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Noncommutative Symmetries and Gravity ⋆

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Abstract

Spacetime geometry is twisted (deformed) into noncommutative spacetime geometry, where functions and tensors are now star-multiplied. Consistently, spacetime diffeomorhisms are twisted into noncommutative diffeomorphisms. Their deformed Lie algebra structure and that of infinitesimal Poincaré transformations is defined and explicitly constructed.

This allows to construct a noncommutative theory of gravity.

This article is based on common work with Christian Blohmann, Marija Dimitrijević, Frank Meyer, Peter Schupp and Julius Wess.

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1 Introduction

The study of gravity on noncommutative spacetime, where spacetime uncertainty relations and nonlocal effects naturally arise, is an interesting arena for the study of spacetime at Planck length, where quantum gravity effects are non-negligible. This line of thought has been pursued since the early days of quantum mechanics [1], and more recently in [2] - [14], (see also the recent review [15]).

This work is based on [13] and [14], where we study the algebra of diffeomorphisms on noncommutative spacetime and a noncommutative gravity theory covariant under diffeomorphisms. In [13] we study the case of constant noncommutativity, $x^\mu \star x^\nu - x^\nu \star x^\mu = \theta^{\mu\nu}$, while in [14] we consider a quite general class of noncommutative manifolds obtained via Drinfeld twists; there generally the commutator $x^\mu \star x^\nu - x^\nu \star x^\mu$ is a nonconstant function. We here emphasize the twist approach to noncommutative gravity, and we develop the notion of noncommutative Lie algebra, including a detailed study of the noncommutative Poincaré Lie algebra. For pedagogical reasons we treat just the case of constant noncommutativity (and we assume that commutative spacetime as a manifold is $\mathbb{R}^4$).

In Section 2 we introduce the twist $\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}$. The general notion of twist is well known [16, 17]. Multiparametric twists appear in [18]. In the context of deformed Poincaré group and Minkowski space geometry twists have been studied in [19], [20], [21] (multiparametric deformations), and in [22]- [28] (Moyal-Weyl deformations).

Given a twist $\mathcal{F}$ we state the general principle that allows to construct noncommutative products by composing commutative products with the twist $\mathcal{F}$. In this way we obtain the algebrae of noncommutative functions, tensorfields, exterior forms and diffeomorphisms. Noncommutative diffeomorphisms are then shown to naturally act on tensorfields and forms. We study in detail the notion of infinitesimal diffeomorphism, and the corresponding notion of deformed Lie algebra.

In Section 3 we present the example of the Poincaré symmetry, give explicitly the infinitesimal generators and their deformed Lie bracket, and explain the geometric origin of the latter. The generators and the bracket differ from the ones usually considered in the literature.

In Section 4 we use the noncommutative differential geometry formalism introduced in Section 2 and develop the notion of covariant derivative, and of torsion, curvature and Ricci curvature tensors. In Section 5 a metric on noncommutative space is introduced. The corresponding unique torsion free metric compatible connection is used to construct the Ricci tensor and obtain the Eintein equations for gravity on noncommutative spacetime.

In the Appendix we show that the algebra of differential operators is not a Hopf algebra, and we relate it to the Hopf algebra of infinitesimal diffeomorphisms.

2 Deformation by twists

A quite general procedure in order to construct noncommutative spaces and noncommutative field theories is that of a twist. The ingredients are:

I) a Lie algebra g.

II) an action of the Lie algebra on the space one wants to deform.

III) a twist element $\mathcal{F}$, constructed with the generators of the Lie algebra g.

Concerning III), a twist element $\mathcal{F}$ is an invertible element in $Ug \otimes Ug$, where Ug is the universal enveloping algebra of g. Ug is a Hopf algebra, in particular there is a linear map, called coproduct

$$\Delta : Ug \to Ug \otimes Ug \ . \tag{2.1}$$

For every Lie algebra element $t \in g$ we have

$$\Delta(t) = t \otimes 1 + 1 \otimes t \, . \tag{2.2}$$

The coproduct Δ is extended to all Ug by defining

$$\Delta(tt') := \Delta(t)\Delta(t') = tt' \otimes 1 + t \otimes t' + t' \otimes t + 1 \otimes tt'$$

and more generally $\Delta(tt' \ldots t'') = \Delta(t)\Delta(t') \ldots \Delta(t'')$. A main property $\mathcal{F}$ has to satisfy is the cocycle condition

$$(\mathcal{F} \otimes 1)(\Delta \otimes id)\mathcal{F} = (1 \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F} \ . \tag{2.3}$$

If g is the Lie algebra of vectorfields on spacetime $M = \mathbb{R}^4$, or simply the subalgebra spanned by the commuting vectorfields $\partial/\partial x^\mu$, we can consider the twist

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}} \ , \tag{2.4}$$

where $\theta^{\mu\nu}$ is a real antisymmetric matrix of dimensionful constants. We consider $\theta^{\mu\nu}$ fundamental physical constants, like the velocity of light c, or like $\hbar$. The symmetries of our physical system will leave $\theta^{\mu\nu}$, c and $\hbar$ invariant. The inverse of $\mathcal{F}$ is

$$\mathcal{F}^{-1} = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}} \ .$$

This twist satisfies condition (2.3) because the Lie algebra of partial derivatives is abelian.

The star-product between functions can be obtained from the usual pointwise product via the action of the twist operator, namely,

$$f \star g := \mu \circ \mathcal{F}^{-1}(f \otimes g) \ , \tag{2.5}$$

where μ is the usual pointwise product between functions, $\mu(f \otimes g) = fg$.

We shall frequently use the notation (sum over α understood)

$$\mathcal{F} = \mathrm{f}^{\,\alpha} \otimes \mathrm{f}_{\,\alpha} \quad , \qquad \mathcal{F}^{-1} = \overline{\mathrm{f}}^{\,\alpha} \otimes \overline{\mathrm{f}}_{\,\alpha} \ , \tag{2.6}$$

so that

$$f \star g := \overline{\mathrm{f}}^{\,\alpha}(f)\overline{\mathrm{f}}_{\,\alpha}(g) \ . \tag{2.7}$$

Explicitly we have

$$\mathcal{F}^{-1} = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}} = \sum \frac{1}{n!}\left(\frac{i}{2}\right)^n \theta^{\mu_1\nu_1}\dots\theta^{\mu_n\nu_n}\partial_{\mu_1}\dots\partial_{\mu_n}\otimes\partial_{\nu_1}\dots\partial_{\nu_n} = \overline{\mathrm{f}}^{\,\alpha} \otimes \overline{\mathrm{f}}_{\,\alpha} \ , \tag{2.8}$$

so that α is a multi-index. We also introduce the universal $\mathcal{R}$-matrix

$$\mathcal{R} := \mathcal{F}_{21}\mathcal{F}^{-1} \tag{2.9}$$

where by definition $\mathcal{F}_{21} = \mathrm{f}_{\,\alpha} \otimes \mathrm{f}^{\,\alpha}$. In the sequel we use the notation

$$\mathcal{R} = R^\alpha \otimes R_\alpha \quad , \qquad \mathcal{R}^{-1} = \overline{R}^\alpha \otimes \overline{R}_\alpha \ . \tag{2.10}$$

In the present case we simply have $\mathcal{R} = \mathcal{F}^{-2}$ but for more general twists this is no more the case. The $\mathcal{R}$-matrix measures the noncommutativity of the $\star$-product. Indeed it is easy to see that

$$h \star g = \overline{R}^\alpha(g) \star \overline{R}_\alpha(h) \ . \tag{2.11}$$

We now use the twist to deform the commutative geometry on spacetime (vectorfields, 1-forms, exterior algebra, tensor algebra, symmetry algebras, covariant derivatives etc.) into the twisted noncommutative one. The guiding principle is the observation that every time we have a bilinear map

$$\mu \, : X \times Y \to Z$$

where X, Y, Z are vectorspaces, and where there is an action of the Lie algebra g (and therefore of $\mathcal{F}^{-1}$) on X and Y we can combine this map with the action of the twist. In this way we obtain a deformed version $\mu_\star$ of the initial bilinear map μ:

$$\mu_\star := \mu \circ \mathcal{F}^{-1} \ , \tag{2.12}$$

$$\mu_\star : X \times Y \to Z$$
$$(\mathsf{x},\mathsf{y}) \mapsto \mu_\star(\mathsf{x},\mathsf{y}) = \mu(\overline{\mathrm{f}}^\alpha(\mathsf{x}), \overline{\mathrm{f}}_\alpha(\mathsf{y})) \ .$$

The cocycle condition (2.3) implies that if μ is an associative product then also $\mu_\star$ is an associative product.

Algebra of Functions $A_\star$. If $X = Y = Z = Fun(M)$ where $A \equiv Fun(M)$ is the space of functions on spacetime M, we obtain the star-product formulae (2.5), (2.7). The $\star$-product is associative because of the cocycle condition (2.3). We denote by $A_\star$ the noncommutative algebra of functions with the $\star$-product. Notice that to define the $\star$-product we need condition II), the action of the Lie algebra on functions. In this case it is obvious; the action of ∂_μ on a function h is just $\partial_\mu h$, i.e. the Lie derivative of ∂_μ on h. In the sequel we will always use the Lie derivative action.

Vectorfields $\Xi_\star$. We now deform the product $\mu : A \otimes \Xi \to \Xi$ between the space $A = Fun(M)$ of functions on spacetime M and vectorfields. A generic vectorfield is $v = v^\nu \partial_\nu$. Partial derivatives acts on vectorfields via the Lie derivative action

$$\partial_\mu(v) = [\partial_\mu, v] = \partial_\mu(v^\nu)\partial_\nu \ . \tag{2.13}$$

According to (2.12) the product $\mu : A \otimes \Xi \to \Xi$ is deformed into the product

$$h \star v = \overline{\mathrm{f}}^\alpha(h)\overline{\mathrm{f}}_\alpha(v) \ . \tag{2.14}$$

Iterated use of (2.13) gives

$$h \star v = \overline{\mathrm{f}}^\alpha(h)\overline{\mathrm{f}}_\alpha(v) = \overline{\mathrm{f}}^\alpha(h)\overline{\mathrm{f}}_\alpha(v^\nu)\partial_\nu = (h \star v^\nu)\partial_\nu \ . \tag{2.15}$$

It is then easy to see that $h \star (g \star v) = (h \star g) \star v$. We have thus constructed the $A_\star$ module of vectorfields. We denote it by $\Xi_\star$. As vectorspaces $\Xi = \Xi_\star$, but Ξ is an A module while $\Xi_\star$ is an $A_\star$ module.

