# Dimensions of recurrence times and minimal subshifts

Henk Bruin
Department of Mathematics
California Institute of Technology
bruin@cco.caltech.edu

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**Abstract**

Examples are presented of minimal subshifts with positive entropy, and their Afraimovitch-Pesin capacities are computed. It is shown that the lower capacity can be strictly smaller than the entropy.

## 1 Introduction

Let $T$ be a continuous transformation of a metric space $(X, d)$. In [1], Afraimovich proposes the following fractal dimension-like strategy. For a set $U \subset X$, let

$$\tau(U) = \inf\{n > 0; T^n(U) \cap U \neq \emptyset\}$$

be the return time of $U$ to itself. (If $T^n(U) \cap U = \emptyset$ for all $n$, we set $\tau(U) = \infty$ by convention.) This quantity can be used to compute a measure of the space:

$$m_\alpha(X) = \lim_{\varepsilon \to 0} \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} e^{-\tau(U)\alpha},$$

where the infimum is taken over all covers $\mathcal{U}$ whose elements $U$ have diameter $\mathrm{diam}(U) = \sup_{x,y \in U} d(x, y) < \varepsilon$. (By convention $e^{-\infty\alpha} = 0$.) The critical dimension $\alpha_c$ of the space $X$ is defined as

$$\alpha_c = \sup\{\alpha; m_\alpha(X) = \infty\}.$$

This setup is analogous to the definition of Hausdorff dimension, and fits in Caratheodory's construction, see Pesin's book [10]. It is noticed that $\alpha_c$ in many cases coincides with the topological entropy of $(X, d, T)$, although $e^{-\tau(U)}$ is not the same quantity as $\eta(U)$ in [10, page 68], which was shown to lead to a dimension-theoretic definition of entropy.

The motivation for this note were discussions with and questions raised by Penné, Saussol and Vaienti. In [9] they study the properties of $\alpha_c$ (which they call the Afraimovich-Pesin or AP-dimension of $X$). Let us make a few remarks:

- The covers $\mathcal{U}$ are not sufficiently determined; one can think of open covers, closed covers, covers of arbitrary sets, covers of Borel measurable sets, etc. In [9] the first three possibilities are studied. In this note $X$ will be a subshift on two symbols, *i.e.* a compact shift-invariant space of $\{0,1\}^{\mathbb{N}}$ or $\{0,1\}^{\mathbb{Z}}$. The metric used is $d(x,y) = \sum_n \frac{\delta(x_n,y_n)}{2^{-|n|}}$ where $\delta(x_n,x_n) = 0$ if $x_n = y_n$ and 1 otherwise. Therefore it is natural to consider covers consisting of cylinder sets.

- Instead of covers with sets $U$ with $\mathrm{diam}(U) < \varepsilon$, one could take covers with sets $U$ with $\mathrm{diam}(U) = \varepsilon$. This gives rise to the quantities

$$\overline{m}_\alpha(X) = \limsup_{\varepsilon\to 0} \inf_{\mathcal{U},\, diam(U)=\varepsilon} \sum_{U\in\mathcal{U}} e^{-\tau(U)\alpha}$$

  and

$$\underline{m}_\alpha(X) = \liminf_{\varepsilon\to 0} \inf_{\mathcal{U},\, diam(U)=\varepsilon} \sum_{U\in\mathcal{U}} e^{-\tau(U)\alpha},$$

  and the Afraimovich-Pesin upper and lower capacities

$$\overline{\alpha}_c(X) = \sup\{\alpha; \overline{m}_\alpha(X) = \infty\},$$

  and

$$\underline{\alpha}_c(X) = \sup\{\alpha; \underline{m}_\alpha(X) = \infty\}.$$

  We have the obvious inequality

$$\alpha_c(X) \le \underline{\alpha}_c(X) \le \overline{\alpha}_c(X). \tag{1}$$

- In [9] the special role of periodic points of $(X,T)$ becomes apparent. If $x \in X$ is $n$-periodic, then $\tau(U) \le n$ for any $U \ni x$, irrespective its diameter. This causes $m_\alpha(X)$ to be strictly positive for $\alpha > \alpha_c$. This is unlike the situation encountered in Caratheodory's construction, cf [10, page 12 A2.], and it is the reason why $\alpha_c$ cannot be defined as $\inf\{\alpha; m_\alpha(X) = 0\}$.

