ESI **The Erwin Schrödinger International Institute for Mathematical Physics** Boltzmanngasse 9 A-1090 Wien, Austria

Unique Expansions of Real Numbers

Martijn de Vries
Vilmos Komornik

Vienna, Preprint ESI 1810 (2006) June 29, 2006

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at

UNIQUE EXPANSIONS OF REAL NUMBERS

MARTIJN DE VRIES AND VILMOS KOMORNIK

ABSTRACT. It was discovered some years ago that there exist non-integer real numbers $q > 1$ for which only one sequence (c_i) of integers $0 \le c_i < q$ satisfies the equality $\sum_{i=1}^{\infty} c_i q^{-i} = 1$. The set $\mathcal{U}$ of such "univoque numbers" has a rich topological structure, and its study revealed a number of unexpected connections with measure theory, fractals, ergodic theory and Diophantine approximation.

For each fixed $q > 1$ consider the set $\mathcal{U}_q$ of real numbers x having a unique expansion of the form $\sum_{i=1}^{\infty} c_i q^{-i} = x$ with integers $0 \le c_i < q$. We carry out a detailed topological study of these sets. In particular, we characterize their closures, and we determine those bases q for which $\mathcal{U}_q$ is closed or even a Cantor set.

1. Introduction

Following a seminal paper of Rényi [R] many works were devoted to probabilistic, measure-theoretical and number theoretical aspects of developments in non-integer bases; see, e.g., Frougny and Solomyak [FS], Pethő and Tichy [PT], Schmidt [Sc]. A new research field was opened when Erdős, Horváth and Joó [EHJ] discovered many non-integer real numbers $q > 1$ for which only one sequence (c_i) of integers $0 \le c_i < q$ satisfies the equality

$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} = 1.$$

(They considered only the case $1 < q < 2$.) Subsequently, the set $\mathcal{U}$ of such *univoque* numbers was characterized in [EJK1], [KL3], its smallest element was determined in [KL1], and its topological structure was described in [KL3]. On the other hand, the investigation of numbers q for which there exist continuum many such sequences, including sequences containing all possible finite variations of the integers $0 \le c < q$ revealed close connections to Diophantine approximations; see, e.g., [EJK1], [EJK3], [EK], [KLP], Borwein and Hare [BH1], [BH2], Komatsu [K], and Sidorov [Si2].

For any fixed real number $q > 1$, we may also introduce the set $\mathcal{U}_q$ of real numbers x having exactly one expansion of the form

$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} = x$$

where the integer coefficients c_i are subject to the conditions $0 \le c_i < q$. If q is an integer, these sets are well known. However, their structure is more complex if q is a non-integer, see, e.g., Daróczy and Kátai [DK1], [DK2], Glendinning and Sidorov [GS], and Kallós [K1], [K2]. The purpose of this paper is to give a complete topological description of the sets $\mathcal{U}_q$: they have a different nature for different

Date: June 29, 2006.

1991 *Mathematics Subject Classification.* Primary:11A63, Secondary:11B83.

Key words and phrases. Greedy expansions, beta-expansions, univoque sequences, univoque sets, Cantor sets.

classes of the numbers q. Our investigations also provide new results concerning the univoque set $\mathcal{U}$.

In order to state our results we need to introduce some notation and terminology. In this paper a *sequence* always means a sequence of nonnegative integers. A sequence is called *infinite* if it contains infinitely many nonzero elements; otherwise it is called *finite*. Given a real number $q > 1$, an expansion in base q of a number x is a sequence (c_i) such that

$$0 \le c_i < q \text{ for all } i \ge 1 \quad \text{and} \quad x = \sum_{i=1}^{\infty} \frac{c_i}{q^i}.$$

This definition only makes sense if x belongs to the interval

$$J := \left[0, \frac{\lceil q \rceil - 1}{q - 1}\right]$$

where $\lceil q \rceil$ denotes the upper integer part of q. Note that $[0,1] \subset J$ for all $q > 1$.

A *sequence* (c_i) satisfying $0 \le c_i < q$ for each $i \ge 1$ is called *univoque* in base q if

$$x = \sum_{i=1}^{\infty} \frac{c_i}{q^i}$$

is an element of $\mathcal{U}_q$. The *greedy* expansion $(b_i(x)) = (b_i)$ of a number $x \in J$ in base q is the largest expansion of x in lexicographical order. It is well known that the greedy expansion of any $x \in J$ exists; [P], [EJK1], [EJK2]. A *sequence* (b_i) is called *greedy* in base q if (b_i) is the greedy expansion of

$$x = \sum_{i=1}^{\infty} \frac{b_i}{q^i}.$$

The *quasi-greedy* expansion $(a_i(x)) = (a_i)$ of a number $x \in J \setminus \{0\}$ in base q is the largest *infinite* expansion of x in lexicographical order. Observe that we have to exclude the number 0 since there do not exist infinite expansions of $x = 0$ at all. On the other hand, the largest infinite expansion of any $x \in J \setminus \{0\}$ exists, as we shall prove in the next section. In order to simplify some statements below, the quasi-greedy expansion of the number $0 \in J$ is defined to be $0^\infty = 00\ldots$. Note that this is the only expansion of $x = 0$. A *sequence* (a_i) is called *quasi-greedy* in base q if (a_i) is the quasi-greedy expansion of

$$x = \sum_{i=1}^{\infty} \frac{a_i}{q^i}.$$

We denote the quasi-greedy expansion of the number 1 in base q by (α_i). Since $\alpha_1 = \lceil q \rceil - 1$ the digits of an expansion (c_i) satisfy

$$c_i \in \{0, \ldots, \alpha_1\} \quad \text{for all} \quad i \ge 1.$$

Hence, we consider expansions with coefficients or digits in the alphabet $A := \{0, \ldots, \alpha_1\}$ of numbers $x \in [0, \alpha_1/(q-1)]$.

Of course, whether a sequence is univoque, greedy or quasi-greedy depends on the base q. However, if q is understood, we simply speak of univoque sequences and (quasi)-greedy sequences. Furthermore, we shall write $\overline{a} := \alpha_1 - a \, (a \in A)$, unless stated otherwise. Finally, we set $a^+ := a + 1$ and $a^- := a - 1$, where $a \in A$.

The following important theorem, which is essentially due to Parry (see [P]), plays a crucial role in the proofs of our main results:

Theorem 1.1. *Fix $q > 1$.*

(i) *A sequence* $(b_i) = b_1b_2 \ldots \in \{0, \ldots, \alpha_1\}^{\mathbb{N}}$ *is a greedy sequence in base* q *if and only if*

$$b_{n+1}b_{n+2}\ldots < \alpha_1\alpha_2\ldots \quad \textit{whenever} \quad b_n < \alpha_1.$$

(ii) *A sequence* $(c_i) = c_1c_2 \ldots \in \{0, \ldots, \alpha_1\}^{\mathbb{N}}$ *is a univoque sequence in base* q *if and only if*

$$c_{n+1}c_{n+2}\ldots < \alpha_1\alpha_2\ldots \quad \textit{whenever} \quad c_n < \alpha_1$$

and

$$\overline{c_{n+1}c_{n+2}\ldots} < \alpha_1\alpha_2\ldots \quad \textit{whenever} \quad c_n > 0.$$

Note that if $q \in \mathcal{U}$, then (α_i) is the unique expansion of 1 in base q. Hence, replacing the sequence (c_i) in Theorem 1.1 (ii) by the sequence (α_i), one obtains a lexicographical characterization of $\mathcal{U}$.

Recently, the authors of [KL3] studied the topological structure of the set $\mathcal{U}$. In particular, they showed that $\mathcal{U}$ is *not* closed and they obtained the following characterization of its closure $\overline{\mathcal{U}}$:

Theorem 1.2. $q \in \overline{\mathcal{U}}$ *if and only if the quasi-greedy expansion of the number* 1 *in base* q *satisfies*

$$\overline{\alpha_{k+1}\alpha_{k+2}\ldots} < \alpha_1\alpha_2\ldots \quad \textit{for all} \quad k \ge 1.$$

Remark. In the definition of $\mathcal{U}$ given in [KL3] the integers were excluded; however, $\overline{\mathcal{U}}$ is the same in both cases. Our definition simplifies some statements. For example it will follow from the theorems below that

$$\mathcal{U}_q = \overline{\mathcal{U}_q} \quad \Longleftrightarrow \quad q \in (1,\infty) \setminus \overline{\mathcal{U}}$$

where $\overline{\mathcal{U}_q}$ denotes the closure of $\mathcal{U}_q$.

Now we are ready to state our main results.

Theorem 1.3. *Let* $q \in \overline{\mathcal{U}}$ *and* $x \in J$. *Denote the quasi-greedy expansion of* x *by* (a_i). *Then,*

$$x \in \overline{\mathcal{U}_q} \Longleftrightarrow \quad \overline{a_{n+1}a_{n+2}\ldots} \le \alpha_1\alpha_2\ldots \quad \textit{whenever} \quad a_n > 0.$$

Theorem 1.4. *Suppose that* $q \in \overline{\mathcal{U}}$. *Then,*

(i) $|\overline{\mathcal{U}_q} \setminus \mathcal{U}_q| = \aleph_0$ *and* $\overline{\mathcal{U}_q} \setminus \mathcal{U}_q$ *is dense in* $\overline{\mathcal{U}_q}$.
(ii) *If* $q \in \mathcal{U}$, *then each element* $x \in \overline{\mathcal{U}_q} \setminus \mathcal{U}_q$ *has* 2 *expansions.*
(iii) *If* $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$, *then each element* $x \in \overline{\mathcal{U}_q} \setminus \mathcal{U}_q$ *has* $\aleph_0$ *expansions.*

Remarks.

- Our proof of part (i) yields the following more precise results where for $q \in \overline{\mathcal{U}}$ we set

$$A_q = \left\{x \in \overline{\mathcal{U}_q} \setminus \mathcal{U}_q : x \text{ has a finite greedy expansion}\right\}$$

  and

$$B_q = \left\{x \in \overline{\mathcal{U}_q} \setminus \mathcal{U}_q : x \text{ has an infinite greedy expansion}\right\} :$$

  - If $q \in \overline{\mathcal{U}} \setminus \mathbb{N}$, then both A_q and B_q are countably infinite and dense in $\overline{\mathcal{U}_q}$. Moreover, the greedy expansion of a number $x \in B_q$ ends with $\overline{\alpha_1\alpha_2\ldots}$.
  - If $q = 2, 3, \ldots$, then $B_q = \varnothing$.
- For each $x \in \overline{\mathcal{U}_q} \setminus \mathcal{U}_q$, the proof of parts (ii) and (iii) also provide the list of all expansions of x in terms of its greedy expansion.

Motivated by Theorem 1.3, we introduce for a general real number $q > 1$, the set $\mathcal{V}_q$, defined by

$$\mathcal{V}_q = \left\{ x \in J : \overline{a_{n+1}(x)a_{n+2}(x)\ldots} \le \alpha_1\alpha_2\ldots \quad \text{whenever} \quad a_n(x) > 0 \right\}.$$

It follows from the above theorems that $\mathcal{U}_q \subsetneq \overline{\mathcal{U}_q} = \mathcal{V}_q$ if $q \in \overline{\mathcal{U}}$. It is natural to study the relationship between the sets $\mathcal{U}_q$, $\overline{\mathcal{U}_q}$ and $\mathcal{V}_q$ in case $q \notin \overline{\mathcal{U}}$. In order to do so, we introduce the set $\mathcal{V}$, consisting of those numbers $q > 1$, for which the quasi-greedy expansion of the number 1 in base q satisfies

$$\overline{\alpha_{k+1}\alpha_{k+2}\ldots} \le \alpha_1\alpha_2\ldots \quad \text{for each} \quad k \ge 1.$$

It follows from Theorem 1.1 and Theorem 1.2 that $\mathcal{U} \subset \overline{\mathcal{U}} \subset \mathcal{V}$ and $\mathcal{U}_q \subset \mathcal{V}_q$ for all $q > 1$. The following results imply that $\mathcal{U}_q$ is closed if $q \notin \overline{\mathcal{U}}$ and that the set $\mathcal{V}_q$ is closed for each number $q > 1$.

Theorem 1.5. *Suppose that $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$. Then,*

(i) *The sets $\mathcal{U}_q$ and $\mathcal{V}_q$ are closed.*
(ii) *$|\mathcal{V}_q \setminus \mathcal{U}_q| = \aleph_0$ and $\mathcal{V}_q \setminus \mathcal{U}_q$ is a discrete set, dense in $\mathcal{V}_q$.*
(iii) *Each element $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has $\aleph_0$ expansions, and a finite greedy expansion.*

Remark. Our proof also provides the list of all expansions of all elements $x \in \mathcal{V}_q \setminus \mathcal{U}_q$.

Theorem 1.6. *Suppose that $q \in (1, \infty) \setminus \mathcal{V}$. Then,*

$$\mathcal{U}_q = \overline{\mathcal{U}_q} = \mathcal{V}_q.$$

Remarks.

- In view of the above results, Theorem 1.1 already gives us a lexicographical characterization of $\overline{\mathcal{U}_q}$ in case $q \notin \overline{\mathcal{U}}$ because in this case $\mathcal{U}_q = \overline{\mathcal{U}_q}$.
- It is well-known that the set $\mathcal{U}$ has Lebesgue measure zero; [EHJ], [KK]. In [KL3] it was shown that the set $\overline{\mathcal{U}} \setminus \mathcal{U}$ is countably infinite. It follows from the above results that $\mathcal{U}_q$ is closed for almost every $q > 1$.
- Let $q > 1$ be a non-integer. In [DDV] it has been proved that almost every $x \in J$ has a continuum of expansions in base q (see also [Si1]). It follows from the above results that the set $\overline{\mathcal{U}_q}$ has Lebesgue measure zero. Hence, the set $\mathcal{U}_q$ is nowhere dense for any non-integer $q > 1$.
- Let $q > 1$ be an integer. In this case, the quasi-greedy expansion of 1 in base q is given by $(\alpha_i) = \alpha_1^\infty = (q-1)^\infty$. It follows from Theorem 1.1 that $J \setminus \mathcal{U}_q$ is countable and each element in $J \setminus \mathcal{U}_q$ has only two expansions, one of them being finite while the other one ends with an infinite string of $(q-1)$'s.
- In [KL1] it was shown that the smallest element of $\mathcal{U}$ is given by $q' \approx 1.787$, and the unique expansion of 1 in base q' is given by the truncated Thue-Morse sequence $(\tau_i) = \tau_1\tau_2\ldots$, which can be defined recursively by setting $\tau_{2^N} = 1$ for $N = 0, 1, 2, \ldots$ and

  $$\tau_{2^N+i} = 1 - \tau_i \quad \text{for } 1 \le i < 2^N,\ N = 1, 2, \ldots.$$

  Subsequently, Glendinning and Sidorov [GS] proved that $\mathcal{U}_q$ is countable if $1 < q < q'$ and has the cardinality of the continuum if $q \in [q', 2)$. Moreover, they showed that $\mathcal{U}_q$ is a set of positive Hausdorff dimension if $q' < q < 2$, and they described a method to compute its Hausdorff dimension (see also [DK2], [K1], [K2]).

