SUBCONTINUA OF FIBONACCI-LIKE INVERSE LIMIT SPACES.

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ABSTRACT. We study the subcontinua of inverse limit spaces of Fibonacci-like unimodal maps. Under certain combinatorial constraints no other subcontinua than points, arcs and $\sin \frac{1}{x}$-curves are shown to exist. From the way these $\sin \frac{1}{x}$-curves accumulate onto each other, a method of partially distinguishing the Fibonacci-like inverse limit spaces is proposed.

1. INTRODUCTION

The classification of inverse limit spaces of unimodal map is a tenacious problem. The main conjecture, posed by Ingram, is

> If f and $\tilde{f}$ are two non-conjugate unimodal maps, then the corresponding inverse limit spaces (I, f) and $(I, \tilde{f})$ are non-homeomorphic.

Let us restrict the unimodal map $f : I \to I$ to its core $I = [c_2, c_1]$, where c is the turning point and $c_k = f^k(c)$. We assume that f is locally eventually onto. Any such map is conjugate to a tent map T_s with slope $\pm s$ (where $\log s = h_{top}(f)$), and the resulting inverse limit space (I, f) is an indecomposable continuum.

The classification (and an affirmative answer to Ingram conjecture) have been obtained for maps with a finite critical orbit [9, 10, 12, 3], but for the case that $\text{orb}(c)$ is infinite, and especially when c is recurrent, few results are known, see [1, 4, 2, 11].

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In this paper, we extend the study of one example in [4], *the Fibonacci map*, and use subcontinua as a tool for partial classification of Fibonacci-like unimodal inverse limit spaces.

Fibonacci-like unimodal maps are defined combinatorially by a condition on their kneading maps Q, or equivalently their *cutting times* $\{S_k\}_{k\geq 0}$. Cutting times are those iterates n of the map such that the central branch of f^n maps onto c; they satisfy a recursive relation $S_0 = 1$ and $S_k = S_{k-1} + S_{Q(k)}$, where $Q : \mathbb{N} \to \mathbb{N} \cup \{0\}$ is called the *kneading map*. In the sequel it will be more convenient to use

$$R(k) := Q(k+1), \quad \text{so} \quad S_{k+1} = S_k + S_{R(k)}.$$

If $Q(k) = \max\{k-2, 0\}$, then the Fibonacci map is obtained (and the S_ks are the Fibonacci numbers, hence the name). We will consider unimodal maps are like Fibonacci map in the sense that $Q(k) \to \infty$. For such maps, it is known that c is recurrent and the critical omega-limit set $\omega(c)$ is a minimal Cantor set [6]. As a result, (I, f) will have uncountably many end-points which densely fill a Cantor set, but away from this Cantor set, (I, f) is locally homeomorphic to a Cantor set cross and arc, see e.g. [7]. We will make further restrictions, reducing the complexity of the subcontinua in (I, f) even more, but there remains a rich structure and variety in the (arrangement of) subcontinua, so that some coarse classifications is still possible.

It is well-known that if $orb(c)$ is finite, then the only proper subcontinua of (I, f) are points and arcs. The same is true if f is *long-branched*, see [4]. In this paper we will mainly encounter *arc + ray continuum* which consist of a ray (or half-ray) and two (or one) arcs. The simplest such continuum is the $\sin \frac{1}{x}$-curve, but more complicated arc + ray subcontinua can be found in many unimodal inverse limit spaces, see [4]. The arc(s) of an arc + ray continua are also called the *bar(s)* of the continuum $\mathcal{H}$. We will denote this bar by $\text{bar}(\mathcal{H})$.

Theorem 1.1. *If f is a unimodal map such that*

$$Q(k) \to \infty \text{ and } R(1+k) > 1 + R^2(k) \tag{1}$$

for all k sufficiently large, then the only proper subcontinua of (I, f) are points, arcs and $\sin \frac{1}{x}$*-curves.*

Let $\hat{f}$ denote the induced homeomorphism on (I, f).

Theorem 1.2. *If* (1) *is satisfied, then there is a one-to-one correspondence between* $\hat{f}$*-orbits of* $\sin\frac{1}{x}$*-curves in* (I, f) *and infinite backward* R*-orbits in* $\mathbb{N}$.

The following corollaries are immediate.

Corollary 1.1. *If* R *is eventually surjective, then there are only countably many* $\sin\frac{1}{x}$*-curves in* (I, f)*. In particular, if* $Q(k) = \max\{0, k-d\}$*, then there are* $d-1$ *distinct* $\hat{f}$*-orbits of* $\sin\frac{1}{x}$*-curves.*

However, we do not know if (I, f) are distinct for different values of d.

Corollary 1.2. *If* $\#R^{-1}(k) \geq 2$ *for all sufficiently large* k*, then are uncountably many* $\hat{f}$*-orbits of* $\sin\frac{1}{x}$*-curves in* (I, f)*.*

Although the subcontinua presented here are all $\sin\frac{1}{x}$-curves, they can be used, to some extent, to distinguish inverse limit spaces by the way they (or rather their bars) accumulate onto each other.

