Adding machines and wild attractors

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September 16, 1996

Abstract

We investigate the dynamics of unimodal maps f of the interval restricted to the omega limit set X of the critical point for cases where X is a Cantor set. In particular many cases where X is a measure attractor of f are included. We give two classes of examples of such maps, both generalizing unimodal Fibonacci maps [LM, BKNS]. In all cases $f_{|X}$ is a continuous factor of a generalized odometer (an adding machine–like dynamical system), and at the same time $f_{|X}$ factors onto an irrational circle rotation. In some of the examples we obtain irrational rotations on more complicated groups as factors.

1 Introduction

Let $f : [0,1] \to [0,1]$ be a C^2–unimodal map with critical point c, and denote by X the omega limit set of c, *i.e.* the set of all accumulation points of the forward orbit of c. It is well known that X is either

- a *cycle of exact intervals*, *i.e.* a finite union of intervals that are cyclically permuted under the action of f and for which the restriction of f^p to each of these intervals is topologically exact (p denotes the period of the cycle), or
- a zero–dimensional set. In this case X may be
  - a *periodic orbit* (attracting or repelling),

*Supported by the Netherlands Organization for Scientific Research (NWO) and the Deutsche Forschungsgemeinschaft (DFG)
**Supported by the Deutsche Forschungsgemeinschaft (DFG)

- a *solenoidal attractor*, *i.e.* the intersection of an infinite nested sequence of cycles of intervals, or
- a *Cantor set* that has no arbitrarily small invariant neighbourhoods. Although in such a situation the set of points with omega limit equal to X is of first category (*i.e.* a topological null set), there are some examples where this set has full Lebesgue measure [BKNS, Bru]. In this case we call X a *wild attractor*.

If X is a wild attractor, then $f_{|X}$ is topologically minimal [BL, M]. The same is true, of course, if X is a limit cycle or a solenoidal attractor. Throughout this paper, a strong version of minimality of $f_{|X}$ ($Q(k) \to \infty$, see below) will be our basic assumption on f, although a few assertions are valid without it.

The simplest case of a map with a solenoidal attractor is the Feigenbaum map. It has a nested sequence of interval cycles of length 2^n, and it is not hard to see that $f_{|X}$ is conjugate to the dyadic adding machine, *i.e.* to the addition of 1 in the group of dyadic integers. Similarly, the dynamics on other solenoidal attractors are conjugate to generalized adding machines. In the case of wild attractors, a particularly well studied example is the Fibonacci map. By a Fibonacci map we mean a unimodal map from a particular topological conjugacy class that we describe in some detail in Section 3. It was proved in [BKNS] that Fibonacci maps have a wild attractor provided the order of the critical point is sufficiently high. The topological dynamics of $f_{|X}$ for Fibonacci maps was clarified before [LM]: $f_{|X}$ is conjugate to a symbolic dynamical system (Ω, T) which is similar to an adding machine, although it fails to be invertible at the critical point c where it has two preimages. Indeed (see [GLT]), (Ω, T) is just the S*–odometer* built from the sequence of Fibonacci numbers or, equivalently, the *adic transformation* on the golden mean subshift of finite type, and it has the circle rotation by the golden ratio $\frac{\sqrt{5}-1}{2}$ as a continuous factor. This factor map is in fact nearly a conjugacy, because it is invertible except on the backwards trajectory of c under $f_{|X}$. In order to investigate the dynamics of $f_{|X}$ in other cases, we study a particular class of (generally nonstationary) adic transformations in Section 2 and relate them to the dynamics of $f_{|X}$. A closer look at two ways to generalize the Fibonacci case is taken in the next two sections. First we construct examples based on properties of *Pisot–Vijayaraghavan (PV) numbers*, and in a second construction we exploit convergence properties of continued fractions. Our main results can be summarized as follows:
We construct examples of S–odometers *corresponding to unimodal maps f for which X is a wild attractor.* These examples have the following spectral properties:

1. For $d = 1, 2, 3$ we find S–odometers generalizing the Fibonacci case which are measure theoretically isomorphic to irrational rotations on the d–dimensional torus. The isomorphism is continuous from the sequence space Ω onto the torus. For $d \geq 4$, however, this construction leads to weakly mixing S–odometers (cf. Theorems 3 and 4).