In the following theorem we characterize those $q > 1$ for which $\mathcal{U}_q$ or $\overline{\mathcal{U}_q}$ is a Cantor set, i.e., a non-empty closed set having no interior or isolated points. We recall from [KL3] that

- $\mathcal{V}$ is closed and $\mathcal{U}$ is closed from above,
- $|\overline{\mathcal{U}} \setminus \mathcal{U}| = \aleph_0$ and $\overline{\mathcal{U}} \setminus \mathcal{U}$ is dense in $\overline{\mathcal{U}}$,
- $|\mathcal{V} \setminus \overline{\mathcal{U}}| = \aleph_0$ and $\mathcal{V} \setminus \overline{\mathcal{U}}$ is a discrete set, dense in $\mathcal{V}$.

Since the set $(1,\infty) \setminus \mathcal{V}$ is open, we can write $(1,\infty) \setminus \mathcal{V}$ as the union of countably many disjoint open intervals (q_1, q_2): its connected components. Let us denote by L and R the set of left (respectively right) endpoints of the intervals (q_1, q_2).

Theorem 1.7.

(i) $L = \mathbb{N} \cup (\mathcal{V} \setminus \mathcal{U})$ *and* $R = \mathcal{V} \setminus \overline{\mathcal{U}}$. *Hence,* $R \subset L$ *and*

$$(1,\infty) \setminus \overline{\mathcal{U}} = \cup (q_1, q_2]$$

where the union runs over the connected components (q_1, q_2) *of* $(1,\infty) \setminus \mathcal{V}$.

(ii) *If* $q \in \{2, 3, \ldots\}$, *then neither* $\mathcal{U}_q$ *nor* $\overline{\mathcal{U}_q}$ *is a Cantor set.*

(iii) *If* $q \in \overline{\mathcal{U}} \setminus \mathbb{N}$, *then* $\mathcal{U}_q$ *is not a Cantor set, but its closure* $\overline{\mathcal{U}_q}$ *is a Cantor set.*

(iv) *If* $q \in (q_1, q_2]$, *where* (q_1, q_2) *is a connected component of* $(1,\infty) \setminus \mathcal{V}$, *then* $\mathcal{U}_q$ *is a Cantor set if and only if* $q_1 \in \{3, 4, \ldots\} \cup (\overline{\mathcal{U}} \setminus \mathcal{U})$.

Remark. We also describe the set of endpoints of the connected components (p_1, p_2) of $(1,\infty) \setminus \overline{\mathcal{U}}$: denoting by L' and R' the set of left (respectively right) endpoints of the intervals (p_1, p_2), we have

$$L' = \mathbb{N} \cup (\overline{\mathcal{U}} \setminus \mathcal{U}) \quad \text{and} \quad R' \subset \mathcal{U};$$

see the remarks at the end of Section 7.

The above theorem enables us to give a new characterization of the *stable bases*, introduced and investigated by Daróczy and Kátai ([DK1], [DK2]). Let us denote by $\mathcal{U}'_q$ and $\mathcal{V}'_q$ the sets of quasi-greedy expansions of the numbers $x \in \mathcal{U}_q$ and $x \in \mathcal{V}_q$. We recall that a number $q > 1$ is *stable from above* (respectively *stable from below*) if there exists a number $s > q$ (respectively $1 < s < q$) such that

$$\mathcal{U}'_q = \mathcal{U}'_s.$$

Furthermore, we say that an interval $I \subset (1,\infty)$ is a *stability interval* if $\mathcal{U}'_q = \mathcal{U}'_s$ for all $q, s \in I$.

Theorem 1.8. *The maximal stability intervals are given by the singletons* $\{q\}$ *where* $q \in \overline{\mathcal{U}}$ *and the intervals* $(q_1, q_2]$ *where* (q_1, q_2) *is a connected component of* $(1,\infty) \setminus \mathcal{V}$. *Moreover, if* $q_1 \in \mathcal{V} \setminus \mathcal{U}$, *then*

$$\mathcal{U}'_q = \mathcal{V}'_{q_1} \quad \textit{for all} \quad q \in (q_1, q_2].$$

Remark. The proof of Theorem 1.8 yields a new characterization of the sets $\overline{\mathcal{U}}$ and $\mathcal{V}$ (see Lemma 7.9).

We recall from [LM] that a set $A \subset \{0, \ldots, \alpha_1\}^{\mathbb{N}}$ is called a *subshift of finite type* if there exists a finite set $\mathcal{F} \subset \cup_{k=1}^{\infty} \{0, \ldots, \alpha_1\}^k$ such that a sequence $(c_i) \in \{0, \ldots, \alpha_1\}^{\mathbb{N}}$ belongs to A if and only if each "word" in $\mathcal{F}$ does not appear in (c_i). The following theorem characterizes those $q > 1$ for which $\mathcal{U}'_q$ is a subshift of finite type.

Theorem 1.9. *Let* $q > 1$ *be a real number. Then,*

$$\mathcal{U}'_q \quad \textit{is a subshift of finite type} \iff q \in (1,\infty) \setminus \overline{\mathcal{U}}.$$

Finally, we determine the cardinality of $\mathcal{U}_q$ for all $q > 1$. We recall that for $q \in (1, 2)$ this has already been done by Glendinning and Sidorov ([GS]), using a different method. Denote by q'' the smallest element of $\mathcal{U} \cap (2, 3)$. The unique expansion of 1 in base q'' is given in [KL2].

Theorem 1.10. *Let $q > 1$ be a real number.*

(i) *If $q \in (1, (1+\sqrt{5})/2]$, then $\mathcal{U}_q$ consists merely of the endpoints of J.*
(ii) *If $q \in ((1+\sqrt{5})/2, q') \cup (2, q'')$, then $|\mathcal{U}_q| = \aleph_0$.*
(iii) *If $q \in [q', 2] \cup [q'', \infty)$, then $|\mathcal{U}_q| = 2^{\aleph_0}$.*

Remark. We also determine the unique expansion of 1 in base $q^{(n)}$ for $n \in \{3, 4, \ldots\}$ where $q^{(n)}$ denotes the smallest element of $\mathcal{U} \cap (n, n+1)$ (see the remarks at the end of Section 7).

For the reader's convenience we recall some properties of quasi-greedy expansions in the next section. These properties are also stated in [BK] and are closely related to some important results, first established in the seminal works by Rényi [R] and Parry [P]. Sections 3 and 4 are then devoted to the proof of Theorem 1.3. Theorem 1.4 is proved in Section 5, Theorem 1.5 and Theorem 1.6 are proved in Section 6, and our final Theorems 1.7, 1.8, 1.9 and 1.10 are established in Section 7.

2. Quasi-greedy expansions

Let $q > 1$ be a real number and let $m = \lceil q \rceil - 1$. In the previous section we defined the quasi-greedy expansion as the largest infinite expansion of $x \in (0, m/(q-1)]$. In order to prove that this notion is well defined, we introduce the *quasi-greedy algorithm*: if $a_i = a_i(x)$ is already defined for $i < n$, then a_n is the largest element of the set $\{0, \ldots, m\}$ that satisfies

$$\sum_{i=1}^{n} \frac{a_i}{q^i} < x.$$

Of course, this definition only makes sense if $x > 0$. In the following proposition we show that this algorithm generates an expansion of x, for all $x \in (0, m/(q-1)]$. It follows that the quasi-greedy expansion is generated by the quasi-greedy algorithm.

Proposition 2.1. *Let $x \in (0, m/(q-1)]$. Then,*

$$x = \sum_{i=1}^{\infty} \frac{a_i}{q^i}.$$

Proof. If $x = m/(q-1)$, then the quasi-greedy algorithm provides $a_i = m$ for all $i \geq 1$ and the desired equality follows.

Suppose that $x \in (0, m/(q-1))$. Then, by definition of the quasi-greedy algorithm, there exists an index n such that $a_n < m$.

First assume that $a_n < m$ for infinitely many n. For any such n, we have by definition

$$0 < x - \sum_{i=1}^{n} \frac{a_i}{q^i} \leq \frac{1}{q^n}.$$

Letting $n \to \infty$, we obtain

$$x = \sum_{i=1}^{\infty} \frac{a_i}{q^i}.$$

Next assume there exists a largest n such that $a_n < m$. Then,

$$\sum_{i=1}^{n} \frac{a_i}{q^i} + \sum_{i=n+1}^{N} \frac{m}{q^i} < x \leq \sum_{i=1}^{n} \frac{a_i}{q^i} + \frac{1}{q^n},$$

for each $N > n$. Hence,

$$\sum_{i=n+1}^{\infty} \frac{m}{q^i} \leq x - \sum_{i=1}^{n} \frac{a_i}{q^i} \leq \frac{1}{q^n}.$$

Note that

$$\frac{1}{q^n} \leq \sum_{i=n+1}^{\infty} \frac{m}{q^i},$$

for any $q > 1$ and

$$\frac{1}{q^n} = \sum_{i=n+1}^{\infty} \frac{m}{q^i},$$

if and only if $q = m + 1$. Hence, the existence of a largest n such that $a_n < m$ is only possible if q is an integer, in which case $a_{n+i} = m$ for all $i \geq 1$ and

$$x = \sum_{i=1}^{n} \frac{a_i}{q^i} + \sum_{i=n+1}^{\infty} \frac{m}{q^i} = \sum_{i=1}^{\infty} \frac{a_i}{q^i}. \qquad \square$$

Now we consider the quasi-greedy expansion (α_i) of $x = 1$. Note that $\alpha_1 = m = \lceil q \rceil - 1$. If $\alpha_n < \alpha_1$, then

$$\sum_{i=1}^{n} \frac{\alpha_i}{q^i} + \frac{1}{q^n} \geq 1,$$

by definition of the quasi-greedy algorithm. The following lemma states that this inequality holds for each $n \geq 1$.

Lemma 2.2. *For each $n \geq 1$, the inequality*

$$\sum_{i=1}^{n} \frac{\alpha_i}{q^i} + \frac{1}{q^n} \geq 1 \tag{2.1}$$

holds.

Proof. The proof is by induction on n. For $n = 1$, the inequality holds, since $\alpha_1 + 1 \geq q$. Assume the inequality is valid for some $n \in \mathbb{N}$. If $\alpha_{n+1} < \alpha_1$, then (2.1) with n replaced by $n+1$ follows from the definition of the quasi-greedy algorithm. If $\alpha_{n+1} = \alpha_1$, then the same conclusion follows from the induction hypothesis and the inequality $\alpha_1 + 1 \geq q$. $\square$

Proposition 2.3. *The map $q \mapsto (\alpha_i)$ is a strictly increasing bijection from the open interval $(1, \infty)$ onto the set of all infinite sequences satisfying*

$$\alpha_{k+1}\alpha_{k+2}\ldots \leq \alpha_1\alpha_2\ldots \quad \textit{for all} \quad k \geq 1. \tag{2.2}$$

Proof. By definition of the quasi-greedy algorithm and Proposition 2.1 the map $q \mapsto (\alpha_i)$ is strictly increasing. According to the preceding lemma,

$$\sum_{i=1}^{n} \frac{\alpha_i}{q^i} + \frac{1}{q^n} \geq \sum_{i=1}^{\infty} \frac{\alpha_i}{q^i}$$

for every $n \geq 1$, whence

$$\frac{\alpha_{n+1}}{q} + \frac{\alpha_{n+2}}{q^2} + \cdots \leq 1. \tag{2.3}$$

Since (α_{n+i}) is infinite and (α_i) is the largest infinite sequence satisfying (2.3), inequality (2.2) follows.

Conversely, let (α_i) be an infinite sequence satisfying (2.2). Solving the equation

$$\sum_{i=1}^{\infty} \frac{\alpha_i}{q^i} = 1,$$

we obtain a number $q > 1$. In order to prove that (α_i) is the quasi-greedy expansion of 1 in base q, it suffices to prove that for each n satisfying $\alpha_n < \alpha_1$, the inequality

$$\sum_{i=n+1}^{\infty} \frac{\alpha_i}{q^i} \le \frac{1}{q^n}$$

holds.

Starting with $k_0 := n$ and using (2.2), we try to define a sequence

$$k_0 < k_1 < \cdots$$

satisfying for $j = 1, 2, \ldots$ the conditions

$$\alpha_{k_{j-1}+i} = \alpha_i \quad \text{for } i = 1, \ldots, k_j - k_{j-1} - 1 \quad \text{and } \alpha_{k_j} < \alpha_{k_j - k_{j-1}}.$$

If we obtain in this way an infinite number of indices, then we have

$$\begin{aligned} \sum_{i=n+1}^{\infty} \frac{\alpha_i}{q^i} &\le \sum_{j=1}^{\infty} \left(\sum_{i=1}^{k_j - k_{j-1}} \frac{\alpha_i}{q^{k_{j-1}+i}} - \frac{1}{q^{k_j}} \right) \\ &< \sum_{j=1}^{\infty} \left(\frac{1}{q^{k_{j-1}}} - \frac{1}{q^{k_j}} \right) = \frac{1}{q^n}. \end{aligned}$$

If we only obtain a finite number of indices, then there exists a first nonnegative integer N such that $(\alpha_{k_N+i}) = (\alpha_i)$ and we have

$$\begin{aligned} \sum_{i=n+1}^{\infty} \frac{\alpha_i}{q^i} &\le \sum_{j=1}^{N} \left(\sum_{i=1}^{k_j - k_{j-1}} \frac{\alpha_i}{q^{k_{j-1}+i}} - \frac{1}{q^{k_j}} \right) + \sum_{i=1}^{\infty} \frac{\alpha_i}{q^{k_N+i}} \\ &\le \sum_{j=1}^{N} \left(\frac{1}{q^{k_{j-1}}} - \frac{1}{q^{k_j}} \right) + \sum_{i=1}^{\infty} \frac{\alpha_i}{q^{k_N+i}} = \frac{1}{q^n}. \end{aligned}$$ □

The proofs of the following propositions are almost identical to the proof of Proposition 2.3 and are therefore omitted.

Proposition 2.4. *Fix $q > 1$ and denote by (α_i) the quasi-greedy expansion of 1 in base q. Then the map $x \mapsto (a_i)$ is a strictly increasing bijection from $(0, \alpha_1/(q-1)]$ onto the set of all infinite sequences, satisfying*

$$0 \le a_n \le \alpha_1 \quad \textit{for all} \quad n \ge 1$$

and

$$a_{n+1}a_{n+2}\ldots \le \alpha_1\alpha_2\ldots \quad \textit{whenever} \quad a_n < \alpha_1. \tag{2.4}$$

Proposition 2.5. *Fix $q > 1$ and denote by (α_i) the quasi-greedy expansion of 1 in base q. Then the map $x \mapsto (b_i)$ is a strictly increasing bijection from $[0, \alpha_1/(q-1)]$ onto the set of all sequences, satisfying*

$$0 \le b_n \le \alpha_1 \quad \textit{for all} \quad n \ge 1$$

and

$$b_{n+1}b_{n+2}\ldots < \alpha_1\alpha_2\ldots \quad \textit{whenever} \quad b_n < \alpha_1. \tag{2.5}$$

Remarks.

- A sequence (c_i) is univoque if and only if (c_i) is greedy and $(\alpha_1 - c_i)$ is greedy. Hence, Theorem 1.1 is a consequence of Proposition 2.5.

- The greedy expansion of $x \in [0, \alpha_1/(q-1)]$ is generated by the *greedy algorithm*: if $b_i = b_i(x)$ is already defined for $i < n$, then b_n is the largest element of A such that
$$\sum_{i=1}^{n} \frac{b_i}{q^i} \le x.$$
The proof of this assertion goes along the same lines as the proof of Proposition 2.1.
- Note that the greedy expansion of a number $x \in (0, \alpha_1/(q-1)]$ coincides with the quasi-greedy expansion if and only if the greedy expansion of x is infinite. If the greedy expansion (b_i) of $x \in J \setminus \{0\}$ is finite and b_n is its last nonzero element, then the quasi-greedy expansion of x is given by
$$(a_i) = b_1 \dots b_{n-1} b_n^- \alpha_1 \alpha_2 \dots,$$
as follows from the inequalities (2.2), (2.4) and (2.5).

