bar{k}$) into three orbits

$$V_{2(n-r)+1} = \{0\} \cup Z_0 \cup Z_1 \tag{2.6}$$

where $Z_0$ is the set of all nonzero isotropic vectors with respect to the bilinear form and $Z_1$ is the set of all anisotropic vectors with respect to the bilinear form. It is clear that $Z_1$ is the Zariski open orbit.

It is important to remark that the above discussion works as well if one replaces the ground field $\mathbb{C}$ by the algebraic closure of a number field or a local field of characteristic zero.

In the sequel, we denote by $k$ a number field or a local field of characteristic zero and by $\bar{k}$ the algebraic closure of $k$.

We will use the notion of $k$-stable orbits to discuss the $k$-rationality of the nilpotent orbits and parameterization. For a detailed discussion of $k$-stable orbits, we refer to [Ktw82]. The $k$-stable nilpotent orbits associated to the $\bar{k}$-nilpotent orbit $\mathcal{O}_{[2r+1,1^{2(n-r)}]}(\bar{k})$ are in one to one correspondence with the $k$-stable orbits associated to the $\bar{k}$-orbit $Z_1(\bar{k})$. Note that the $k$-rational orbit $Z_0(k)$ is $k$-stable, since Witt's theorem holds for number fields or local fields of characteristic zero.

If $r \neq n$, in the Zariski open $\bar{k}$-orbit $Z_1(\bar{k})$, one has the following parameterization of the $k$-stable orbits of $\mathrm{GL}_1(k) \times \mathrm{SO}_{2(n-r)+1}(k)$ in terms of the square classes of $k^\times$:

$$Z_1(k) = \cup_{\lambda \in k^\times/(k^\times)^2} Z_1^\lambda, \tag{2.7}$$

where the set $Z_1^\lambda$ is defined by

$$Z_1^\lambda := \{v \in V_{2(n-r)+1}(k) \mid (v,v) = \lambda \mod (k^\times)^2\}.$$

It is clear that the set $Z_1^\lambda$ is a single $k$-rational orbit of $\mathrm{GL}_1(k)\times\mathrm{SO}_{2(n-r)+1}(k)$. Note that $Z_0(k)$ is a single $k$-rational orbit of $\mathrm{GL}_1(k) \times \mathrm{SO}_{2(n-r)+1}(k)$.

If $r = n$, then $V_{2(n-r)+1}$ is one dimensional and has the decomposition

$$V_1(k) = \{0\} \cup k^\times. \tag{2.8}$$

Hence $Z_1(k) = k^\times$ is a single $\mathrm{GL}_1(k)$-orbit. Therefore the $k$-stable regular nilpotent orbit $\mathcal{O}_{[2n+1]}(k)$ is a single $k$-rational orbit.

**Proposition 2.1.** *Assume that $r \in \{1, 2, \cdots, n-1\}$. Let $Z_1^\lambda$ be defined as in (2.7). Given a vector $u_\lambda \in Z_1^\lambda$, the subspace $u_\lambda^\perp$ of $V_{2(n-r)+1}$ has dimension $2(n-r)$. The Witt index $\iota(u_\lambda^\perp)$ of $u_\lambda^\perp$ is given by the following formula*

$$\iota(u_\lambda^\perp) = \begin{cases} n-r, & \text{if } \lambda \equiv d \mod (k^\times)^2; \\ n-r-1, & \text{otherwise.} \end{cases}$$

*Here the number $d \in k^\times$ is given in (2.1).*

*Proof.* By (2.3), the quadratic subspace $V_{2(n-r)+1}$ is provided with the quadratic form defined by (2.1) with $n$ replaced by $n-r$. Hence the Witt index of $V_{2(n-r)+1}$ is $n-r$.

If $\lambda \equiv d \mod (k^\times)^2$, then $Z_1^\lambda = Z_1^d$. We may choose

$$u_\lambda = e_0.$$

Hence the subspace $u_\lambda^\perp$ is isomorphic to $e_0^\perp$ as subspaces of $V_{2(n-r)+1}$. It follows that the Witt index of $u_\lambda^\perp$ is $n-r$.

If $\lambda \not\equiv d \mod (k^\times)^2$, we may choose a pair of orthogonal vectors $u_\lambda, v_\lambda$ in the orthogonal complement of $e_0$ in $V_{2(n-r)+1}$ such that $\{u_\lambda, v_\lambda\}$ generates a hyperbolic plane and $u_\lambda \in Z_1^\lambda$. For example, we may choose

$$u_\lambda = e_n + \frac{\lambda}{2} e_{-n}, \qquad v_\lambda = e_n - \frac{\lambda}{2} e_{-n}.$$

Then we may express the quadratic space $V_{2(n-r)+1}$ as

$$V_{2(n-r)+1} = (ku_\lambda \perp kv_\lambda) \perp V'_{2(n-r-1)+1}$$

where the subspace $V'_{2(n-r-1)+1}$ is orthogonal to $ku_\lambda \perp kv_\lambda$, has the Witt index $n-r-1$, and is isomorphic to $V_{2(n-r-1)+1}$ (as in (2.3)) as quadratic spaces. Hence the orthogonal complement $u_\lambda^\perp$ of $u_\lambda$ in $V_{2(n-r)+1}$ is $kv_\lambda \perp V'_{2(n-r-1)+1}$. Since $\lambda \not\equiv d \mod (k^\times)^2$, the pair $\{v_\lambda, e_0\}$ forms an anisotropic two-dimensional quadratic subspace in $u_\lambda^\perp$. Therefore the Witt index of $u_\lambda^\perp$ is $n-r-1$ in this case. The proof is completed. $\square$

The $k$-rational property of $Z_1^\lambda$ has in fact interesting implications to cuspidal automorphic forms, which will be discussed next.

## 2.2 Bessel models of automorphic forms.

As in §2.1, we have the partial polar decomposition for each $r \in \{1, 2, \cdots, n\}$:

$$V_{2n+1} = X_r \oplus V_{2(n-r)+1} \oplus X_r^*$$

Then $\mathrm{GL}(X_r)$ is isomorphic to $\mathrm{GL}(r)$. Let $U_r$ be the standard maximal unipotent subgroup of $\mathrm{GL}(r)$. Recall from §2.1 that elements in $N_n^r$ (the unipotent radical, corresponding to $\mathfrak{n}_n^r$,) can be written as

$$n = \begin{pmatrix} u & x & z \\ & I_{2(n-r)+1} & x^* \\ & & u^* \end{pmatrix} \in \mathrm{SO}_{2n+1}$$

where $u \in U_r$. Let $\mathbb{A}$ be the ring of adeles of $k$ and let $\psi$ be a nontrivial additive character of $\mathbb{A}$ which is trivial on $k$. We define a character $\psi_{n,r;\lambda}$ of $N_n^r(\mathbb{A})$ by

$$\psi_{n,r;\lambda}(n) := \psi(u_{1,2} + \cdots + u_{r-1,r})\psi((x \cdot u_\lambda, e_r)). \tag{2.9}$$

Recall that we may view $x$ as an element in $\mathrm{Hom}(V_{2(n-r)+1}, X_r)$. Here, $u_\lambda$ is an element of $Z_1^\lambda$, i.e. it is an anisotropic vector in $V_{2(n-r)+1}$ such that

$$(u_\lambda, u_\lambda) = \lambda.$$

To define the character $\psi_{n,r;\lambda}$, we have to choose such a $u_\lambda$ first and fix it. If one chooses any other vector in the $k$-rational orbit $Z_1^\lambda$, one may replace $\lambda$ in the character $\psi_{n,r;\lambda}$ by another number in the square class of $\lambda$. It is clear that $\psi_{n,r;\lambda}$ is trivial on $N_n^r(k)$.

Now we are ready to define the Fourier coefficient of automorphic forms attached to the nilpotent orbit $\mathcal{O}_{[2r+1,1^{2(n-r)}]}$. Let $\varphi$ be an automorphic form on $\mathrm{SO}_{2n+1}(\mathbb{A})$. We define

$$\mathcal{F}^{\psi_{n,r;\lambda}}(g;\varphi) := \int_{N_n^r(k)\backslash N_n^r(\mathbb{A})} \varphi(ng)\psi_{n,r;\lambda}^{-1}(n)dn \tag{2.10}$$

If the integral is not identically zero, we say that the automorphic form $\varphi$ has a nonzero Fourier coefficient attached to the nilpotent orbit $\mathcal{O}_{[2r+1,1^{2(n-r)}]}$.