2. For any $d \geq 1$ we find S–odometers which have irrational rotations on d–dimensional tori as continuous factors and are measure theoretically isomorphic to them (Theorem 5).

3. For each irrational ρ with continued fraction of bounded type[1)] we find an S–odometer with the rotation by ρ on the circle as a continuous factor. Again the factor map is a measure theoretical isomorphism (Theorem 6).

4. We find an S–odometer which has simultaneously an irrational rotation on a solenoidal group and the addition of 1 in the dyadic adding machine as continuous factors (Theorem 7).

In all these examples the continuous factor maps from (Ω, T) onto the group rotations factorize over $(X, f_{|X})$ in such a way that $f_{|X}$ has the same spectral properties as (Ω, T). The factor maps from $(X, f_{|X})$ onto the group rotations are measure theoretical isomorphisms in cases 1.–3.

During our research we learned that Marco Martens, using a different method, obtained similar results.

Acknowledgments: We would like to thank the referee whose valuable comments significantly helped to improve the previous version of this paper. We are also greatful to Prof. Schinzel for drawing our attention to reference [Sel].

2 Adic transformations associated with unimodal maps

Let $f : [0,1] \to [0,1]$ be a unimodal map. We construct a tower by defining its level sets D_n as $D_1 = [c, c_1]$ and for $n \geq 2$,

$$D_n = \begin{cases} f(D_{n-1}) & \text{if} \quad D_{n-1} \not\ni c \ , \\ [c_n, c_1] & \text{if} \quad D_{n-1} \ni c \ . \end{cases}$$

The numbers n such that $c \in D_n$ are called the *cutting times* $(S_k)_{k \geq 0}$. Observe that $S_0 = 1$ and that $D_n = [c_n, c_{n-S_{k-1}}]$ for $S_{k-1} < n \leq S_k$.

It is well known (see *e.g.* [H], or for a more recent exposition [MS, p.108]) that there is a mapping $Q : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ (called the *kneading map* of f) such that $S_k - S_{k-1} = S_{Q(k)}$. For every n, $Q(n) < n$, and Q satisfies the *admissibility condition*

$$\{Q(k+j)\}_{j \geq 1} \succeq \{Q(Q^2(k)+j)\}_{j \geq 1} \tag{1}$$

1)The boundedness of the continued fraction is not necessary for the construction of the factor. It is only used to guarantee that the corresponding f may have X as a wild attractor.

for all k, where $\succeq$ denotes the lexicographic order on sequences of integers. On the other hand, each map $Q : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ satisfying $Q(n) < n$ and the admissibility condition (1) can be realized within each full family of unimodal maps, in particular so by some map from each of the families $(f_{a,\ell} : 0 < a \leq 1)$ for $\ell = 2, 3, 4, 5, \ldots$, where $f_{a,\ell}(x) = 1 - a2^\ell |x - \frac{1}{2}|^\ell$.

Recall now that X denotes the omega limit set of the critical point of f. If $Q(k) \to \infty$, then c is recurrent and $f_{|X}$ is topologically minimal, see *e.g.* [Bru]. Moreover $|D_n| \to 0$. In this paper we are interested in a particular class of kneading maps that can be realized by unimodal maps with wild attractors.

Theorem 0 ([Bru]) *Suppose Q is a kneading map and there are $N, k_1 \in \mathbb{N}$ such that for all $k \geq k_1$ holds:*

$$Q(k) \geq k - N \quad and \quad Q(k+1) > Q^2(k) + 1 \ .$$

Then there is $\ell(N) \in \mathbb{N}$ such that for $\ell > \ell(N)$ there is a parameter $a = a(\ell, Q)$ for which X is a wild attractor of the map $f_{a,\ell}$.