3. Proof of the necessity part of Theorem 1.3

Let $q > 1$ be a real number.

Lemma 3.1. *Let $(b_i) = b_1 b_2 \dots$ be a greedy sequence. Then the truncated sequence $b_1 \dots b_n 0^\infty$ is also a greedy sequence, where $n \ge 1$ is an arbitrary positive integer.*

Proof. The statement follows at once from Proposition 2.5. □

Lemma 3.2. *Let $(b_i) \neq \alpha_1^\infty$ be a greedy sequence and let N be a positive integer. Then there exists a greedy sequence $(c_i) > (b_i)$ such that*
$$c_1 \dots c_N = b_1 \dots b_N.$$

Proof. Since $(b_i) \neq \alpha_1^\infty$, it follows from (2.5) that $b_n < \alpha_1$ for infinitely many n. Hence, we may assume, by enlarging N if necessary, that $b_N < \alpha_1$. Let
$$I = \{1 \le i \le N : b_i < \alpha_1\} =: \{i_1, \dots, i_n\}.$$
Since $N \in I, I \neq \varnothing$. Note that for $i_r \in I$,
$$\sum_{j=1}^{\infty} \frac{b_{i_r+j}}{q^j} = \sum_{j=1}^{N-i_r} \frac{b_{i_r+j}}{q^j} + \frac{1}{q^{N-i_r}} \sum_{i=1}^{\infty} \frac{b_{N+i}}{q^i} < 1$$
because (b_i) is greedy and $b_{i_r} < \alpha_1$. Choose y_{i_r} such that
$$\sum_{i=1}^{\infty} \frac{b_{N+i}}{q^i} < y_{i_r} \le \alpha_1/(q-1) \tag{3.1}$$
and
$$\sum_{j=1}^{N-i_r} \frac{b_{i_r+j}}{q^j} + \frac{1}{q^{N-i_r}} y_{i_r} < 1. \tag{3.2}$$
Let $y = \min\{y_{i_1}, \dots, y_{i_n}\}$ and denote the greedy expansion of y by $d_1 d_2 \dots$. Finally, let $(c_i) = b_1 \dots b_N d_1 d_2 \dots$. From (3.1) we infer that $(c_i) > (b_i)$. It remains to show that (c_i) is a greedy sequence, i.e., we need to show that
$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} < 1 \quad \text{whenever} \quad c_n < \alpha_1. \tag{3.3}$$
If $c_n < \alpha_1$ and $n \le N$, then (3.3) follows from (3.2). If $c_n < \alpha_1$ and $n > N$, then (3.3) follows from the fact that (d_i) is a greedy sequence. □

Lemma 3.3. *Let (b_i) be the greedy expansion of some $x \in [0, \alpha_1/(q-1)]$ and suppose that for some $n \geq 1$, $b_n > 0$ and*

$$\overline{b_{n+1}b_{n+2}\dots} > \alpha_1\alpha_2\dots.$$

Then,

(i) *There exists a number $z > x$ such that $[x,z] \cap \mathcal{U}_q = \varnothing$.*
(ii) *If $b_j > 0$ for some $j > n$, then there exists a number $y < x$ such that $[y,x] \cap \mathcal{U}_q = \varnothing$.*

Proof. (i) Choose a positive integer $M > n$ such that

$$\overline{b_{n+1}\dots b_M} > \alpha_1\dots\alpha_{M-n}.$$

Applying Lemma 3.2, choose a greedy sequence $(c_i) > (b_i)$ such that $c_1\dots c_M = b_1\dots b_M$. Then, (c_i) is the greedy expansion of some $z > x$. If (d_i) is the greedy expansion of some element in $[x,z]$, then (d_i) also begins with $b_1\dots b_M$ and hence

$$\overline{d_{n+1}\dots d_M} > \alpha_1\dots\alpha_{M-n}.$$

In particular, we have that $d_n > 0$ and

$$\overline{d_{n+1}d_{n+2}\dots} > \alpha_1\alpha_2\dots.$$

We infer from Theorem 1.1 that $[x,z] \cap \mathcal{U}_q = \varnothing$.

(ii) It follows from Lemma 3.1 that $(c_i) := b_1\dots b_n 0^\infty$ is the greedy expansion of some $y < x$. If (d_i) is the greedy expansion of some element in $[y,x]$, then $(c_i) \leq (d_i) \leq (b_i)$ and $d_1\dots d_n = b_1\dots b_n$. Therefore,

$$\overline{d_{n+1}d_{n+2}\dots} \geq \overline{b_{n+1}b_{n+2}\dots} > \alpha_1\alpha_2\dots$$

and $d_n = b_n > 0$. It follows from Theorem 1.1 that $[y,x] \cap \mathcal{U}_q = \varnothing$. □

Proof of the necessity part of Theorem 1.3. If $x \in \mathcal{U}_q$, then the quasi-greedy expansion (a_i) and the greedy expansion (b_i) of x coincide. Hence, the stronger implication

$$a_m > 0 \Longrightarrow \overline{a_{m+1}a_{m+2}\dots} < \alpha_1\alpha_2\dots \tag{3.4}$$

follows from Theorem 1.1. Suppose now that $x \in \overline{\mathcal{U}_q} \setminus \mathcal{U}_q$. According to Theorem 1.1, there exists a *smallest* positive integer n for which

$$b_n > 0 \quad \text{and} \quad \overline{b_{n+1}b_{n+2}\dots} \geq \alpha_1\alpha_2\dots.$$

First assume that

$$\overline{b_{n+1}b_{n+2}\dots} > \alpha_1\alpha_2\dots.$$

Applying Lemma 3.3 we conclude that $b_i = 0$ for $i > n$. Hence, the quasi-greedy expansion of x is given by

$$(a_i) = b_1\dots b_n^-\alpha_1\alpha_2\dots.$$

We must show that

$$a_m > 0 \Longrightarrow \overline{a_{m+1}a_{m+2}\dots} \leq \alpha_1\alpha_2\dots.$$

Instead, we prove the stronger implication (3.4).
If $m > n$, then (3.4) follows from Theorem 1.2 and our assumption $q \in \overline{\mathcal{U}}$.
If $m = n$, then (3.4) follows from $\overline{\alpha_1} = 0 < \alpha_1$.
Now assume that $m < n$ and $a_m > 0$. Since $a_m = b_m$, we have that $b_m > 0$ and by minimality of n,

$$\overline{b_{m+1}b_{m+2}\dots} < \alpha_1\alpha_2\dots.$$

Equivalently,

$$\overline{b_{m+1}\dots b_n}\alpha_1^\infty < \alpha_1\alpha_2\dots.$$

Hence,

$$\overline{b_{m+1}\dots b_n} < \alpha_1 \dots \alpha_{n-m},$$

from which it follows that

$$\overline{a_{m+1}\dots a_n} \le \alpha_1 \dots \alpha_{n-m}.$$

Moreover, according to Theorem 1.2,

$$\overline{a_{n+1}a_{n+2}\dots} = \overline{\alpha_1\alpha_2\dots} < \alpha_{n-m+1}\alpha_{n-m+2}\dots,$$

proving (3.4).

Next assume that

$$\overline{b_{n+1}b_{n+2}\dots} = \alpha_1\alpha_2\dots. \tag{3.5}$$

If q is an integer, then $(\alpha_i) = \alpha_1^\infty = (q-1)^\infty$ and the implication (3.4) follows from the fact that (a_i) is infinite by definition. If q is a non-integer and (3.5) holds, then (b_i) is infinite and therefore $(a_i) = (b_i)$. Hence, we need to show that the implication

$$b_m > 0 \Longrightarrow \overline{b_{m+1}b_{m+2}\dots} \le \alpha_1\alpha_2\dots \tag{3.6}$$

holds. If $m \ge n$, then (3.6) follows from

$$\overline{b_{m+1}b_{m+2}\dots} = \alpha_{m-n+1}\alpha_{m-n+2}\dots \le \alpha_1\alpha_2\dots.$$

If $m < n$, then (3.6) follows from the minimality of n. □

4. Proof of the sufficiency part of Theorem 1.3

Fix $q \in \overline{\mathcal{U}}$ and denote the quasi-greedy expansion of $x \in [0, \alpha_1/(q-1)]$ by $(a_i) = (a_i(x))$. Suppose that

$$\overline{a_{n+1}a_{n+2}\dots} \le \alpha_1\alpha_2\dots \quad \text{whenever} \quad a_n > 0. \tag{4.1}$$

In this section we prove that such an element x belongs to the set $\overline{\mathcal{U}_q}$.

It follows from Theorem 1.2 and Proposition 2.3 that the quasi-greedy expansion of 1 in base q satisfies

$$\alpha_{k+1}\alpha_{k+2}\dots \le \alpha_1\alpha_2\dots \quad \text{for all} \quad k \ge 1, \tag{4.2}$$

and

$$\overline{\alpha_{k+1}\alpha_{k+2}\dots} < \alpha_1\alpha_2\dots \quad \text{for all} \quad k \ge 1. \tag{4.3}$$

Note that a sequence satisfying (4.2) and (4.3) is automatically infinite. Hence, a sequence satisfying (4.2) and (4.3) is the quasi-greedy expansion of 1 in base q for some $q \in \overline{\mathcal{U}}$. The following two lemmas are obtained in [KL3].

Lemma 4.1. *If (α_i) is a sequence satisfying (4.2) and (4.3), then there exist arbitrary large integers m such that $\alpha_m > 0$ and*

$$\overline{\alpha_{k+1}\dots\alpha_m} < \alpha_1\dots\alpha_{m-k} \quad \textit{for all} \quad 0 \le k < m. \tag{4.4}$$

Lemma 4.2. *Let (γ_i) be a sequence satisfying*

$$\gamma_{k+1}\gamma_{k+2}\dots \le \gamma_1\gamma_2\dots$$

and

$$\overline{\gamma_{k+1}\gamma_{k+2}\dots} \le \gamma_1\gamma_2\dots$$

for all $k \ge 1$, with $\overline{\gamma_j} := \gamma_1 - \gamma_j$. If

$$\overline{\gamma_{n+1}\dots\gamma_{2n}} \ge \gamma_1\dots\gamma_n$$

for some $n \ge 1$, then in fact

$$(\gamma_i) = (\gamma_1\dots\gamma_n\overline{\gamma_1\dots\gamma_n})^\infty.$$

Now we are able to prove the sufficiency part of Theorem 1.3. For a fixed base $q \in \overline{\mathcal{U}}$, we will distinguish between $x \in J$ with a finite greedy expansion and $x \in J$ with an infinite greedy expansion.

Lemma 4.3. *Fix $q \in \overline{\mathcal{U}}$. Suppose that $x \in [0, \alpha_1/(q-1)]$ has a finite greedy expansion (b_i) and suppose that the quasi-greedy expansion (a_i) of x satisfies the condition* (4.1)*. Then, $x \in \overline{\mathcal{U}_q}$.*

Proof. Note that $0 \in \mathcal{U}_q$. Hence, we may assume that $x \in (0, \alpha_1/(q-1)]$. If b_n is the last nonzero element of (b_i), then

$$(a_i) = b_1 \dots b_n^- \alpha_1 \alpha_2 \dots .$$

According to Lemma 4.1, there exists a sequence $1 \le m_1 < m_2 < \cdots$, such that (4.4) is satisfied with $m = m_i$ for all $i \ge 1$. We may assume that $m_i > n$ for all $i \ge 1$. Consider for each $i \ge 1$ the sequence (b_j^i), given by

$$(b_j^i) = b_1 \dots b_n^- (\alpha_1 \dots \alpha_{m_i} \overline{\alpha_1 \dots \alpha_{m_i}})^\infty .$$

Define for $i \ge 1$, the number x_i by

$$x_i = \sum_{j=1}^{\infty} \frac{b_j^i}{q^j}.$$

Note that the sequence $(x_i)_{i \ge 1}$ converges to x as i goes to infinity. It remains to show that $x_i \in \mathcal{U}_q$ for all $i \ge 1$. According to Theorem 1.1 it suffices to verify that

$$b_{m+1}^i b_{m+2}^i \dots < \alpha_1 \alpha_2 \dots \quad \text{whenever} \quad b_m^i < \alpha_1 \tag{4.5}$$

and

$$\overline{b_{m+1}^i b_{m+2}^i \dots} < \alpha_1 \alpha_2 \dots \quad \text{whenever} \quad b_m^i > 0. \tag{4.6}$$

According to (4.3),

$$\overline{\alpha_{m_i+1} \dots \alpha_{2m_i}} \le \alpha_1 \dots \alpha_{m_i}.$$

Note that this inequality cannot be an equality, for otherwise it would follow from Lemma 4.2 that

$$(\alpha_i) = (\alpha_1 \dots \alpha_{m_i} \overline{\alpha_1 \dots \alpha_{m_i}})^\infty .$$

However, this sequence does not satisfy (4.3) for $k = m_i$. Therefore,

$$\overline{\alpha_{m_i+1} \dots \alpha_{2m_i}} < \alpha_1 \dots \alpha_{m_i}.$$

Equivalently,

$$\overline{\alpha_1 \dots \alpha_{m_i}} < \alpha_{m_i+1} \dots \alpha_{2m_i}. \tag{4.7}$$

If $m \ge n$, then (4.5) and (4.6) follow from (4.2), (4.4) and (4.7).
Now assume that $m < n$. If $b_m^i < \alpha_1$, then

$$\begin{aligned} b_{m+1}^i \dots b_n^i &= b_{m+1} \dots b_n^- \\ &< b_{m+1} \dots b_n \\ &\le \alpha_1 \dots \alpha_{n-m}, \end{aligned}$$

where the last inequality follows from the fact that (b_i) is a greedy expansion and $b_m = b_m^i < \alpha_1$. Hence,

$$b_{m+1}^i b_{m+2}^i \dots < \alpha_1 \alpha_2 \dots .$$

Suppose that $b_m^i = a_m > 0$. Since by assumption,

$$\overline{a_{m+1} a_{m+2} \dots} \le \alpha_1 \alpha_2 \dots$$

and since $b_{m+1}^i \dots b_n^i = a_{m+1} \dots a_n$, it suffices to verify that

$$\overline{b_{n+1}^i b_{n+2}^i \dots} < \alpha_{n-m+1} \alpha_{n-m+2} \dots .$$

This is equivalent to

$$\overline{\alpha_{n-m+1}\alpha_{n-m+2}\dots} < (\alpha_1\dots\alpha_{m_i}\overline{\alpha_1\dots\alpha_{m_i}})^\infty. \tag{4.8}$$

Since $n < m_i$ for each $i \geq 1$, we infer from (4.4) that

$$\overline{\alpha_{n-m+1}\dots\alpha_{m_i}} < \alpha_1\dots\alpha_{m_i-(n-m)}$$

and (4.8) follows. □

Lemma 4.4. *Fix $q \in \overline{\mathcal{U}}$. Suppose that $x \in [0, \alpha_1/(q-1)]$ has an infinite greedy expansion (b_i) and suppose that the quasi-greedy expansion (a_i) of x satisfies the condition* (4.1)*. Then, $x \in \overline{\mathcal{U}_q}$.*

Proof. We may assume that $x \notin \mathcal{U}_q$. Note that $(a_i) = (b_i)$, since the greedy expansion of x is infinite by assumption. Since $x \notin \mathcal{U}_q$, there exists a *first* positive integer n such that

$$b_n > 0 \quad \text{and} \quad \overline{b_{n+1}b_{n+2}\dots} \geq \alpha_1\alpha_2\dots.$$

According to (4.1) this last inequality is in fact an equality.