Let $D_\lambda$ be the connected component of the identity of the stabilizer of the vector

$$v = (1, 1, \cdots, 1; u_\lambda)$$

in the Levi subgroup $M_n^r$. Then $D_\lambda$ is isomorphic to the algebraic group $\mathrm{SO}(u_\lambda^\perp)$, where as before $u_\lambda^\perp$ is the orthogonal complement of $u_\lambda$ in $V_{2(n-r)+1}$. Hence $D_\lambda$ is a $k$-rational form of $\mathrm{SO}_{2(n-r)}$.

Again if $r = 0$, then $N_n^0 = \{I_{2n+1}\}$, and the integral in (2.10) reduces to evaluation at $g$ for $g \in \mathrm{SO}_{2n+1}(\mathbb{A})$, i.e.

$$\mathcal{F}^{\psi_{n,0;\lambda}}(g;\varphi) = \varphi(g).$$

In this case we define $D_\lambda$ to be $\mathrm{SO}_{2n}$.

If $r = n$, then the Fourier coefficient $\mathcal{F}^{\psi_{n,n;\lambda}}(g;\varphi)$ is the usual Whittaker-Fourier coefficient. Since the $k$-stable regular nilpotent orbit $\mathcal{O}_{[2n+1]}(k)$ is a single $k$-rational orbit, the non-vanishing of the Whittaker-Fourier coefficient $\mathcal{F}^{\psi_{n,n;\lambda}}(g;\varphi)$ is independent of the choice of $\lambda$. In this case, $D_\lambda$ is the identity.

In general, the Fourier coefficient $\mathcal{F}^{\psi_{n,r;\lambda}}(g;\varphi)$ is left $D_\lambda(k)$-invariant, and hence it produces an automorphic function on $D_\lambda(k)\backslash D_\lambda(\mathbb{A})$.

Let $\phi$ be an automorphic form on $D_\lambda(\mathbb{A})$. Then one can formally define

$$\mathcal{B}^{\psi_{n,r;\lambda}}(\varphi,\phi) := \int_{D_\lambda(k)\backslash D_\lambda(\mathbb{A})} \mathcal{F}^{\psi_{n,r;\lambda}}(h;\varphi)\phi(h)dh. \tag{2.11}$$

If this integral is not identically zero, we say that $\varphi$ has a nonzero Bessel model of type $(D_\lambda, \phi, \psi_{n,r;\lambda})$. Of course we have to assume the convergence of the integral in order to define the Bessel model by such an integral.

## 2.3 Fourier coefficients $\mathcal{F}^{\psi_{n,r;\lambda}}$ for $\lambda \equiv d \mod (k^\times)^2$.

Recall that the group $G_n = \mathrm{SO}_{2n+1}$ is $k$-split, and associated to the symmetric matrix (2.1)

$$J_{2n+1}(d) = \begin{pmatrix} & & J_n \\ & d & \\ J_n & & \end{pmatrix}.$$

It is clear that the number $d$ is uniquely determined up to square classes in $k^\times$.

**Proposition 2.2.** *Assume that $\lambda \in k^\times$ satisfies $\lambda \equiv d \mod (k^\times)^2$ and $r = n-1$. An irreducible cuspidal automorphic representation $\pi$ of $G_n(\mathbb{A})$ has a non-zero Fourier coefficient $\mathcal{F}^{\psi_{n,n-1;\lambda}}(\cdot, \varphi_\pi)$ attached to the nilpotent orbit $\mathcal{O}_{[2n-1,1^2]}$ if and only if $\pi$ is generic, i.e. $\pi$ has a non-zero Whittaker-Fourier coefficient. Note that the Whittaker-Fourier coefficient is the one attached to the regular nilpotent orbit $\mathcal{O}_{[2n+1]}$.*

*Proof.* The proof is based on the Rankin-Selberg integrals given by Novodvorski in [N75], which represent the tensor product $L$-function for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_1$. These integrals have the form $\mathcal{B}^{\psi_{n,n-1;\lambda}}(\varphi_\pi, \nu \otimes |\cdot|^s)$ (see (2.11) for $r = n-1$), and $D_\lambda \cong k^\times$. Hence automorphic forms $\phi$ on $D_\lambda(\mathbb{A})$ are of the form $\nu \otimes |\cdot|^s$, where $\nu$ is a character of $k^\times \backslash \mathbb{A}^\times$. The main identity which makes these integrals Eulerian is

$$\begin{aligned}
&\mathcal{B}^{\psi_{n,n-1;\lambda}}(\varphi_\pi, \nu \otimes |\cdot|^s) \\
= &\int_{k^\times \backslash \mathbb{A}^\times} \mathcal{F}^{\psi_{n,n-1;\lambda}}(m(\begin{pmatrix} & I_{n-1} \\ 1 & \end{pmatrix} \cdot \begin{pmatrix} x & \\ & I_{n-1} \end{pmatrix}), \varphi_\pi)\nu(x)|x|^{s-\frac{1}{2}} d^\times x \\
= &\int_{\mathbb{A}^\times} \int_{\mathbb{A}^{n-1}} W_{\varphi_\pi}^{\psi_{n,n;d}}(m(\begin{pmatrix} 1 & 0 \\ y & I_{n-1} \end{pmatrix})m(\begin{pmatrix} x & \\ & I_{n-1} \end{pmatrix}))dy\nu(x)|x|^{s-\frac{1}{2}} d^\times x
\end{aligned}$$

.

Here the real part of $s$ is sufficiently large, and $W_{\varphi_\pi}^{\psi_{n,n;d}}(g)$ is the Whittaker-Fourier coefficient of the automorphic form $\varphi_\pi$ with respect to the generic character $\psi = \psi_{n,n;d}$. For $g \in \mathrm{GL}_n$, we denote $m(g) = \begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix}$. If $\mathcal{F}^{\psi_{n,n-1;\lambda}}$ is non-zero, then there exists a character $\nu$, such that the Mellin transform $\mathcal{B}^{\psi_{n,n-1;\lambda}}(\varphi_\pi, \nu \otimes |\cdot|^s)$ is non-zero, for some $\varphi_\pi$. Hence the last integral is non-zero, and so is $W_{\varphi_\pi}^{\psi_{n,n;d}}$. Therefore, $\pi$ is generic.

Conversely, if $W_{\varphi_\pi}^{\psi_{n,n;d}}$ is non-trivial, then we use the fact that, for decomposable data, the last integral is a product of the corresponding local integrals over all places. For each local integral, and for each fixed real value $s_0$, which we may take to be very large, we may choose data, at that local place, so that the local integral is holomorphic and nonzero at $s_0$.The local integrals, at the places $\omega$, where all data are unramified, are equal to the local $L$-function $L(s, \pi_\omega \otimes \nu)$. Thus, for a suitable choice of data, we get that the last integral equals $a_S(s, \varphi_\pi) L^S(s, \pi \otimes \nu)$, where S is a finite set of places, outside of which all data are unramified, and $a_S(s, \varphi_\pi)$ is the product of the local integrals at $S$, which, as we just mentioned, is holomorphic and non-zero at $s_0$, by our choice of data. The partial $L$-function $L^S(s, \pi \otimes \nu)$ is, of course, non-zero at $s_0$. We conclude that $\mathcal{B}^{\psi_{n,n-1;\lambda}}(\varphi_\pi, \nu \otimes |\cdot|^{s_0})$ is non-zero. Hence the Fourier coefficient $\mathcal{F}^{\psi_{n,n-1;\lambda}}$ is non-zero. □

**Remark 2.3.** *From the last proof, it is clear that if $\pi$ is generic and the central L-value $L^S(\frac{1}{2}, \pi \otimes \nu)$ is non-zero, then $\pi$ has a Bessel model of type $(R_\lambda, \nu, \psi_\lambda)$, where $\lambda \equiv d \mod (k^\times)^2$, and $\nu$ is viewed as a character of $D_\lambda$ in the case of $r = n - 1$.*