Proposition 1.3. *For each* $k \in \mathbb{N}$ *that appears in infinitely many backward* R*-orbits there is a* $\sin\frac{1}{x}$*-curve whose bar is the limit of the bars of other* $\sin\frac{1}{x}$*-curves.*

The orbit structure of R can be expressed by a tree with vertices labelled by $\mathbb{N}$ and arrows $k \to l$ if $l = R(k)$, see e.g. Figure 1. The tree is rooted at 0. Although $R(0) = 0$, we do not write the arrow $0 \to 0$. A backward R-orbit is an infinite backward path in this tree.

Based on an idea of Raines [11], and using Proposition 1.3, we can define the equivalent of Cantor-Bendixson depth for $\sin\frac{1}{x}$-curves. We say that the *depth* of a $\sin\frac{1}{x}$-curve is 0 if its bar B is not the accumulation of a sequence of bars of other $\sin\frac{1}{x}$-curves (even though a sequence of bars may accumulate to a proper subset of B). We continue inductively, saying that a $\sin\frac{1}{x}$-curve has *depth* d if, after removing all $\sin\frac{1}{x}$-curves of lower depth have been removed, its bar is not the accumulation of a sequence of bars of other $\sin\frac{1}{x}$-curves. The $\sin\frac{1}{x}$-curves that are never removed in this process are said to have *depth* ∞.

Example 1: Figure 1 gives an example of a backward tree of the map R, and indicates the depths of some of its $\sin\frac{1}{x}$-curves. The vertices in the lower half of the picture (including the leftmost

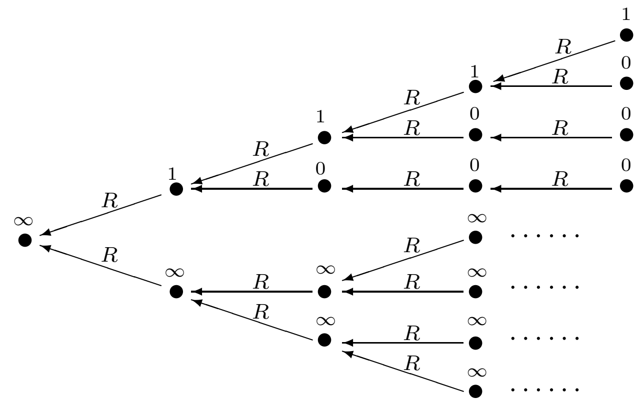

FIGURE 1. Example of a backward tree of R with the depths of each vertex indicated.

vertex) all have two preimages, and therefore infinitely many backward R-orbits. Hence all the subcontinua corresponding to such a backward R-orbit have depth ∞. The vertices in the upper half, excluding the uppermost path, have only one backward R-orbit; the corresponding subcontinua have depth 0. After removing these subcontinua (and hence the corresponding backward R-orbits), the vertices in the remaining uppermost have only one backward R-orbit left, so their depths are 1.

Example 2: Let $k_1 = 1$, $k_2 = 3$ and $k_i = k_{i-1} + i = \frac{i(i+1)}{2}$ be the i-th triangular number. Define two unimodal maps f and $\tilde{f}$ by means of their functions R and $\tilde{R}$ respectively:

$$\begin{cases} R(1) = R(2) = 0; \\ R(k_i) = k_{i-1} \text{ for } i \geq 2; \\ R(k_i + l) = k_{i-1} + l \text{ for } 1 \leq l < i; \\ R(k_i + i) = k_{i-1} + i - 1, \end{cases}$$

and

$$\begin{cases} \tilde{R}(1) = \tilde{R}(2) = 0; \\ \tilde{R}(k_i) = k_{i-1} \text{ for } i \geq 2; \\ \tilde{R}(k_i + l) = k_{i-1} \text{ for } 1 \leq l < i; \\ \tilde{R}(k_i + i) = k_{i-1} + i - 1. \end{cases}$$

Then (I, f) and $(I, \tilde{f})$ are non-homeomorphic. Indeed (I, f) has $\sin \frac{1}{x}$-curves of Cantor-Bendixson depth 0 and 1, whereas $(I, \tilde{f})$ only

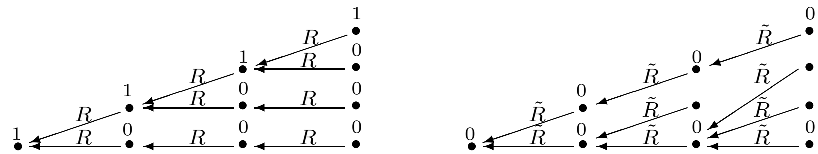

FIGURE 2. The backward trees of R and $\tilde{R}$ with the depths of each vertex indicated.

has $\sin \frac{1}{x}$-curves of Cantor-Bendixson depth 0. This can be seen from the trees of the backward orbits of R and $\tilde{R}$ respectively, see Figure 2.