As before, let $1 \leq m_1 < m_2 < \cdots$ be a sequence such that (4.4) is satisfied with $m = m_i$ for all $i \geq 1$. Again, we may assume that $m_i > n$ for all $i \geq 1$. Consider for each $i \geq 1$ the sequence (b^i_j), given by

$$(b^i_j) = b_1\dots b_n(\overline{\alpha_1\dots\alpha_{m_i}}\alpha_1\dots\alpha_{m_i})^\infty$$

and define for $i \geq 1$, the number y_i by

$$y_i = \sum_{j=1}^{\infty} \frac{b^i_j}{q^j}.$$

Then the sequence $(y_i)_{i\geq 1}$ converges to x as i goes to infinity. It remains to show that $y_i \in \mathcal{U}_q$ for all $i \geq 1$.

If $m \geq n$, then (4.5) and (4.6) follow as in the proof of the preceding lemma.

Now assume that $m < n$. If $b^i_m = b_m < \alpha_1$, then by (2.5),

$$b^i_{m+1}\dots b^i_n = b_{m+1}\dots b_n \leq \alpha_1\dots\alpha_{n-m}.$$

Hence, it suffices to verify that

$$b^i_{n+1}b^i_{n+2}\dots = (\overline{\alpha_1\dots\alpha_{m_i}}\alpha_1\dots\alpha_{m_i})^\infty < \alpha_{n-m+1}\alpha_{n-m+2}\dots,$$

which is already done (cf. (4.8)). Finally, suppose that $b^i_m = b_m > 0$. We must verify that

$$\overline{b^i_{m+1}\dots b^i_n b^i_{n+1}\dots} < \alpha_1\alpha_2\dots.$$

By minimality of n, we have

$$\overline{b_{m+1}\dots b_n b_{n+1}\dots} < \alpha_1\alpha_2\dots,$$

i.e.,

$$\overline{b_{m+1}\dots b_n\overline{\alpha_1\alpha_2\dots}} < \alpha_1\alpha_2\dots. \tag{4.9}$$

Therefore,

$$\overline{b^i_{m+1}\dots b^i_n} = \overline{b_{m+1}\dots b_n} < \alpha_1\dots\alpha_{n-m},$$

for otherwise

$$\overline{b_{m+1}\dots b_n} = \alpha_1\dots\alpha_{n-m}$$

and it would follow from (4.9) that

$$\alpha_1\alpha_2\dots < \alpha_{n-m+1}\alpha_{n-m+2}\dots,$$

which contradicts (4.2). □

5. Proof of Theorem 1.4

In order to prove Theorem 1.4 we start with some preliminary lemmas.

Lemma 5.1. *Fix $q > 1$. If $(b_i) \neq \alpha_1^\infty$ is the greedy expansion of a number $x \in J$, i.e., if $0 \le x < \alpha_1/(q-1)$, then there exists a sequence $1 \le n_1 < n_2 < \cdots$ such that for each $i \ge 1$,*

$$b_{n_i} < \alpha_1 \quad \text{and} \quad b_{m+1} \dots b_{n_i} < \alpha_1 \dots \alpha_{n_i - m} \quad \text{if } m < n_i \text{ and } b_m < \alpha_1. \tag{5.1}$$

Proof. We define a sequence $(n_i)_{i \ge 1}$ satisfying the requirements by induction.

Let r be the first positive integer for which $b_r < \alpha_1$. Then, (5.1) with n_i replaced by r holds trivially. Set $n_1 := r$.

Suppose we have already defined $n_1 < \cdots < n_\ell$, such that for each $1 \le j \le \ell$,

$$b_{n_j} < \alpha_1 \quad \text{and} \quad b_{m+1} \dots b_{n_j} < \alpha_1 \dots \alpha_{n_j - m} \quad \text{if } m < n_j \text{ and } b_m < \alpha_1.$$

Since (b_i) is greedy and $b_{n_\ell} < \alpha_1$, there exists a *first* integer $n_{\ell+1} > n_\ell$ such that

$$b_{n_\ell + 1} \dots b_{n_{\ell+1}} < \alpha_1 \dots \alpha_{n_{\ell+1} - n_\ell}. \tag{5.2}$$

Note that $b_{n_{\ell+1}} < \alpha_{n_{\ell+1} - n_\ell} \le \alpha_1$. It remains to verify that for all $1 \le m < n_{\ell+1}$ for which $b_m < \alpha_1$, we have that

$$b_{m+1} \dots b_{n_{\ell+1}} < \alpha_1 \dots \alpha_{n_{\ell+1} - m}. \tag{5.3}$$

If $m < n_\ell$, then (5.3) follows from the induction hypothesis. If $m = n_\ell$, then (5.3) reduces to (5.2). If $n_\ell < m < n_{\ell+1}$, then

$$b_{n_\ell + 1} \dots b_m = \alpha_1 \dots \alpha_{m - n_\ell},$$

by minimality of $n_{\ell+1}$, and thus by (5.2),

$$\begin{aligned} b_{m+1} \dots b_{n_{\ell+1}} &< \alpha_{m - n_\ell + 1} \dots \alpha_{n_{\ell+1} - n_\ell} \\ &\le \alpha_1 \dots \alpha_{n_{\ell+1} - m}. \end{aligned}$$

□

The following lemma has been established in [KL3]:

Lemma 5.2. *If $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$, then the greedy expansion (β_i) of 1 is finite and all expansions of 1 are given by*

$$(\alpha_i) \quad \text{and} \quad (\alpha_1 \dots \alpha_m)^N \alpha_1 \dots \alpha_{m-1} \alpha_m^+ 0^\infty, \quad N = 0, 1, 2, \dots, \tag{5.4}$$

where m is such that β_m is the last nonzero element of (β_i).

Note that if the greedy expansion (β_i) of 1 is finite with last nonzero element β_m, then the quasi-greedy expansion of 1 is given by $(\alpha_i) = (\beta_1 \dots \beta_m^-)^\infty = (\alpha_1 \dots \alpha_m)^\infty$. Hence, as a consequence of Lemma 5.2, for each $q \in \overline{\mathcal{U}}$, the quasi-greedy expansion (α_i) of 1 is also the smallest expansion of 1 in lexicographical order.

Proof of Theorem 1.4. (ia) We establish that $|\overline{\mathcal{U}_q} \setminus \mathcal{U}_q| = \aleph_0$. More specifically, if $q \in \overline{\mathcal{U}} \setminus \mathbb{N}$, then the sets A_q and B_q (introduced in a remark following the statement of Theorem 1.4) are countably infinite. Moreover, the greedy expansion of a number $x \in B_q$ ends with $\overline{\alpha_1 \alpha_2 \dots}$. If $q \in \{2, 3, \dots\}$, then $A_q = \overline{\mathcal{U}_q} \setminus \mathcal{U}_q$.

Fix $q \in \overline{\mathcal{U}}$. Denote the greedy expansion of a number $x \in \overline{\mathcal{U}_q} \setminus \mathcal{U}_q$ by (b_i). Since $x \notin \mathcal{U}_q$, there exists a number n such that $b_n > 0$ and

$$\overline{b_{n+1} b_{n+2} \dots} \ge \alpha_1 \alpha_2 \dots .$$

If this inequality is strict, then $b_i = 0$ for all $i > n$ (cf. Lemma 3.3). Otherwise, the sequence (b_i) ends with $\overline{\alpha_1 \alpha_2 \dots}$, which is infinite unless q is an integer. It follows from Theorem 1.1, Theorem 1.2 and Theorem 1.3 that a sequence of the form $0^n 1 0^\infty$ for $n \ge 0$, is the finite greedy expansion of $1/q^{n+1} \in \overline{\mathcal{U}_q} \setminus \mathcal{U}_q$. Moreover, if $q \in \overline{\mathcal{U}} \setminus \mathbb{N}$, then a sequence of the form $\alpha_1^n \overline{\alpha_1 \alpha_2 \ldots}$ for $n \geq 1$, is the infinite greedy expansion of a number $x \in \overline{\mathcal{U}_q} \setminus \mathcal{U}_q$. These observations conclude the proof.

(ib) We show that if $q \in \overline{\mathcal{U}}$, then A_q is dense in $\overline{\mathcal{U}_q}$.

Fix $q \in \overline{\mathcal{U}}$. For each $x \in \mathcal{U}_q$, we will define a sequence $(x_i)_{i \geq 1}$ of numbers in $A_q \subset \overline{\mathcal{U}_q} \setminus \mathcal{U}_q$ that converges to x. We have seen in the proof of part (i) that $1/q^n \in A_q$ for each $n \geq 1$. Hence, there exists a sequence of numbers in A_q that converges to $0 \in \mathcal{U}_q$. Suppose now that $x \in \mathcal{U}_q \setminus \{0\}$ and denote by (c_i) the unique expansion of x. Since $\overline{c_1 c_2 \ldots} \neq \alpha_1^\infty$ is greedy, we infer from Lemma 5.1 that there exists a sequence $1 \leq n_1 < n_2 < \cdots$, such that for each $i \geq 1$,

$$c_{n_i} > 0 \quad \text{and} \quad \overline{c_{m+1} \ldots c_{n_i}} < \alpha_1 \ldots \alpha_{n_i - m} \quad \text{if } m < n_i \text{ and } c_m > 0. \tag{5.5}$$

Now consider for each $i \geq 1$ the sequence (b_j^i), given by

$$(b_j^i) = c_1 \ldots c_{n_i} 0^\infty$$

and define the number x_i by

$$x_i = \sum_{j=1}^{\infty} \frac{b_j^i}{q^j}.$$

According to Lemma 3.1, the sequences (b_j^i) are the finite greedy expansions of the numbers x_i, $i \geq 1$. Moreover, the sequence $(x_i)_{i \geq 1}$ converges to x as i goes to infinity. We claim that $x_i \in A_q$ for each $i \geq 1$. Note that $x_i \notin \mathcal{U}_q$ because the quasi-greedy expansion (a_j^i), given by

$$c_1 \ldots c_{n_i}^- \alpha_1 \alpha_2 \ldots$$

is another expansion of x_i. According to Theorem 1.3, it remains to prove that

$$a_j^i > 0 \Longrightarrow \overline{a_{j+1}^i a_{j+2}^i \ldots} \leq \alpha_1 \alpha_2 \ldots. \tag{5.6}$$

If $j < n_i$ and $a_j^i > 0$, then

$$\overline{a_{j+1}^i \ldots a_{n_i}^i} = \overline{c_{j+1} \ldots c_{n_i}^-} \leq \alpha_1 \ldots \alpha_{n_i - j},$$

by (5.5) and

$$\overline{a_{n_i+1}^i a_{n_i+2}^i \ldots} = \overline{\alpha_1 \alpha_2 \ldots} < \alpha_{n_i - j+1} \alpha_{n_i - j+2} \ldots,$$

by Theorem 1.2. If $j = n_i$, then (5.6) follows from $\overline{\alpha_1} = 0 < \alpha_1$. Finally, if $j > n_i$, then (5.6) follows again from Theorem 1.2.

(ic) We show that if $q \in \overline{\mathcal{U}} \setminus \mathbb{N}$, then the set B_q is dense in $\overline{\mathcal{U}_q}$.

Fix $q \in \overline{\mathcal{U}} \setminus \mathbb{N}$. For each $x \in \mathcal{U}_q$, we will define a sequence $(x_i)_{i \geq 1}$ of numbers in $B_q \subset \overline{\mathcal{U}_q} \setminus \mathcal{U}_q$ that converges to x. It follows from Theorem 1.1, Theorem 1.2 and Theorem 1.3 that a sequence of the form $0^i \alpha_1 \overline{\alpha_1 \alpha_2 \ldots}$ for $i \geq 0$, is the infinite greedy expansion of a number $x_i \in \overline{\mathcal{U}_q} \setminus \mathcal{U}_q$. Note that the sequence $(x_i)_{i \geq 1}$ converges to $0 \in \mathcal{U}_q$. Therefore, we may again assume that $x \in \mathcal{U}_q \setminus \{0\}$. Let (c_i) be the unique expansion of a number $x \in \mathcal{U}_q \setminus \{0\}$, and let $1 \leq n_1 < n_2 < \cdots$ be a sequence of integers satisfying (5.5). Arguing as in the proof of part (ib), one finds that for each $i \geq 1$, the sequence

$$(b_j^i) = c_1 \ldots c_{n_i} \overline{\alpha_1 \alpha_2 \ldots}$$

is the infinite greedy expansion of a number $x_i \in \overline{\mathcal{U}_q} \setminus \mathcal{U}_q$.

(ii) and (iii) Fix $q \in \overline{\mathcal{U}}$ and let (b_i) be the greedy expansion of some number $x \in \overline{\mathcal{U}_q} \setminus \mathcal{U}_q$. Let n be the *smallest* positive integer for which $b_n > 0$ and $\overline{b_{n+1} b_{n+2} \ldots} \geq \alpha_1 \alpha_2 \ldots$. Let (d_i) be another expansion of x. Then, $(d_i) < (b_i)$ and hence there exists a smallest integer j for which $d_j < b_j$. First we show that $j \geq n$. Suppose by contradiction that $j < n$. Then, $b_j > 0$ and by minimality of n, we have

$$b_{j+1} b_{j+2} \ldots > \overline{\alpha_1 \alpha_2 \ldots}.$$

Since $\alpha_1\alpha_2\ldots$ is the smallest expansion of 1, we have that $\overline{\alpha_1\alpha_2\ldots}$ is the largest expansion of the number $\sum_{i\geq 1}\overline{\alpha_i}/q^i$, and therefore,

$$\sum_{i=1}^{\infty}\frac{b_{j+i}}{q^i} > \sum_{i=1}^{\infty}\frac{\overline{\alpha_i}}{q^i}.$$

But then,

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{d_{j+i}}{q^i} &= \sum_{i=1}^{\infty}\frac{b_{j+i}}{q^i} + b_j - d_j\\ &> \sum_{i=1}^{\infty}\frac{\overline{\alpha_i}}{q^i} + 1\\ &= \sum_{i=1}^{\infty}\frac{\alpha_1}{q^i},\end{aligned}$$

which is clearly impossible. If $j = n$, then $d_n = b_n^-$, for otherwise we have again

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{d_{n+i}}{q^i} &\geq \sum_{i=1}^{\infty}\frac{b_{n+i}}{q^i} + 2\\ &> \sum_{i=1}^{\infty}\frac{b_{n+i}}{q^i} + \sum_{i=1}^{\infty}\frac{\alpha_i}{q^i} + \sum_{i=1}^{\infty}\frac{\overline{\alpha_i}}{q^i}\\ &\geq \sum_{i=1}^{\infty}\frac{\alpha_1}{q^i},\end{aligned}$$

where the second inequality follows from the fact that (α_i) is the smallest expansion of 1 and the inequality

$$\overline{\alpha_1\alpha_2\ldots} < \alpha_1\alpha_2\ldots.$$

Now we distinguish between two cases:

If $j = n$ and

$$\overline{b_{n+1}b_{n+2}\ldots} > \alpha_1\alpha_2\ldots, \tag{5.7}$$

then by Lemma 3.3, we have $b_r = 0$ for $r > n$, from which it follows that (d_{n+i}) is an expansion of 1. Hence, if $q \in \mathcal{U}$ and (5.7) holds, then the only expansion of x starting with $b_1\ldots b_n^-$ is given by $(c_i) := b_1\ldots b_n^-\alpha_1\alpha_2\ldots$. If $q \in \overline{\mathcal{U}}\setminus\mathcal{U}$ and (5.7) holds, then any expansion (c_i) starting with $b_1\ldots b_n^-$ is an expansion of x if and only if (c_{n+i}) is one of the expansions listed in (5.4).