**Remark 2.4.** *By using Ginzburg's work ([G90]), which generalizes [N75] to represent the tensor product L-functions for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_m$ (generic cases) and the same argument used in the proof of Proposition 2.2, we can show that for $\lambda \in k^\times$ with $\lambda \equiv d \mod (k^\times)^2$ and $r \in \{1, 2, \cdots, n-2\}$, an irreducible cuspidal automorphic representation $\pi$ of $G_n(\mathbb{A})$ is generic if and only if $\pi$ has a non-zero Bessel model of type $(D_\lambda, E(\cdot, \Phi_{\tau,s}); \psi_{n,r;\lambda})$ for the real part of $s$ large, where $E(\cdot, \Phi_{\tau,s})$ is the Eisenstein series of $\mathrm{SO}_{2(n-r)}(\mathbb{A})$ with cuspidal datum $(P_{n-r}, \tau)$ supported on the Siegel parabolic subgroup $P_{n-r}$ of $\mathrm{SO}_{2(n-r)}$. Such Bessel models involve the central value of the tensor product L-function for generic cuspidal automorphic representations $\pi$ of $\mathrm{SO}_{2n+1}$ and $\tau$ of $\mathrm{GL}_{n-r}$. For more detailed discussion in this aspect, we refer to [GJR04] and [GJR].*

# 3 Rigidity of generic cuspidal data in Conjecture 1.1

In this section, we prove that generic cuspidal data as in Conjecture 1.1 are invariants for the classification of automorphic representations up to near equivalence.

It is a well-known theorem that every irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ is generic (i.e. having non-zero Whittaker-Fourier coefficients). This follows from the Whittaker-Fourier expansion of cuspidal automorphic forms of $\mathrm{GL}_n(\mathbb{A})$. Moreover, Jacquet and Shalika proved the following theorem.

**Theorem 3.1 (Theorem 4.4 [JS81]).** *Let $(P;\sigma)$ and $(Q;\tau)$ be two pairs of cuspidal data of $\mathrm{GL}(n)$. If the two induced representations $\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})}(\sigma)$ and $\mathrm{Ind}_{Q(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})}(\tau)$ share the same irreducible unramified local constituents at almost all places, then the two pairs of cuspidal data are associate.*

It follows that there are no CAP automorphic representations on $\mathrm{GL}_n(\mathbb{A})$. We prove an analogue of Jacquet-Shalika's Theorem for $G_n := \mathrm{SO}_{2n+1}$ with generic cuspidal data. For the parabolic subgroup $P = G_n$, this was proved in [JngS03] and in [GRS01].

**Theorem 3.2.** *Let $(P;\sigma)$ and $(Q;\tau)$ be two pairs of generic cuspidal data of $G_n(\mathbb{A})$. If the two induced representations $\mathrm{Ind}_{P(\mathbb{A})}^{G_n(\mathbb{A})}(\sigma)$ and $\mathrm{Ind}_{Q(\mathbb{A})}^{G_n(\mathbb{A})}(\tau)$ share the same irreducible unramified local constituent at almost all places, then $(P;\sigma)$ and $(Q;\tau)$ are associate.*

*Proof.* We shall use the Langlands functorial lift from $\Pi^{gc}(G_m)$, the set of equivalence classes of irreducible generic cuspidal automorphic representations of $\mathrm{SO}_{2m+1}(\mathbb{A})$, to $\Pi(\mathrm{GL}_{2m})$, the set of equivalence classes of irreducible unitary automorphic representations of $\mathrm{GL}_{2m}$.

Write the Levi decomposition $P = MN$ with

$$M \cong \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r} \times G_{n_0}, \tag{3.1}$$

and similarly, $Q = LU$ with

$$L \cong \mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_l} \times G_{m_0}. \tag{3.2}$$

Then we have

$$\sigma = |\cdot|^{s_1} \cdot \sigma_1 \otimes \cdots \otimes |\cdot|^{s_r} \cdot \sigma_r \otimes \sigma_0 \tag{3.3}$$

$$\tau = |\cdot|^{t_1} \cdot \tau_1 \otimes \cdots \otimes |\cdot|^{t_l} \cdot \tau_l \otimes \tau_0. \tag{3.4}$$

Without loss of generality, we may assume that the $s_i$'s and $t_j$'s are real numbers and satisfy

$$s_1 \geq \cdots \geq s_r \geq 0, \tag{3.5}$$

$$t_1 \geq \cdots \geq t_l \geq 0, \tag{3.6}$$

and the representations $\sigma_i$'s and $\tau_j$'s are irreducible, unitary, automorphic, cuspidal, and generic.

By the explicit Langlands functorial lift from $\Pi^{gc}(G_m)$ to $\Pi(\mathrm{GL}_{2m})$ (see [GRS01] and [CKPSS01]), there are irreducible self-dual unitary cuspidal automorphic representations $\xi_1, \cdots, \xi_p$ and $\eta_1, \cdots, \eta_q$ such that (1) $\xi_i \ncong \xi_j$, and $\eta_i \ncong \eta_j$ if $i \neq j$; (2) the exterior square $L$-functions $L(\xi_i, \Lambda^2, s)$ and $L(\eta_j, \Lambda^2, s)$ have a pole at $s = 1$ for $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$; and (3) the images of the Langlands functorial lifts of $\sigma_0$ and $\tau_0$ are

$$\xi_1 \boxplus \cdots \boxplus \xi_p := \mathrm{Ind}_{P_{e_1,\cdots,e_p}(\mathbb{A})}^{\mathrm{GL}_{2n_0}(\mathbb{A})}(\xi_1 \otimes \cdots \otimes \xi_p) \tag{3.7}$$

and

$$\eta_1 \boxplus \cdots \boxplus \eta_q := \mathrm{Ind}_{P_{f_1,\cdots,f_q}(\mathbb{A})}^{\mathrm{GL}_{2m_0}(\mathbb{A})}(\eta_1 \otimes \cdots \otimes \eta_q), \tag{3.8}$$

respectively. Hence we may lift the induced representations $\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{SO}_{2n+1}(\mathbb{A})}(\sigma)$ and $\mathrm{Ind}_{Q(\mathbb{A})}^{\mathrm{SO}_{2n+1}(\mathbb{A})}(\tau)$ to the representations of $\mathrm{GL}_{2n}(\mathbb{A})$ parabolically induced from

$$|\cdot|^{s_1}\sigma_1 \otimes \cdots \otimes |\cdot|^{s_r}\sigma_r \otimes \xi_1 \otimes \cdots \otimes \xi_p \otimes |\cdot|^{-s_r}\sigma_r^\vee \otimes \cdots \otimes |\cdot|^{-s_1}\sigma_1^\vee, \tag{3.9}$$

and

$$|\cdot|^{t_1}\tau_1 \otimes \cdots \otimes |\cdot|^{t_l}\tau_l \otimes \eta_1 \otimes \cdots \otimes \eta_q \otimes |\cdot|^{-t_l}\tau_l^\vee \otimes \cdots \otimes |\cdot|^{-t_1}\tau_1^\vee, \tag{3.10}$$

respectively.

By assumption, we know that the last two induced representations of $\mathrm{GL}_{2n}(\mathbb{A})$ ((3.9) and (3.10)) have the same irreducible local unramified constituents at almost all local places. By the Theorem of Jacquet and Shalika, we know that the last two induced representations are equivalent, in the following sense: the two corresponding parabolic subgroups $P_{n_1,\cdots,n_r;e_1,\cdots,e_p;n_r,\cdots,n_1}$ and $P_{m_1,\cdots,m_l;f_1,\cdots,f_q;m_l,\cdots,m_1}$ and the two cuspidal representations in (3.9) and (3.10), respectively, are associate. This means that $r = l$, $p = q$; and, after a reordering, $n_i = m_i$, $s_i = t_i$, and $\sigma_i \cong \tau_i$ for $i = 1, 2, \cdots, r$; and $e_j = f_j$ and $\xi_j \cong \eta_j$ for $j = 1, 2, \cdots, p$.