1-forms $\Omega_\star$. The space of 1-forms Ω becomes also an $A_\star$ module, with the product between functions and 1-forms given again by following the general prescription (2.12):

$$h \star \omega := \overline{\mathrm{f}}^\alpha(h)\overline{\mathrm{f}}_\alpha(\omega) \ . \tag{2.16}$$

The action of $\overline{\mathrm{f}}_\alpha$ on forms is given by iterating the Lie derivative action of the vectorfield ∂_μ on forms. Explicitly, if $\omega = \omega_\nu dx^\nu$ we have

$$\partial_\mu(\omega) = \partial_\mu(\omega_\nu)dx^\nu \tag{2.17}$$

and $\omega = \omega_\nu dx^\nu = \omega_\mu \star dx^\mu$.

Functions can be multiplied from the left or from the right, if we deform the multiplication from the right we obtain the new product

$$\omega \star h := \overline{\mathrm{f}}^{\alpha}(\omega)\overline{\mathrm{f}}_{\alpha}(h) \tag{2.18}$$

and we "move h to the right" with the help of the R-matrix,

$$\omega \star h = \overline{R}^{\alpha}(h) \star \overline{R}_{\alpha}(\omega)\ . \tag{2.19}$$

We have defined the $A_\star$-bimodule of 1-forms.

Tensorfields $\mathcal{T}_\star$. Tensorfields form an algebra with the tensorproduct $\otimes$. We define $\mathcal{T}_\star$ to be the noncommutative algebra of tensorfields. As vectorspaces $\mathcal{T} = \mathcal{T}_\star$ the noncommutative tensorproduct is obtained by applying (2.12):

$$\tau \otimes_\star \tau' := \overline{\mathrm{f}}^{\alpha}(\tau) \otimes \overline{\mathrm{f}}_{\alpha}(\tau')\ . \tag{2.20}$$

Associativity of this product follows from the cocycle condition (2.3).

Exterior forms $\Omega^\bullet_\star = \oplus_p \Omega^p_\star$. Exterior forms form an algebra wth product $\wedge :\ \Omega^\bullet \times \Omega^\bullet \to \Omega^\bullet$. We $\star$-deform the wedge product into the $\star$-wedge product,

$$\vartheta \wedge_\star \vartheta' := \overline{\mathrm{f}}^{\alpha}(\vartheta) \wedge \overline{\mathrm{f}}_{\alpha}(\vartheta')\ . \tag{2.21}$$

We denote by $\Omega^\bullet_\star$ the linear space of forms equipped with the wedge product $\wedge_\star$.

As in the commutative case exterior forms are totally $\star$-antisymmetric contravariant tensorfields. For example the 2-form $\omega \wedge_\star \omega'$ is the $\star$-antisymmetric combination

$$\omega \wedge_\star \omega' = \omega \otimes_\star \omega' - \overline{R}^{\alpha}(\omega') \otimes_\star \overline{R}_{\alpha}(\omega)\ . \tag{2.22}$$

The usual exterior derivative $d : A \to \Omega$ satisfies the Leibniz rule $d(h \star g) = dh \star g + h \star dg$ and is therefore also the $\star$-exterior derivative.

$\star$-Pairing between 1-forms and vectorfields. We now consider the bilinear map

$$\langle\ ,\ \rangle :\ \Xi \times \Omega_\star \to A\ , \tag{2.23}$$

$$(v, \omega)\ \mapsto \langle v, \omega\rangle = \langle v^\mu \partial_\mu, \omega_\nu dx^\nu\rangle = v^\mu \omega_\mu\ . \tag{2.24}$$

Always according to the general prescription (2.12) we deform this pairing into

$$\langle\ ,\ \rangle_\star :\ \Xi_\star \times \Omega_\star \to A_\star\ , \tag{2.25}$$

$$(\xi, \omega)\ \mapsto \langle \xi, \omega\rangle_\star := \langle \overline{\mathrm{f}}^{\alpha}(\xi), \overline{\mathrm{f}}_{\alpha}(\omega)\rangle\ . \tag{2.26}$$

It is easy to see that the $\star$-pairing satisfies the $A_\star$-linearity properties

$$\langle h \star u, \omega \star k\rangle_\star = h \star \langle u, \omega\rangle_\star \star k \ , \tag{2.27}$$

$$\langle u, h \star \omega\rangle_\star = \overline{R}^\alpha(h) \star \langle \overline{R}_\alpha(u), \omega\rangle_\star \ . \tag{2.28}$$

Notice that $\langle \partial_\mu, dx^\nu\rangle_\star = \langle \partial_\mu, dx^\nu\rangle = \delta_\mu^\nu$.

Using the pairing $\langle \ , \ \rangle_\star$ we associate to any 1-form ω the left $A_\star$-linear map $\langle \ , \omega\rangle_\star$. Also the converse holds: any left $A_\star$-linear map $\Phi : \Xi_\star \to A_\star$ is of the form $\langle \ , \omega\rangle_\star$ for some ω (explicitly $\omega = \Phi(\partial_\mu) dx^\mu$).

$\star$-Hopf algebra of diffeomorphisms $U\Xi_\star$. We recall that the (infinite dimensional) linear space Ξ of smooth vectorfields on spacetime M becomes a Lie algebra through the map

$$\begin{aligned} [\ \ ] : \qquad & \Xi \times \Xi \to \Xi \\ & (u, v) \mapsto [u\ v]. \end{aligned} \tag{2.29}$$

The element $[u\ v]$ of Ξ is defined by the usual Lie bracket

$$[u\ v](h) = u(v(h)) - v(u(h)), \tag{2.30}$$

where h is a function on spacetime.

The Lie algebra of vectorfields (i.e. the algebra of infinitesimal diffeomorphisms) can also be seen as an abstract Lie algebra without referring to the action of vectorfields on functions. The universal enveloping algebra $U\Xi$ of this abstract Lie algebra is the associative algebra (over $\mathbb{C}$) generated by the elements of Ξ and the unit element 1 and where the element $[u\ v]$ is given by the commutator $uv - vu$, i.e. $uv - vu = [u\ v]$. Here uv and vu denotes the product in $U\Xi$. The algebra $U\Xi$ is the universal enveloping algebra of infinitesimal diffeomorphisms, we shall denote its elements by the letter ξ, ζ, η,. . . .

The undeformed algebra $U\Xi$ has a natural Hopf algebra structure [35]. On the generators $u \in \Xi$ the coproduct map Δ the antipode and the counit ε are defined by

$$\begin{aligned} &\Delta(u) = u \otimes 1 + 1 \otimes u \ , \\ &\varepsilon(u) = 0 \ , \\ &S(u) = -u \ . \end{aligned} \tag{2.31}$$

(and $\Delta(1) = 1 \otimes 1 \ , \varepsilon(1) = 1 \ , S(1) = 1$). The maps Δ and ε are then extended as algebra homomorphisms and S as antialgebra homomorphism to the full enveloping algebra, $\Delta : U\Xi \to U\Xi \otimes U\Xi$, $\varepsilon : U\Xi \to \mathbb{C}$ and $S : U\Xi \to U\Xi$,

$$\begin{aligned} \Delta(\xi\zeta) &:= \Delta(\xi)\Delta(\zeta) \ , \\ \varepsilon(\xi\zeta) &:= \varepsilon(\xi)\varepsilon(\zeta) \ , \\ S(\xi\zeta) &:= S(\zeta)S(\xi) \ . \end{aligned} \tag{2.32}$$

The extensions of Δ, ε and S are well defined because they are compatible with the relations $uv - vu = [u\, v]$ (for ex. $S(uv - vu) = S(v)S(u) - S(u)S(v) = -[u\, v] = S[u\, v]$).

On the generators, the coproduct encodes the Leibniz rule property $u(hg) = u(h)g + hu(g)$, the antipode expresses the fact that the inverse of the group element e^u is e^{-u}, while the counit associates to every element e^u the identity 1.

In order to construct the deformed algebra of diffeomorphisms we apply the recepy (2.12) and deform the product in $U\Xi$ into the new product

$$\xi \star \zeta = \overline{\mathrm{f}}^{\alpha}(\xi)\overline{\mathrm{f}}_{\alpha}(\zeta)\ . \tag{2.33}$$

We call $U\Xi_\star$ the new algebra with product $\star$. As vectorspaces $U\Xi = U\Xi_\star$. Since any sum of products of vectorfields in $U\Xi$ can be rewritten as sum of $\star$-products of vectorfields via the formula $u\, v = \mathrm{f}^{\alpha}(u) \star \mathrm{f}_{\alpha}(v)$, vectorfields u generate the algebra $U\Xi_\star$.

It turns out [14] that $U\Xi_\star$ has also a natural Hopf algebra structure. We describe it by giving the coproduct, the counit and the antipode[1] on the generators u of $U\Xi_\star$:

$$\Delta_\star(u) = u \otimes 1 + \overline{R}^{\alpha} \otimes \overline{R}_{\alpha}(u)\ , \tag{2.34}$$

$$\varepsilon_\star(u) = \varepsilon(u) = 0\ , \tag{2.35}$$

$$S_\star(u) = -\overline{R}^{\alpha}(u)\, \overline{R}_{\alpha}\ . \tag{2.36}$$

In the appendix we prove for example coassociativity of the coproduct $\Delta_\star$. We here show that the coproduct definition (2.34) can be inferred from a deformed Leibniz rule.