If $j = n$ and

$$\overline{b_{n+1}b_{n+2}\ldots} = \alpha_1\alpha_2\ldots, \tag{5.8}$$

then

$$\sum_{i=1}^{\infty}\frac{d_{n+i}}{q^i} = \sum_{i=1}^{\infty}\frac{b_{n+i}}{q^i} + 1 = \sum_{i=1}^{\infty}\frac{\alpha_1}{q^i}.$$

Hence, if (5.8) holds, then the only expansion of x starting with $b_1\ldots b_n^-$ is given by $b_1\ldots b_n^-\alpha_1^\infty$.

Finally, if $j > n$, then

$$\overline{b_{n+1}b_{n+2}\ldots} = \alpha_1\alpha_2\ldots,$$

for otherwise $(b_{n+i}) = 0^\infty$ and $d_j < b_j$ is impossible. Note that in this case $q \notin \mathcal{U}$, because otherwise (b_{n+i}) is the unique expansion of $\sum_{i\geq 1}\overline{\alpha_i}/q^i$ and thus $(d_{n+i}) = (b_{n+i})$ which is impossible since $j > n$. Hence, if $q \in \mathcal{U}$, then (b_i) is the only expansion of x starting with $b_1\ldots b_n$. If $q \in \overline{\mathcal{U}}\setminus\mathcal{U}$ and (5.8) holds, then any expansion (c_i) starting with $b_1 \dots b_n$ is an expansion of x if and only if (c_{n+i}) is one of the conjugates of the expansions listed in (5.4).

The statements of parts (ii) and (iii) follow directly from the above considerations. □

Remark. Fix $q \in \overline{\mathcal{U}}$. It follows from Theorem 1.4 (i) that each $x \in \overline{\mathcal{U}_q} \setminus \mathcal{U}_q$ has either a finite expansion or an expansion that ends with $\overline{\alpha_1 \alpha_2 \dots}$, i.e., x can be written as

$$x = \frac{b_1}{q} + \dots + \frac{b_n}{q^n} + \frac{1}{q^n}\left(\frac{\alpha_1}{q-1} - 1\right).$$

Moreover, according to Lemma 5.2, the greedy expansion of 1 in base q is finite if $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$. Hence, if $q \in \mathcal{U}$ is transcendental, then each $x \in \overline{\mathcal{U}_q} \setminus \mathcal{U}_q$ is a transcendental number. If $q \in \mathcal{U}$ is an algebraic number or if $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$, then each $x \in \overline{\mathcal{U}_q} \setminus \mathcal{U}_q$ is also algebraic.

6. Proof of Theorem 1.5 and Theorem 1.6

Fix $q > 1$. It follows from Proposition 2.3 and Proposition 2.5 that a sequence (b_i) is greedy if and only if $0 \le b_n \le \alpha_1$ for all $n \ge 1$, and

$$b_{n+k+1} b_{n+k+2} \dots < \alpha_1 \alpha_2 \dots \quad \text{for all } k \ge 0, \quad \text{whenever} \quad b_n < \alpha_1. \tag{6.1}$$

Lemma 6.1. *Assume that $q \notin \overline{\mathcal{U}}$. Then a greedy sequence (b_i) cannot end with $\overline{\alpha_1 \alpha_2 \dots}$.*

Proof. Assume by contradiction that for some n,

$$b_{n+1} b_{n+2} \dots = \overline{\alpha_1 \alpha_2 \dots}.$$

Since in this case $b_{n+1} = \overline{\alpha_1} = 0 < \alpha_1$, it would follow from (6.1) that

$$\overline{\alpha_{k+1} \alpha_{k+2} \dots} < \alpha_1 \alpha_2 \dots \quad \text{for all} \quad k \ge 1.$$

But this contradicts our assumption that $q \notin \overline{\mathcal{U}}$. □

Lemma 6.2. *Assume that $q \notin \overline{\mathcal{U}}$. Then,*

(i) *The set $\mathcal{U}_q$ is closed.*
(ii) *Each element $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has a finite greedy expansion.*

Proof. (i) Let $x \in J \setminus \mathcal{U}_q$ and denote the greedy expansion of x in base q by (b_i). According to Theorem 1.1, there exists a positive integer n such that

$$b_n > 0 \quad \text{and} \quad \overline{b_{n+1} b_{n+2} \dots} \ge \alpha_1 \alpha_2 \dots.$$

Applying Lemma 3.3 and Lemma 6.1 we conclude that

$$[x, z] \cap \mathcal{U}_q = \varnothing,$$

for some number $z > x$. It follows that $\mathcal{U}_q$ is closed from above. Note that the set $\mathcal{U}_q$ is symmetric in the sense that

$$x \in \mathcal{U}_q \Longleftrightarrow \alpha_1/(q-1) - x \in \mathcal{U}_q,$$

as follows from Theorem 1.1. Hence, the set $\mathcal{U}_q$ is also closed from below.

(ii) Let $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ and suppose that $(a_i(x)) = (b_i(x))$. Then, it would follow that for some positive integer n,

$$\overline{b_{n+1} b_{n+2} \dots} = \alpha_1 \alpha_2 \dots,$$

contradicting Lemma 6.1. □

Lemma 6.3. *Let (a_i) be the quasi-greedy expansion of some $x \in [0, \alpha_1/(q-1)]$. Furthermore, let M be an arbitrary positive integer. Then $a_1 \dots a_M 0^\infty$ is a greedy sequence.*

Proof. If $x = 0$, then there is nothing to prove. If $x \neq 0$, then the statement follows from Proposition 2.4 and Proposition 2.5. □

Recall from the introduction that the set $\mathcal{V}$ consists of those numbers $q > 1$ for which the quasi-greedy expansion (α_i) of 1 in base q satisfies

$$\overline{\alpha_{k+1}\alpha_{k+2}\ldots} \leq \alpha_1\alpha_2\ldots \quad \text{for all} \quad k \geq 1. \tag{6.2}$$

Note that the quasi-greedy expansion of 1 in base q for $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$ is of the form

$$(\alpha_i) = (\alpha_1 \ldots \alpha_k \overline{\alpha_1 \ldots \alpha_k})^\infty, \tag{6.3}$$

where $k \geq 1$ is the first integer for which equality holds in (6.2). In particular, such a sequence is periodic. Any sequence of the form $(1^n0^n)^\infty$, where n is a positive integer, is infinite and satisfies (2.2) and (6.2) but not (4.3). On the other hand, there are only countably many periodic sequences. Hence, the set $\mathcal{V}\setminus\overline{\mathcal{U}}$ is countably infinite. Note that $\alpha_k > 0$, for otherwise it would follow from (6.2) and (6.3) that

$$\overline{\alpha_k\alpha_{k+1}\ldots\alpha_{2k-1}} = \alpha_1\alpha_1\alpha_2\ldots\alpha_{k-1} \leq \alpha_1\ldots\alpha_{k-1}0,$$

which is impossible, because $\alpha_1 > 0$. The following lemma ([KL3]) implies that the number of expansions of 1 is countably infinite in case $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$. Moreover, all expansions of the number 1 in base q are determined explicitly.

Lemma 6.4. *If $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$, then all expansions of 1 are given by (α_i), and the sequences*

$$(\alpha_1 \ldots \alpha_{2k})^N \alpha_1 \ldots \alpha_{2k-1}\alpha_{2k}^+ 0^\infty, \quad N = 0, 1, \ldots$$

and

$$(\alpha_1 \ldots \alpha_{2k})^N \alpha_1 \ldots \alpha_{k-1}\alpha_k^- \alpha_1^\infty, \quad N = 0, 1, \ldots.$$

It follows from the above lemma that for each $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$, the greedy expansion of 1 in base q is given by $(\beta_i) = \alpha_1 \ldots \alpha_{2k-1}\alpha_{2k}^+0^\infty$ and the smallest expansion of 1 in base q is given by $\alpha_1 \ldots \alpha_{k-1}\alpha_k^-\alpha_1^\infty$.

Now we are ready to prove Theorem 1.5 and Theorem 1.6. Throughout the proof of Theorem 1.5, $q \in \mathcal{V}\setminus\overline{\mathcal{U}}$ is fixed but arbitrary, and $k \geq 1$ is the first positive integer for which equality holds in (6.2).

Proof of Theorem 1.5. (i) We show that the sets $\mathcal{U}_q$ and $\mathcal{V}_q$ are closed. In view of Lemma 6.2, it remains to prove that $\mathcal{V}_q$ is closed.

Fix $x \in J \setminus \mathcal{V}_q$ and let $(a_i(x)) = (a_i)$ be the quasi-greedy expansion of x. Then, there exists an integer $n > 0$, such that

$$a_n > 0 \quad \text{and} \quad \overline{a_{n+1}a_{n+2}\ldots} > \alpha_1\alpha_2\ldots.$$

Let m be such that

$$\overline{a_{n+1}\ldots a_{n+m}} > \alpha_1\ldots\alpha_m, \tag{6.4}$$

and let

$$y = \sum_{i=1}^{n+m} \frac{a_i}{q^i}.$$

According to Lemma 6.3, the greedy expansion of y is given by $a_1 \ldots a_{n+m}0^\infty$. Therefore, the quasi-greedy expansion of each number $v \in (y, x]$ starts with the block $a_1 \ldots a_{n+m}$. It follows from (6.4) that

$$(y, x] \cap \mathcal{V}_q = \varnothing.$$

Consider now the sequence

$$(d_i) = a_1 \ldots a_n\overline{\alpha_1\alpha_2\ldots}.$$

It follows from (2.4) and (6.2) that (d_i) is the quasi-greedy expansion of

$$\sum_{i\geq 1} d_i/q^i = z > x.$$

Note that the quasi-greedy expansion (v_i) of an element $v \in [x, z)$ satisfies

$$v_n = a_n > 0 \quad \text{and} \quad v_{n+1}v_{n+2}\ldots < \overline{\alpha_1\alpha_2\ldots}.$$

Hence,

$$[x, z) \cap \mathcal{V}_q = \varnothing,$$

from which the claim follows.

(iia) We prove that $|\mathcal{V}_q \setminus \mathcal{U}_q| = \aleph_0$. The set $\mathcal{V}_q \setminus \mathcal{U}_q$ is countable, because each element $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has a finite greedy expansion (cf. Lemma 6.2). On the other hand, for each $n \geq 1$, the sequence $\alpha_1^n 0^\infty$ is the greedy expansion of an element $x \in \mathcal{V}_q \setminus \mathcal{U}_q$, from which the claim follows.

(iib) In order to show that $\mathcal{V}_q \setminus \mathcal{U}_q$ is dense in $\mathcal{V}_q$, one can argue in the same way as in the proof of Theorem 1.4 (ib). Instead of applying Theorem 1.2 one should now apply the inequality (6.2).

(iic) Finally, we show that all elements of $\mathcal{V}_q \setminus \mathcal{U}_q$ are isolated points of $\mathcal{V}_q$. Let $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ and let b_n be the last nonzero element of the greedy expansion (b_i) of x. Choose m such that $\alpha_m < \alpha_1$. Note that this is possible because $q \notin \mathbb{N}$. According to Lemma 3.2, there exists a greedy sequence $(c_i) > (b_i)$, such that

$$c_1 \ldots c_{n+m} = b_1 \ldots b_n 0^m.$$

If we set

$$z = \sum_{i=1}^{\infty} \frac{c_i}{q^i},$$

then the quasi-greedy expansion (v_i) of a number $v \in (x, z]$ starts with $b_1 \ldots b_n 0^m$. Hence, $v_n = b_n > 0$, and

$$\overline{v_{n+1}\ldots v_{n+m}} = \alpha_1^m > \alpha_1 \ldots \alpha_m.$$

Therefore,

$$(x, z] \cap \mathcal{V}_q = \varnothing.$$

In order to show that there exists also a number $y < x$, such that

$$(y, x) \cap \mathcal{V}_q = \varnothing,$$

we introduce for $m \geq 1$ the sequences (b_j^m), given by

$$(b_j^m) = b_1 \ldots b_n^- (\alpha_1 \ldots \alpha_k \overline{\alpha_1 \ldots \alpha_k})^m 0^\infty$$

and we define the numbers x_m by

$$x_m = \sum_{j=1}^{\infty} \frac{b_j^m}{q^j}.$$

The sequences (b_j^m) are all greedy by Lemma 6.3. Moreover, $x_m \uparrow x$ as m goes to infinity. Let $v \in (x_m, x_{m+1}]$ for some $m \geq 1$, and let the quasi-greedy expansion of v be given by (d_i). Then,

$$d_1 \ldots d_n d_{n+1} \ldots d_{2km+n} = b_1 \ldots b_n^- (\alpha_1 \ldots \alpha_k \overline{\alpha_1 \ldots \alpha_k})^m$$

and

$$d_{2km+n+1} \ldots d_{2k(m+1)+n} < \alpha_1 \ldots \alpha_k \overline{\alpha_1 \ldots \alpha_k}.$$

Therefore,

$$d_{2k(m-1)+n+k} = \alpha_k > 0$$

and

$$\begin{aligned}\overline{d_{2k(m-1)+n+k+1}\dots d_{2k(m+1)+n}} &> \alpha_1\dots\alpha_k\overline{\alpha_1\dots\alpha_k}\alpha_1\dots\alpha_k\\ &= \alpha_1\dots\alpha_{3k}.\end{aligned}$$

Hence, $v\notin\mathcal{V}_q$, i.e.,

$$(x_1,x)\cap\mathcal{V}_q=\varnothing.$$

(iii) We already know from Lemma 6.2 that each $x\in\mathcal{V}_q\setminus\mathcal{U}_q$ has a finite greedy expansion. It remains to show that each element $x\in\mathcal{V}_q\setminus\mathcal{U}_q$ has $\aleph_0$ expansions. Let $x\in\mathcal{V}_q\setminus\mathcal{U}_q$ and let b_n be the last nonzero element of its greedy expansion (b_i). If $j<n$ and $b_j=a_j>0$, then

$$\overline{a_{j+1}\dots a_n}=\overline{b_{j+1}\dots b_n^-}\le\alpha_1\dots\alpha_{n-j},$$

because $x\in\mathcal{V}_q$. Therefore,

$$b_{j+1}\dots b_n>\overline{\alpha_1\dots\alpha_{n-j}}. \tag{6.5}$$

Let (d_i) be another expansion of x and let j be the *smallest* positive integer for which $d_j\ne b_j$. Since (b_i) is greedy, we have $d_j<b_j$ and $j\in\{1,\dots,n\}$. First we show that $j\in\{n-k,n\}$. Suppose by contradiction that $j\notin\{n-k,n\}$.

First assume that $n-k<j<n$. Then, $b_j>0$ and by (6.5),

$$b_{j+1}\dots b_n0^\infty>\overline{\alpha_1\dots\alpha_{n-j}}\overline{\alpha_{n-j+1}\dots\alpha_k^-}0^\infty.$$

Since $\alpha_1\dots\alpha_k^-\alpha_1^\infty$ is the smallest expansion of 1 in base q, $\overline{\alpha_1\dots\alpha_k^-}0^\infty$ is the greedy expansion of $\alpha_1/(q-1)-1$, and therefore,

$$\sum_{i=1}^\infty\frac{b_{j+i}}{q^i}>\alpha_1/(q-1)-1.$$

But then,

$$\begin{aligned}\sum_{i=1}^\infty\frac{d_{j+i}}{q^i} &= \sum_{i=1}^\infty\frac{b_{j+i}}{q^i}+b_j-d_j\\ &> \alpha_1/(q-1),\end{aligned}$$

which is impossible.