One can show that $Q(k+1) > Q^2(k)+1$ eventually, if $Q(k)$ is eventually nondecreasing and f is not infinitely renormalizable (no solenoidal attractor).

In order to understand the dynamics of $f_{|X}$, we relate this dynamical system to a symbolic system (Ω, T) defined in terms of the sequence $S = (S_n)_{n>0}$ and called an S–*adic machine* or S–*odometer* in [GLT]. We follow the presentation from [GLT] but note that there is a strong connection to the work of Vershik [Ve] and others [VL, L1, So1].

Given a nonnegative integer n let k be the unique integer satisfying $S_k \leq n < S_{k+1}$. There exist integers $\langle n \rangle_k$ and $r_k(n)$ with $n = \langle n \rangle_k \cdot S_k + r_k(n)$ and $0 \leq r_k(n) < S_k$, and by iteration we get the S–expansion of n

$$n = \langle n \rangle_0 S_0 + \cdots + \langle n \rangle_k S_k \tag{2}$$

where the digits $\langle n \rangle_j$ satisfy $0 \leq \langle n \rangle_j < \frac{S_{j+1}}{S_j}$. This is the greedy algorithm (see *e.g.* [Fr]). As for our particular sequences $\frac{S_{j+1}}{S_j} \leq 2$, the digits take only the values 0 and 1. The expansion (2) is uniquely determined provided $\sum_{i=0}^{j} \langle n \rangle_i S_i < S_{j+1}$ for $0 \leq j \leq k$. Setting $\langle n \rangle_j = 0$ for $j > k$ we obtain a sequence $\langle n \rangle = (\langle n \rangle_j)_{j>0}$ in $\{0,1\}^{\mathbb{N}}$. Let $\langle \mathbb{N} \rangle := \{\langle n \rangle : n \in \mathbb{N}\}$, and denote by Ω the closure of $\langle \mathbb{N} \rangle$ in $\{0,1\}^{\mathbb{N}}$. Then

$$\Omega = \{\omega \in \{0,1\}^{\mathbb{N}} : \sum_{i=0}^{j} \omega_i S_i < S_{j+1} \ \forall j \geq 0\} \ .$$

and we have

Lemma 1

$$\Omega = \{\omega = (\omega_i)_{i\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}} : \forall j \geq 0 \ : \omega_j = 1 \Rightarrow \omega_i = 0 \ (Q(j+1) \leq i \leq j-1)\} \ ,$$

and if $\sum_{i=0}^{k} \omega_i S_i = S_{k+1} - 1$ *for some* $k \geq 0$ *with* $Q(k+1) > 0$*, then* $\sum_{i=0}^{Q(k+1)-1} \omega_i S_i = S_{Q(k+1)} - 1$.

Proof: To prove "$\subseteq$" just observe that $\sum_{i=0}^{j-1} \omega_i S_i < S_{j+1} - S_j = S_{Q(j+1)}$ for each $\omega \in \Omega$. For the reverse inclusion fix an index j, let $j' = Q(j+1) - 1$, and suppose that $\omega_j = 1$. Then

$$\sum_{i=0}^{j} \omega_i S_i = \sum_{i=0}^{j'} \omega_i S_i + \sum_{i=Q(j+1)}^{j} \omega_i S_i = \sum_{i=0}^{j'} \omega_i S_i + S_j = \sum_{i=0}^{j'} \omega_i S_i - S_{j'+1} + S_{j+1} \ . \qquad (3)$$

If $\sum_{i=0}^{k} \omega_i S_i < S_{k+1}$ for all $k < j$, this continues to be true for $k = j$. If $\omega_j = 0$, then $\sum_{i=0}^{j} \omega_i S_i = \sum_{i=0}^{j-1} \omega_i S_i < S_j < S_{j+1}$. As $\omega_0 S_0 < S_1$, the induction is complete.