Finally, since both irreducible generic cuspidal automorphic representations $\sigma_0$ and $\tau_0$ share the same image

$$\xi_1 \boxplus \cdots \boxplus \xi_p$$

under the Langlands functorial lifting from $\mathrm{SO}_{2n+1}$ to $\mathrm{GL}_{2n}$, we conclude that $\sigma_0$ must be equivalent to $\tau_0$ by Theorem 5.2 of [JngS03], i.e. the Langlands functorial lifting is injective when restricted to the irreducible generic cuspidal automorphic representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$. Therefore, the cuspidal data $(P;\sigma)$ and $(Q;\tau)$ are associate. □

Theorem 3.2 has the following consequences which are important to the understanding of the CAP conjecture (Conjecture 1.1) and the structure of the discrete spectrum of $\mathrm{SO}_{2n+1}(\mathbb{A})$.

**Corollary 3.3.** *With notations as above, we have*

*(1) Any irreducible generic cuspidal automorphic representation $\pi$ of $G_n(\mathbb{A})$ can not be a CAP with respect to a generic, proper, cuspidal datum.*

*(2) If two pairs of generic cuspidal data $(P;\sigma)$ and $(Q;\tau)$, are nearly associate, i.e. their local components are associate at almost all local places, then they are globally associate.*

*(3) The generic cuspidal datum $(P;\sigma)$ in Conjecture 1.1 is an invariant for irreducible automorphic representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$ up to near equivalence.*

# 4 Non-generic Automorphic Forms

As remarked at the end of §2, the $k$-rational structure of the nilpotent orbit with which the Fourier coefficient is defined is expected to have deep impact on the structure of irreducible cuspidal automorphic representations. We consider here the case when the nilpotent orbit is of subregular type.

We first investigate irreducible cuspidal automorphic representations with Bessel models of special type and their relation with generic cuspidal automorphic representations.

We recall from 2.2 that a Bessel model of an irreducible cuspidal automorphic representation $\pi$ of $G_n(\mathbb{A})$ attached to the sub-regular nilpotent orbit $\mathcal{O}_{[2n-1,1^2]}$ is given by the integral in (2.11) for $r = n-1$. In this case, $D_\lambda$ is a $k$-rational form of SO(2), and the Bessel model is called of type $(D_\lambda, \nu, \psi_{n,n-1;\lambda})$, where $\nu$ is a character of $D_\lambda(\mathbb{A})$ that is trivial on $D_\lambda(k)$. When the character $\nu$ is trivial, the corresponding Bessel model is called *of special type*, i.e. of type $(D_\lambda, 1, \psi_{n,n-1;\lambda})$. Because of Proposition 3.4, we assume from now on that $\lambda \not\equiv d \mod (k^\times)^2$.

## 4.1 Bessel models of special type (nearly generic case).

Denote by $\alpha_\lambda$ the Hilbert symbol $(\cdot, \lambda)$, and by $\chi_\lambda$ the composition of $\alpha_\lambda$ with the spinor norm on $G_l = \mathrm{SO}_{2l+1}$. We shall show that an irreducible cuspidal automorphic representation of $G_n(\mathbb{A})$ with a Bessel model of special type $(D_\lambda, 1, \psi_{n,n-1;\lambda})$, where $\lambda \not\equiv d \mod (k^\times)^2$, has significant properties related to various theta liftings to metaplectic groups $\widetilde{\mathrm{Sp}}_{2m}(\mathbb{A})$ for $m \geq 1$.

The proof of these properties uses both global and local Howe correspondences. We recall first some basic facts on Weil representations and theta liftings in the special cases needed for the proof. The ideas go back to [PSS87], [F95], and [JngS03].

We consider the reductive dual pair $(\mathrm{O}_{2n+1}, \mathrm{Sp}_{2m})$ in a symplectic group $\mathrm{Sp}_{2m(2n+1)}$. Note that we only consider the $k$-split orthogonal group $\mathrm{O}_{2n+1}$. For a non-trivial additive character $\psi$ of $k\backslash\mathbb{A}$, one has the Weil representation $\omega_\psi$ of $\widetilde{\mathrm{Sp}}_{2m(2n+1)}(\mathbb{A})$ (the metaplectic double cover of $\mathrm{Sp}_{2m(2n+1)}(\mathbb{A})$), acting in the Schrödinger model $\mathcal{S}(\mathbb{A}^{(2n+1)m})$, the space of all Bruhat-Schwartz functions on $\mathbb{A}^{(2n+1)m}$. For basic formulas of the action of certain subgroups, we refer to §2.4.2 in [F95]. For a Bruhat-Schwartz function $\phi$ in $\mathcal{S}(\mathbb{A}^{(2n+1)m})$, one defines a theta function

$$\theta^\psi(x, \phi) := \sum_{\xi \in k^{(2n+1)m}} \omega_\psi(x)\phi(\xi)$$

for $x \in \widetilde{\mathrm{Sp}}_{2m(2n+1)}(\mathbb{A})$. For $(g, h) \in \mathrm{O}_{2n+1}(\mathbb{A}) \times \widetilde{\mathrm{Sp}}_{2m}(\mathbb{A})$, denote

$$\theta^\psi((g, h), \phi) := \sum_{\xi \in k^{(2n+1)m}} \omega_\psi(g, h)\phi(\xi).$$

Let $(\pi, V_\pi)$ be an irreducible cuspidal automorphic representation of $\mathrm{O}_{2n+1}(\mathbb{A})$. The following integral

$$\int_{\mathrm{O}_{2n+1}(k)\backslash \mathrm{O}_{2n+1}(\mathbb{A})} \theta^\psi((g, h), \phi)\varphi_\pi(g)dg,$$

where $\varphi_\pi \in V_\pi$, converges and defines an automorphic function on $\widetilde{\mathrm{Sp}}_{2m}(\mathbb{A})$. By varying $\phi$ in $\mathcal{S}(\mathbb{A}^{(2n+1)m})$ and $\varphi_\pi$ in $V_\pi$, the integrals generate a space of a genuine automorphic representation of $\widetilde{\mathrm{Sp}}_{2m}(\mathbb{A})$, which is denoted by $\Theta_{n,m}(\pi, \psi)$ and is called the $\psi$-theta lifting of $\pi$. We denote by $m_{0,\psi}(\pi)$ the first occurrence of $\pi$ in the Witt tower $\widetilde{\mathrm{Sp}}_{2m}$ for $m \geq 1$, that is, the $\psi$-theta lifting of $\pi$ to $\widetilde{\mathrm{Sp}}_{2m_0}(\mathbb{A})$ is non-zero and cuspidal, but the $\psi$-theta lifting to $\widetilde{\mathrm{Sp}}_{2r}(\mathbb{A})$ is zero for all $r \leq m_0 - 1$.

Similarly, we consider the theta liftings from $\widetilde{\mathrm{Sp}}_{2m}$ to the $k$-split Witt tower $\mathrm{O}_{2n+1}$ for $n \geq 1$. Let $(\widetilde{\pi}, V_{\widetilde{\pi}})$ be an irreducible genuine cuspidal automorphic representation of $\widetilde{\mathrm{Sp}}_{2m}(\mathbb{A})$. For any $\varphi_{\widetilde{\pi}} \in V_{\widetilde{\pi}}$ and $\phi \in \mathcal{S}(\mathbb{A}^{(2n+1)m})$, the integral

$$\int_{\mathrm{Sp}_{2m}(k)\backslash\widetilde{\mathrm{Sp}}_{2m}(\mathbb{A})} \theta^{\psi}((g,h),\phi)\varphi_{\widetilde{\pi}}(h)dh$$

converges and defines an automorphic function over $\mathrm{O}_{2n+1}(\mathbb{A})$. The space generated by all such integrals is denoted by $\Theta_{m,n}(\widetilde{\pi}, \psi^{-1})$, which is an automorphic representation of $\mathrm{O}_{2n+1}(\mathbb{A})$. We call $\Theta_{m,n}(\widetilde{\pi}, \psi^{-1})$ the $\psi^{-1}$-theta lifting of $\widetilde{\pi}$.

The first theorem we want to prove is stated as follows.