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2. Preliminaries

The tent map $T_s : [0,1] = [0,1]$ with slope $\pm s$ is defined as

$$T_s(x) = \begin{cases} sx & \text{if } x \leq \frac{1}{2}, \\ s(1-x) & \text{if } x > \frac{1}{2}. \end{cases}$$

We fix $s \in (\sqrt{2}, 2]$ and call $f = T_s$. Write $c = \frac{1}{2}$ and $c_k = f^k(c)$. Let $I = [c_2, c_1]$ be the *core* of the map. It is well-known that f is *locally eventually onto*, i.e., for every non-degenerate interval J, there is n such that $f^n(J) = I$.

The inverse limit space (I, f) is

$$(I, f) = \{x = (x_0, x_1, x_2, \dots) : I \ni x_i = f(x_{i+1}) \text{ for all } i \geq 0\},$$

equipped with metric $d(x, y) = \sum_{n \geq 1} |x_n - y_n| 2^{-n}$ and induced (or shift) homeomorphism

$$\hat{f}(x_0, x_1, x_2, \dots) = (f(x_0), x_0, x_1, x_2, \dots).$$

Let $\pi_k : (I, f) \to I$, $\pi_k(x) = x_k$ be the k-th projection map.

To describe the combinatorial structure of f, we recall the definition of cutting times and kneading map from [8, 6]. If J is a maximal (closed) interval on which f^n is monotone, then $f^n : J \to f^n(J)$ is called a *branch*. If $c \in \partial J$, $f^n : J \to f^n(J)$ is a *central branch*. Obviously f^n has two central branches, and they have the same image if n is sufficiently large. Denote this image (or the largest of the two) by D_n. If $D_n \ni c$, then n is called a *cutting time*.

Denote the cutting times by $\{S_i\}_{i\geq 0}$, $S_0 < S_1 < S_2 < \dots$ If the slope $s > 1$, then $S_0 = 1$ and $S_1 = 2$. It can be shown that there is a map $Q : \mathbb{N} \to \mathbb{N} \cup \{0\}$, called *kneading map* such that $S_k - S_{k-1} = S_{Q(k)}$ for all $k \geq 1$. This map satisfies $Q(k) < k$ plus other conditions that will not be of our concern in this paper. Recall that $R(k) = Q(k+1)$.

In this paper we are interested in maps with kneading map $Q(k) \to \infty$. In this case, the critical omega-limit set $\omega(c)$ is a minimal Cantor set [6]. We call z a *closest precritical point* if $f^n(z) = c$ for some $n \geq 1$ and $f^m((c,z)) \not\ni c$ for $m < n$. It is not hard to show (see [6]), that the integers n for which this happens are exactly the cutting times, so we write z_k when $n = S_k$. Moreover, $z_{R(k)}$ is the closest precritical point with smallest index in $[c, c_{S_k}]$. (There is a closest precritical point of the same index at either side of c; we will denote the one in $[c, c_{S_k}]$ by $z_{R(k)}$ and the other by $\hat{z}_{R(k)}$.) Since $|c - c_{S_k}| \leq |c - z_{R(k)-1}| \to 0$ as $R(k) \to \infty$, the condition $\lim_{k\to\infty} R(k) = \infty$ obviourly implies that $|c - c_{S_k}| \to 0$ as $k \to \infty$.

We will use the following terminology: A *ray* is a continuous copy of $(0,1)$; a *half-ray* is a continuous copy of $[0,1)$. A *continuum* is a compact connected metric space and a *subcontinuum* $\mathcal{H}$ is subset of a continuum which is closed and connected itself. The $\sin \frac{1}{x}$-curve is the homeomorphic image of the graph $\{(t, sin\frac{1}{t}) : t \in (0,1]\}$ together with the arc (called *bar*) $\{0\}\times[-1,1]$. Given a subcontinuum $\mathcal{H}$ of (I,f), its *critical projections times* are the integers n such that $\mathcal{H}_n \ni c$.

Lemma 2.1. *If $\mathcal{H} \subset (I,f)$ is a subcontinuum that is not an arc or point, there there is an infinite sequence of critical projections times $\{n_i\}_{i\geq 0}$ such that*

(1) *If $\mathcal{H}$ is a proper subcontinuum, then $|\mathcal{H}_n| \to 0$ as $n \to \infty$.*
(2) *$n_i - n_{i-1} = S_{k_i}$ for some k_i and $k_i \to \infty$ as $i \to \infty$.*
(3) *$\pi_{n_{i-1}}(\mathcal{H}) \supset [c, c_{S_{k_i}}]$.*
(4) *$k_{i-1} \leq R(k_i) \leq k_i$ for all i, so $\{k_i\}_{i\geq 1}$ is non-decreasing.*

Proof. If there are only finitely many critical projections times, say n is the largest, then $\mathcal{H}$ can be parametrised by $t \in \mathcal{H}_n$, so that $\mathcal{H}$ is a point or an arc. Therefore subcontinua other than arcs or points have infinitely many critical projections times. Let us prove the other statements:
1. The first statement follows because f is locally eventually onto.