To prove the additional assertion, fix an index k and let $j \leq k$ be the maximal index with $\omega_j = 1$. If $\sum_{i=0}^{k} \omega_i S_i = S_{k+1} - 1$, then it follows from (3) that

$$S_{k+1} - 1 = \sum_{i=0}^{Q(j+1)-1} \omega_i S_i - S_{Q(j+1)} + S_{j+1} \leq S_{j+1} - 1 \leq S_{k+1} - 1$$

with equality if and only if $j = k$ and $\sum_{i=0}^{Q(j+1)-1} \omega_i S_i = S_{Q(j+1)} - 1$. □

Next we define an "addition of 1" on $\langle\mathbb{N}\rangle$:

$$T\langle n\rangle := \langle n+1\rangle \ .$$

Lemma 1 suggests a way how this addition can be described in a purely symbolic way by means of an "add & carry" operation: Given an $\omega \in \langle\mathbb{N}\rangle$ proceed as follows:

add one : Only one of the first two digits of ω can be one. If the first digit is 0, replace it by 1, if it is 1, shift it to the right (i.e., replace 10 by 01). In general the result will be no element of Ω anymore.

carry : Beginning with the smallest l such that $\omega_l = 1$ take the following steps: There is at most one $k \in \mathbb{N}$ with $Q(k+1) = l$ and $\omega_k = 1$. If such a k exists replace ω_k and ω_l by 0 and ω_{k+1} by 1. (In this case k equals $\min\{i > l : \omega_i = 1\}$) Then restart the carry operation with $l = k+1$. If there is no such k, then the procedure stops.

In the next lemma we collect some facts about T proved in [GLT].

Lemma 2 *If $Q(k) \to \infty$ then T extends uniquely to a continuous map $T : \Omega \to \Omega$. Moreover T^{-1} is well defined on $\Omega \setminus \langle 0 \rangle$, T is surjective, and T is minimal.*

Proof: If $Q(n) \to \infty$ there are, for each fixed j, only finitely many k such that $Q(k+1) - 1 = j$. Therefore the continuity of T follows from [GLT, Theorem 1] and the rest from Proposition 1(iii) and Theorem 2 of the same reference. □

Remark 1 The map T defined in the two preceeding lemmas is just the adic transformation on Ω in the sense of Vershik [Ve], see [GLT, Proposition 2].

Let $\rho \in [0, 1]$ and $R_\rho : \mathbb{T}^1 \to \mathbb{T}^1$ be the rotation by angle ρ. Define $\Pi_\rho : \Omega \to \mathbb{T}^1$ by

$$\Pi_\rho(\omega) = \sum_{k \geq 0} \omega_k S_k \rho \pmod 1$$

For $x \in \mathbb{R}$ let $\|x\|$ denote the distance of x to the closest point in $\mathbb{Z}$. Variants of the following lemma, whose short proof we include for completeness, are well known.

Lemma 3 1. *If $\sum_k \|\rho S_k\| < \infty$, then Π_ρ is well defined, continuous and $\Pi_\rho \circ T = R_\rho \circ \Pi_\rho$.*

2. *If $\phi : (\Omega, T) \to (\mathbb{T}^1, R_\rho)$ is a nonconstant continuous homomorphism and if $\phi\langle 0 \rangle = 0$, then $\sum_k \|\rho S_k\| < \infty$ and $\phi = \Pi_\rho$.*