**Theorem 4.1.** *Let $\pi$ be an irreducible cuspidal automorphic representation of $G_n(\mathbb{A})$ with a Bessel model of special type $(D_\lambda, 1, \psi_{n,n-1;\lambda})$, where $\lambda \not\equiv d$* mod $(k^\times)^2$. *Then $\pi$ enjoys the following four properties.*

*(1) The first occurrence of $\pi$ in the Witt tower $\widetilde{\mathrm{Sp}}_{2m}(\mathbb{A})$ for $m \geq 1$ satisfies*

$$n - 1 \leq m_{0,\psi}(\pi) \leq n.$$

*(2) Assume that $m_{0,\psi}(\pi) = n$, i.e. the $\psi$-theta lift $\Theta_{n,n}(\pi, \psi)$ of $\pi$ to $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ is cuspidal. Then either $\pi$ is nearly equivalent to an irreducible, generic, cuspidal, automorphic representation of $G_n(\mathbb{A})$, or $\pi$ is CAP with respect to the cuspidal data*

$$(P_1; (\alpha_\lambda \cdot |\cdot|^{\frac{1}{2}}) \otimes \sigma),$$

*where $P_1$ is the standard parabolic subgroup whose Levi part is isomorphic to $\mathrm{GL}_1 \times \mathrm{SO}_{2n-1}$, and $\sigma$ is a cuspidal representation of $\mathrm{SO}_{2n-1}(\mathbb{A})$, such that $\sigma \otimes \chi_\lambda$ is in the image of the $\psi^{-1}$-theta correspondence from a certain $\psi$-generic, cuspidal representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$.*

*(3) Assume that $\pi$ is generic at one finite place. Then $m_{0,\psi}(\pi) = n$.*

*(4) Assume that $\pi$ is generic at a finite odd place $\nu$, where $\pi_\nu$ is unramified and $\lambda$ is a unit. Then $\pi$ is nearly equivalent to a cuspidal generic representation. In particular, $\pi$ is locally generic almost everywhere.*

This theorem will be proved step by step in the rest of this section.

We prove first Part (1) of Theorem 4.1, which we single out as a separate theorem.

**Theorem 4.2.** *Let $\pi$ be an irreducible cuspidal automorphic representation of $G_n(\mathbb{A})$ with a Bessel model of special type $(D_\lambda, 1, \psi_{n,n-1;\lambda})$, where $\lambda \not\equiv d \mod (k^\times)^2$. Then $\Theta_{n,m}(\pi,\psi) = 0$, for $m < n-1$, and $\Theta_{n,n}(\pi,\psi) \neq 0$. Thus, the first occurrence, $m_{0,\psi}(\pi)$, of the theta lift of $\pi$ to the Witt tower $\widetilde{\mathrm{Sp}}_{2m}(\mathbb{A})$ satisfies*

$$n-1 \leq m_{0,\psi}(\pi) \leq n.$$

*Proof.* By Proposition 1, [F95], the space $\Theta_{n,n}(\pi,\psi)$ is $\psi_\lambda$-generic. In particular, $\Theta_{n,n}(\pi,\psi)$ is a non-zero automorphic representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. Thus, $m_{0,\psi}(\pi) \leq n$. Here, the character $\psi_\lambda$ is as follows. Let $\widetilde{B} = \widetilde{T} \cdot U$ be the Borel subgroup of $\widetilde{\mathrm{Sp}}_{2n}$ with $U$ the maximal unipotent subgroup. If we denote, for $u \in U_{\mathbb{A}}$, by $x_i$ its coordinate associated to the $i$-th simple root $\alpha_i$ $(i = 1, 2, \cdots, n)$, then the character $\psi_\lambda$ is defined by

$$\psi_\lambda(u) := \psi(x_1 + \cdots + x_{n-1} + \frac{\lambda}{2} x_n).$$

Next we will show that $\Theta_{n,m}(\pi,\psi) = 0$, for $m < n-1$. Assume, for simplicity, that $d = 1$. The proof is based on the local analog of this statement, which is an extension of our previous work in [JngS03]. Fix a finite place $\nu$, and let $m < n-1$. We will show that there is no irreducible representation $\tau_\nu$ of $\widetilde{\mathrm{Sp}}_{2m}(k_\nu)$, such that

$$Hom(\omega_{\psi_\nu} \otimes \tau_\nu, \pi_\nu) \neq 0.$$

Here $\omega_{\psi_\nu}$ is the local Weil representation, of $\widetilde{\mathrm{Sp}}_{2m(2n+1)}(k_\nu)$, at the place $\nu$, restricted to the dual pair $(\mathrm{O}_{2n+1}(k_\nu), \widetilde{\mathrm{Sp}}_{2m}(k_\nu))$. The proof is entirely similar to [JngS03], p.754-759.

As in [JngS03], denote by $W_m$ the $2m$-dimensional symplectic space over $k_\nu$, on which $\widetilde{\mathrm{Sp}}_{2m}(k_\nu)$ acts. We take a right action. Choose two transversal Lagrangian subspaces,

$$Y_m := W_m^+, \quad \textit{and} \quad Y_m^* := W_m^-$$

of $W_m$, and choose a basis $f_1, \cdots, f_m$ of $Y_m$ and a dual basis $f_{-1}, \cdots, f_{-m}$ of $Y_m^*$. We realize the Weil representation $\omega_{\psi_\nu}$ in the mixed model

$$\mathcal{S}(X_{n-1}^* \otimes W_m \oplus V_3 \otimes Y_m).$$

We also use the notations in the beginning of Section 2. Let $\ell$ be a non-trivial linear Bessel functional of special type $(D_\lambda, 1, \psi_{n,n-1;\lambda})$, on the space $V_{\pi_\nu}$ of $\pi_\nu$, i.e. $l$ satisfies

$$\ell(\pi_\nu(tu)v) = \psi_{n,n-1;\lambda}(u)\ell(v)$$

Here, $t \in D_\lambda = D_\lambda(k_\nu), u \in N_n^{n-1} = N_n^{n-1}(k_\nu)$, and $v \in V_{\pi_\nu}$. We abbreviate $\psi_\nu$ to $\psi$. Composing $\ell$ with a nontrivial element of $\mathrm{Hom}(\omega_{\psi_\nu} \otimes \tau_\nu, \pi_\nu)$, we get, as in [JngS03], (2.9), a non-trivial bilinear form $\beta$ on

$$\mathcal{S}(X_{n-1}^* \otimes W_m \oplus V_3 \otimes Y_m) \otimes V_{\tau_\nu}$$

(where $V_{\tau_\nu}$ is the space of $\tau_\nu$), which satisfies

$$\beta(\omega_\psi(tu, h)\varphi, \xi) = \psi_{n,n-1;\lambda}(u)\beta(\varphi, \xi)$$

for $t \in D_\lambda, u \in N_n^{n-1}, h \in \widetilde{\mathrm{Sp}}_{2m}(k_\nu), \xi \in V_{\tau_\nu}$, and $\varphi$ is a function in the mixed model. Let us abbreviate the notation for $\varphi$ as follows:

$$\begin{aligned} & \varphi((w_1, \cdots, w_{n-1}); (y_1, y_0, y_2)) \\ =: \ & \varphi(e_{-n+1} \otimes w_1 + \cdots + e_{-1} \otimes w_{n-1} + e_n \otimes y_1 + e_0 \otimes y_0 + e_{-n} \otimes y_2) \end{aligned}$$

for $w_i \in W_m, y_j \in Y_m$. Consider the subgroup of $\mathrm{O}_{2n+1}$ consisting of matrices

$$z' = \begin{pmatrix} I & 0 & z \\ & I_3 & 0 \\ & & I \end{pmatrix} \in \mathrm{O}_{2n+1}.$$

Then we have the following formula

$$\begin{aligned} & \omega_\psi(z', 1)\varphi((w_1, \cdots, w_{n-1}); (y_1, y_0, y_2)) = \\ & \psi(\frac{1}{2}\mathrm{Gr}(w_1, \cdots, w_{n-1})\omega z)\varphi((w_1, \cdots, w_{n-1}); (y_1, y_0, y_2)). \end{aligned}$$

Here $\omega$ is the (standard) long Weyl element in $\mathrm{GL}(n-1)$, and Gr denotes the Gram matrix. As in [JngS03], p. 755, we conclude that for each fixed $\xi$, $\beta$ is supported on

$$C_0 = \{((w_1, \cdots, w_{n-1}); (y_1, y_0, y_2)) | (w_i, w_j) = 0, \forall\ i, j < n\}.$$