There is a natural action (Lie derivative) of $\Xi_\star$ on the space of functions $A_\star$. It is given once again by combining the usual Lie derivative on functions $\mathcal{L}_u(h) = u(h)$ with the twist $\mathcal{F}$ as in (2.12),

$$\mathcal{L}^\star_u(h) := \overline{\mathrm{f}}^{\alpha}(u)(\overline{\mathrm{f}}_{\alpha}(h))\ . \tag{2.37}$$

By recalling that every vectorfield can be written as $u = u^\mu \star \partial_\mu = u^\mu \partial_\mu$ we have

$$\begin{aligned}\mathcal{L}^\star_u(h) &= \overline{\mathrm{f}}^{\alpha}(u^\mu\partial_\mu)(\overline{\mathrm{f}}_{\alpha}(h)) = \overline{\mathrm{f}}^{\alpha}(u^\mu)\,\partial_\mu(\overline{\mathrm{f}}_{\alpha}(h))\\ &= u^\mu \star \partial_\mu(h)\ ,\end{aligned} \tag{2.38}$$

where in the second equality we have considered the explicit expression (2.8) of $\overline{\mathrm{f}}^{\alpha}$ in terms of partial derivatives, and we have iteratively used the property $[\partial_\nu, u^\mu\partial_\mu] = \partial_\nu(u)\,\partial_\mu$. In the last equality we have used that the partial derivatives contained in $\overline{\mathrm{f}}_{\alpha}$ commute with the partial derivative ∂_μ.

[1]Notice that because of the antisymmetry of $\theta^{\mu\nu}$ we have $\overline{R}^{\alpha}(u)\,\overline{R}_{\alpha} = \overline{R}_{\alpha}\ \ \overline{R}^{\alpha}(u)$. Since $S_\star(\partial_\nu) = -\partial_\nu$ it is then easy to prove that $S^2_\star = id$. It is also easy to check that $\mu(S \otimes id)\Delta(u) = \mu(id \otimes S)\Delta(u) = \varepsilon(u)1 = 0$. This last property uniquely defines the antipode.

In accordance with the coproduct formula (2.34) the differential operator $\mathcal{L}^\star_u$ satisfies the deformed Leibniz rule

$$\mathcal{L}^\star_u(h \star g) = \mathcal{L}^\star_u(h) \star g + \overline{R}^\alpha(h) \star \mathcal{L}^\star_{\overline{R}_\alpha(u)}(g) \ . \tag{2.39}$$

Indeed recalling that $u = u^\mu \star \partial_\mu = u^\mu \partial_\mu$ we have

$$\begin{aligned} \mathcal{L}^\star_u(h \star g) = u^\mu \star \partial_\mu(h \star g) &= u^\mu \star \partial_\mu(h) \star g + u^\mu \star h \star \partial_\mu(g) \\ &= \mathcal{L}^\star_u(h) \star g + \overline{R}^\alpha(h) \star \overline{R}_\alpha(u^\mu) \star \partial_\mu(g) \\ &= \mathcal{L}^\star_u(h) \star g + \overline{R}^\alpha(h) \star \mathcal{L}^\star_{\overline{R}_\alpha(u)}(g) \ . \end{aligned} \tag{2.40}$$

From (2.38) it is also immediate to check the compatibility condition

$$\mathcal{L}^\star_{f \star u}(h) = f \star \mathcal{L}^\star_u(h) \ , \tag{2.41}$$

that shows that the action $\mathcal{L}^\star$ is the one compatible with the $A_\star$ module structure of vectorfields.

The action $\mathcal{L}^\star$ of $\Xi_\star$ on $A_\star$ can be extended to all $U\Xi_\star$. We recall that the action of $U\Xi$ on the space of functions can be defined by extending the Lie derivative. For any function $h \in A = Fun(M)$, we define the Lie derivative of a product of generators $u...vz$ in $U\Xi$ to be the compositon of the Lie derivatives of the generators,

$$(u...vz)(h) = u(\ldots v(z(h))...) \ . \tag{2.42}$$

Then by linearity we know the Lie derivative along any element ξ of $U\Xi$. We then define

$$\mathcal{L}^\star_\xi(h) := \overline{\mathrm{f}}^\alpha(\xi)(\overline{\mathrm{f}}_\alpha(h)) \ . \tag{2.43}$$

The map $\mathcal{L}^\star$ is an action of $U\Xi_\star$ on $A_\star$, i.e. it represents the algebra $U\Xi_\star$ as differential operators on functions because

$$\mathcal{L}^\star_u(\mathcal{L}^\star_v(h)) = \mathcal{L}^\star_{u \star v}(h) \ . \tag{2.44}$$

$\star$-Lie algebra of vectorfields $\Xi_\star$. We now turn our attention to the issue of determining the Lie algebra $\Xi_\star$ of $U\Xi_\star$. In the undeformed case the Lie algebra of the universal enveloping algebra $U\Xi$ is the linear subspace Ξ of $U\Xi$ of primitive elements, i.e. of elements u that have coproduct:

$$\Delta(u) = u \otimes 1 + 1 \otimes u \ . \tag{2.45}$$

Of course Ξ generates $U\Xi$ and Ξ is closed under the usual commutator bracket [,],

$$[u, v] = uu - vu \in \Xi \quad \text{for all } u, v \in \Xi \ . \tag{2.46}$$

The geometric meaning of the bracket $[u, v]$ is that it is the adjoint action of Ξ on Ξ,

$$[u, v] = ad_u \, v \tag{2.47}$$

$$ad_u \, v := u_1 v S(u_2) \tag{2.48}$$

where we have used the notation $\Delta(u) = u_1 \otimes u_2$, where a sum over u_1 and u_2 is understood. Recalling that $\Delta(u) = u \otimes 1 + 1 \otimes u$ and that $S(u) = -u$, from (2.48) we immediately obtain (2.47). In other words, the commutator $[u, v]$ is the Lie derivative of the left invariant vectorfield u on the left invariant vectorfield v. More in general the adjoint action of $U\Xi$ on $U\Xi$ is given by

$$ad_\xi \, \zeta = \xi_1 \zeta S(\xi_2) \ , \tag{2.49}$$

where we used the notation (sum understood)

$$\Delta(\xi) = \xi_1 \otimes \xi_2 \ .$$

For example $ad_{uv} \, \zeta = [u, [v, \zeta]]$.

In the deformed case the coproduct is no more cocommutative and we cannot identify the Lie algebra of $U\Xi_\star$ with the primitive elements of $U\Xi$, they are too few[2]. There are three natural conditions that according to [29] the $\star$-Lie algebra of $U\Xi_\star$ has to satisfy, see [30, 31], and [32] p. 41. It has to be a linear subspace $\Xi_\star$ of $U\Xi_\star$ such that

$$i) \quad \Xi_\star \text{ generates } U\Xi_\star \ , \tag{2.50}$$

$$ii) \quad \Delta_\star(\Xi_\star) \subset \Xi_\star \otimes 1 + U\Xi_\star \otimes \Xi_\star \ , \tag{2.51}$$

$$iii) \quad [\Xi_\star, \Xi_\star]_\star \subset \Xi_\star \ . \tag{2.52}$$

Property *ii)* implies a minimal deformation of the Leibnitz rule. Property *iii)* is the closure of $\Xi_\star$ under the adjoint action:

$$[u, v]_\star = ad^\star_u \, v = u_{1_\star} \star v \star S(u_{2_\star}) \ , \tag{2.53}$$

here we have used the coproduct notation $\Delta_\star(u) = u_{1_\star} \otimes u_{2_\star}$. More in general the adjoint action is given by

$$ad^\star_\xi \, \zeta := \xi_{1_\star} \star \zeta \star S_\star(\xi_{2_\star}) \ , \tag{2.54}$$

where we used the coproduct notation $\Delta_\star(\xi) = \xi_{1_\star} \otimes \xi_{2_\star}$.

[2]This can already be seen at the semiclassical level, where we are left with the symplectic structure. Primitive elements then correspond to symplectic infinitesimal transformations. Instead of restricting the set of transformations to those compatible with the bivector $\theta^{\mu\nu}$ we want to properly generalize/relax the notion of infinitesimal automorphism. In this way we do not consider $\theta^{\mu\nu}$ as the components of a bivector, but as a set of constant coefficients.

In the case the deformation is given by a twist we have a natural candidate for the Lie algebra of the Hopf algebra $U\Xi_\star$. We apply the recepy (2.12) and deform the Lie algebra product $[\;\;]$ given in (2.29) into

$$\begin{aligned} [\;\;]_\star : \qquad & \Xi \times \Xi \rightarrow \Xi \\ & (u,v) \mapsto [u\; v]_\star := [\overline{\mathrm{f}}^\alpha(u)\; \overline{\mathrm{f}}_\alpha(v)] \; . \end{aligned} \tag{2.55}$$

In $U\Xi_\star$ this $\star$-Lie bracket can be realized as a deformed commutator

$$\begin{aligned} [u\; v]_\star &= [\overline{\mathrm{f}}^\alpha(u)\; \overline{\mathrm{f}}_\alpha(v)] = \overline{\mathrm{f}}^\alpha(u)\overline{\mathrm{f}}_\alpha(v) - \overline{\mathrm{f}}_\alpha(v)\overline{\mathrm{f}}^\alpha(u) \\ &= u \star v - \overline{R}^\alpha(v) \star \overline{R}_\alpha(u) \; . \end{aligned} \tag{2.56}$$

It is easy to see that the bracket $[\;\;]_\star$ has the $\star$-antisymmetry property

$$[u\; v]_\star = -[\overline{R}^\alpha(v)\; \overline{R}_\alpha(u)]_\star \; . \tag{2.57}$$

This can be shown as follows

$$[u\; v]_\star = [\overline{\mathrm{f}}^\alpha(u)\; \overline{\mathrm{f}}_\alpha(v)] = -[\overline{\mathrm{f}}_\alpha(v)\; \overline{\mathrm{f}}^\alpha(u)] = -[\overline{R}^\alpha(v)\; \overline{R}_\alpha(u)]_\star \; .$$

A $\star$-Jacoby identity can be proven as well

$$[u\; [v\; z]_\star]_\star = [[u\; v]_\star\; z]_\star + [\overline{R}^\alpha(v)\; [\overline{R}_\alpha(u)\; z]_\star]_\star \; . \tag{2.58}$$

The appearence of the R-matrix $\mathcal{R}^{-1} = \overline{R}^\alpha \otimes \overline{R}_\alpha$ is not unexpected. We have seen that $\mathcal{R}^{-1}$ encodes the noncommutativity of the $\star$-product $h \star g = \overline{R}^\alpha(g) \star \overline{R}_\alpha(h)$ so that $h \star g$ do $\mathcal{R}^{-1}$-commute. Then it is natural to define $\star$-commutators using the $\mathcal{R}^{-1}$-matrix. In other words, the representation of the permutation group to be used on twisted noncommutative spaces is the one given by the $\mathcal{R}^{-1}$ matrix.