If $\liminf_n |\mathcal{H}_n| := \varepsilon > 0$, then there is an interval J that belongs to infinitely many $\mathcal{H}_n$. Take N such that $f^N(J) = I$. Then $I \in \mathcal{H}_{n-N} = f^N(\mathcal{H})$ infinitely often, so $\mathcal{H} = (I, f)$.
2. Let k_i be the lowest index of closest precritical points in $\mathcal{H}_{n_i}$. Then the next iterate such that $\mathcal{H}_n \ni c$ is $n_{i-1} = n_i - S_{k_i}$ iterates.
3. Follows immediately from 2.
4. As $\mathcal{H}_{n_{i-1}} \supset [c, c_{S_{k_i}}] \ni z_{R(k_i)}$, $S_{k_{i-1}} = n_i - n_{i-1} \leq S_{R(k_i))}$, and 4. follows. □

For each i there are closed intervals M_{n_i} and L_{n_i} such that

$$\mathcal{H}_{n_i} = M_{n_i} \cup L_{n_i},\ M_{n_i} \cap L_{n_i} = \{c\} \text{ and } f^{n_i - n_{i-1}}(M_{n_i}) = \mathcal{H}_{n_{i-1}}$$

Lemma 2.2. *Every subcontinuum $\mathcal{H}$ contains a dense ray*

$$\mathrm{ray}(\mathcal{H}) := \{x \in \mathcal{H} : x_{n_i} \in M_{n_i} \textit{ for all } i \textit{ sufficiently large}\}.$$

Proof. This proof is given in [4], but we include it for completeness. We will define a parametrisation $\varphi : \mathbb{R} \to \mathcal{H}$ of a subset of $\mathcal{H}$ and show that $\varphi(\mathbb{R})$ lies dense in $\mathcal{H}$. For the construction of φ, it suffices to construct

$$\varphi_i : \mathbb{R} \to \mathcal{H}_{n_i} \text{ such that } \varphi_j = f^{n_i - n_j} \circ \varphi_i$$

for all $1 \leq j \leq i$. We do this inductively. First let $\varphi_0 : [-1, 1] \to \mathcal{H}_{n_0}$ be any homeomorphism, and set $a_0 = -1$ and $b_0 = 1$.

If $\varphi_{i-1} : [a_{i-1}, b_{i-1}] \to \mathcal{H}_{n_{i-1}}$ is defined, let $\varphi_i[a_{i-1}, b_{i-1}] \to M_{n_i}$ be such that $\varphi_{i-1} = f^{n_i - n_{i-1}} \circ \varphi_i$. Then either $\varphi_i(a_{i-1}) = c$ or $\varphi_i(b_{i-1}) = c$.

- If $\varphi_i(a_{i-1}) = c$, then set $a_i = a_{i-1} - 1$ and $b_i = b_{i-1}$. Extend φ to $[a_i, a_{i-1}]$ such that it maps homeomorphically on L_{n_i}. Then going inductively downwards from $j = i - 1$ to 1, define $\varphi_j : [a_i, a_{i-1}] \to \mathcal{H}_{n_j}$ such that $\varphi_j = f^{n_{j+1} - n_j} \circ \varphi_j$.
- If $\varphi_i(b_{i-1}) = c$, then set $b_i = b_{i-1} + 1$ and $a_i = a_{i-1}$. Extend φ to $[b_{i-1}, b_i]$ such that it maps homeomorphically on L_{n_i}. Then going inductively downwards from $j = i - 1$ to 1, define $\varphi_j : [b_{i-1}, b_i] \to \mathcal{H}_{n_j}$ such that $\varphi_j = f^{n_{j+1} - n_j} \circ \varphi_j$.

If $a_i \to -\infty$ and $b_i \to \infty$, then this defines the ray. If $\inf a_i > -\infty$ and/or $\sup b_i < \infty$, then φ parametrises a half-ray or arc. In this case, we can restrict φ to $(\inf a_i, \sup b_i)$.

To show that $\varphi(\mathbb{R})$ is dense in $\mathcal{H}$, take $\varepsilon > 0$ and i so large that $2^{-n_i} < \varepsilon$. Now, for any $x \in \mathcal{H}$, take $t \in \mathbb{R}$ such that $\varphi_i(t) = x_{n_i}$.

Then $\varphi(t)_n = x_n$ for $n \le n_i$ and $d(x, \varphi(t)) \le \sum_{m > n_i} 2^{-m} = 2^{-n_i} < \varepsilon$. $\square$

3. Proof of the Main Results

Proof of Theorem 1.1. Let $\mathcal{H}$ be a proper subcontinuum of (I, f) which is more complicated than an arc, so it has infinitely many critical projections times $\{n_i\}_{i \ge 1}$ and $S_{k_i} = n_i - n_{i-1}$. By applying $\hat{f}^{-1}$ repeatedly (which has the effect of subtracting a large fixed number from the critical projection times n_i), we can assume that (1) holds for all $k \ge k_1$.