The same proof as in [JngS03], p. 756-758 shows that, for $\beta$ to be nonzero, it has to be supported in the subvariety of $C_0$, where $w_1, \cdots, w_{n-1}$ are linearly independent. This is impossible when $m < n - 1$. $\square$

If we continue the last proof, for $m=n-1$, as we did in [JngS03], Cor. 2.1, we find that if $\beta$ is non-zero, then $\tau_\nu$ needs to be $\psi_\lambda$-generic. Choose then a $\psi_\lambda$-Whittaker functional $w_\lambda$ on $V_{\tau_\nu}$. The bilinear form $\beta$ is then determined uniquely, up to scalar multiples by

$$\beta(\varphi,\xi)=\int_{S\backslash \mathrm{Sp}_{2n-2}(k_\nu)}\omega_\psi(1,h)\varphi(f_{-(n-1)},\cdots,f_{-1};e(\lambda)\otimes f_{n-1})w_\lambda(h)dh.$$

Here $S$ is the Siegel radical and $e(\lambda)=e_n-\frac{\lambda}{2}e_{-n}$. In order to get the last result, we have to use the action in our mixed model of

$$n(x,z)=\begin{pmatrix} I & x & z\\ & I_3 & x^*\\ & & I\end{pmatrix}\in \mathrm{O}_{2n+1}$$

on $\varphi((w_1,\cdots,w_{n-1});(y_1,y_0,y_2))$, where $w_1,\cdots,w_{n-1}$ lie in $Y_m^*$. We have

$$\begin{aligned}&\omega_\psi(n(x,z),1)\varphi((w_1,\cdots,w_{n-1});(y_1,y_0,y_2))\\ =\ &\psi(\sum_{i=1}^{n-1}(w_{n-i},x_{i1}y_1+x_{i2}y_0+x_{i3}y_2))\varphi((w_1,\cdots,w_{n-1});(y_1,y_0,y_2)).\end{aligned}$$

Let us record this local result in the following theorem.

**Theorem 4.3.** *Let $F$ be a finite extension of a p-adic number field $\mathbb{Q}_p$. Let $\pi$ be an irreducible representation of $\mathrm{SO}_{2n+1}(F)$, which has a non-trivial Bessel functional of special type $(D_\lambda,1,\psi_{n,n-1;\lambda})$. Assume that $\tau$ is an irreducible representation of $\widetilde{\mathrm{Sp}}_{2n-2}(F)$ which is a non-trivial local $\psi$-Howe lift of $\pi$ to $\widetilde{\mathrm{Sp}}_{2n-2}(F)$. Then $\tau$ is $\psi_\lambda$-generic.*

Assume now that $m_{0,\psi}(\pi)=n$, i.e. the $\psi$-theta lift of $\pi$ to $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ is cuspidal. This is certainly the case if $\pi$ is locally generic at one finite place. Indeed, By Prop. 2.1 of [JngS03], if $\nu$ is a finite place, where $\pi_\nu$ is generic, then $\pi$ has no non-trivial local $\psi$-Howe lifts to $\widetilde{\mathrm{Sp}}_{2m}(k_\nu)$, for any $m<n$. This also proves (3) of Theorem 4.1. By [F95], $\Theta_{n,n}(\pi,\psi)$ is $\psi_\lambda$-generic. Consider now the $\psi_\lambda$-theta lift $\Theta_{n,n}(\pi,\psi_\lambda)$ of $\pi$. This representation is also cuspidal, since it is obtained from the former by an outer conjugation by an element of $\mathrm{GSp}_{2n}(k)$, which has similitude factor $\lambda$. See [MVW87], p. 36. Again, by [F95], $\Theta_{n,n}(\pi,\psi_\lambda)$ is $\psi_1$-generic.

Let $\widetilde{\sigma}$ be an irreducible $\psi_1$-generic summand of $\Theta_{n,n}(\pi, \psi_\lambda)$. Then $\widetilde{\sigma}$ is, of course, cuspidal. Now, consider the $\psi^{-1}$-theta lift of $\widetilde{\sigma}$ to $\mathrm{SO}_{2n+1}(\mathbb{A})$. By Proposition 3 of [F95], $\Theta_{n,n}(\widetilde{\sigma}, \psi^{-1})$ is generic. This is just due to the fact that $\widetilde{\sigma}$ is $\psi_1$-generic. The question is whether $\Theta_{n,n}(\widetilde{\sigma}, \psi^{-1})$ is cuspidal. Let $n_{0,\psi^{-1}}(\widetilde{\sigma})$ be the first occurrence of $\widetilde{\sigma}$ in the $k$-split Witt tower $\mathrm{SO}_{2n'+1}(\mathbb{A})$ for $n' \geq 1$ of the $\psi^{-1}$-theta lifts of $\widetilde{\sigma}$. Then we have

**Lemma 4.4.** *Let $\widetilde{\sigma}$ be an irreducible, cuspidal, (genuine) automorphic representation of $\widetilde{\mathrm{Sp}}_{2m}(\mathbb{A})$. Assume that $\widetilde{\sigma}$ is $\psi_1$-generic. Then*

$$m - 1 \leq n_{0,\psi^{-1}}(\widetilde{\sigma}) \leq m.$$

*Proof.* We already noted that $n_{0,\psi^{-1}}(\widetilde{\sigma}) \leq m$. The second inequality follows from Cor. 2.2 in [JngS03]. ☐

Thus, there are two possible cases. Either $\Theta_{n,n}(\widetilde{\sigma}, \psi^{-1})$ is cuspidal, or $\Theta_{n,n-1}(\widetilde{\sigma}, \psi^{-1})$ is (non-zero) cuspidal. Recall that in the second case, the constant term of $\Theta_{n,n}(\widetilde{\sigma}, \psi^{-1})$, along the unipotent radical of the parabolic subgroup $P_1$ is

$$|\cdot|^{1/2} \otimes \Theta_{n,n-1}(\widetilde{\sigma}, \psi^{-1}),$$

where $P_1$ has the Levi subgroup isomorphic to $\mathrm{GL}_1 \times \mathrm{SO}_{2n-1}$. This is Rallis' theory of the tower structure of theta correspondences ([R84]). Let us consider each case separately.

Assume that $n_{0,\psi^{-1}}(\widetilde{\sigma}) = n$. Then $\Theta_{n,n}(\widetilde{\sigma}, \psi^{-1})$ is cuspidal, and generic by [F95]. Let $\chi_\lambda$ be the quadratic character obtained by composing the spinor norm with the Hilbert symbol $\alpha_\lambda(x) = (x, \lambda)$. We will soon show that each irreducible summand of $\Theta_{n,n}(\widetilde{\sigma}, \psi^{-1}) \otimes \chi_\lambda$ is nearly equivalent to $\pi$. This will imply that $\pi$ is nearly equivalent to a generic cuspidal representation, i.e. the first possibility in Part (2) of Theorem 4.1.

Assume next that $n_{0,\psi^{-1}}(\widetilde{\sigma}) = n - 1$. Let $\sigma'$ be an irreducible summand of $\Theta_{n,n-1}(\widetilde{\sigma}, \psi^{-1})$. Then, by Rallis' tower theory ([R84]), the irreducible constituents of $\Theta_{n,n}(\widetilde{\sigma}, \psi^{-1})$ are locally isomorphic, at almost all local places, (nearly equivalent) to those of

$$\mathrm{Ind}_{P_1(\mathbb{A})}^{G_n(\mathbb{A})}(|\cdot|^{1/2} \otimes \sigma').$$

As in the previous case, $\pi$ is nearly equivalent to any irreducible constituent of $\Theta_{n,n}(\widetilde{\sigma}, \psi^{-1}) \otimes \chi_\lambda$. Let $\sigma = \sigma' \otimes \chi_\lambda$. Thus $\pi$ is CAP with respect to the cuspidal data $(P_1; \alpha_\lambda |\cdot|^{1/2} \otimes \sigma)$. This proves Part (2) of Theorem 4.1.

Let us recall the determination of the local unramified components at almost all local places of the irreducible summands of $\Theta_{n,n}(\widetilde{\sigma}, \psi^{-1}) \otimes \chi_\lambda$, and show that they are isomorphic to the local unramified components to $\pi$, at these local places.