Proof: The first assertion is obvious. We prove the second one: Let (n_j) be any strictly increasing sequence of integers such that $\langle n_j \rangle \to \langle 0 \rangle$ as $j \to \infty$. Then by continuity of ϕ,

$$\sum_k \langle n_j \rangle_k S_k \rho = n_j \rho = R_\rho^{n_j}(0) = R_\rho^{n_j}(\phi\langle 0 \rangle) = \phi(T^{n_j}\langle 0 \rangle) = \phi\langle n_j \rangle \to \phi\langle 0 \rangle = 0$$

as $j \to \infty$. As this is true for any such sequence (n_j), it follows that $\sum_k \|\rho S_k\| < \infty$. In particular, Π_ρ is well defined. Furthermore, as $(\phi - \Pi_\rho) \circ T^n = \phi - \Pi_\rho$ for all n, $\phi - \Pi_\rho$ is constant by continuity and topological transitivity of (Ω, T), and as $\phi\langle 0 \rangle = 0 = \Pi_\rho\langle 0 \rangle$, we conclude that $\phi = \Pi_\rho$. □

In later sections we discuss classes of kneading maps Q which satisfy $\sum_{k \geq 0} \|\rho S_k\| < \infty$ for suitable numbers ρ. But first we relate the dynamics of such S–odometers to those of the corresponding $f_{|X}$.

Lemma 4 *If $\phi : (\Omega, T) \to (Y, S)$ is a continuous factor map onto some invertible dynamical system (Y, S) and if ν is an ergodic, invariant Borel–probability for S, then the cardinality $\#\phi^{-1}(y)$ is the same for ν–a.e. $y \in Y$ (it may be infinite).*

Proof: Denote by $\mathcal{Z}_n$ the family of cylinder sets of rank n of Ω. Then $Y = \phi(\Omega) = \cup_{Z \in \mathcal{Z}_n} \phi(Z)$. Define $F_n : Y \to \mathbb{Z}$ as $F_n = \sum_{Z \in \mathcal{Z}_n} 1_{\phi(Z)}$. As all $\phi(Z)$ are compact, the F_n are measurable. Obviously $F_1 \leq F_2 \leq F_3 \leq \dots$ such that $F_\infty := \sup_n F_n$ is measurable and for each $y \in Y$ and $k \in \mathbb{Z}$ holds: $F_\infty(y) \geq k$ if and only if $\#\phi^{-1}(y) \geq k$. Hence $\#\phi^{-1}(y) = F_\infty(y)$ is a measurable function of y. Furthermore,

$$\#\phi^{-1}(Sy) \leq \#T^{-1}\phi^{-1}(Sy) = \#\phi^{-1}S^{-1}(Sy) = \#\phi^{-1}(y) ,$$

whence $S^{-1}\{y : \#\phi^{-1}(y) \geq k\} \subseteq \{y : \#\phi^{-1}(y) \geq k\}$ for each $k \in \mathbb{Z}$. By ergodicity of ν all these sets have either ν–measure 0 or 1, which means that $\#\phi^{-1}(y)$ is ν–a.s. constant. □

Let us introduce some more dynamical notions. A point z is called a *closest precritical point* if (z, c) is the maximal interval on which f^n is monotone for some n. Clearly z is indeed a precritical point. Taking also n maximal, then $f^n[z, c] \ni c$. It follows from our tower construction that $f^n[z, c] = D_{S_k}$ for some k and $z \in f^{-S_{k-1}}(c)$. We can enumerate the precritical points as $z_i \in f^{-S_i}(c)$ and $z_0 < z_1 < \dots < c$. Of course, the symmetric points $\hat{z}_i$ are closest precritical points too.

Because $S_{k+1} - S_k = S_{Q(k+1)}$ is the maximal iterate such that $f^{S_{Q(k+1)}}|_{[c, c_{S_k}]}$ is still monotone, it follows that

$$c_{S_k} \in [z_{Q(k+1)-1}, z_{Q(k+1)}] \cup [\hat{z}_{Q(k+1)}, \hat{z}_{Q(k+1)-1}] .$$