Let i be arbitrary; we know that $(c, c_{S_{k_i}}]$ is one component of $\mathcal{H}_{n_{i-1}} \setminus \{c\}$. Suppose n_{i-1} is such that $M_{n_{i-1}} = [c, c_{S_{k_i}}]$. Then $n_{i-1} - n_{i-2} = S_{k_{i-1}} = S_{R(k_i)}$, and

$$(2)\quad \mathcal{H}_{n_{i-2}} = [c_{S_{k_{i-1}}}, c_{S_{1+k_i}}] = [c_{S_{R(k_i)}}, c_{S_{1+k_i}}] \supset \{z_{R^2(k_i)}, z_{R(1+k_i)}\}.$$

Therefore $k_{i-2} = \min\{R^2(k_i), R(1+k_i)\} = R^2(k_i)$ by (1). It follows that $M_{n_{i-2}} = [c, c_{S_{R(k_i)}}]$ and $L_{n_{i-2}} = [c, c_{S_{1+k_i}}]$. We obtain by induction that $M_{n_j} = [c, c_{S_{k_{j+1}}}]$ and $k_j = R(k_{j+1})$ for all $j < i$.

This leaves us with two cases:

Case A: There are i arbitrarily large such that $M_{n_{i-1}} = [c, c_{S_{k_i}}]$, and therefore $M_{n_{i-1}} = [c, c_{S_{k_i}}]$ for all i, or

Case B: $L_{n_{i-1}} = [c, c_{S_{k_i}}]$ for all i sufficiently large.

We tackle Case B first. Let us call $\mathcal{T}_{n_{i-1}} = [z_{R(k_i)}, c_{S_{k_i}}]$ the *tip* of $\mathcal{H}_{n_{i-1}}$. We claim that $f^{n_{i-1}}$ is monotone on $\mathcal{T}_{n_{i-1}}$.

If $k_{i-1} = R(k_i)$, then

$$z_{R(1+k_i)} \in [c, c_{S_{1+k_i}}] = f^{S_{k_{i-1}}}(\mathcal{T}_{n_{i-1}}) \subset \mathcal{H}_{n_{i-1}}.$$

is the closest precritical point with lowest index in $f^{S_{k_{i-1}}}(\mathcal{T}_{n_{i-1}})$. So to prove the claim, we need to show that $S_{R(1+k_i)} \ge n_{i-2}$. Using (1) and Lemma 2.1 (part 4) respectively, we find for any j,

$$\begin{aligned} S_{2+k_j} &= S_{1+k_j} + S_{R(1+k_j)} \\ &\ge S_{1+k_j} + S_{2+R^2(k_j)} \ge S_{1+k_j} + S_{2+k_{j-2}}. \end{aligned}$$

Using this repeatedly, we obtain

$$\begin{array}{rcl} S_{2+k_{i-2}} & \geq & S_{1+k_{i-2}} + S_{2+k_{i-4}} \\ & \geq & S_{1+k_{i-2}} + S_{1+k_{i-4}} + S_{2+k_{i-6}} \\ & & \quad\vdots \qquad\quad \vdots \\ & \geq & S_{1+k_{i-2}} + S_{1+k_{i-4}} + S_{1+k_{i-6}} + S_{1+k_{i-8}} + \dots \end{array}$$

Therefore, using Lemma 2.1 and (1) once more,

$$\begin{array}{rcl} n_{i-2} & = & S_{k_{i-2}} + S_{k_{i-3}} + S_{k_{i-4}} + S_{k_{i-5}} + S_{k_{i-6}} + S_{k_{i-7}} \dots \\ & \leq & S_{k_{i-2}} + S_{R(k_{i-2})} + S_{k_{i-4}} + S_{R(k_{i-4})} + S_{k_{i-6}} + S_{R(k_{i-6})} \dots \\ & \leq & S_{1+k_{i-2}} + S_{1+k_{i-4}} + S_{1+k_{i-6}} \dots \\ & \leq & S_{2+k_{i-2}} \leq S_{1+R^2(k_i)} \leq S_{R(1+k_i)}, \end{array}$$

as claimed. If $k_{i-1} < R(k_i)$, then the same computation gives strict inequality $n_{i-2} < S_{R(1+k_i)}$.

We can picture the behaviour of $f^{n_i - n_1} : L_{n_i} \to I$ as a strip of paper that is folded over and over again, see Figure 3 with the property that whenever a piece of strip is folded, only one subpiece is folded again.