We now show that the subspace $\Xi_\star$ (that as vectorspace equals Ξ) has all the three properties *i*), *ii*), *iii*). It satisfies *i*) because any sum of products of vectorfields in $U\Xi$ can be rewritten as sum of $\star$-products of vectorfields via the formula $u\, v = \mathrm{f}^{\,\alpha}(u) \star \mathrm{f}_{\,\alpha}(v)$, and therefore $\star$-vectorfields generate the algebra. It obviously satisfies *ii*), and finally in the appendix we prove that it satisfies *iii*) by showing that the bracket $[u\; v]_\star$ is indeed the adjoint action, $ad^\star_u v = [u\; v]_\star$.

We stress that the geometrical –and therefore physical– interpretation of $\Xi_\star$ as infinitesimal diffeomorphisms is due to the deformed Leibniz rule property *ii*) and to the closure of $\Xi_\star$ under the adjoint action. Property *ii*) will be fundamental in order to define covariant derivatives (cf. (4.3)).

Note 1. The Hopf algebra $U\Xi_\star$ can be described via the generators $X_u := \mathrm{f}^{\,\alpha}(u)\mathrm{f}_{\,\alpha}$ rather than via the u generators. The action of X_u on functions is the differential operator $X^\star_u \equiv \mathcal{L}^\star_{X_u}$, we have $X^\star_u(f) \equiv \mathcal{L}^\star_{X_u}(f) = u(f)$, compare with eq. (5.2) in [13], see also [34]. The generators X_u satisfy the commutation relations $X_u \star X_v - X_v \star X_u = X_{[u,v]}$ and their coproduct is $\Delta_\star(X_u) = \mathcal{F}(X_u \otimes 1 + 1 \otimes X_u)\mathcal{F}^{-1}$. We see that $U\Xi_\star$ is the abstract Hopf algebra of diffeomorphisms considered in [13], end of Section 5. Since the elements X_u generate $U\Xi_\star$, invariance under the diffeomorphisms algebra $U\Xi_\star$ is equivalently shown by proving invariance under the X_u or the u generators. Since $X_{\partial_\mu} = \partial_\mu$ partial derivatives belong to both sets of generators. We also have $\mathcal{L}^\star_{\partial_\mu}(f) = \partial_\mu(f) = \partial^\star_\mu \triangleright f$.

2.1 Relation between $U\Xi_\star$ and $U\Xi^\mathcal{F}$

In the previous four pages, using the twist $\mathcal{F}$ and the general prescription (2.12) we have described the Hopf algebra

$$(U\Xi_\star, \star, \Delta_\star, S_\star, \varepsilon)$$

and its Lie algebra $(\Xi_\star, [\quad]_\star)$. These are a deformation of the cocommutative Hopf algebra

$$(U\Xi, \cdot, \Delta, S, \varepsilon)$$

and its Lie algebra $(\Xi, [\quad])$. Usually given a twist $\mathcal{F}$ one deforms the Hopf algebra $(U\Xi, \cdot\,\Delta, S, \varepsilon)$ into the Hopf algebra

$$(U\Xi^\mathcal{F}, \cdot, \Delta^\mathcal{F}, S^\mathcal{F}, \varepsilon)$$

where the coproduct is deformed via

$$\Delta^\mathcal{F}(\xi) := \mathcal{F}\Delta(\xi)\mathcal{F}^{-1} \ , \tag{2.59}$$

while product, antipode and counit are undeformed $\cdot^\mathcal{F} = \cdot$, $S^\mathcal{F} = S$, $\varepsilon^\mathcal{F} = \varepsilon$ ($S^\mathcal{F} = S$ only for abelian antisymmetric twists).

The Hopf algebras $U\Xi_\star$ and $U\Xi^\mathcal{F}$ are isomorphic. As vectorspaces $U\Xi_\star = U\Xi = U\Xi^\mathcal{F}$. The Hopf algebra isomorphism is given by the linear map $D : U\Xi^\star \to U\Xi^\star$

$$D(\xi) = \bar{\mathrm{f}}^\alpha(\xi)\bar{\mathrm{f}}_\alpha \ . \tag{2.60}$$

The inverse of the map D is

$$D^{-1} \equiv X : \ \xi \longmapsto X_\xi = \mathrm{f}^{\,\alpha}(\xi)\mathrm{f}_{\,\alpha} \ ,$$

indeed $D(X_\xi) = \bar{\mathrm{f}}^\beta(\mathrm{f}^{\,\alpha}(\xi)\mathrm{f}_{\,\alpha})\bar{\mathrm{f}}_\beta = \bar{\mathrm{f}}^\beta(\mathrm{f}^{\,\alpha}(\xi))\mathrm{f}_{\,\alpha}\bar{\mathrm{f}}_\beta = (\bar{\mathrm{f}}^\beta\mathrm{f}^{\,\alpha})(\xi)\,\mathrm{f}_{\,\alpha}\bar{\mathrm{f}}_\beta = \xi$ where we used that partial derivatives commute among themselves and in the last line we used $\mathcal{F}^{-1}\mathcal{F} = 1 \otimes 1$. Explicitly the Hopf algebra isomorphisms between $U\Xi^\star$ and $U\Xi^\mathcal{F}$ is, [14]

$$D(\xi \star \zeta) = D(\xi)D(\zeta) \ , \tag{2.61}$$

$$\Delta_\star = (D^{-1} \otimes D^{-1}) \circ \Delta^\mathcal{F} \circ D \ , \tag{2.62}$$

$$S_\star = D^{-1} \circ S^\mathcal{F} \circ D \ . \tag{2.63}$$

Under this isomorphism the Lie algebra $\Xi_\star$ is mapped into the Lie algebra $\Xi^{\mathcal{F}} := D(\Xi_\star)$ of all elements

$$u^{\mathcal{F}} := D(u) = \overline{\mathrm{f}}^\alpha(u)\overline{\mathrm{f}}_\alpha \ .$$

The bracket in $\Xi^{\mathcal{F}}$ is the deformed commutator

$$[u^{\mathcal{F}}, v^{\mathcal{F}}]_{\mathcal{F}} = u^{\mathcal{F}} v^{\mathcal{F}} - \overline{R}^\alpha(v^{\mathcal{F}})\overline{R}_\alpha(u^{\mathcal{F}}) \tag{2.64}$$

and it equals the adjoint action in $U\Xi^{\mathcal{F}}$,

$$[u^{\mathcal{F}}, v^{\mathcal{F}}]_{\mathcal{F}} = ad^{\mathcal{F}}_{u^{\mathcal{F}}} v^{\mathcal{F}} = u^{\mathcal{F}}_{1_{\mathcal{F}}}\, v\, S(u^{\mathcal{F}}_{2_{\mathcal{F}}}) \ , \tag{2.65}$$

where we used the notation $\Delta^{\mathcal{F}}(\xi) = \xi_{1_{\mathcal{F}}} \otimes \xi_{2_{\mathcal{F}}}$. The usual Lie algebra Ξ of vectorfields with the usual bracket $[u, v] = uv - vu$ is not properly a Lie algebra of $U\Xi^{\mathcal{F}}$ because the commutator fails to be the adjoint action and the Leibniz rule is not of the type *ii*). In particular the vectofields u have not the geometric interpretation of infinitesimal diffeomorphisms.

2.2 $\star$-Diffeomorphisms Symmetry

In the commutative case the diffeomorphisms algebra $U\Xi$ acts on the algebra of functions and more in general on the algebra of tensorfields via the Lie derivative. The Rimemann curvature, the Ricci tensor and the curvature scalar are tensors and therefore they transform covariantly under the diffeomorphisms action. In the twisted case, the $\star$-diffeomorphisms algebra $U\Xi_\star$ acts on the $\star$-algebra of functions $A_\star$ and more in general on the $\star$-algebra of tensorfields $\mathcal{T}_\star$. The action on functions is given by the $\star$-Lie derivative defined in (2.37). Similarly the action on tensors is given, according to (2.12), by

$$\mathcal{L}^\star_u(\tau) := \overline{\mathrm{f}}^\alpha(u)(\overline{\mathrm{f}}_\alpha(\tau)) \ . \tag{2.66}$$

This expression defines an action because $\mathcal{L}^\star_u(\mathcal{L}^\star_v(\tau)) = \mathcal{L}^\star_{u\star v}(\tau)$. In particular the $\star$-Lie derivative is a representation of the Lie algebra of infinitesimal diffeomorphisms $\Xi_\star$,

$$\mathcal{L}^\star_u\, \mathcal{L}^\star_v - \mathcal{L}^\star_{\overline{R}^\alpha(v)}\, \mathcal{L}^\star_{\overline{R}_\alpha(u)} = \mathcal{L}^\star_{[u\ \ v]_\star} \ , \tag{2.67}$$

where $\mathcal{L}^\star_u\, \mathcal{L}^\star_v = \mathcal{L}^\star_u \circ \mathcal{L}^\star_v$ i s the usual composition of operators. The coproduct in $U\Xi$ is compatible with the product in the tensorfields algebra because

$$\mathcal{L}^\star_u(\tau \otimes_\star \tau') = \mathcal{L}^\star_u(\tau) \star \tau' + \overline{R}^\alpha(\tau) \star \mathcal{L}^\star_{\overline{R}_\alpha(u)}(\tau') \ . \tag{2.68}$$

In Section 4 we introduce the noncommutative Riemann tensor and Ricci curvature, and show that they are indeed tensors. Then they transform covariantly under the action of the $\star$-diffeomorphism algebra. The corresponding noncommutative Einstein equations satisfy the symmetry principle of noncommutative general covariance, i.e. they are covariant under $\star$-diffeomorphism symmetry.