Define for $2 \leq S_{k-1} < n \leq S_k$ the *lower interval* of D_n as

$$\beta(D_n) = D_{n - S_{k-1}} .$$

Lemma 5 *If $n > 2$, then $\beta(D_n) \supseteq D_n$.*

Proof: Let additionally $z_{-1} := 0$. First we claim that $D_{S_k} \subseteq D_{S_{Q(k)}}$ for all $k > 1$. Indeed, $f^{S_{k-1}}(c) \in [z_{Q(k)-1}, c]$ or $[c, \hat{z}_{Q(k)-1}]$, whence $[f^{S_{k-1}}(c), c] \subseteq [z_{Q(k)-1}, c]$ or $[c, \hat{z}_{Q(k)-1}]$. Taking the $S_{Q(k)}$–th iterate on this inclusion, we obtain

$$\begin{aligned} D_{S_k} &= [f^{S_{Q(k)}+S_{k-1}}(c), f^{S_{Q(k)}}(c)] = [f^{S_k}(c), f^{S_{Q(k)}}(c)] \subseteq [f^{S_{Q(k)}}(z_{Q(k)-1}), f^{S_{Q(k)}}(c)] \\ &= [f^{S_{Q(k)} - S_{Q(k)-1}}(c), f^{S_{Q(k)}}(c)] = [f^{S_{Q^2(k)}}(c), f^{S_{Q(k)}}(c)] = D_{S_{Q(k)}} , \end{aligned}$$

which yields the claim.

Now let $S_{k-1} < n < S_k$ and assume by contradiction that $\beta(D_n) \subset D_n$ properly. Take the $(S_k - n)$–th iterate of both intervals. Then

$$D_{S_k} = f^{S_k - n}(D_n) \supset f^{S_k - n}(\beta(D_n)) = \beta(D_{S_k}) = D_{S_{Q(k)}} .$$

This contradicts the claim. □

We can also define β as a map on $\langle\mathbb{N}\rangle$ (or $\mathbb{N}$) as follows: If $\omega \in \langle\mathbb{N}\rangle \setminus \{\langle 0\rangle\}$, then $\beta(\omega) \in \langle\mathbb{N}\rangle$ is the sequence that appears after changing the last 1 of ω into a 0. On $\mathbb{N} \setminus \{0\}$ this is equivalent to $\beta(n) = n - S_{k-1}$ for $S_{k-1} \leq n < S_k$. So $\beta(n) = 0$ if n is a cutting time, and

$$\beta(D_n) = D_{\beta(n)} \quad \text{for} \quad n \notin \{S_k : k \geq 0\} \ . \tag{4}$$

This relation will be used in the next theorem to determine the position of c_n for n large: We can construct a nested sequence of intervals $D_n = D_{n_1} \subset D_{n_2} \subset \ldots \subset D_{S_k}$ for some k such that $\beta(n_i) = n_{i+1}$. We proceed to define a factor map $\pi : \Omega \to X$.

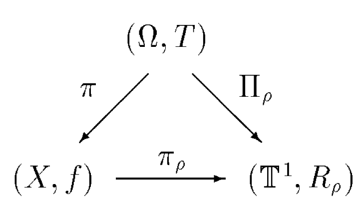
Theorem 1 *Define $\pi : \langle\mathbb{N}\rangle \to orb(c)$ as $\pi(\langle k\rangle) = c_k$. Then π is uniformly continuous and has a continuous extension $\pi : \Omega \to X = \omega(c)$ such that (X, f) is a continuous factor of (Ω, T). Moreover $\pi^{-1}(c) = \langle 0\rangle$.*

Proof: Take $\epsilon > 0$ arbitrary. Because $|D_n| \to 0$, there exists r_0 such that $|D_n| < \epsilon/2$ for all $n \geq S_{r_0}$. Let m be arbitrary, and let U be an ϵ-neighbourhood of $c_m = \pi(\langle m\rangle)$. Take $i \in \mathbb{N}$ minimal such that $\beta^i(m) =: k < S_{r_0}$. So $\langle k\rangle$ coincides with $\langle m\rangle$ up to entry $r_0 - 1$, and has only 0's elsewhere. If $m < S_{r_0}$, i.e. $i = 0$, then simply $k = m$, and $c_m \in D_{k+S_{r_0}}$. If $m \geq S_{r_0}$, then $\beta^{i-1}(m) = k + S_r$ for some $r \geq r_0$. It follows by (4) that $\beta^{i-1}(D_m) = D_{k+S_r}$, and by Lemma 5, $c_m \in D_{k+S_r}$.