It follows that the number of branches of $f^{n_i-n_1}|L_{n_i}$ grows linear in i. Let us distinguish between "long branches" (i.e., the ones whose image contains c, and "short ones", whose image does not contain c. Then there is an easy recurrence relation between their cardinalities l_i and s_i:

$$\begin{pmatrix} l_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} l_i \\ s_i \end{pmatrix} = \begin{pmatrix} l_i + 1 \\ l_i + s_{i-1} - 1 \end{pmatrix}.$$

This gives $l_i + s_i = i + \frac{(i-1)(i-2)}{2} = \frac{i(i-1)}{2} + 1$. Furthermore, if $k_j = R(k_{j+1})$ for all $j < i$, then the parts of L_{n_i} that are folded at step n_j map to

$$f^{n_{j+1}}(\mathcal{T}_{n_{j+1}}) = f^{n_j}([c, c_{S_{1+k_{j+2}}}]) = [c_{S_{Q(2+k_{j+2})}}, c_{S_{2+k_{j+2}}}].$$

These are nested intervals, and their diameters tend to 0 as $j \to \infty$. Therefore the limit set of ray($\mathcal{H}$) is a spiralling arc; it is the set of points $x \in \mathcal{H}$ that belong to L_{n_i} infinitely often (in fact, every other i).

This shows that $\mathcal{H}$ is an arc + ray continuum. We postpone the proof that $\mathcal{H}$ are $\sin\frac{1}{x}$-curves until we have treated Case A:

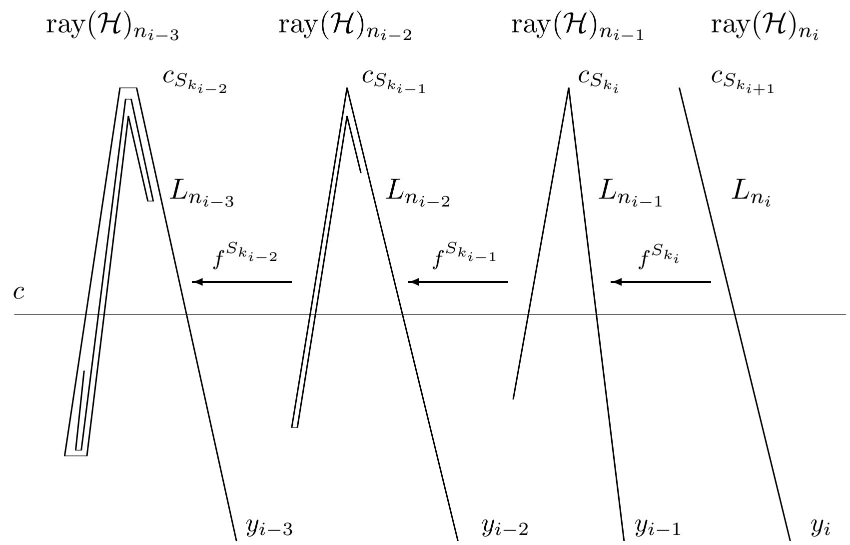

FIGURE 3. Impression how the critical projections of ray($\mathcal{H}$) in Case B are parametrised. The sequence of points $\{y_i\}$ is explained in the proof of Theorem 1.2.

$M_{n_{i-1}} = [c, c_{S_{k_i}}]$ (and hence $k_{i-1} = R(k_i)$) for all i. By (2), $L_{n_{i-1}} = [c, c_{S_{1+k_{i+1}}}]$, see Figure 4. We claim that $f^{n_{i-1}}|L_{n_{i-1}}$ is monotone. Since $z_{R(1+k_{i+1})}$ is the closest precritical point of lowest index in $L_{n_{i-1}}$, this claim follows from $S_{R(1+k_{i+1})} \geq n_{i-1}$, but this the same as the computation of Case B with $i-2$ replaced by $i-1$. The image $f^{S_{R(1+k_{i+1})}}(L_{n_{i-1}}) = [c_{S_{R(1+k_{i+1})}}, c_{S_{2+k_{i+1}}}]$ is an interval whose length tends to 0 as $i \to \infty$. Since $f^{n_{i-1}}(L_{n_{i-1}})$ is a preimage of this, also $|f^{n_{i-1}}(L_{n_{i-1}})| \to 0$ as $i \to \infty$. This shows (cf. Figure 4), that $\{f^{n_i}(L_{n_i})\}_{i \text{ even}}$ and $\{f^{n_i}(L_{n_i})\}_{i \text{ odd}}$ are nested sequence of intervals converging to points as $i \to \infty$. Using a parametrisation analogous to the one of Lemma 2.2, we find that $\mathcal{H}$ is a (spiralling) arc after all.