  Another aspect of periodic points is its relation to entropy. The growthrate of the number of periodic points is given by $\zeta = \limsup_n \frac{1}{n}\log\#\{x; T^n(x) = x\}$. In many systems (*e.g.* subshifts of finite type, continuous interval maps) $\zeta$ coincides with the topological entropy. On the other hand, it is shown [9, Proposition 4.1], that $\zeta = \alpha_c(X)$, provided one works with covers of arbitrary sets, while if $(X,d,T)$ is a subshift of finite type and $\mathcal{U}$ are open covers,

$$\zeta = \alpha_c(X) = h_{top}(X),$$

  see [9, Theorem 5.1].

This raised the question how $\alpha_c$ and $h_{top}$ are related if there are no periodic points. We will give subshift examples. Let $\Sigma_2$ denote the one or two-sided shift of 2 symbols.

**Theorem 1 (Main).** *For all subshifts $\Sigma$ of $\Sigma_2$, $\alpha_c(\Sigma) \leq h_{top}(\Sigma)$. However, there exist minimal subshifts $\Sigma$ of $\Sigma_2$ such that $\alpha_c(\Sigma) \neq h_{top}(\Sigma)$.*

The Theorem of Jewett-Krieger [7, 8] assures the existence of minimal (even strictly ergodic) subshifts of positive entropy. Concrete examples were given in [6] and (more simple) [3]. In [4] and [5] Grillenberger and Shields present examples with additional properties (K-automorphism, Bernoulli). See also [11, Section 4.4] and references therein.

*Proof:* This follows directly from Theorem 2 and Proposition 3.2 below, and formula (1). □

# 2 The Upper Bound

We will work with a one-sided shift space $\Sigma_2 = \{0,1\}^{\mathbb{N}}$, but the methods are easily seen to apply to two-sided shifts as well. Let $\sigma$ denote the left-shift. A subshift $\Sigma$ is a closed shift-invariant subspace of $\Sigma_2$. A block $B$ is a string of symbols; its length is denoted by $|B|$. If $|B| = l$, $B$ is called an $l$-block. By abuse of terminology we will treat block and cylinder as synonyms. In this setting

$$h_{top} = \lim_l \frac{1}{l} \log \#\{\text{different } l\text{-blocks in } \Sigma\}$$

is the usual definition of topological entropy.

**Theorem 2.** *For any subshift $\Sigma \subset \Sigma_2$, the Afraimovitch-Pesin upper capacity $\overline{\alpha}_c \leq h_{top}$.*

*Proof:* Let $h = h_{top}$ and let $\varepsilon > 0$ be arbitrary. Let $N_l$ be the number of different $l$-blocks in $\Sigma$. The sequence $\{\log N_l\}$ is subadditive, so

$$h = \lim_l \frac{1}{l} \log N_l = \liminf_l \frac{1}{l} \log N_l = \inf_l \frac{1}{l} \log N_l.$$

Therefore there exists $l_0$ such that $N_l < e^{(h+\varepsilon)l}$ for all $l \geq l_0$. Obviously, if $\mathcal{U} = \{U\}$ is a cover of $l$-cylinders of $\Sigma$,

$$\sum_{U \in \mathcal{U}} e^{-\tau(U)\alpha} \leq \sum_{m=1}^{l} e^{-m\alpha} \#\{U \in \mathcal{U}; \min(\tau(U), l) = m\}. \tag{2}$$

If $\tau(U) = m < l$, then for $x \in U$, $x_{m+i} = x_i$ for every $1 \leq i \leq l - m$. It follows that $\#\{U \in \mathcal{U}; \tau(U) = m\} \leq N_m$. This gives for the right hand side of (2):

$$\begin{aligned} \sum_{m=1}^{l} e^{-m\alpha} \#\{U \in \mathcal{U}; \min(\tau(U), l) = m\} &\leq \sum_{m=1}^{l} e^{-m\alpha} N_m \\ &\leq \sum_{m=1}^{l_0} e^{-m\alpha} 2^m + \sum_{m=l_0+1}^{l} e^{(h+\varepsilon-\alpha)m}. \end{aligned}$$

This is finite independently of $l$ whenever $\alpha > h + \varepsilon$. As $\varepsilon$ is arbitrary, $\overline{\alpha}_c \leq h_{top}$. □

# 3 Constructions

We start by giving a general method to build minimal subshifts of positive entropy. It resembles Grillenberger's [3] construction; because we do not aim for the precise value of the entropy, nor for a strictly ergodic subshift, our construction is simpler. It consists of a three step algorithm:

- Let $\mathcal{E}_1$ be a collection of $n_1$ different $l_1$-blocks. Let $A_1$ be one of these blocks.
- Given the collection $\mathcal{E}_{i-1}$ of $n_{i-1}$ different $l_{i-1}$-blocks, there are $n_i = n_{i-1}!$ ways to concatenate the $l_{i-1}$-blocks into an $l_i$-block ($l_i = l_{i-1}n_{i-1}$), such that this $l_i$-block contains each $l_{i-1}$-block precisely once. Let $\mathcal{E}_i$ be the collections of these concatenations, and let $A_i$ be one of them.
- Let $a$ be the concatenations $A_1A_2A_3A_4\ldots$, and let $\Sigma = \omega_\sigma(a)$, *i.e.* the set of accumulation points of $\{\sigma^n(a); n \in \mathbb{N}\}$.