3 Poincaré Symmetry

The considerations about the undeformed Hopf algebra $U\Xi$, and the Hopf algebras $U\Xi_\star$ and $U\Xi^{\mathcal{F}}$ hold independently from Ξ being the Lie algebra of infinitesimal diffeomorphisms. In this section we study the case of the deformed Poincaré algebra. It can be seen as an abstract algebra or also as a subalgebra of infinitesimal diffeomorphisms Ξ.

3.1 $\star$-Poincaré algebra

We start by recalling that the usual Poincaré Lie algebra $iso(3,1)$:

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \ , \\ [P_\rho, M_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \ , & (3.1) \\ [M_{\mu\nu}, M_{\rho\sigma}] &= -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}) \ , & (3.2) \end{aligned}$$

is not a symmetry of θ-noncommutative space because the relations

$$x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu} \tag{3.3}$$

are not compatible with Poincaré transformations. Indeed consider the standard representation of the Poincaré algebra on functions $h(x)$,

$$P_\mu(h) = i\partial_\mu(h) \ , \ M_{\mu\nu}(h) = i(x_\mu\partial_\nu - x_\nu\partial_\mu)(h) \ , \tag{3.4}$$

then we have $M_{\rho\sigma}(\theta^{\mu\nu}) = 0$ while $M_{\rho\sigma}(x^\mu \star x^\nu - x^\nu \star x^\mu) \neq 0$. This is so because we use the undeformed Leibniz rule $M_{\rho\sigma}(x^\mu \star x^\nu - x^\nu \star x^\mu) = M_{\rho\sigma}(x^\mu) \star x^\nu + x^\mu \star M_{\rho\sigma}(x^\nu)$. In other words the Hopf algebra $U(iso(3,1))$ generated by the Poincaré Lie algebra and with usual coproducts

$$\Delta(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu \quad , \quad \Delta(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} \tag{3.5}$$

is not a symmetry of noncommutative spacetime.

One approach to overcome this problem is to just deform the coproduct Δ into the new coproduct $\Delta^{\mathcal{F}}(M_{\mu\nu}) = \mathcal{F}\Delta(M_{\mu\nu})\mathcal{F}^{-1}$ (see next subsection).

Another approach is to observe first that the action of $M_{\rho\sigma}$ on $h\star g$ is hybrid, indeed it mixes ordinary products with $\star$-products: $M_{\mu\nu}(h\star g) = ix_\mu\partial_\nu(h\star g) - ix_\nu\partial_\mu(h\star g)$. This is cured by considering a different action of the generators P_μ and $M_{\mu\nu}$ on noncommutative spacetime. The $\mathcal{L}^\star$ action defined in (2.37), accordingly with the general prescription (2.12), exactly replaces the ordinary product with the $\star$-product. For any function $h(x)$ we have,

$$\begin{aligned} \mathcal{L}^\star_{P_\mu}(h) &= i\partial_\mu(h) \\ \mathcal{L}^\star_{M_{\mu\nu}}(h) &= ix_\mu \star \partial_\nu(h) - ix_\nu \star \partial_\mu(h) \ . & (3.6) \end{aligned}$$

This action of the Poincaré generators on functions can be extended to an action of the universal enveloping algebra $U(iso(3,1))$ if $U(iso(3,1))$ is endowed with the new $\star$-product

$$\begin{aligned}\xi \star \zeta &:= \overline{\mathrm{f}}^{\alpha}(\xi)\overline{\mathrm{f}}_{\alpha}(\zeta) \qquad (3.7)\\ &= \sum \frac{1}{n!}\left(\frac{-i}{2}\right)^{n} \theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}[P_{\rho_1}\dots[P_{\rho_n},\xi]\dots]\,[P_{\sigma_1}\dots[P_{\sigma_n},\zeta]\dots]\,,\end{aligned}$$

for all ξ and ζ in $U(iso(3,1))$. For example it is easy to see that

$$\mathcal{L}^{\star}_{M_{\mu\nu}\star M_{\rho\sigma}}(h) = \mathcal{L}^{\star}_{M_{\mu\nu}}(\mathcal{L}^{\star}_{M_{\rho\sigma}}(h))\ . \qquad (3.8)$$

In formula (3.7) we have identified the Lie algebra of partial derivatives with the Lie algebra of momenta P_μ, so that

$$\mathcal{F} = e^{\frac{i}{2}\theta^{\mu\nu}P_\mu\otimes P_\nu} \quad , \quad \mathcal{R}^{-1} = e^{i\theta^{\mu\nu}P_\mu\otimes P_\nu}\ . \qquad (3.9)$$

This identification is uniquely fixed by the representation (3.4): $P_\mu = i\partial_\mu$. Since products of the generators P_μ and $M_{\mu\nu}$ can be rewritten as sum of $\star$-products via the formula $\xi\zeta = \mathrm{f}^{\alpha}(\xi)\star\mathrm{f}_{\alpha}(\zeta)$, the elements P_μ and $M_{\mu\nu}$ generate the algebra $U_\star(iso(3,1))$.

The coproduct compatible with noncommutative spacetime is inferred from the Leibniz rule

$$\begin{aligned}x_\mu\star\partial_\nu(h\star g) &= x_\mu\star\partial_\nu(h)\star g + x_\mu\star h\star\partial_\nu(g)\\ &= x_\mu\star\partial_\nu(h)\star g + \overline{R}^{\alpha}(h)\star\overline{R}_{\alpha}(x_\mu)\star\partial_\nu(g)\ . \qquad (3.10)\end{aligned}$$

The coproduct that implements this Leibniz rule is (cf. (2.34))

$$\Delta_\star(M_{\mu\nu}) = M_{\mu\nu}\otimes 1 + \overline{R}^{\alpha}\otimes\overline{R}_{\alpha}(M_{\mu\nu})\ . \qquad (3.11)$$

Explicitly the coproduct on the generators P_μ and $M_{\mu\nu}$ reads

$$\begin{aligned}\Delta_\star(P_\mu) &= P_\mu\otimes 1 + 1\otimes P_\mu\ ,\\ \Delta_\star(M_{\mu\nu}) &= M_{\mu\nu}\otimes 1 + 1\otimes M_{\mu\nu} + i\theta^{\alpha\beta}P_\alpha\otimes[P_\beta, M_{\mu\nu}]\ . \qquad (3.12)\end{aligned}$$

The counit and antipode on the generators can be calculated from (2.35) and (2.36), they are

$$\varepsilon(P_\mu) = \varepsilon(M_{\mu\nu}) = 0 \ \ ,\quad S_\star(P_\mu) = -P_\mu \ \ ,\quad S_\star(M_{\mu\nu}) = -M_{\mu\nu} - i\theta^{\rho\sigma}[P_\rho, M_{\mu\nu}]P_\sigma\ . \qquad (3.13)$$

We have constructed the Hopf algebra $U_\star(iso(3,1))$.

We recall that there are three natural conditions that the $\star$-Poincaré Lie algebra $iso_\star(3,1)$ has to satisfy. It has to be a linear subspace of $U_\star(iso(3,1))$ such that if $\{t_i\}_{i=1,...n}$ is a basis of $iso_\star(3,1)$, we have (sum understood on repeated indices)

$$\begin{aligned} i) &\quad \{t_i\} \text{ generates } U_\star(iso(3,1)) \\ ii) &\quad \Delta_\star(t_i) = t_i \otimes 1 + f_i{}^j \otimes t_j \\ iii) &\quad [t_i, t_j]_\star = C_{ij}{}^k t_k \end{aligned}$$

where $C_{ij}{}^k$ are structure constants and $f_i{}^j \in U^{\mathcal{F}}(iso(3,1))$ $(i,j = 1,...n)$. In the last line the bracket $[\ ,\ ]_\star$ is the adjoint action (we use the notation $\Delta_\star(t) = t_{1_\star} \otimes t_{2_\star}$):

$$[t, t']_\star := ad^\star_t\, t' = t_{1_\star} \star t' \star S_\star(t_{2_\star}) \ . \tag{3.14}$$

We have seen that the elements P_μ and $M_{\mu\nu}$ generate $U_\star(iso(3,1))$. They are deformed *infinitesimal* generators because they satisfy the Leibniz rule $ii)$ and because they close under the adjoint action $iii)$. In order to prove property $iii)$ we perform a short calculation and obtain the explicit expression of the adjoint action (3.14),

$$\begin{aligned} [P_\mu, P_\nu]_\star &= [P_\mu, P_\nu] \ , \\ [P_\rho, M_{\mu\nu}]_\star &= [P_\rho, M_{\mu\nu}] = -[M_{\mu\nu}, P_\rho]_\star \ , \\ [M_{\mu\nu}, M_{\rho\sigma}]_\star &= M_{\mu\nu} \star M_{\rho\sigma} - M_{\rho\sigma} \star M_{\mu\nu} - i\theta^{\alpha\beta}[P_\alpha, M_{\rho\sigma}][P_\beta, M_{\mu\nu}] = [M_{\mu\nu}, M_{\rho\sigma}] \ . \end{aligned}$$

Notice that this result shows that the adjoint action (3.14) equals the deformed commutator

$$t \star t' - \overline{R}^\alpha(t') \star \overline{R}_\alpha(t) \ .$$

Property $iii)$, i.e. closure under the adjoint action, explicitly reads

$$\begin{aligned} [P_\mu, P_\nu]_\star &= 0 \ , \\ [P_\rho, M_{\mu\nu}]_\star &= i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu) \ , \\ [M_{\mu\nu}, M_{\rho\sigma}]_\star &= -i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}) \ . \end{aligned} \tag{3.15}$$

We notice that the structure constants are the same as in the undeformed case, however the adjoint action $[M_{\mu\nu}, M_{\rho\sigma}]_\star$ is not the commutator $M_{\mu\nu} \star M_{\rho\sigma} - M_{\rho\sigma} \star M_{\mu\nu}$ anymore, it is a deformed commutator quadratic in the generators and antisymmetric.