Remark: In fact, the arc of Case A is the bar of the arc + ray subcontinuum of Case B. This is because the subcontinua of Case A and B are the only ones with infinitely many critical projection times, and any Case A arc has a ray converging onto it as explained

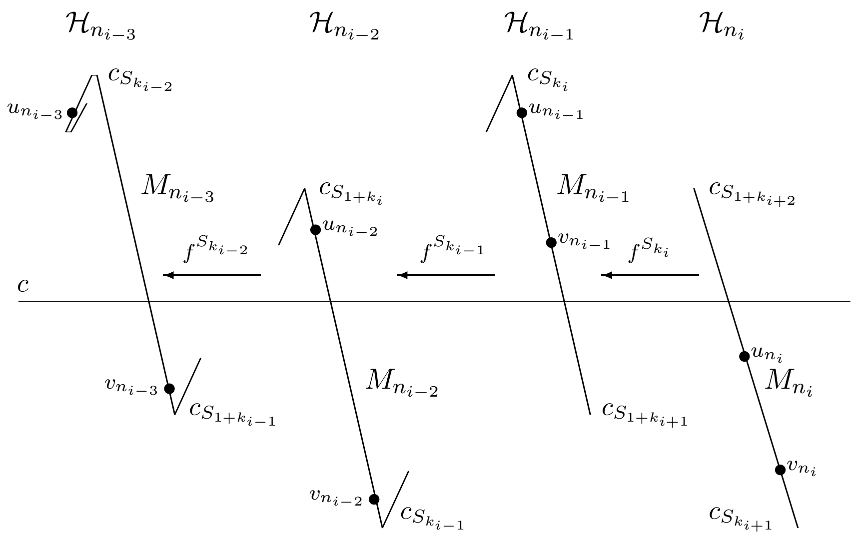

FIGURE 4. Impression of the critical projections of $\mathcal{H}$ in Case A. The points u_{n_j} and v_{n_j} feature in the proof that the subcontinuum of Case B is a $\sin \frac{1}{x}$-curve.

in the proof of Theorem 1.2 below.

We finish by proving that the subcontinuum $\mathcal{H}$ from Case B is indeed a $\sin \frac{1}{x}$-curve. A $\sin \frac{1}{x}$-curve $\mathcal{H}$ can be characterised (and thus distinguished from other arc + ray continua) by the following property: For all $u \neq v$ in the interior of $B := \mathrm{bar}(\mathcal{H})$ and all disjoint neighbourhoods $U \ni u$ and $V \ni v$, there are neighbourhoods U', U'' with $u \in U'' \subset U' \subset U$ such that for every $x \in \mathrm{ray}(\mathcal{H}) \cap U''$, if we follow the ray from x in at least one direction, we visit V before returning to U'. In other words, we can parametrise $\mathrm{ray}(\mathcal{H})$ by $\varphi : \mathbb{R} \to \mathcal{H}$ such that $x = \varphi(0)$, and if $t_0 = \inf_{t>0} \varphi(t) \notin U'$, and $t_1 = \sup_{t>t_0} \varphi([t_0, t]) \cap U' = \emptyset$, then $\varphi([t_0, t_1]) \cap V \neq \emptyset$.

Take $u \neq v$ in the interior of B. Then there is i such that at critical projection $\pi_{n_i}(B)$, there are no "folds" overlapping at u_{n_i} and v_{n_i}, i.e., $\pi_{n_i}^{-1}(u_{n_i}) \cap B = u$ and $\pi_{n_i}^{-1}(v_{n_i}) \cap B = v$, see Figure 4. Take neighbourhoods $U \ni u$ and $V \ni v$ disjoint but arbitrary otherwise. Let $J \subset B_{n_i}$ be a neighbourhood of u_{n_i} such that $\pi_{n_i}^{-1}(\overline{J}) \cap B \subset U$ and $\#(\pi_{n_i}^{-1}(y) \cap B) = 1$ for each $y \in J$. Construct a "tubular neighbourhood" of $\pi_{n_i}^{-1}(J)$ by setting

$$U' = \{x \in \mathcal{H} : \pi_{n_i}(x) \in J \text{ and } d(x, \pi_{n_i}^{-1}(J)) < \varepsilon\},$$

where ε is so small that $U' \subset U$ and

$$\{x \in \mathcal{H} : \pi_{n_i}(x) = v_{n_i} \text{ and } d(x, v) < \varepsilon\} \subset V.$$

Next let $U'' \ni u$ be so small that for every $x \in U'' \cap \text{ray}(\mathcal{H})$, when following the ray from x in either direction, we visit U' at least once more.

Note that the "folds" of both B and $\text{ray}(\mathcal{H})$ (taken sufficiently close to the bar) project to the same points under π_{n_i}, namely points of the form $f^{n_j - n_i}(c)$ for $j > i$. There are no such points between u_{n_i} and v_{n_i}, so $\pi_{n_i}^{-1}([v_{n_i}, u_{n_i}]) \cap \{x \in \mathcal{H} : d(x, B) < \varepsilon\}$ consists of a countable collection of arcs, each of which projects onto $[u_{n_i}, v_{n_i}]$ under π_{n_i}. It follows that for each $x \in \text{ray}(\mathcal{H}) \cap U''$, when following the ray from x in the direction of v, we visit V before returning to U''. This completes the proof. □

Proof of Theorem 1.2. Let $\mathcal{H}$ be a proper subcontinuum that is not a point or an arc. From Theorem 1.1 we know that its sequence k_i satisfies $R(k_i) = k_{i-1}$ for all i sufficiently large. By applying $\hat{f}^{-1}$ sufficiently often, we can assume that $R(k_i) = k_{i-1}$ for all i. Hence $\mathcal{H}$ corresponds to a backward R-orbit.