Assume that the local unramified component $\pi_v$ of $\pi$ is the (unique) irreducible spherical constituent of the following induced representation

$$\mathrm{Ind}_{B(k_v)}^{\mathrm{SO}_{2n+1}(k_v)}(\eta_v)$$

where the character $\eta_v$ is defined as follows:

$$\eta_v(\mathrm{diag}(t_1, \cdots, t_n, 1, t_n^{-1}, \cdots, t_1^{-1})) = \eta_{1,v}(t_1) \cdots \eta_{n,v}(t_n)$$

for some unramified characters $\eta_{1,v}, \cdots, \eta_{n,v}$ of $k_v^\times$. It is clear that both unramified $\pi_v$ and $\mathrm{Ind}_{B(k_v)}^{\mathrm{SO}_{2n+1}(k_v)}(\eta_v)$ can be viewed as unramified representations of $\mathrm{O}_{2n+1}(k_v)$. In other words, one has an irreducible spherical representation $\pi_v$ which is realized in the induced representation

$$\mathrm{Ind}_{B(k_v)}^{\mathrm{O}_{2n+1}(k_v)}(\eta_v).$$

By the explicit local Howe correspondence for spherical representations (see [Kdl86] and [Kdl96] for instance), the local spherical component $\widetilde{\sigma}_v$ of $\widetilde{\sigma}$ is equal to the corresponding local spherical component of an irreducible $\psi_1$-generic summand of the $\psi_1$-generic cuspidal automorphic representation $\Theta_{n,n}(\pi, \psi^\lambda)$, where the character $\psi^\lambda$ is given by

$$\psi^\lambda(x) = \psi(\lambda \cdot x).$$

It follows that $\widetilde{\sigma}_v$ can be realized as the irreducible spherical constituent of the induced representation

$$\mathrm{Ind}_{\widetilde{B}(k_v)}^{\widetilde{\mathrm{Sp}}_{2n}(k_v)}(\widetilde{\eta}_v)$$

where $\widetilde{\eta}_v$ is defined by

$$\widetilde{\eta}(\mathrm{diag}(t_1, \cdots, t_n, t_n^{-1}, \cdots, t_1^{-1}), \epsilon) = \eta_{1,v}(t_1) \cdots \eta_{n,v}(t_n) \gamma(t_1 \cdots t_n, \psi_v^\lambda).$$

The factor $\gamma(a, \psi_v)$ is defined by

$$\gamma(a, \psi_v) := \frac{\gamma(\psi_v^a)}{\gamma(\psi_v)} \tag{4.1}$$

where $\psi_v^a(x) = \psi_v(ax)$ and $\gamma(\psi_v)$ is the Weil index associated to the second degree character given by $\psi_v(x^2)$. The basic properties of $\gamma(\psi_v)$ and $\gamma(a, \psi_v)$ are given in Lemma 4.1, [Kdl96]. Similarly, if we assume that the unramified local component of $\widetilde{\sigma}$ is given as above, the local spherical component of $\Theta_{n,n}(\widetilde{\sigma}, \psi_1^{-1})$ can be realized as the irreducible spherical component of the induced representation

$$\mathrm{Ind}_{B(k_v)}^{\mathrm{O}_{2n+1}(k_v)}(\eta_{\lambda,v})$$

where $\eta_{\lambda,v}$ is given by

$$\eta_{\lambda,v}(\mathrm{diag}(t_1, \cdots, t_n, 1, t_n^{-1}, \cdots, t_1^{-1})) = \eta_{1,v}(t_1) \cdots \eta_{n,v}(t_n) \frac{\gamma(t_1 \cdots t_n, \psi_v^\lambda)}{\gamma(t_1 \cdots t_n, \psi_{1,v})}.$$

By the basic properties of $\gamma(a, \psi_v)$ as stated in Lemma 4.1, [Kdl96], one has

$$\frac{\gamma(t_1 \cdots t_n, \psi_v^\lambda)}{\gamma(t_1 \cdots t_n, \psi_{1,v})} = (t_1 \cdots t_n, \lambda)_{k_v}.$$

This shows that the spherical local component at $v$ of an irreducible summand of $\Theta_{n,n}(\widetilde{\sigma}, \psi_1^{-1}) \otimes \chi_\lambda$ is the same as that of $\pi$ as representations of $\mathrm{O}_{2n+1}$ or of $\mathrm{SO}_{2n+1}$.

Finally, assume that $\pi_\nu$ is generic and spherical at a given finite, odd place $\nu$, where $\lambda$ is a unit. We have already noted before that $m_{0,\psi}(\pi) = n$. Let us show, using the notation above, that $n_{0,\psi^{-1}}(\widetilde{\sigma}) = n$. If this is not the case, then $n_{0,\psi^{-1}}(\widetilde{\sigma}) = n - 1$, and by what we proved before, $\pi_\nu$ is the spherical constituent of a representation of the form

$$\mathrm{Ind}_{P_1(k_\nu)}^{G_n(k_\nu)}(\alpha_{\lambda_\nu}| \cdot |^{1/2} \otimes \sigma_\nu),$$

where $\sigma_\nu$ is an unramified representation of $G_{n-1}(k_\nu)$, say induced from the unramified characters $\beta_{1,v}, \cdots, \beta_{n-1,v}$. It is easy to see that the spherical constituent of the last induced representation is equal to the spherical constituent of

$$\mathrm{Ind}_{Q(k_\nu)}^{G_n(k_\nu)}(\beta_{1,v} \otimes \cdots \otimes \beta_{n-1,v} \otimes \chi_{\lambda,v}),$$

where $Q$ is the standard parabolic subgroup of $G_n$, whose Levi part is isomorphic to $(\mathrm{GL}_1)^{n-1} \times \mathrm{SO}_3$, and here $\chi_{\lambda,v}$ denotes the composition of the spinor norm on $G_1 = \mathrm{SO}_3$ with the character $\alpha_{\lambda,v}$ (the local Hilbert symbol). Now it is clear that this spherical component can not be generic. This is a contradiction. Hence Part (4) of Theorem 4.1 follows. This completes the proof of Theorem 4.1.

**Remark 4.5.** (1) *By assuming that the work of Moeglin ([M97a] and [M97b]) can be extended to the dual reductive pair* $(\mathrm{SO}_{2n+1}, \widetilde{\mathrm{Sp}}_{2m})$, *we expect that the cuspidal datum in Part (2) of Theorem 4.1 is generic. This confirms the CAP conjecture (Conjecture 1.1) for irreducible cuspidal automorphic representations of* $\mathrm{SO}_{2n+1}(\mathbb{A})$ *with Bessel model of special type and the first occurrence* $m_{0,\psi}(\pi) = n$. *The other case, i.e.* $m_{0,\psi}(\pi) = n-1$ *will be discussed in §4.2.*

(2) *We expect that if* $\pi$ *is locally generic at one place, then* $\pi$ *is nearly equivalent to an irreducible, generic, cuspidal, automorphic representation. We will consider this improvement of Part (4) of Theorem 4.1 in our future work.*

## 4.2 Bessel models of special type (nongeneric case).

In this section, we address the remaining case for $\pi$, i.e. $\pi$ (as in the previous section) is such that $m_{0,\psi}(\pi) = n-1$. Thus, $\Theta_{n,n-1}(\pi,\psi)$ is cuspidal and non-zero. Note that by Proposition 2.1 in [JngS03] or Part (3) of Theorem 4.1, $\pi$ is nowhere generic at all finite places. Let $\widetilde{\tau}$ be an irreducible summand of $\Theta_{n,n-1}(\pi,\psi)$. We have seen, in Theorem 4.3, that $\widetilde{\tau}$ is $\psi_\lambda$-generic at all finite places. We expect that $\widetilde{\tau}$ should be also globally $\psi_\lambda$-generic, i.e. has a non-zero $\psi_\lambda$-Whittaker Fourier coefficient (as in Remark 4.5). In the following we will construct examples with such properties. Let us start with an irreducible, cuspidal, automorphic representation $\widetilde{\tau}$ of $\widetilde{\mathrm{Sp}}_{2n-2}(\mathbb{A})$, which is globally $\psi_\lambda$-generic, and its $\psi^{-1}$-theta lift to $\mathrm{SO}_{2n+1}(\mathbb{A})$, denoted by $\Theta_{n-1,n}(\widetilde{\tau},\psi^{-1})$, is cuspidal. Then, an analogous repetition of the proof of Theorem 4.3 shows that $\Theta_{n-1,n}(\widetilde{\tau},\psi^{-1})$ is non-trivial, and has a non-zero Bessel model of special type with respect to $\lambda$ . Let $\pi$ be an irreducible summand of $\Theta_{n-1,n}(\widetilde{\tau},\psi^{-1})$ with a non-zero Bessel model of special type. As before, the $\psi_\lambda^{-1}$-theta lift of $\widetilde{\tau}$ to $\mathrm{SO}_{2n-1}(\mathbb{A})$ is non-trivial and generic. We already know, by Lemma 4.4, that the following inequalities hold:

$$n-2 \le n_{0,\psi_\lambda}(\widetilde{\tau}) \le n-1.$$

Assume first, that $n_{0,\psi_\lambda}(\widetilde{\tau}) = n-1$, i.e. the $\psi_\lambda^{-1}$-theta lift, $\Theta_{n-1,n-1}(\widetilde{\tau},\psi_\lambda^{-1})$, is cuspidal. Let $\pi'$ be an irreducible summand of $\Theta_{n-1,n-1}(\widetilde{\tau},\psi_\lambda^{-1})$, which is generic. Then by the computations of unramified parameters carried out at end of the last section, it is clear that $\pi$ is CAP with respect to the cuspidal data $(P_1; |\cdot|^{1/2} \otimes (\pi' \otimes \chi_\lambda))$.

Consider now the case $n_{0,\psi_\lambda}(\widetilde{\tau}) = n-2$. Let $\pi''$ be an irreducible summand of $\Theta_{n-1,n-2}(\widetilde{\tau}, \psi_\lambda^{-1})$. By Rallis theory of theta towers, any constituent of $\Theta_{n-1,n-1}(\widetilde{\tau}, \psi_\lambda^{-1})$ is a constituent of the following induced representation

$$\mathrm{Ind}_{P'_1(\mathbb{A})}^{G_{n-1}(\mathbb{A})}(|\cdot|^{1/2} \otimes \pi''),$$

where $P'_1$ is the standard parabolic subgroup of $\mathrm{SO}_{2n-1}$, whose Levi part is isomorphic to $\mathrm{GL}_1 \times G_{n-2}$. We conclude, by the computations of unramified parameters carried out at the end of the last section, that $\pi$ is CAP with respect to the cuspidal data

$$(P_{1,1}; |\cdot|^{1/2} \otimes \alpha_\lambda |\cdot|^{1/2} \otimes (\pi'' \otimes \chi_\lambda)),$$

where $P_{1,1}$ is the standard parabolic subgroup of $G_n$, whose Levi part is isomorphic to

$$\mathrm{GL}_1 \times \mathrm{GL}_1 \times G_{n-2}.$$

We proved

**Theorem 4.6.** *Let $\widetilde{\tau}$ be an irreducible, cuspidal, (genuine) automorphic representation of $\widetilde{\mathrm{Sp}}_{2n-2}(\mathbb{A})$, which is globally $\psi_\lambda$-generic ($\lambda$ is not in the square class of $d$ as assumed in Theorem 4.1). Then the $\psi^{-1}$-theta lift of $\widetilde{\tau}$ to $\mathrm{SO}_{2n+1}(\mathbb{A})$, $\Theta_{n-1,n}(\widetilde{\tau}, \psi^{-1})$, is non-trivial, and has a Bessel model of special type $(D_\lambda, 1, \psi_{n,n-1;\lambda})$. Assume that $\Theta_{n-1,n}(\widetilde{\tau}, \psi^{-1})$ is cuspidal. Let $\pi$ be an irreducible summand of $\Theta_{n-1,n}(\widetilde{\tau}, \psi^{-1})$, which has a Bessel model of special type $(D_\lambda, 1, \psi_{n,n-1;\lambda})$. Then $\pi$ is a CAP representation, with respect to either the cuspidal data*

$$(P_1; |\cdot|^{1/2} \otimes \sigma),$$

*where $\sigma$ is an irreducible, generic, cuspidal, automorphic representation of $G_{n-1}(\mathbb{A})$, or the cuspidal data*

$$(P_{1,1}; |\cdot|^{1/2} \otimes \alpha_\lambda |\cdot|^{1/2} \otimes \sigma'),$$

*where $\sigma'$ is an irreducible, cuspidal, automorphic representation of $G_{n-2}(\mathbb{A})$.*

**Remark 4.7.** *The representation $\sigma'$ in the last theorem is locally generic at all finite places. This follows from [JngS03], Cor. 2.2(3). $\sigma'$ should be globally generic, and the proof would have a global analogue of loc. cit provided we know that $\widetilde{\tau}$ is the $\psi_\lambda$-theta lift of $\pi''$ (notation of the proof above). We expect to deal with the generalization of Theorem 4.6 in our forthcoming work by establishing the work of Moeglin for the reductive dual paris $(\mathrm{SO}_{2n+1}, \widetilde{\mathrm{Sp}}_{2n})$.* .

Finally, let us give examples of representations $\widetilde{\tau}$ as in the last theorem. Fix a finite place $\nu_0$ of $k$. Let $\widetilde{\tau}_0$ be an irreducible, supercuspidal (genuine) representation of $\widetilde{\mathrm{Sp}}_{2n-2}(k_{\nu_0})$. Assume that $\widetilde{\tau}_0$ is $\psi_\lambda$-generic and that $\lambda$ is not in the square class of $d$ at $\nu_0$. Assume that $\widetilde{\tau}_0$ has a non-trivial $\psi_{\nu_0}$-Howe lift to $\mathrm{SO}'_{2n-1}(k_{\nu_0})$, the quasi-split special orthogonal group corresponding to the quadratic extension of $k_{\nu_0}$ generated by $\sqrt{\lambda}$. By the dichotomy principle of Kudla-Rallis ([KR04]), which is valid for supercuspidal representations, we know that the first occurrence of $\psi$-Howe lifts of $\widetilde{\tau}_0$ in the $k_{\nu_0}$-split Witt tower of orthogonal groups $G_m(k_{\nu_0})$ is $m = n$.

Let $\widetilde{\tau}$ be any irreducible, $\psi_\lambda$-generic, cuspidal, (genuine) automorphic representation of $\widetilde{\mathrm{Sp}}_{2n-2}(\mathbb{A})$, whose local component at $\nu_0$ is $\widetilde{\tau}_0$. Such a $\tau$ can be constructed using the Poincare series method (see [Sh90] for instance). By Theorem 4.6, we know that $\Theta_{n-1,n}(\widetilde{\tau}, \psi^{-1})$ is non-trivial. It is also cuspidal, since otherwise, by the Rallis theory of the theta towers, we obtain that $\Theta_{n-1,n-1}(\widetilde{\tau}, \psi^{-1})$ is non-trivial, and, in particular, this will imply that $\widetilde{\tau}_0$ has a non-trivial $\psi_{\nu_0}$-Howe lift to $G_{n-1}(k_{\nu_0})$, which is impossible by our choice of $\widetilde{\tau}_0$.

Let $\pi$ be an irreducible summand of $\Theta_{n-1,n}(\widetilde{\tau}, \psi^{-1})$. Of course, we can pick one with a Bessel model of special type, as above. Then $\pi$ is CAP with respect to the cuspidal data

$$(P_1; |\cdot|^{1/2} \otimes \sigma),$$

where $\sigma$ is an irreducible, generic, cuspidal, automorphic representation of $G_{n-1}(\mathbb{A})$. To see this, we follow our proof above. We have to show that $n_{0,\psi_{-\lambda}}(\widetilde{\tau}) = n - 1$. Indeed, it is enough to show that the first occurrence of the local $\psi_{-\lambda}$-Howe lift of $\widetilde{\tau}_0$ in the tower $G_m(k_{\nu_0})$ is $m = n - 1$, and this follows from [JngS03], Part (2) of Theorem 2.2. Thus, $\Theta_{n-1,n-1}(\tau, \psi_{-\lambda})$ is cuspidal and generic. Let $\pi'$ be a generic summand. Then $\sigma = \pi' \otimes \chi_\lambda$, which is generic.

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