Let $\langle n\rangle$ be such that it coincides with $\langle m\rangle$ up to the $(r_0 - 1)$–th position. Then there exists i' such that $\beta^{i'}(n) = \beta^i(m) = k$. This means that also $\beta^{i'-1}(n) = k + S_{r'}$ for some $r' \geq r_0$. By (4), $\beta^{i'-1}(D_n) = D_{k+S_{r'}}$ and by Lemma 5, $c_n \in D_{k+S_{r'}}$. So $c_m \in D_{k+S_r}$ and $c_n \in D_{k+S_{r'}}$ for $r, r' \geq r_0$. Since D_{k+S_r} and $D_{k+S_{r'}}$ have c_k as common boundary point, this yields $c_n \in U$. Hence π is uniformly continuous. The existence and continuity of the extension follows.

Now suppose $\langle 0\rangle \neq \omega = \omega_0\omega_1\ldots \in \Omega$ and $\pi(\omega) = c$. Let r, s be the first two integers such that $\omega_r, \omega_s = 1$. Clearly s exists, otherwise $\pi(\omega) = c_r$. It follows from Lemma 5 that $\pi(\omega) \in D_{S_r+S_s}$. So if $\pi(\omega) = c$, then $c \in D_{S_r+S_s}$, and $S_r + S_s$ is a cutting time, say S_t. But then $s = t - 1$ and $r = Q(t)$, which is forbidden by the construction of Ω.
□

The final goal of this section is to define a factor map $\pi_\rho : X \to \mathbb{T}^1$ which makes the following diagram commute:



Theorem 2 *Suppose that $\sum_k \|\rho S_k\| < \infty$ such that Π_ρ is well defined and continuous by Lemma 3. Then $\#\Pi_\rho(\pi^{-1}\{x\}) = 1$ for all $x \in X$, and if the unique element of this set is denoted by $\pi_\rho(x)$ this defines a continuous factor map $\pi_\rho : (X, f) \to (\mathbb{T}^1, R_\rho)$.*

Proof: Let $M := \{x \in X : \#\Pi_\rho(\pi^{-1}\{x\}) > 1\}$. We claim that if $x \in M$ then the closure of the orbit of x is contained in M. Since (X, f) is minimal it follows that M is either void or equal to X, and as $c \notin M$ by Theorem 1, this proves that π_ρ is well defined.

Suppose $x \in M$. Then there exist $\omega, \omega' \in \Omega$ such that: $\pi\omega = \pi\omega' = x$ and $\Pi_\rho\,\omega \neq \Pi_\rho\,\omega'$. Then

$$\begin{array}{rcccccl} \Pi_\rho\,(T^n\omega) & = & R_\rho^n \pi\,\omega & \neq & R_\rho^n \pi\,\omega' & = & \Pi_\rho\,(T^n\omega') \\ \pi(T^n\omega) & = & f^n\pi\omega & = & f^n\pi\omega' & = & \pi(T^n\omega')\ , \end{array}$$

whence $\mathrm{orb}(x) \subset M$. Now let y be any limit point of $\mathrm{orb}(x)$ and a sequence $(n_k)_{k\in\mathbb{N}}$ be given with $f^{n_k}x \to y$. Since Ω is compact we can assume that $T^{n_k}\omega$ and $T^{n_k}\omega'$ converge simultaneously against some ω_0 resp. ω_0' in Ω. From continuity of π follows that $\pi\omega_0 = \pi\omega_0' = y$ and from continuity of Π_ρ together with the fact that R_ρ is an isometry we get $\mathrm{dist}(\Pi_\rho\,\omega_0, \Pi_\rho\,\omega_0') = \mathrm{dist}(\Pi_\rho\,\omega, \Pi_\rho\omega') \neq 0$. This shows that $y \in M$ and proves the claim.