Conversely, given a backward R-orbit $\{k_i\}$, construct the subcontinuum $\mathcal{H}$ of Case B with $n_1 = 0$, $n_i = \sum_{j=1}^{i} S_{k_j}$ (so $n_i - n_{i-1} = S_{k_i}$), and

$$\mathcal{H}_{n_i} = [c_{S_{k_{i+1}}}, y_i] \ni c$$

such that $L_{n_i} = [c, c_{S_{k_{i+1}}}]$ and $M_{n_i} = [c, y_i]$. The points y_i will be chosen inductively to satisfy

- $|y_i - c| > |c_{S_{k_{i+1}}} - c|$, so M_{n_i} is larger than L_{n_i}, and $M_{n_i} \ni \hat{z}_{R(k_{i+1})}$;
- $\{z_{R(k_{i+1})-1}, \hat{z}_{R(k_{i+1})-1}\} \notin [c, y_i]$, so that $\mathcal{H}_{n_i}$ maps indeed homeomorphically for $S_{k_i} = S_{R(k_{i+1})}$ iterates;
- $f^{S_{k_i}}(y_i) = y_{i-1}$.

Because Q is non-decreasing and $Q(k) \to \infty$,

$$|c_{S_{1+k_{i+1}}} - c| < |c_{S_{k_i}} - c| = |\hat{c}_{S_{k_i}} - c| < |c_{S_{Q(k_i)}} - c|.$$

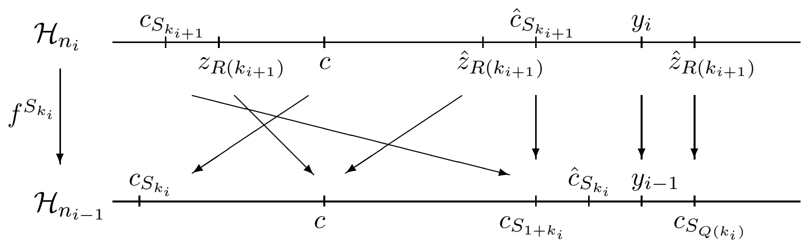

FIGURE 5. Points in $\mathcal{H}_{n_i}$ and their images under $f^{S_{k_i}}$.

The interval $[\hat{c}_{S_{1+k_{i+1}}}, \hat{z}_{R(k_{i+1})}]$ is mapped homeomorphically onto $[c_{S_{1+k_i}}, c_{S_{Q(k_i)}}]$ by $f^{S_{k_i}}$, see Figure 5. Therefore, for any point $y_{i-1} \in [\hat{c}_{S_{k_i}}, c_{S_{Q(k_i)}}]$, there is a point $y_i \in [\hat{c}_{S_{1+k_{i+1}}}, c_{S_{Q(k_{i+1})}}]$ such that $f^{S_{k_i}}(y_i) = y_{i-1}$. Thus we can indeed find a sequence $\{y_i\}$ satisfying the conditions above. The resulting subcontinuum $\mathcal{H}$ is a $\sin \frac{1}{x}$-curve according to Theorem 1.1. □

Proposition 1.3. Since $R(k) \to \infty$, $\#R^{-1}(k) < \infty$ for each k. Suppose that k_1 appears in infinitely many backward R-orbits. Using the pigeon hole principle, we can find an infinite backward R-orbit $k_1 \overset{R}{\leftarrow} k_2 \overset{R}{\leftarrow} \dots$ such that for each j, there is another infinite backward R-orbit that coincides with it up to at least k_j. Let $\mathcal{H}$ and $\{\mathcal{H}^j\}_{j \in \mathbb{N}}$ be the corresponding type B subcontinua, as constructed in Theorem 1.2. Then $\mathcal{H}$ and $\mathcal{H}^j$ have the same critical projections times up till n_{j-1}. As $n_{j-1} - n_{j-2} \to \infty$, and hence $|\mathcal{H}_{n_k}|, |\mathcal{H}^j_{n_k}| \to 0$, this means that $\text{bar}(\mathcal{H}^j) \to \text{bar}(\mathcal{H})$.

Conversely, if there is a sequence $\{\mathcal{H}^j\}_j$ of disjoint subcontinua (having distinct sequences of critical projection times), such that $\text{bar}(\mathcal{H}^j) \to \text{bar}(\mathcal{H})$, then for each k, there is j_0 such that the critical projection times of $\mathcal{H}$ and $\mathcal{H}^j$ coincide up to k for all $j \geq j_0$. This implies that k_1 belongs to infinitely many different backward R-orbits. □

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