From (3.15) we immediately obtain the Jacoby identities:

$$[t\,, [t', t'']_\star]_\star + [t'\,, [t'', t]_\star]_\star + [t''\,, [t, t']_\star]_\star = 0 \ , \tag{3.16}$$

for all $t, t', t'' \in iso_\star(3,1)$.

It can be proven that the Hopf algebra $U_\star(iso(3,1))$ is the algebra freely generated by P_μ and $M_{\mu\nu}$ (we denote the product by $\star$) modulo the relations iii).

Note 2. In [19] we studied quantum Poincaré groups (in any dimension) obtained via abelian twists $\mathcal{F}$ different from the one considered here. Their Lie algebra is described according to i), ii), iii) (see for ex. eq. (6.65),(7.36),(7.6),(7.7) in the first paper in [19]). Because of these three properties the Lie algebra defines a differential calculus on the quantum Poincaré group manifold that respects the quantum Poincaré symmetry (i.e. that is bicovariant).

3.2 Twisted Poincaré algebra

The Poincaré Hopf algebra $U^{\mathcal{F}}(iso(3,1))$ is another deformation of $U(iso(3,1))$. As algebras $U^{\mathcal{F}}(iso(3,1)) = U(iso(3,1))$; but $U^{\mathcal{F}}(iso(3,1))$ has the new coproduct

$$\Delta^{\mathcal{F}}(\xi) = \mathcal{F}\Delta(\xi)\mathcal{F}^{-1} , \tag{3.17}$$

for all $\xi \in U(iso(3,1))$. In order to write the explicit expression for $\Delta^{\mathcal{F}}(P_\mu)$ and $\Delta^{\mathcal{F}}(M_{\mu\nu})$, we use the Hadamard formula

$$Ad_{e^X}Y = e^X\ Y\ e^{-X} = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[X,[X,...[}_{n}X,Y]] = \sum_{n=0}^{\infty} \frac{(adX)^n}{n!}\ Y$$

and the relation $[P \otimes P', M \otimes 1] = [P, M] \otimes P'$, and thus obtain [24], [23]

$$\begin{aligned} \Delta^{\mathcal{F}}(P_\mu) &= P_\mu \otimes 1 + 1 \otimes P_\mu\ , \\ \Delta^{\mathcal{F}}(M_{\mu\nu}) &= M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} \\ &\quad - \frac{1}{2}\theta^{\alpha\beta}\left((\eta_{\alpha\mu}P_\nu - \eta_{\alpha\nu}P_\mu) \otimes P_\beta + P_\alpha \otimes (\eta_{\beta\mu}P_\nu - \eta_{\beta\nu}P_\mu)\right) . \end{aligned} \tag{3.18}$$

We have constructed the Hopf algebra $U^{\mathcal{F}}(iso(3,1)$: it is the algebra generated by $M_{\mu\nu}$ and P_μ modulo the relations (3.1), and with coproduct (3.18) and counit and antipode that are as in the undeformed case:

$$\varepsilon(P_\mu) = \varepsilon(M_{\mu\nu}) = 0 \quad , \quad S(P_\mu) = -P_\mu \quad , \quad S(M_{\mu\nu}) = -M_{\mu\nu}\ . \tag{3.19}$$

This Hopf algebra is a symmetry of noncommutative spacetime provided that we consider the "hybrid" action $M_{\mu\nu}(h \star g) = ix_\mu\partial_\nu(h \star g) - ix_\nu\partial_\mu(h \star g)$.

The Poincaré Lie algebra $iso^{\mathcal{F}}(3,1)$ must be a linear subspace of $U^{\mathcal{F}}(iso(3,1))$ such that if $\{t_i\}_{i=1,...n}$ is a basis of $iso^{\mathcal{F}}(3,1)$, we have (sum understood on repeated indices)

i) $\{t_i\}$ generates $U^{\mathcal{F}}(iso(3,1))$

ii) $\Delta^{\mathcal{F}}(t_i) = t_i \otimes 1 + f_i{}^j \otimes t_j$

iii) $[t_i, t_j]_{\mathcal{F}} = C_{ij}{}^k t_k$

where ${C_{ij}}^k$ are structure constants and ${f_i}^j \in U^{\mathcal{F}}(iso(3,1))$ $(i,j = 1,...n)$. In the last line the bracket $[\ ,\ ]_{\mathcal{F}}$ is the adjoint action:

$$[t,t']_{\mathcal{F}} := ad_t^{\mathcal{F}}\, t' = t_{1_{\mathcal{F}}} t' S(t_{2_{\mathcal{F}}}) \ . \tag{3.20}$$

The statement that the Lie algebra of $U^{\mathcal{F}}(iso(3,1))$ is the undeformed Poincaré Lie algebra (3.1) is not correct because conditions $ii)$ and $iii)$ are not met by the generators P_μ and $M_{\mu\nu}$. There is a canonical procedure in order to obtain the Lie algebra $iso^{\mathcal{F}}(3,1)$ of $U^{\mathcal{F}}(iso(3,1))$. We use the Hopf algebra isomorphism

$$\begin{aligned} D\,:\, U_\star(iso(3,1)) &\to U^{\mathcal{F}}(iso(3,1)) \\ \xi &\mapsto \overline{\mathrm{f}}^\alpha(\xi)\overline{\mathrm{f}}_\alpha \end{aligned}$$

and define

$$iso^{\mathcal{F}}(3,1) := D(iso_\star(3,1)) \ .$$

Explicitly consider the elements

$$P_\mu^{\mathcal{F}} := \overline{\mathrm{f}}^\alpha(P_\mu)\overline{\mathrm{f}}_\alpha = P_\mu \ , \tag{3.21}$$

$$\begin{aligned} M_{\mu\nu}^{\mathcal{F}} := \overline{\mathrm{f}}^\alpha(M_{\mu\nu})\overline{\mathrm{f}}_\alpha &= M_{\mu\nu} - \frac{i}{2}\theta^{\rho\sigma}[P_\rho, M_{\mu\nu}]P_\sigma \\ &= M_{\mu\nu} + \frac{1}{2}\theta^{\rho\sigma}(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu)P_\sigma \end{aligned} \tag{3.22}$$

Their coproduct is

$$\begin{aligned} \Delta^{\mathcal{F}}(P_\mu) &= P_\mu \otimes 1 + 1 \otimes P_\mu \ , \\ \Delta^{\mathcal{F}}(M_{\mu\nu}^{\mathcal{F}}) &= M_{\mu\nu}^{\mathcal{F}} \otimes 1 + 1 \otimes M_{\mu\nu}^{\mathcal{F}} + i\theta^{\alpha\beta}P_\alpha \otimes [P_\beta, M_{\mu\nu}] \ . \end{aligned} \tag{3.23}$$

The counit and antipode are

$$\varepsilon(P_\mu) = \varepsilon(M_{\mu\nu}^{\mathcal{F}}) = 0 \quad , \quad S(P_\mu) = -P_\mu \quad , \quad S(M_{\mu\nu}^{\mathcal{F}}) = -M_{\mu\nu}^{\mathcal{F}} - i\theta^{\rho\sigma}[P_\rho, M_{\mu\nu}]P_\sigma \ . \tag{3.24}$$

The elements $P_\mu^{\mathcal{F}}$ and $M_{\mu\nu}^{\mathcal{F}}$ are generators because they satisfy condition $i)$ (indeed $M_{\mu\nu} = M_{\mu\nu}^{\mathcal{F}} + \frac{i}{2}\theta^{\rho\sigma}[P_\rho, M_{\mu\nu}^{\mathcal{F}}]P_\sigma$). They are deformed *infinitesimal* generators because they satisfy the Leibniz rule $ii)$ and because they close under the Lie bracket $iii)$. Explicitly

$$\begin{aligned} [P_\mu, P_\nu]_{\mathcal{F}} &= 0 \ , \\ [P_\rho, M_{\mu\nu}^{\mathcal{F}}]_{\mathcal{F}} &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \, , \\ [M_{\mu\nu}^{\mathcal{F}}, M_{\rho\sigma}^{\mathcal{F}}]_{\mathcal{F}} &= -i(\eta_{\mu\rho}M_{\nu\sigma}^{\mathcal{F}} - \eta_{\mu\sigma}M_{\nu\rho}^{\mathcal{F}} - \eta_{\nu\rho}M_{\mu\sigma}^{\mathcal{F}} + \eta_{\nu\sigma}M_{\mu\rho}^{\mathcal{F}}) \ . \end{aligned} \tag{3.25}$$

We notice that the structure constants are the same as in the undeformed case, however the adjoint action $[M^{\mathcal{F}}_{\mu\nu}, M^{\mathcal{F}}_{\rho\sigma}]_{\mathcal{F}}$ is not the commutator anymore, it is a deformed commutator quadratic in the generators and antisymmetric:

$$\begin{aligned} [P_\mu, P_\nu]_{\mathcal{F}} &= [P_\mu, P_\nu]\,, \\ [P_\rho, M^{\mathcal{F}}_{\mu\nu}]_{\mathcal{F}} &= [P_\rho, M^{\mathcal{F}}_{\mu\nu}]\,, \\ [M^{\mathcal{F}}_{\mu\nu}, M^{\mathcal{F}}_{\rho\sigma}]_{\mathcal{F}} &= [M^{\mathcal{F}}_{\mu\nu}, M^{\mathcal{F}}_{\rho\sigma}] - i\theta^{\alpha\beta}[P_\alpha, M_{\rho\sigma}][P_\beta, M_{\mu\nu}]\ . \end{aligned} \tag{3.26}$$

From (3.25) we immediately obtain the Jacoby identities:

$$[t\,, [t', t'']_{\mathcal{F}}]_{\mathcal{F}} + [t'\,, [t'', t]_{\mathcal{F}}]_{\mathcal{F}} + [t''\,, [t, t']_{\mathcal{F}}]_{\mathcal{F}} = 0\ , \tag{3.27}$$

for all $t, t', t'' \in iso^{\mathcal{F}}(3,1)$.