We claim that $(\Sigma, \sigma)$ is a minimal shift space of positive entropy. By construction, each $l_i$-block returns within $l_{i+2}$ iterations of $\sigma$, so $(\Sigma, \sigma)$ is uniformly recurrent. This is equivalent (see [2]) to $(\Sigma, \sigma)$ being a minimal system. As for the entropy, we will calculate a lower bound of $\frac{1}{l} \log N_l$. Indeed, by Sterling's formula

$$\begin{aligned} \frac{1}{l_i} \log N_{l_i} \geq \frac{1}{l_i} \log \#\mathcal{E}_i &= \frac{1}{l_i} \log n_i = \frac{1}{l_{i-1}n_{i-1}} \log n_{i-1}! \\ &\geq \frac{1}{l_{i-1}} \log n_{i-1} e^{-1} \geq \frac{1}{l_{i-1}} \log \#\mathcal{E}_{l_{i-1}} - \frac{1}{l_{i-1}}. \end{aligned}$$

By taking $n_1$ large enough compared to $l_1$, and using the fact that $l_i \to \infty$ very rapidly, we get that $\lim_i \frac{1}{l_i} \log \#E_i > 0$. Because $\{\log N_l\}$ is subadditive, we get $h_{top} = \lim \frac{1}{l} \log N_l \geq \lim_i \frac{1}{l_i} \log \#\mathcal{E}_i > 0$.

**Lemma 3.1.** *Let $\mathcal{A}$ be an alphabet of $n$ letters. Let $\mathcal{F} \subset \mathcal{A}^n$ be the set of $n$-blocks such that if $B = b_1 \dots b_n \in \mathcal{F}$, there exists $j$, $1 \le j \le n$ such that $b_{j+1} = b_k$ for some $k \le j$ and $b_s \neq b_t$ whenever $1 \le s < t \le j$ or $j+1 \le s < t \le n$. (If $j = n$, then $B$ is just a permutation of the letters of $\mathcal{A}$.) The cardinality $\#\mathcal{F} = n!2^{n-1}$.*

*Proof:* There are $\frac{n!}{(n-j)!}$ choices for the first $j$ letters. If $j < n$, then we have $j$ possibilities for the $j+1$-th letter and $\frac{(n-1)!}{((n-1)-(n-(j+1)))!} = \frac{(n-1)!}{j!}$ choices for the remaining letters. This adds up to

$$\begin{aligned}
\#\mathcal{F} &= n! + \sum_{j=1}^{n-1} \frac{n!}{(n-j)!} \cdot j \cdot \frac{(n-1)!}{j!} \\
&= n! + n! \sum_{j=1}^{n-1} \binom{n-1}{j-1} = n! \left[ 1 + \sum_{j=0}^{n-2} \binom{n-1}{j} \right] \\
&= n! \sum_{j=0}^{n-1} \binom{n-1}{j} = n!2^{n-1}.
\end{aligned}$$

This proves the lemma. □

**Proposition 3.1.** *The above example satisfies $\overline{\alpha}_c = h_{top}$.*

*Proof:* Let $i$ be arbitrary. For each $B \in \mathcal{E}_i$, $\tau(B) \le |B| = l_i$. This is because $\mathcal{E}_{i+1}$ contains blocks $C_1, C_2$ ending, respectively starting with $B$. Hence $\mathcal{E}_{i+2}$ contains a block in which $C_1C_2$ and therefore $BB$ appear as subblocks.

Let $\mathcal{F}_i$ be the concatenations $B$ of $n_i$ blocks from $\mathcal{E}_i$ (not necessarily different) that appear in $\Sigma$. If we picture the blocks in $\mathcal{F}_i$ as $n_{i-1}$ letter words with the $l_{i-1}$-blocks of $\mathcal{E}_{i-1}$ as letters, *i.e.*

$$B = b_1 \dots b_{n_{i-1}} \quad (b_j \in \mathcal{E}_{i-1}),$$

then there exists a unique $j$, $1 \le j \le n_{i-1}$, such that $b_k = b_{j+1}$ for some $k \le j$ and $b_s \neq b_t$ whenever $1 \le s < t \le j$ or $j+1 \le s < t \le n_{i-1}$. Note that $B \in \mathcal{E}_i$ if and only if $j = n_{i-1}$. Hence we are in the situation of Lemma 3.1 which gives $\#F_i = n_{i-1}!2^{n_{i-1}-1} = \#\mathcal{E}_i 2^{n_{i-1}-1}$.