Next assume that $1\le j<n-k$. Rewriting (6.5), one gets

$$\overline{b_{j+1}\dots b_n}<\alpha_1\dots\alpha_{n-j}.$$

If

$$\overline{b_{j+1}\dots b_{j+k}}=\alpha_1\dots\alpha_k,$$

then

$$\overline{b_{j+k+1}\dots b_n}<\alpha_{k+1}\dots\alpha_{n-j}.$$

Hence,

$$\begin{aligned}b_{j+k+1}b_{j+k+2}\dots &> \overline{\alpha_{k+1}\alpha_{k+2}\dots}\\ &= \alpha_1\alpha_2\dots.\end{aligned}$$

Since in this case $b_{j+k}=\overline{\alpha_k}<\alpha_1$, the last inequality contradicts the fact that (b_i) is a greedy sequence. Hence, if $j<n-k$, then

$$\overline{b_{j+1}\dots b_{j+k}}<\alpha_1\dots\alpha_k.$$

Equivalently,

$$b_{j+1}\dots b_{j+k}\ge\overline{\alpha_1\dots\alpha_k^-}.$$

Since $n > j + k$ and $b_n > 0$, it follows that

$$b_{j+1}b_{j+2}\ldots > \overline{\alpha_1 \ldots \alpha_k^-}0^\infty,$$

which leads to the same contradiction as at the beginning of the proof. It remains to investigate what happens if $j \in \{n-k, n\}$.

If $j = n - k$, then it follows from (6.5) that

$$b_{n-k+1}\ldots b_n \geq \overline{\alpha_1 \ldots \alpha_k^-}.$$

Equivalently,

$$b_{n-k+1}b_{n-k+2}\ldots = b_{n-k+1}\ldots b_n 0^\infty \geq \overline{\alpha_1 \ldots \alpha_k^-}0^\infty,$$

and thus

$$\sum_{i=1}^{\infty} \frac{d_{n-k+i}}{q^i} \geq \sum_{i=1}^{\infty} \frac{b_{n-k+i}}{q^i} + 1 \geq \alpha_1/(q-1), \tag{6.6}$$

where both inequalities in (6.6) are equalities if and only if

$$d_{n-k} = b_{n-k}^-,\ b_{n-k+1}\ldots b_n = \overline{\alpha_1 \ldots \alpha_k^-}, \text{ and } d_{n-k+1}d_{n-k+2}\ldots = \alpha_1^\infty.$$

Hence, $d_{n-k} < b_{n-k}$ is only possible in case $b_{n-k} > 0$ and $b_{n-k+1}\ldots b_n = \overline{\alpha_1 \ldots \alpha_k^-}$.

Finally, if $j = n$, then $d_n = b_n^-$, for otherwise

$$\sum_{i=1}^{\infty} \frac{d_{n+i}}{q^i} \geq 2 > \sum_{i=1}^{\infty} \frac{\alpha_i}{q^i} + \sum_{i=1}^{\infty} \frac{\overline{\alpha_i}}{q^i} = \alpha_1/(q-1),$$

because

$$\overline{\alpha_1\alpha_2\ldots} < \alpha_1 \ldots \alpha_k^- \alpha_1^\infty.$$

In this case (d_{n+i}) is one of the expansions listed in Lemma 6.4. $\square$

Remark. Fix $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$. According to Lemma 6.4, the number 1 has a finite greedy expansion in base q. Hence, each element $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$ is algebraic. Because each $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has a finite greedy expansion in base q, it follows that the set $\mathcal{V}_q \setminus \mathcal{U}_q$ consists entirely of algebraic numbers.

Lemma 6.5. *Let (α_i) be the quasi-greedy expansion of 1 in some base $q > 1$ and assume there exists a positive integer k such that*

$$\overline{\alpha_{k+1}\alpha_{k+2}\ldots} > \alpha_1\alpha_2\ldots$$

Then there exists a positive integer m such that $\alpha_m > 0$ and

$$\overline{\alpha_{m+1}\alpha_{m+2}\ldots} > \alpha_1\alpha_2\ldots.$$

Proof. Let $m = \max\{1 \leq i \leq k : \alpha_m > 0\}$. Note that m is well defined, since $\alpha_1 > 0$. Then, $\alpha_{m+1}\ldots\alpha_k = 0\ldots0$. Hence,

$$\overline{\alpha_{m+1}\alpha_{m+2}\ldots} > \alpha_1\alpha_2\ldots. \qquad \square$$

Proof of Theorem 1.6. Fix $q \notin \mathcal{V}$. In view of Lemma 6.2, it remains to prove that a number $x \in J \setminus \{0\}$ with a finite greedy expansion does *not* belong to $\mathcal{V}_q$.

Let $x \in J \setminus \{0\}$ be an element with a finite greedy expansion. Since $q \notin \mathcal{V}$, there exists a positive integer k, such that

$$\overline{\alpha_{k+1}\alpha_{k+2}\ldots} > \alpha_1\alpha_2\ldots.$$

According to Lemma 6.5, we may assume that $\alpha_k > 0$. Since the quasi-greedy expansion of each element $x \in J \setminus \{0\}$ with a finite greedy expansion ends with $\alpha_1\alpha_2\ldots$, we conclude that $x \notin \mathcal{V}_q$. $\square$

7. Proof of Theorem 1.7–Theorem 1.10

In this section we will complete our study of the univoque set $\mathcal{U}_q$ for numbers $q > 1$. The results proved in the preceding sections were mainly concerned with various properties of the sets $\mathcal{U}_q$ for numbers $q \in \mathcal{V}$. Now we will use these properties to describe the topological structure of $\mathcal{U}_q$ for all numbers $q > 1$.

Since the set $\mathcal{V}$ is closed, we may write $(1, \infty) \setminus \mathcal{V}$ as the union of countably many disjoint open intervals (q_1, q_2): the connected components of $(1, \infty) \setminus \mathcal{V}$. In order to determine the endpoints of these components we recall from [KL3] that

- $\mathcal{V}$ is closed.
- $\mathcal{V} \setminus \overline{\mathcal{U}}$ is dense in $\mathcal{V}$.
- all elements of $\mathcal{V} \setminus \overline{\mathcal{U}}$ are isolated in $\mathcal{V}$.

In fact, for each element $q \in \overline{\mathcal{U}}$ there exists a sequence $(q_m)_{m \geq 1}$ of numbers in $\mathcal{V} \setminus \overline{\mathcal{U}}$ such that $q_m \uparrow q$, as can be seen from the proof of Theorem 2.6 in [KL3].

Proposition 7.1.

(i) *The set R of right endpoints q_2 of the connected components (q_1, q_2) is given by $R = \mathcal{V} \setminus \overline{\mathcal{U}}$.*

(ii) *The set L of left endpoints q_1 of the connected components (q_1, q_2) is given by $L = \mathbb{N} \cup (\mathcal{V} \setminus \mathcal{U})$.*

Proof of Proposition 7.1 (i). Note that $\mathcal{V} \setminus \overline{\mathcal{U}} \subset R$ because the set $\mathcal{V} \setminus \overline{\mathcal{U}}$ is discrete. As we have already observed in the preceding paragraph, each element $q \in \overline{\mathcal{U}}$ can be approximated from below by elements in $\mathcal{V} \setminus \overline{\mathcal{U}}$. Hence, $R = \mathcal{V} \setminus \overline{\mathcal{U}}$. $\square$

The proof of part (ii) of Proposition 7.1 requires more work. We will prove a number of technical lemmas first. In the remainder of this section $q \sim (\alpha_i)$ indicates that the quasi-greedy expansion of 1 in base q is given by (α_i). For convenience we also write $1 \sim 0^\infty$, and we occasionally refer to 0^∞ as the quasi-greedy expansion of the number 1 in base 1.

Let $q_2 \in \mathcal{V} \setminus \overline{\mathcal{U}}$ and suppose that

$$q_2 \sim (\alpha_i) = (\alpha_1 \dots \alpha_k \overline{\alpha_1 \dots \alpha_k})^\infty$$

where k is chosen to be minimal.

Remark. The minimality of k implies that the smallest period of (α_i) equals $2k$. Indeed, if j is the smallest period of (α_i), then $\alpha_j = \alpha_{2k} = \overline{\alpha_k} < \alpha_1$ because j divides $2k$. Hence, $\alpha_1 \dots \alpha_j^+ 0^\infty$ is an expansion of 1 in base q_2 which contradicts Lemma 6.4 if $j < 2k$.

Lemma 7.2. *For all $0 \leq i < k$, we have*

$$\overline{\alpha_{i+1} \dots \alpha_k} < \alpha_1 \dots \alpha_{k-i}.$$

Proof. For $i = 0$, the inequality follows from the relation $\overline{\alpha_1} = 0 < \alpha_1$. Hence, assume that $1 \leq i < k$. Since $q_2 \in \mathcal{V}$,

$$\overline{\alpha_{i+1} \dots \alpha_k} \leq \alpha_1 \dots \alpha_{k-i}.$$

Suppose that for some $1 \leq i < k$,

$$\overline{\alpha_{i+1} \dots \alpha_k} = \alpha_1 \dots \alpha_{k-i}.$$

If $k \geq 2i$, then

$$\alpha_1 \dots \alpha_{2i} = \alpha_1 \dots \alpha_i \overline{\alpha_1 \dots \alpha_i},$$

and it would follow from Lemma 4.2 that

$$(\alpha_i) = (\alpha_1 \dots \alpha_i \overline{\alpha_1 \dots \alpha_i})^\infty,$$

contradicting the minimality of k. If $i < k < 2i$, then

$$\begin{aligned}\overline{\alpha_{i+1}\dots\alpha_{2i}} &= \overline{\alpha_{i+1}\dots\alpha_k}\alpha_1\dots\alpha_{2i-k}\\ &= \alpha_1\dots\alpha_{k-i}\alpha_1\dots\alpha_{2i-k}\\ &\ge \alpha_1\dots\alpha_{k-i}\alpha_{k-i+1}\dots\alpha_i\\ &= \alpha_1\dots\alpha_i,\end{aligned}$$

leading to the same contradiction. □

Let q_1 be the largest element of $\mathcal{V}\cup\{1\}$ that is smaller than q_2. This element exists because the set $\mathcal{V}\cup\{1\}$ is closed and the elements of $\mathcal{V}\setminus\overline{\mathcal{U}}$ are isolated points of $\mathcal{V}\cup\{1\}$. The next lemma provides the quasi-greedy expansion of 1 in base q_1.

Lemma 7.3. $q_1 \sim (\alpha_1\dots\alpha_k^-)^\infty$.

Proof. Let $q_1 \sim (v_i)$. If $k = 1$, then $q_2 \sim (\alpha_1 0)^\infty$ and $(v_i) = (\alpha_1^-)^\infty$ because q_2 is the smallest element of $\mathcal{V}\cap(\alpha_1, \alpha_1+1)$. Hence, we may assume that $k \ge 2$. Observe that

$$v_1\dots v_k \le \alpha_1\dots\alpha_k.$$

If $v_1\dots v_k = \alpha_1\dots\alpha_k$, then

$$v_{k+1}\dots v_{2k} \le \overline{\alpha_1\dots\alpha_k},$$

i.e.,

$$\begin{aligned}\overline{v_{k+1}\dots v_{2k}} &\ge \alpha_1\dots\alpha_k,\\ &= v_1\dots v_k\end{aligned}$$

(here we need that $k \ge 2$, for otherwise the conjugate bars on both sides have a different meaning) and it would follow from Lemma 4.2 that $q_1 = q_2$. Hence,

$$v_1\dots v_k \le \alpha_1\dots\alpha_k^-.$$

It follows from Proposition 2.3 that $(w_i) = (\alpha_1\dots\alpha_k^-)^\infty$ is the largest quasi-greedy expansion of 1 in some base $q > 1$ that starts with $\alpha_1\dots\alpha_k^-$. Therefore, it suffices to show that the sequence (w_i) satisfies inequality (6.2) for all $k \ge 1$. Since the sequence (w_i) is periodic with period k, it suffices to verify that

$$\overline{w_{j+1}w_{j+2}\dots} \le w_1w_2\dots \quad \text{for all } 0 \le j < k. \tag{7.1}$$

If $j = 0$, then (7.1) is true because $\overline{w_1} = 0 < w_1$; hence assume that $1 \le j < k$. Then, according to the preceding lemma,

$$\overline{\alpha_{j+1}\dots\alpha_k} < \alpha_1\dots\alpha_{k-j},$$

and

$$\overline{\alpha_1\dots\alpha_j} < \alpha_{k-j+1}\dots\alpha_k.$$

Hence,

$$\begin{aligned}\overline{w_{j+1}\dots w_{j+k}} &= \overline{\alpha_{j+1}\dots\alpha_k^-}\,\alpha_1\dots\alpha_j\\ &\le \alpha_1\dots\alpha_{k-j}\overline{\alpha_1\dots\alpha_j}\\ &< \alpha_1\dots\alpha_k,\end{aligned}$$

so that

$$\overline{w_{j+1}\dots w_{j+k}} \le w_1\dots w_k.$$

Since the sequence $(w_{j+i}) = w_{j+1}w_{j+2}\dots$ is also periodic with period k, the inequality (7.1) follows. □

Lemma 7.4. *Fix $q > 1$ and denote by (β_i) the greedy expansion of the number 1 in base q. For any positive integer n, we have*

$$\beta_{n+1}\beta_{n+2}\ldots \le \beta_1\beta_2\ldots.$$

Proof. It follows from (6.1) that

$$\beta_{n+1}\beta_{n+2}\ldots < \alpha_1\alpha_2\ldots \le \beta_1\beta_2\ldots,$$

whenever there exists a positive integer $j \le n$ satisfying $\beta_j < \beta_1 = \alpha_1$. If such an integer j does not exist, then either $(\beta_i) = \alpha_1^\infty$ or there exists an integer $j > n$ for which $\beta_j < \alpha_1$. In both these cases the desired inequality readily follows as well. □

Now we consider a number $q_1 \in \mathcal{V} \setminus \mathcal{U}$. Recall from Lemma 5.2 and Lemma 6.4 that the greedy expansion (β_i) of 1 in base q_1 is finite. Denote its last nonzero element by β_m.

Lemma 7.5.

(i) *The smallest element q_2 of $\mathcal{V}$ that is larger than q_1 exists. Moreover,*

$$q_2 \sim (\beta_1\ldots\beta_m\overline{\beta_1\ldots\beta_m})^\infty.$$

(ii) *The greedy expansion of 1 in base q_2 is given by $(\gamma_i) = \beta_1\ldots\beta_m\overline{\beta_1\ldots\beta_m^-}0^\infty$.*

Proof. (i) First of all, note that

$$q_1 \sim (\alpha_i) = (\beta_1\ldots\beta_m^-)^\infty.$$

Moreover,

$$(\beta_1\ldots\beta_m^-)^\infty$$

is the largest quasi-greedy expansion of 1 in some base $q > 1$ that starts with $\beta_1\ldots\beta_m^-$. Hence, in view of Lemma 4.2, it suffices to show that the sequence

$$(w_i) = (\beta_1\ldots\beta_m\overline{\beta_1\ldots\beta_m})^\infty$$

satisfies the inequalities

$$w_{k+1}w_{k+2}\ldots \le w_1w_2\ldots \tag{7.2}$$

and

$$\overline{w_{k+1}w_{k+2}\ldots} \le w_1w_2\ldots \tag{7.3}$$

for all $k \ge 0$. Observe that (7.2) for $k+m$ is equivalent to (7.3) for k and (7.3) for $k+m$ is equivalent to (7.2) for k. Since both relations are obvious for $k = 0$, we only need to verify (7.2) and (7.3) for $1 \le k < m$. Fix $1 \le k < m$.