4 Covariant Derivative, Torsion and Curvature

By now we have acquired enough knowledge on $\star$-noncommutative differential geometry to develop the formalism of covariant derivative, torsion, curvature and Ricci tensors just by following the usual classical formalism.

We define a $\star$-covariant derivative $\nabla^\star_u$ along the vector field $u \in \Xi$ to be a linear map $\nabla^\star_u : \Xi_\star \to \Xi_\star$ such that for all $u, v, z \in \Xi_\star,\ h \in A_\star$:

$$\nabla^\star_{u+v} z = \nabla^\star_u z + \nabla^\star_v z\ , \tag{4.1}$$

$$\nabla^\star_{h\star u} v = h \star \nabla^\star_u v\ , \tag{4.2}$$

$$\nabla^\star_u (h \star v)\ =\ \mathcal{L}^\star_u(h) \star v + \overline{R}^\alpha(h) \star \nabla^\star_{\overline{R}_\alpha(u)} v \tag{4.3}$$

Notice that in the last line we have used the coproduct formula (2.34), $\Delta_\star(u) = u \otimes 1 + \overline{R}^\alpha \otimes \overline{R}_\alpha(u)$. Epression (4.3) is well defined because $\overline{R}_\alpha(u)$ is again a vectorfield.

The (noncommutative) connection coefficients ${\Gamma_{\mu\nu}}^\sigma$ are given by

$$\nabla^\star_\mu \partial_\nu = {\Gamma_{\mu\nu}}^\sigma \star \partial_\sigma = {\Gamma_{\mu\nu}}^\sigma\, \partial_\sigma\ , \tag{4.4}$$

where $\nabla^\star_\mu = \nabla^\star_{\partial_\mu}$. They uniquely determine the connection, indeed let $z = z^\mu \star \partial_\mu$, $u = u^\nu \star \partial_\nu$, then

$$\begin{aligned} \nabla^\star_z u &= z^\mu \star \nabla^\star_\mu (u^\nu \star \partial_\nu) \\ &= z^\mu \star \partial_\mu (u^\nu)\, \partial_\nu + z^\mu \star u^\nu \star \nabla^\star_\mu \partial_\nu \\ &= z^\mu \star \partial_\mu (u^\nu)\, \partial_\nu + z^\mu \star u^\nu \star {\Gamma_{\mu\nu}}^\sigma\, \partial_\sigma\ ; \end{aligned} \tag{4.5}$$

these equalities are equivalent to the connection properties (4.2) and (4.3).

The covariant derivative is extended to tensorfields using the deformed Leibniz rule

$$\nabla^\star_u(v \otimes_\star z) = \nabla^\star_u(v) \otimes_\star z + \overline{R}^\alpha(v) \otimes_\star \nabla^\star_{\overline{R}_\alpha(u)} z \ .$$

Requiring compatibility of the covariant derivative with the contraction operator gives the covariant derivative on 1-forms, we have $\nabla^\star_z = z^\mu \star \nabla^\star_\mu$, and

$$\nabla^\star_\mu(\omega_\rho dx^\rho) = \partial_\mu(\omega_\rho)\, dx^\rho - \Gamma_{\mu\rho}{}^\nu \star \omega_\nu \, dx^\rho \ . \tag{4.6}$$

The torsion T and the curvature R associated to a connection $\nabla^\star$ are the linear maps $\mathsf{T} : \Xi_\star \times \Xi_\star \to \Xi_\star$, and $\mathsf{R}^\star : \Xi_\star \times \Xi_\star \times \Xi_\star \to \Xi_\star$ defined by

$$\mathsf{T}(u, v) := \nabla^\star_u v - \nabla^\star_{\overline{R}^\alpha(v)} \overline{R}_\alpha(u) - [u\ v]_\star \ , \tag{4.7}$$

$$\mathsf{R}(u, v, z) := \nabla^\star_u \nabla^\star_v z - \nabla^\star_{\overline{R}^\alpha(v)} \nabla^\star_{\overline{R}_\alpha(u)} z - \nabla^\star_{[u\,v]_\star} z \ , \tag{4.8}$$

for all $u, v, z \in \Xi_\star$. From the antisymmetry property of the bracket $[\ \ ]_\star$, see (2.57), it easily follows that the torsion T and the curvature R have the following $\star$-antisymmetry property

$$\mathsf{T}(u, v) = -\mathsf{T}(\overline{R}^\alpha(v), \overline{R}_\alpha(u)) \ ,$$
$$\mathsf{R}(u, v, z) = -\mathsf{R}(\overline{R}^\alpha(v), \overline{R}_\alpha(u), z) \ .$$

The presence of the R-matrix in the definition of torsion and curvature insures that T and R are left $A_\star$-linear maps [14], [33], i.e.

$$\mathsf{T}(f \star u, v) = f \star \mathsf{T}(u, v) \quad , \quad \mathsf{T}(\partial_\mu, f \star v) = f \star \mathsf{T}(\partial_\mu, v)$$

(for any ∂_μ), and similarly for the curvature. We have seen that any left $A_\star$-linear map $\Xi_\star \to A_\star$ is identified with a tensor, precisely a 1-form (recall comments after (2.28)), similarly the $A_\star$-linearity of T and R insures that we have well defined the torsion tensor and the curvature tensor.

One can also prove (twisted) first and second Bianchi identities [14], [33].

The coefficients $\mathsf{T}_{\mu\nu}{}^\rho$ and $\mathsf{R}_{\mu\nu\rho}{}^\sigma$ with respect to the partial derivatives basis $\{\partial_\mu\}$ are defined by

$$\mathsf{T}(\partial_\mu, \partial_\nu) = \mathsf{T}_{\mu\nu}{}^\rho \partial_\rho \quad , \qquad \mathsf{R}(\partial_\mu, \partial_\nu, \partial_\rho) = \mathsf{R}_{\mu\nu\rho}{}^\sigma \partial_\sigma \tag{4.9}$$

and they explicitly read

$$\mathsf{T}_{\mu\nu}{}^\rho = \Gamma_{\mu\nu}{}^\rho - \Gamma_{\nu\mu}{}^\rho \ ,$$
$$\mathsf{R}_{\mu\nu\rho}{}^\sigma = \partial_\mu \Gamma_{\nu\rho}{}^\sigma - \partial_\nu \Gamma_{\mu\rho}{}^\sigma + \Gamma_{\nu\rho}{}^\beta \star \Gamma_{\mu\beta}{}^\sigma - \Gamma_{\mu\rho}{}^\beta \star \Gamma_{\nu\beta}{}^\sigma \ . \tag{4.10}$$

As in the commutative case the Ricci tensor is a contraction of the curvature tensor,

$$\mathsf{Ric}_{\mu\nu} = \mathsf{R}_{\rho\mu\nu}{}^\rho . \tag{4.11}$$

A definition of the Ricci tensor that is independent from the $\{\partial_\mu\}$ basis is also possible.

5 Metric and Einstein Equations

In order to define a $\star$-metric we need to define $\star$-symmetric elements in $\Omega_\star \otimes_\star \Omega_\star$. Recalling the $\star$-antisymmetry of the wedge $\star$-product (2.22) we see that $\star$-symmetric elements are of the form

$$\omega \otimes_\star \omega' + \overline{R}^\alpha(\omega') \otimes_\star \overline{R}_\alpha(\omega) \ . \tag{5.1}$$

In particular any symmetric tensor in $\Omega \otimes \Omega$,

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu \ , \tag{5.2}$$

$g_{\mu\nu} = g_{\nu\mu}$, is also a $\star$-symmetric tensor in $\Omega_\star \otimes_\star \Omega_\star$ because

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu = g_{\mu\nu} \star dx^\mu \otimes_\star dx^\nu \tag{5.3}$$

and the action of the R-matrix is the trivial one on dx^ν. We denote by $g^{\star\mu\nu}$ the star inverse of $g_{\mu\nu}$,

$$g^{\star\mu\rho} \star g_{\rho\nu} = g_{\nu\rho} \star g^{\star\rho\mu} = \delta^\mu_\nu \ . \tag{5.4}$$

The metric $g_{\mu\nu}$ can be expanded order by order in the noncommutative parameter $\theta^{\rho\sigma}$. Any commutative metric is also a noncommutative metric, indeed the $\star$-inverse metric can be constructed order by order in the noncommutativity parameter. Contrary to [7], [36], we see that in our approach there are infinitely many metrics compatible with a given noncommutative differential geometry, noncommutativity does not single out a preferred metric.

A connection that is metric compatible is a connection that for any vectorfield u satisfies, $\nabla^\star_u g = 0$, this is equivalent to the equation

$$\nabla^\star_\mu g_{\rho\sigma} - \Gamma_{\mu\rho}{}^\nu \star g_{\nu\sigma} - \Gamma_{\mu\sigma}{}^\nu \star g_{\rho\nu} = 0 \ . \tag{5.5}$$

Proceeding as in the commutative case we obtain that there is a unique torsion free metric compatible connection [13]. It is given by

$$\Gamma_{\mu\nu}{}^\rho = \frac{1}{2}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \star g^{\star\sigma\rho} \tag{5.6}$$

We now construct the curvature tensor and the Ricci tensor using this uniquely defined connection. Finally the noncommutative Einstein equations (in vacuum) are

$$\mathsf{Ric}_{\mu\nu} = 0 \tag{5.7}$$

where the dynamical field is the metric g.

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A Differential Operators and Vectorfields

We briefly describe the algebra of differential operators and show that it is not a Hopf algebra by relating it to the Hopf algebra of infinitesimal diffeomorphisms.