Let $\mathcal{H}_i$ be the set of all $l_i - l_{i-1}$-blocks appearing in $\Sigma$. Each $C \in \mathcal{H}_i$ fits in at least one block $B \in \mathcal{F}_i$, so $\tau(C) \le \tau(B)$, and if $B$ can be chosen in $\mathcal{E}_i$, then $\tau(C) \le l_i$.

By the above arguments, at least a $1/2^{n_{i-1}-1}$ proportion of the blocks in $\mathcal{H}_i$ fits in a block $B \in \mathcal{E}_i$. Let $h = h_{top}$ and $\varepsilon > 0$ be arbitrary. Analogous to the proof of Theorem 2, we can assume that $N_l > e^{(h-\varepsilon)l}$ whenever $l \ge l_i$ and $i$ sufficiently large. Therefore

$$\begin{aligned}
\sum_{U \in \mathcal{H}_i} e^{-\tau(U)\alpha} &\ge \frac{1}{2^{n_{i-1}-1}} \#\mathcal{H}_i e^{-l_i\alpha} \ge \frac{1}{2^{n_{i-1}-1}} N_{l_i - l_{i-1}} e^{-l_i\alpha} \\
&\ge \frac{1}{2^{n_{i-1}-1}} e^{(h-\varepsilon)(l_i - l_{i-1}) - l_i\alpha}.
\end{aligned}$$

Because $\frac{l_{i-1}}{l_i}$ and $\frac{n_{i-1}}{l_i} \to 0$ as $i \to \infty$, the right hand side tends to infinity whenever $\alpha < h - \varepsilon$. Because $\varepsilon$ is arbitrary, we get $\overline{\alpha}_c \geq h$. The other inequality is supplied by Theorem 2. □

We conjecture that for this simple example the AP-dimension and upper and lower capacities all coincide: $\alpha_c = \underline{\alpha}_c = \overline{\alpha}_c = h_{top}$. For the next example, $\underline{\alpha}_c$ is strictly less than the entropy. The example is an adjustment of the previous one.

- Let $\mathcal{E}_1$ be the collection of different $l_1$-blocks $B$ all starting with 001000, and such that the string 00 appears nowhere else in $B$. Assume that $n_1 = \#\mathcal{E}_1$ is even and fix a special block $A_1 \in \mathcal{E}_1$.
- For a block $B$ of any length, let $B'$ be the block that emerges after replacing all strings 001000 into 001100 and vice versa. We call the strings 001000 and 001100 *flags.*

  Given the collection $\mathcal{E}_{i-1}$ of $l_{i-1}$ blocks, say $B_j$, $j = 1, \dots, n_{i-1}$, and special block $B_1 = A_{i-1}$, let $\mathcal{E}_i$ consists of all concatenations of the form

$$B_{\pi(1)} B'_{\pi(2)} B_{\pi(3)} B'_{\pi(4)} \dots B_{\pi(n_{i-1}-1)} B'_{\pi(n_{i-1})},$$

  where $\pi$ denotes a permutation of $\{1, \dots, n_{i-1}\}$ fixing 1. So each block in $\mathcal{E}_i$ starts with $A_{i-1}$. Fix a special block $A_i \in \mathcal{E}_i$.

The rest goes the same as in the previous example, including the minimality proof and the calculation of entropy. Note that we now have $n_i = (n_{i-1} - 1)!$ and $l_i = n_{i-1} l_{i-1}$.

**Proposition 3.2.** *The above example satisfies* $2\underline{\alpha}_c \leq \overline{\alpha}_c = h_{top}$.

We start with a lemma and corollary.

**Lemma 3.2.** *For any* $i$ *and any* $B \in \mathcal{E}_i \cup \mathcal{E}'_i$ *holds:* $\tau(B)$ *is a multiple of* $2|B| = 2l_i$.

*Proof:* By induction on $i$. For $i = 1$ the statement is clear because $B \in \mathcal{E}_1 \cup \mathcal{E}'_1$ starts with a flag and in $\Sigma$, the flags 001000 and 001100 appear alternatingly.