The relation (7.3) follows from our assumption that $q_1 \in \mathcal{V}$:

$$\overline{w_{k+1}\ldots w_m} = \overline{\beta_{k+1}\ldots\beta_m} < \overline{\alpha_{k+1}\ldots\alpha_m} \le \alpha_1\ldots\alpha_{m-k} = w_1\ldots w_{m-k}.$$

Since $1 \le m-k < m$, we also have

$$\overline{w_{m-k+1}\ldots w_m} < w_1\ldots w_k.$$

Using Lemma 7.4, we obtain

$$\begin{aligned} w_{k+1}\ldots w_{k+m} &= w_{k+1}\ldots w_m\overline{w_1\ldots w_k} \\ &\le w_1\ldots w_{m-k}\overline{w_1\ldots w_k} \\ &< w_1\ldots w_{m-k}w_{m-k+1}\ldots w_m, \end{aligned}$$

from which (7.2) follows.

(ii) We must show that

$$\gamma_{k+1}\gamma_{k+2}\ldots < w_1w_2\ldots \quad \text{whenever} \quad \gamma_k < w_1. \tag{7.4}$$

If $1 \le k < m$, then (7.4) follows from

$$\gamma_{k+1} \dots \gamma_{k+m} = w_{k+1} \dots w_{k+m} < w_1 \dots w_m.$$

If $k = m$, then (7.4) follows from $\gamma_{m+1} = \overline{\beta_1} = 0 < w_1$.
If $m < k < 2m$, then

$$\gamma_{k+1} \dots \gamma_{2m} = \overline{\beta_{k-m+1} \dots \beta_m^-} \le w_1 \dots w_{2m-k}.$$

Hence,

$$\gamma_{k+1}\gamma_{k+2} \dots = \gamma_{k+1} \dots \gamma_{2m} 0^\infty < w_1 w_2 \dots,$$

because (w_i) is infinite. Finally, if $k \ge 2m$, then $\gamma_{k+1} = 0 < w_1$. $\square$

Proof of Proposition 7.1 (ii). It follows from Lemma 7.5 that $\mathcal{V} \setminus \mathcal{U} \subset L$. If $q_2 \sim (n0)^\infty$ for some $n \in \mathbb{N}$, then (n, q_2) is a connected component of $(1, \infty) \setminus \mathcal{V}$. Hence, $\mathbb{N} \subset L$. It remains to show that $(L \setminus \mathbb{N}) \cap \mathcal{U} = \varnothing$.

If (q_1, q_2) is a connected component of $(1, \infty) \setminus \mathcal{V}$ with $q_2 \sim (\alpha_i)$ and $q_1 \in L \setminus \mathbb{N}$, then by Lemma 7.3, $q_1 \sim (\alpha_1 \dots \alpha_k^-)^\infty$ for some $k \ge 2$. Since $\alpha_1 \dots \alpha_k 0^\infty$ is a larger expansion of 1 in base q_1, we have $q_1 \notin \mathcal{U}$. $\square$

Recall from Section 1 that for $q > 1$, $\mathcal{U}'_q$ and $\mathcal{V}'_q$ denote the sets of quasi-greedy expansions of numbers $x \in \mathcal{U}_q$ and $x \in \mathcal{V}_q$ respectively.

Lemma 7.6. *Let (q_1, q_2) be a connected component of $(1, \infty) \setminus \mathcal{V}$ and suppose that $q_1 \in \mathcal{V} \setminus \mathcal{U}$. Then,*

$$\mathcal{U}'_{q_2} = \mathcal{V}'_{q_1}.$$

Proof. Let us write again

$$q_2 \sim (\alpha_1 \dots \alpha_k \overline{\alpha_1 \dots \alpha_k})^\infty$$

where k is chosen to be minimal. Suppose that a sequence $(c_i) \in \{0, \dots, \alpha_1\}^\infty$ is univoque in base q_2, i.e.,

$$c_{n+1}c_{n+2} \dots < (\alpha_1 \dots \alpha_k \overline{\alpha_1 \dots \alpha_k})^\infty \quad \text{whenever } c_n < \alpha_1 \tag{7.5}$$

and

$$\overline{c_{n+1}c_{n+2} \dots} < (\alpha_1 \dots \alpha_k \overline{\alpha_1 \dots \alpha_k})^\infty \quad \text{whenever } c_n > 0. \tag{7.6}$$

If $c_n < \alpha_1$, then by (7.5),

$$c_{n+1} \dots c_{n+k} \le \alpha_1 \dots \alpha_k.$$

If we had

$$c_{n+1} \dots c_{n+k} = \alpha_1 \dots \alpha_k,$$

then

$$c_{n+k+1}c_{n+k+2} \dots < (\overline{\alpha_1 \dots \alpha_k} \alpha_1 \dots \alpha_k)^\infty,$$

and by (7.6) (note that in this case $c_{n+k} = \alpha_k > 0$),

$$c_{n+k+1}c_{n+k+2} \dots > (\overline{\alpha_1 \dots \alpha_k} \alpha_1 \dots \alpha_k)^\infty,$$

a contradiction. Hence,

$$c_{n+1} \dots c_{n+k} \le \alpha_1 \dots \alpha_k^-.$$

Note that $c_{n+k} < \alpha_1$ in case of equality. It follows by induction that

$$c_{n+1}c_{n+2} \dots \le (\alpha_1 \dots \alpha_k^-)^\infty.$$

Since a sequence (c_i) satisfying (7.5) and (7.6) is infinite unless $(c_i) = 0^\infty$, we may conclude from Proposition 2.4 and Lemma 7.3, that (c_i) is the quasi-greedy expansion of some x in base q_1. Repeating the above argument for the sequence $\overline{c_1 c_2 \dots}$, which is also univoque in base q_2, we conclude that $(c_i) \in \mathcal{V}'_{q_1}$. The converse inclusion follows from the fact that the map $q \mapsto (\alpha_i)$ is strictly increasing. $\square$

Lemma 7.7. *Let (q_1, q_2) be a connected component of $(1,\infty)\setminus\mathcal{V}$ and suppose that $q_1 \in \mathcal{V}\setminus\mathcal{U}$. If $q \in (q_1, q_2]$, then*

(i) $\mathcal{U}'_q = \mathcal{V}'_{q_1}$;

(ii) $\mathcal{U}_q$ *contains isolated points if and only if* $q_1 \in \mathcal{V}\setminus\overline{\mathcal{U}}$. *Moreover, if* $q_1 \in \mathcal{V}\setminus\overline{\mathcal{U}}$, *then each sequence* $(a_i) \in \mathcal{V}'_{q_1}\setminus\mathcal{U}'_{q_1}$ *is the expansion in base q of an isolated point of* $\mathcal{U}_q$ *and each sequence* $(c_i) \in \mathcal{U}'_{q_1}$ *is the expansion in base q of an accumulation point of* $\mathcal{U}_q$.

Proof. (i) Note that

$$\mathcal{U}'_q \subset \mathcal{U}'_r \quad \text{and} \quad \mathcal{V}'_q \subset \mathcal{U}'_r \quad \text{if } q < r \text{ and } \lceil q\rceil = \lceil r\rceil. \tag{7.7}$$

It follows from Lemma 7.6 that $\mathcal{U}'_q = \mathcal{V}'_{q_1}$ for all $q \in (q_1, q_2]$.

(ii) We need the following observation which is a consequence of Lemma 3.1 and Lemma 3.2:

If $x \in J$ has an infinite greedy expansion in base q, then a sequence (x_i) with elements in J converges to x if and only if the greedy expansion of x_i converges (coordinate-wise) to the greedy expansion of x as $i \to \infty$. Moreover, $x_i \downarrow 0$ if and only if the greedy expansion of x_i converges (coordinate-wise) to the sequence 0^∞ as $i \to \infty$.

First assume that $q_1 \in \mathcal{V}\setminus\overline{\mathcal{U}}$. Let $x \in \mathcal{V}_{q_1}\setminus\mathcal{U}_{q_1}$ and denote the quasi-greedy expansion of x in base q_1 by (a_i). Since each element in $\mathcal{V}_{q_1}\setminus\mathcal{U}_{q_1}$ is an isolated point of $\mathcal{V}_{q_1}$ (cf. Theorem 1.5 (ii)), there exists a positive integer n such that the quasi-greedy expansion in base q_1 of an element in $\mathcal{V}_{q_1}\setminus\{x\}$ does *not* start with $a_1\ldots a_n$. Since $(a_i) \in \mathcal{V}'_{q_1} = \mathcal{U}'_q$, it follows from the above observation that the sequence (a_i) is the unique expansion in base q of an isolated point of $\mathcal{U}_q$. If $x \in \mathcal{U}_{q_1}$, then there exists a sequence of numbers (x_i) with $x_i \in \mathcal{V}_{q_1}\setminus\mathcal{U}_{q_1}$ such that the quasi-greedy expansions of the numbers x_i converge to the unique expansion of x, as can be seen from the proof of Theorem 1.5 (iib) (which in turn relies on the proof of Theorem 1.4 (ib)). Hence, the unique expansion of x in base q_1 is the unique expansion in base q of an accumulation point of $\mathcal{U}_q$.

Next assume that $q_1 \in \overline{\mathcal{U}}\setminus\mathcal{U}$. According to Theorem 1.4 (i), the set $\mathcal{U}_{q_1}$ has no isolated points. Hence, for each $x \in \mathcal{U}_{q_1}$, there exists a sequence of numbers (x_i) with $x_i \in \mathcal{U}_{q_1}\setminus\{x\}$ such that $x_i \to x$. In view of the above observation, the unique expansions of the numbers x_i converge to the unique expansion of x. Therefore, the unique expansion of x in base q_1 is the unique expansion in base q of an accumulation point of $\mathcal{U}_q$. If $x \in \mathcal{V}_{q_1}\setminus\mathcal{U}_{q_1} = \overline{\mathcal{U}_{q_1}}\setminus\mathcal{U}_{q_1}$, then there exists a sequence (x_i) of numbers in $\mathcal{U}_{q_1}$ such that the unique expansions of the numbers x_i converge to the quasi-greedy expansion (a_i) of x, as follows from the proof of Lemma 4.3 and Lemma 4.4. Hence, also in this case, (a_i) is the unique expansion in base q of an accumulation point of $\mathcal{U}_q$. Since $\mathcal{U}'_q = \mathcal{V}'_{q_1}$, this completes the proof. $\square$

Lemma 7.8. *Let (q_1, q_2) be a connected component of $(1,\infty)\setminus\mathcal{V}$ and suppose that $q_1 \in \mathbb{N}$. If $q \in (q_1, q_2]$, then $\mathcal{U}'_q = \mathcal{U}'_{q_2}$ and $\mathcal{U}_q$ contains isolated points if and only if $q_1 \in \{1, 2\}$.*

Proof. Note that if $q_1 = n \in \mathbb{N}$, then $q_2 \sim (n0)^\infty$. Suppose that $q \in (n, q_2]$. The verification of the following statements is an easy exercise which we leave to the reader.

A sequence $(a_i) \in \{0,\ldots,n\}^\infty$ is in $\mathcal{U}'_q$ if and only if

$$a_j < n \Longrightarrow a_{j+1} < n,$$

and

$$a_j > 0 \Longrightarrow a_{j+1} > 0.$$

In particular we see that

$$\mathcal{U}'_q = \mathcal{U}'_{q_2}.$$

If $n = 1$, then $\mathcal{U}'_q = \{0^\infty, 1^\infty\}$. If $n = 2$, then

$$\mathcal{U}'_q = \{0^\infty, 2^\infty\} \cup \bigcup_{n=0}^{\infty} \{0^n 1^\infty, 2^n 1^\infty\}.$$

Hence, if $n = 2$, then $\mathcal{U}_q$ is countable and all elements of $\mathcal{U}_q$ are isolated, except for its endpoints. If $n \geq 3$, then $\mathcal{U}_q$ has no isolated points. □

Lemma 7.9. *Let $q > 1$ be a real number.*

(i) *If $q \in \overline{\mathcal{U}}$, then q is neither stable from below nor stable from above.*
(ii) *If $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$, then q is stable from below, but not stable from above.*
(iii) *If $q \notin \mathcal{V}$, then q is stable from below and stable from above.*

Proof. (i) As mentioned at the beginning of this section, if $q \in \overline{\mathcal{U}}$, then there exists a sequence $(q_m)_{m\geq 1}$ with numbers $q_m \in \mathcal{V} \setminus \overline{\mathcal{U}}$, such that $q_m \uparrow q$. Since

$$\mathcal{U}'_{q_m} \subsetneq \mathcal{V}'_{q_m} \subset \mathcal{U}'_q,$$

q is not stable from below. If $q \in \overline{\mathcal{U}} \setminus \mathbb{N}$, then q is not stable from above because

$$\mathcal{U}'_q \subsetneq \mathcal{V}'_q \subset \mathcal{U}'_s$$

for any $s \in (q, \lceil q \rceil]$. If $q \in \{2, 3, \ldots\}$, then q is not stable from above because the sequence $q^\infty \in \mathcal{U}'_s \setminus \mathcal{U}'_q$ for any $s > q$.

(ii) and (iii) If $q \notin \overline{\mathcal{U}}$, then $q \in (q_1, q_2]$, where (q_1, q_2) is a connected component of $(1, \infty) \setminus \mathcal{V}$. From Lemma 7.7 and Lemma 7.8 we conclude that q is stable from below. Note that $q = q_2$ if and only if $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$. Hence, if $q \notin \mathcal{V}$, then q is also stable from above. If $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$, then q is not stable from above because

$$\mathcal{U}'_q \subsetneq \mathcal{V}'_q \subset \mathcal{U}'_s$$

for any $s \in (q, \lceil q \rceil]$. □

Proof of Theorem 1.7. Thanks to Proposition 7.1, we only need to prove parts (ii), (iii) and (iv).

(ii) If $q \in \{2, 3, \ldots\}$, then $\mathcal{U}_q \subsetneq \overline{\mathcal{U}_q} = [0, 1]$. Hence, neither $\mathcal{U}_q$ nor $\overline{\mathcal{U}_q}$ is a Cantor set.

(iii) and (iv) If $q \notin \mathbb{N}$, then $\mathcal{U}_q$ is nowhere dense, according to a remark following the statement of Theorem 1.6 in Section 1. Hence, if $q \notin \mathbb{N}$, then $\mathcal{U}_q$ is a Cantor set if and only if $\mathcal{U}_q$ is closed and does not contain isolated points.

If $q \in \overline{\mathcal{U}} \setminus \mathbb{N}$, then by Theorem 1.4 (i), the set $\mathcal{U}_q$ is not closed and $\overline{\mathcal{U}_q}$ does not contain isolated points from which part (iii) follows.