Differential operator on the space of functions $A = Fun(\mathbb{R}^4)$ are elements of the form $f(x)^{\mu_1 \cdots \mu_n}\partial_{\mu_1} \ldots \partial_{\mu_n}$. They form an algebra, the only nontrivial commutation relations are between functions and partial derivatives,

$$\partial_\mu f = \partial_\mu(f) + f\partial_\mu \ , \tag{A.1}$$

where both ∂_μ and f act on functions (the action of f on the function h is given by the product fh.). Differential operators of zeroth order are functions. Differential operators of first order Diff^1 are derivations of the algebra A of functions (i.e. they satify the Leibniz rule) they are therefore vectorfields Ξ (infinitesimal diffeomorphisms). The isomorphism between vectorfields and first order differential operators is given by the Lie derivative

$$\begin{aligned} \mathcal{L} :\ \Xi &\rightarrow \mathit{Diff}^1 \\ v &\mapsto \mathcal{L}_v \end{aligned} \tag{A.2}$$

where

$$\mathcal{L}_v(f) = v(f) \ .$$

We use the notation $\mathcal{L}_v$ in order to stress that the abstract Lie algebra element $v \in \Xi$ is seen as a differential operator. The Lie derivative can be extended to a map from the universal enveloping algebra of vectorfields $U\Xi$ to all differential operators

$$\begin{aligned} \mathcal{L} :\ \Xi &\rightarrow \mathit{Diff} \\ uv...z &\mapsto \mathcal{L}_u \, \mathcal{L}_v \, ...\mathcal{L}_z \end{aligned} \tag{A.3}$$

Notice that on the left hand side the product uv is in $U\Xi$ (recall the paragraph after (2.30)), while on the right hand side the product $\mathcal{L}_u\,\mathcal{L}_v = \mathcal{L}_u \circ \mathcal{L}_v$ is the usual composition product of operators.

The map $\mathcal{L}$ is an algebra morphism between the algebras $U\Xi$ and *Diff*. It is not surjective because the image of $U\Xi$ does not contain the full space of functions A but only the constant ones (the multiples of the unit of $U\Xi$).

In order to show that the map $\mathcal{L} : U\Xi \to$ *Diff* is not injective we consider the vectorfields

$$\begin{aligned} u = f\partial_\mu \quad &, \qquad v = \partial_\nu \ , \\ u' = f\partial_\nu \quad &, \qquad v' = \partial_\mu \ , \end{aligned}$$

where for example $f = x^\nu$, and we show that

$$uv \neq u'v' \qquad \text{in } U\Xi\,. \tag{A.4}$$

The map $\mathcal{L}$ is then not injective because $f\partial_\mu(\partial_\nu(h)) = f\partial_\nu(\partial_\mu(h))$ for any function h implies

$$\mathcal{L}_{uv} = \mathcal{L}_{u'v'} \ .$$

The algebra $U\Xi$ is a Hopf algebra, in particular there is a well defined coproduct map Δ, and therefore one way to prove the inequalty (A.4) is to prove that $\Delta(uv) \neq \Delta(u'v')$. We calculate

$$\begin{aligned} \Delta(uv) &= \Delta(u)\Delta(v) = uv \otimes 1 + u \otimes v + v \otimes u + 1 \otimes uv \\ &= f\partial_\mu\,\partial_\nu \otimes 1 + f\partial_\mu \otimes \partial_\nu + \partial_\nu \otimes f\partial_\mu + 1 \otimes f\partial_\mu\,\partial_\nu \ , \end{aligned}$$

and

$$\Delta(u'v') = f\partial_\nu\,\partial_\mu \otimes 1 + f\partial_\nu \otimes \partial_\mu + \partial_\mu \otimes f\partial_\nu + 1 \otimes f\partial_\nu\,\partial_\mu \ .$$

These two expressions are different. For example by applying the product map in $U\Xi$, $\cdot : U\Xi \otimes U\Xi \to U\Xi$ and then the map $\mathcal{L} : U\Xi \to$ *Diff* we respectively obtain

$$3f\partial_\mu\,\partial_\nu + \partial_\nu\, f\partial_\mu \neq 3f\partial_\nu\,\partial_\mu + \partial_\mu\, f\partial_\nu \ . \tag{A.5}$$

From this proof we conclude that we cannot equip the algebra of differential operators *Diff* with a coproduct like the one in $U\Xi$. The map defined by $\Delta(\mathcal{L}_u) = \mathcal{L}_u \otimes 1 + 1 \otimes \mathcal{L}_u$, and extended multiplicatively to all *Diff* is not well defined because $\mathcal{L}_{uv} = \mathcal{L}_{u'v'}$, while

$$\Delta(\mathcal{L}_{uv}) = \Delta(\mathcal{L}_u)\Delta(\mathcal{L}_v) \neq \Delta(\mathcal{L}_{u'})\Delta(\mathcal{L}_{v'}) = \Delta(\mathcal{L}_{u'v'}) \ ,$$

as is easily seen by applying the product in *Diff*, $\circ$: *Diff* $\otimes$ *Diff* $\to$ *Diff*, (we obtain again (A.5)).

B Proof that the coproduct $\Delta_\star$ is coassociative

We have to prove that

$$(\Delta_\star \otimes id)\Delta_\star(u) = (id \otimes \Delta_\star)\Delta_\star(u) \ .$$

The left hand side explicitly reads

$$\begin{aligned}(\Delta_\star \otimes id)\Delta_\star(u) &= (\Delta_\star \otimes id)(u \otimes 1 + \overline{R}^\alpha \otimes \overline{R}_\alpha(u)) \\ &= u \otimes 1 \otimes 1 + \overline{R}^\beta \otimes \overline{R}_\beta(u) \otimes 1 + \Delta_\star(\overline{R}^\alpha) \otimes \overline{R}_\alpha(u) \ .\end{aligned}$$

The right hand side is

$$\begin{aligned}(id \otimes \Delta_\star)\Delta_\star(u) &= u \otimes \Delta_\star(1) + \overline{R}^\alpha \otimes \Delta_\star(\overline{R}_\alpha(u)) \\ &= u \otimes 1 \otimes 1 + \overline{R}^\alpha \otimes \overline{R}_\alpha(u) \otimes 1 + \overline{R}^\alpha \otimes \overline{R}^\gamma \otimes \overline{R}_\gamma \overline{R}_\alpha(u) \ .\end{aligned}$$

These two expressions coincide because

$$\begin{aligned}\Delta_\star(\overline{R}^\alpha) \otimes \overline{R}_\alpha &= e^{-i\theta^{\mu\nu}\Delta_\star(\partial_\mu)\otimes\partial_\nu} = e^{-i\theta^{\mu\nu}(\partial_\mu\otimes 1\otimes\partial_\nu + 1\otimes\partial_\mu\otimes\partial_\nu)} \\ &= \overline{R}^\alpha \otimes \overline{R}^\gamma \otimes \overline{R}_\gamma \overline{R}_\alpha \ .\end{aligned} \tag{B.1}$$

C Proof that the bracket $[u\ v]_\star$ is the adjoint action

We have to prove that

$$[u\ v]_\star = ad^\star_u v \ .$$

We know that the backet $[u\ v]_\star$ equals the deformed commutator

$$[u\ v]_\star = u \star v - \overline{R}^\alpha(v) \star \overline{R}_\alpha(u) \ .$$

On the other hand, the adjoint action reads

$$ad^\star_u v = u_{1_\star} \star v \star S_\star(u_{2_\star}) = u \star v + \overline{R}^\alpha \star v \star S_\star(\overline{R}_\alpha(u)) = u \star v - \overline{R}^\alpha \star v \star \overline{R}^\beta(\overline{R}_\alpha(u))\overline{R}_\beta \ .$$

Now the property

$$\partial_\mu \star v = \partial_\mu\, v = \partial_\mu(v) + v\,\partial_\mu \ , \tag{C.1}$$

that using the coproduct $\Delta_\star(\partial_\mu) \equiv \partial_{\mu_{1_\star}} \otimes \partial_{\mu_{2_\star}} = \partial_\mu \otimes 1 + 1 \otimes \partial_\mu$ can be written as

$$\partial_\mu \star v = \partial_\mu\, v = \partial_{\mu_{1_\star}}(v)\,\partial_{\mu_{2_\star}} \ ,$$

implies

$$\overline{R}^\alpha \star v = \overline{R}^\alpha v = \overline{R}^\alpha_{1_\star}(v)\,\overline{R}^\alpha_{2_\star} \ .$$

The coproduct formula (B.1) then implies

$$\begin{aligned}
\overline{R}^{\alpha} \star v \star \overline{R}^{\beta}(\overline{R}_{\alpha}(u))\overline{R}_{\beta} &= \overline{R}^{\alpha}(v)\overline{R}^{\gamma} \star \overline{R}^{\beta}((\overline{R}_{\gamma}\overline{R}_{\alpha})(u))\overline{R}_{\beta} \\
&= \overline{R}^{\alpha}(v) \star \overline{R}^{\beta}((\overline{R}_{\gamma}\overline{R}_{\alpha})(u))\overline{R}_{\beta}\overline{R}^{\gamma} \\
&= \overline{R}^{\alpha}(v) \star (\overline{R}^{\beta}\overline{R}_{\gamma}\overline{R}_{\alpha})(u)\overline{R}_{\beta}\overline{R}^{\gamma} \\
&= \overline{R}^{\alpha}(v) \star \overline{R}_{\alpha}(u)
\end{aligned}$$

where in the second equality we iterated property (C.1) (with $\overline{R}^{\beta}((\overline{R}_{\gamma}\overline{R}_{\alpha})(u))\overline{R}_{\beta}$ insted of v) and used the antysymmetry of $\theta^{\mu\nu}$ in order to cancel the first addend in (C.1). In the last equality we used that $\overline{R}^{\beta}\overline{R}_{\gamma} \otimes \overline{R}_{\beta}\overline{R}^{\gamma} = \mathcal{R}^{-1}\mathcal{R} = 1 \otimes 1$ because of the antysymmetry of $\theta^{\mu\nu}$.

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