Now for the induction step, if $B \in \mathcal{E}_i \cup \mathcal{E}'_i$, then it starts with $A_{i-1}$ (or $A'_{i-1}$; the argument is the same for $A'_{i-1}$). By induction, $A_{i-1}$ returns only at multiples of $2l_{i-1}$. But all other blocks in $B$ are different from $A_{i-1}$, so $A_{i-1}$ can only return after $l_i$ iterates. Therefore $\tau(B)$ must be a multiple of $l_i$, but because of the alternating flagging, it returns actually at a multiple of $2l_i$. □

**Corollary 3.2.1.** *Let* $\mathcal{F}_i$ *be the set of concatenations* $B$ *in* $\Sigma$ *of* $n_{i-1}$ *(not necessarily different) blocks from* $\mathcal{E}_{i-1} \cup \mathcal{E}'_{i-1}$*. For all* $B \in \mathcal{F}_i$, $\tau(B)$ *is a multiple of* $2|B| = 2l_i$.

*Proof:* Each $B \in \mathcal{F}_i$ appearing in $\Sigma$ has the block $A_{i-1}$ or $A'_{i-1}$ at a fixed position. Therefore the previous proof can be used with the obvious adjustments. □

*Proof of Proposition 3.2:* For each $i$, each $l_i + l_{i-1}$-block $B$ in $\Sigma$ contains a block $C \in \mathcal{F}_i$, so $\tau(B) \geq \tau(C) = 2l_i$. Using the notation of the proof of Theorem 2, we get for any $\varepsilon > 0$ and $i$ sufficiently large:

$$\sum_{|B|=l_i+l_{i-1}} e^{-\tau(B)\alpha} \leq e^{(h+\varepsilon)(l_i-l_{i-1})-2l_i\alpha}.$$

Because $\frac{l_{i-1}}{l_i} \to 0$ as $i \to \infty$, this expression tends to 0 whenever $\alpha > \frac{h+\varepsilon}{2}$. Because $\varepsilon$ is arbitrary, $\underline{\alpha}_c \leq \frac{1}{2}h_{top}$.

Now we compute $\overline{\alpha}_c$. For $R \in \mathbb{N}$, let $\mathcal{F}_{R,i}$ be the set of blocks $B$ in $\Sigma$ which consist of $R$ blocks in $\mathcal{E}_{i-1} \cup \mathcal{E}'_{i-1}$. Hence $|B| = Rl_{i-1}$. We claim that if $B \in \mathcal{F}_{R,i}$ consists of $R$ different blocks and none of them is $A_{i-1}$ or $A'_{i-1}$, and also $R < n_i$ is odd, then

$$\tau(B) \leq (R+2)l_{i-1} = \frac{R+2}{R}|B|.$$

Indeed, if $B = C_1C'_2 \dots C'_{R-1}C_R$, $C_j \in \mathcal{E}_{i-1}$, then there are blocks $D_1, D_2 \in \mathcal{E}_i$, $D_2 \neq A_i$, such that $D_1$ ends with $BC'_0$ and $D_2$ starts with $A_{i-1}B'$ for some $C_0 \in \mathcal{E}_i$. If $B = C'_1C_2 \dots C_{R-1}C'_R$, $C_j \in \mathcal{E}_{i-1}$, then there are blocks $D_1, D_2 \in \mathcal{E}_i$, $D_2 \neq A_i$, such that $D_1$ ends with $B$ and $D_2$ starts with $A_{i-1}C'_0B'$ for some $C_0 \in \mathcal{E}_i$. In both cases, the concatenation $D_1D'_2$ contains the block $B$ twice at the right distance. This proves the claim.

If $R \ll n_i$, this claim applies to at least half of the blocks in $\mathcal{F}_{R,i}$. Any $(R-1)l_{i-1}$-block $C$ appearing in $\Sigma$ is contained in a block $B \in \mathcal{F}_{R,i}$, and therefore at least half of them satisfies $\tau(C) \leq \tau(B) \leq (R+2)l_{i-1} = \frac{R+2}{R-1}|C|$. This gives for $\varepsilon > 0$ arbitrary and $i$ sufficiently large:

$$\sum_{|C|=(R-1)l_{i-1}} e^{-\tau(C)\alpha} \geq \frac{1}{2}N_{(R-1)l_{i-1}}e^{-(R+2)l_{i-1}\alpha} \geq \frac{1}{2}e^{(h-\varepsilon)(R-1)l_{i-1}-(R+2)l_{i-1}\alpha}.$$

For any $\alpha < \frac{R-1}{R+2}(h-\varepsilon)$, this tends to infinity as $i \to \infty$. Because $\varepsilon > 0$ and $R \in \mathbb{N}$ are arbitrary, we get $\overline{\alpha}_c \geq h_{top}$. The other inequality is supplied by Theorem 2. □

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