Finally, let $q \in (q_1, q_2]$, where (q_1, q_2) is a connected component of $(1, \infty) \setminus \mathcal{V}$. Since $q \notin \overline{\mathcal{U}}$, the set $\mathcal{U}_q$ is closed. It follows from Lemma 7.7 and Lemma 7.8 that $\mathcal{U}_q$ is a Cantor set if and only if $q_1 \in \{3, 4, \ldots\} \cup (\overline{\mathcal{U}} \setminus \mathcal{U})$. □

Proof of Theorem 1.8. The statements of Theorem 1.8 readily follow from Proposition 7.1 and the Lemmas 7.7, 7.8, and 7.9. □

Proof of Theorem 1.9. First assume that $q \in (1, \infty) \setminus \overline{\mathcal{U}}$. Then, $q \in (q_1, q_2]$, where (q_1, q_2) is a connected component of $(1, \infty) \setminus \mathcal{V}$. Let us write

$$q_2 \sim (\alpha_i) = (\alpha_1 \ldots \alpha_k \overline{\alpha_1 \ldots \alpha_k})^\infty,$$

where k is minimal. Define $\mathcal{F} \subset \{0, \ldots, \alpha_1\}^{k+1}$ by

$$\mathcal{F} = \{ja_1 \ldots a_k \; : \; j < \alpha_1 \quad \text{and} \quad a_1 \ldots a_k \geq \alpha_1 \ldots \alpha_k\}.$$

It follows from Lemma 7.7 (i) and the proof of Lemma 7.6 and Lemma 7.8 that a sequence $(c_i) \in \{0, \ldots, \alpha_1\}^{\mathbb{N}}$ belongs to $\mathcal{U}_q'$ if and only if $c_j \ldots c_{j+k} \notin \mathcal{F}$ and $\overline{c_j \ldots c_{j+k}} \notin \mathcal{F}$ for all $j \geq 1$. Therefore, $\mathcal{U}_q'$ is a subshift of finite type.

Next assume that $q \in \overline{\mathcal{U}}$. It follows from the proof of Lemma 4.3 and Lemma 4.4 that for each $x \in \overline{\mathcal{U}_q} \setminus \mathcal{U}_q$, there exists a sequence (x_i) of numbers in $\mathcal{U}_q$ such that the unique expansions of the numbers x_i converge to the quasi-greedy expansion of x. Hence, the set $\mathcal{U}_q'$ is not a subshift of finite type because it is not closed (in the topology of coordinate-wise convergence). □

Proof of Theorem 1.10. (i) Note that the quasi-greedy expansion of 1 in base $q_2 = (1+\sqrt{5})/2$ is given by $(\alpha_i) = (10)^\infty$. It follows from the proof of Lemma 7.8 that $\mathcal{U}_q' = \{0^\infty, 1^\infty\}$ for all $q \in (1, q_2]$.

(ii) Due to the properties of the set $\mathcal{V} \setminus \overline{\mathcal{U}}$ that we mentioned at the beginning of this section we may write

$$\mathcal{V} \cap (1, q') = \{q_n : n \in \mathbb{N}\} \quad \text{and} \quad \mathcal{V} \cap (2, q'') = \{r_n : n \in \mathbb{N}\},$$

where the q_n's and the r_n's are written in increasing order. Note that $q_1 \sim (10)^\infty$ and $r_1 \sim (20)^\infty$. Moreover, $q_n \uparrow q'$ and $r_n \uparrow q''$. Thanks to (7.7) we only need to verify that $\mathcal{U}_{q_n}$ and $\mathcal{U}_{r_n}$ are countable for all $n \in \mathbb{N}$. This will be carried out by induction.

It follows from the proof of Lemma 7.8 that $\mathcal{U}_{q_1}$ and $\mathcal{U}_{r_1}$ are countable. Suppose now that $\mathcal{U}_{q_n}$ is countable for some $n \geq 1$. By Lemma 7.6,

$$\mathcal{U}_{q_{n+1}}' = \mathcal{V}_{q_n}' = \mathcal{U}_{q_n}' \cup (\mathcal{V}_{q_n}' \setminus \mathcal{U}_{q_n}').$$

According to Theorem 1.5 (ii), the set $\mathcal{V}_{q_n}' \setminus \mathcal{U}_{q_n}'$ is countable, whence $\mathcal{U}_{q_{n+1}}$ is countable as well. It follows by induction that $\mathcal{U}_{q_n}$ is countable for each $n \in \mathbb{N}$. Similarly, $\mathcal{U}_{r_n}$ is countable for each $n \in \mathbb{N}$.

(iii) It follows from Theorem 1.4 (i) that $|\mathcal{U}_{q'}| = 2^{\aleph_0}$ and $|\mathcal{U}_{q''}| = 2^{\aleph_0}$. The relation (7.7) yields that $|\mathcal{U}_q| = 2^{\aleph_0}$ for all $q \in [q', 2] \cup [q'', 3]$. If $q > 3$, then $\mathcal{U}_q'$ contains all sequences consisting of merely ones and twos. Hence, $|\mathcal{U}_q| = 2^{\aleph_0}$. □

We conclude this paper with an example and some remarks.

Example. For $k \in \mathbb{N}$, define the numbers $q^*(k)$ and $q(k)$ by setting

$$q^*(k) \sim (1^{k-1}0)^\infty \quad \text{and} \quad q(k) \sim (1^k 0^k)^\infty.$$

It follows from Lemma 7.5 and Theorem 1.8 that the sets $(q^*(k), q(k)]$ are maximal stability intervals. Moreover, it follows from the proof of Theorem 1.9 that a sequence $(c_i) \in \{0,1\}^{\mathbb{N}}$ belongs to $\mathcal{U}_q'$ for $q \in (q^*(k), q(k)]$ if and only if a zero is never followed by k consecutive ones and a one is never followed by k consecutive zeros. This result was first established by Daróczy and Kátai in [DK1], using a different approach.

Note that the smallest element of $\mathcal{V}$ larger than $q(k)$ is given by $r(k)$, where

$$r(k) \sim (1^k 0^{k-1} 1 0^k 1^{k-1} 0)^\infty,$$

as follows from Lemma 7.5. Therefore, the sets $(q(k), r(k)]$ are also maximal stability intervals. If $q \in (q(k), r(k)]$, then the set $\mathcal{U}_q$ is not a Cantor set because $q(k) \in \mathcal{V} \setminus \overline{\mathcal{U}}$.

Remarks.

- Let us now consider the set of left endpoints L' and the set of right endpoints R' of the connected components of $(1, \infty) \setminus \overline{\mathcal{U}}$. We will show that

$$L' = \mathbb{N} \cup (\overline{\mathcal{U}} \setminus \mathcal{U}) \quad \text{and} \quad R' \subset \mathcal{U}.$$

Fix a number $q \in (1,\infty) \setminus \overline{\mathcal{U}}$. Let q_1 be the smallest element of $\mathcal{V}$ larger or equal than q. Since $q_1 \in \mathcal{V} \setminus \overline{\mathcal{U}}$, the number q_1 is a left endpoint and a right endpoint of a connected component of $(1,\infty) \setminus \mathcal{V}$. Hence, there exists a sequence $q_1 < q_2 < \cdots$ of numbers in $\mathcal{V} \setminus \overline{\mathcal{U}}$, such that

$$(q_i, q_{i+1}) \cap \mathcal{V} = \varnothing \quad \text{for all } i \geq 1.$$

Let β_m be the last nonzero element of the greedy expansion (β_i) of the number 1 in base q_1. We define a sequence (c_i) by induction as follows. First, set

$$c_1 \dots c_m = \beta_1 \dots \beta_m.$$

Then, if $c_1 \dots c_{2^N m}$ is already defined for some nonnegative integer N, set

$$c_{2^N m+1} \dots c_{2^{N+1}m-1} = \overline{c_1 \dots c_{2^N m-1}} \quad \text{and} \quad c_{2^{N+1}m} = \overline{c_{2^N m}} + 1.$$

Note that this construction generalizes that of the truncated Thue–Morse sequence. It follows from Lemma 7.5 that the greedy expansion of 1 in base q_n is given by $c_1 \dots c_{2^{n-1}m} 0^\infty$. Hence, (c_i) is an expansion of 1 in base q^*, where

$$q^* = \lim_{n\to\infty} q_n.$$

Moreover, $q^* \in \mathcal{U}$, as can be seen from the proof of Lemma 4.2 in [KL3]. It follows that $R' \subset \mathcal{U}$.

Now let r_1 be the largest element of $\mathcal{V} \cup \{1\}$ that is smaller than q_1. Let us also write $r_1 \sim (\alpha_i)$ and $q_1 \sim (\eta_i)$. It follows from Lemma 7.3 and the remark preceding Lemma 7.2 that (α_i) has a smaller period than (η_i). Hence, there exists a finite set of numbers $r_k < \cdots < r_1$ in $\mathcal{V} \cup \{1\}$, such that for $1 \leq i < k$,

$$(r_{i+1}, r_i) \cap \mathcal{V} = \varnothing,$$

and such that r_k is a left endpoint of a connected component of $(1,\infty) \setminus \mathcal{V}$, but not a right endpoint. This means that

$$r_k \in \mathbb{N} \cup (\overline{\mathcal{U}} \setminus \mathcal{U}) \quad \text{and} \quad (r_k, q) \cap \overline{\mathcal{U}} = \varnothing.$$

Hence, $r_k \in \overline{\mathcal{U}} \cup \{1\}$ and therefore $r_k \in L'$. We may thus conclude that $L' \subset \mathbb{N} \cup (\overline{\mathcal{U}} \setminus \mathcal{U})$. On the other hand, $L \cap \overline{\mathcal{U}} \subset L'$ because $\overline{\mathcal{U}} \subset \mathcal{V}$. It follows that $L' = \mathbb{N} \cup (\overline{\mathcal{U}} \setminus \mathcal{U})$.

- The analysis of the preceding remark enables us also to determine for each $n \in \mathbb{N}$ the smallest element $q^{(n)}$ of the set $\mathcal{U} \cap (n, n+1)$:

  Fix $n \in \mathbb{N}$, and let q be the smallest element of $\mathcal{V} \cap (n, n+1)$. Then, $q \sim (n0)^\infty$ and the greedy expansion (β_i) of 1 in base q is given by $n10^\infty$. The sequence (c_i) constructed in the preceding remark with $m = 2$ and $c_1 c_2 = n1$ is the unique expansion of the number 1 in base $q^{(n)}$.
- In [KL2] it was shown that for each $n \in \mathbb{N}$, there exists a smallest number $r^{(n)} > 1$ such that the number 1 has only one expansion in base $r^{(n)}$ with coefficients in $\{0, 1, \dots, n\}$. Although this might appear as an equivalent definition of the numbers $q^{(n)}$, there is a subtle difference. For instance, it can be seen from the results in [KL2] that $q^{(n)} = r^{(n)}$ if and only if $n \in \{1, 2\}$.

Acknowledgements. The first author would like to thank the Erwin Schrödinger Institute for Mathematical Physics at Vienna for its hospitality and financial support during the spring of 2006.

References

[BK] Baiocchi, C., Komornik, V. – *Quasi-greedy expansions and lazy expansions in non-integer bases*, manuscript.

[BH1] Borwein, P., Hare, K. G. – *Some computations on the spectra of Pisot and Salem numbers*, Math. Comp. **71** (2002), no. 238, 767–780.

[BH2] Borwein, P., Hare, K. G. – *General forms for minimal spectral values for a class of quadratic Pisot numbers*, Bull. London Math. Soc. **35** (2003), no. 1, 47–54.

[DDV] Dajani, K., de Vries, M. – *Invariant densities for random β-expansions*, J. Eur. Math. Soc. (JEMS), to appear, `http://www.cs.vu.nl/~mdvries`

[DK1] Daróczy, Z., Kátai, I. – *Univoque sequences*, Publ. Math. Debrecen **42** (1993), no. 3–4, 397–407.

[DK2] Daróczy, Z., Kátai, I. – *On the structure of univoque numbers*, Publ. Math. Debrecen **46** (1995), no. 3–4, 385–408.

[EHJ] Erdős, P., Horváth, M., Joó, I. – *On the uniqueness of the expansions* $1 = \sum q^{-n_i}$, Acta Math. Hungar. **58** (1991), no. 3–4, 333–342.

[EJK1] Erdős, P., Joó, I., Komornik, V. – *Characterization of the unique expansions* $1 = \sum_{i=1}^{\infty} q^{-n_i}$ *and related problems*, Bull. Soc. Math. France **118** (1990), no. 3, 377–390.

[EJK2] Erdős, P., Joó, I., Komornik, V. – *On the number of q-expansions*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **37** (1994), 109–118.

[EJK3] Erdős, P., Joó, I., Komornik, V. – *On the sequence of numbers of the form* $\varepsilon_0 + \varepsilon_1 q + \cdots + \varepsilon_n q^n$*,* $\varepsilon_i \in \{0, 1\}$, Acta Arith. **83** (1998), no. 3, 201–210.

[EK] Erdős, P., Komornik, V. – *On developments in noninteger bases*, Acta Math. Hungar. **79** (1998), no. 1–2, 57–83.

[FS] Frougny, Ch., Solomyak, B. – *Finite beta-expansions*, Ergodic Theory Dynam. Systems **12** (1992), no. 4, 713–723.

[GS] Glendinning, P., Sidorov, N. – *Unique representations of real numbers in non-integer bases*, Math. Res. Lett. **8** (2001), no. 4, 535–543.

[K1] Kallós, G. – *The structure of the univoque set in the small case*, Publ. Math. Debrecen **54** (1999), no. 1–2, 153–164.

[K2] Kallós, G. – *The structure of the univoque set in the big case*, Publ. Math. Debrecen **59** (2001), no. 3–4, 471–489.

[KK] Kátai, I., Kallós, G. – *On the set for which* 1 *is univoque*, Publ. Math. Debrecen **58** (2001), no. 4, 743–750.

[K] Komatsu, T. – *An approximation property of quadratic irrationals*, Bull. Soc. Math. France **130** (2002), no. 1, 35–48.

[KL1] Komornik, V., Loreti, P. – *Unique developments in non-integer bases*, Amer. Math. Monthly **105** (1998), no. 7, 636–639.

[KL2] Komornik, V., Loreti, P. – *Subexpansions, superexpansions and uniqueness properties in non-integer bases*, Period. Math. Hungar. **44** (2002), no. 2, 195–216.

[KL3] Komornik, V., Loreti, P. – *On the topological structure of univoque sets*, J. Number Theory, to appear.

[KLP] Komornik, V., Loreti, P., Pedicini, M. – *An approximation property of Pisot numbers*, J. Number Theory **80** (2000), no. 2, 218–237.

[LM] Lind, D., Marcus, B. – *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995.

[P] Parry, W. – *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416.

[PT] Pethő, A., Tichy, R. – *On digit expansions with respect to linear recurrences*, J. Number Theory **33** (1989), no. 2, 243–256.

[R] Rényi, A. – *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. **8** (1957), 477-493.

[Sc] Schmidt, K. – *On periodic expansions of Pisot numbers and Salem numbers*, Bull. London Math. Soc. **12** (1980), no. 4, 269–278.

[Si1] Sidorov, N. – *Almost every number has a continuum of β-expansions*, Amer. Math. Monthly **110** (2003), no. 9, 838–842.

[Si2] Sidorov, N. – *Universal β-expansions*, Period. Math. Hungar. **47** (2003), no. 1–2, 221–231.

Vrije Universiteit Amsterdam, Faculty of Exact Sciences, de Boelelaan 1081, 1081 HV Amsterdam, The Netherlands
E-mail address: `mdvries@cs.vu.nl`

Département de Mathématique, Université Louis Pasteur, 7 rue René Descartes, 67084 Strasbourg Cedex, France
E-mail address: `komornik@math.u-strasbg.fr`