Since the map $\pi : \Omega \to X$ is continuous, surjective and closed (Ω is compact), X carries the final topology of Ω with respect to π. So continuity of π_ρ follows from continuity of $\Pi_\rho = \pi \circ \pi_\rho$. □

Remark 2 Call $p \in \mathbb{N}$ an *eventual divisor* of $(S_n)_n$, if $p \,|\, S_n$ except for at most finitely many n. By Lemma 3, p is an eventual divisor of $(S_n)_n$ if and only if ($\mathbb{Z}_p$, addition of 1) is a factor of (Ω, T). By Theorem 2, $\Pi_{1/p}$ factorizes over (X, f). In particular, $\mathcal{P}_p := \{\pi_{1/p}^{-1}(k) : k = 0, \ldots, p-1\}$ is a finite partition of X into open and closed sets which are cyclically permuted. In fact, p is an eventual divisor of $(S_n)_n$ if and only if (X, f^p) is not minimal. It follows that (X, f) is totally minimal (*i.e.* (X, f^p) is minimal for all $p > 0$) if and only if 1 is the only eventual divisor of $(S_n)_n$.

Suppose now that $\mathbf{p} = (p_1, p_2, p_3, \ldots)$ is a sequence of eventual divisors of $(S_n)_n$ and that $p_i | p_{i+1}$ for all i. Then all ($\mathbb{Z}_{p_i}$, "addition of 1") are factors of (Ω, T), and as $\Pi_{1/p_{i+1}} = \frac{p_{i+1}}{p_i} \cdot \Pi_{1/p_i}$, also the inverse limit of these finite rotations, namely the adding machine ($\mathbb{Z}_{\mathbf{p}}$, "addition of 1"), is a factor of (Ω, T). Denote the factor map by $\Pi_{1/\mathbf{p}}$. As all Π_{1/p_i} factorize over (X, f), the same is true for $\Pi_{1/\mathbf{p}}$. In particular, $\mathcal{P}_{p_{i+1}}$ is finer than $\mathcal{P}_{p_i}$ for all i. We will come back to this in Theorem 7.

3 Examples based on PV–numbers

We start this section with S–odometers given by a kneading map of the type $Q(n+1) = n - d$ for a fixed d.

For $d = 0$ we obtain the classical dyadic adding machine, *i.e.* the addition of 1 on the group of dyadic integers. In the context of unimodal maps it is conjugate to the action of a Feigenbaum map on its attractor.

For $d = 1$ we obtain the Fibonacci S–odometer. It was described by Lyubich and Milnor [LM] who proved that it is conjugate to the action of a Fibonacci map on the omega limit X of its critical point[2)]. Furthermore, the golden circle rotation is a factor of the Fibonacci S–odometer, and the factor map is 1–1 except on the set $\bigcup_{k \geq 1} T^{-k}\{\langle 0 \rangle\}$.

For all $d \geq 1$, there is a close connection with substitution spaces.

Lemma 6 *For $d \geq 1$, Ω is homeomorphic to the shift space Σ_χ on $d+1$ symbols, generated by the substitution $\chi : 0 \to 01, 1 \to 2, \dots, d-1 \to d, d \to 0$.*

Proof: Let $\mathcal{Z}_d = \{Z_{0\dots0}, Z_{10\dots0}, \dots, Z_{0\dots01}\}$ be the collection of cylinder sets of rank d. For each $\omega \in \Omega$, we can define the itinerary $a(\omega) \in \{0, \dots, d\}^{\mathbb{N}}$ with respect to $\mathcal{Z}_d$ as

$$a_i(\omega) = \begin{cases} 0 & \text{if} \quad T^i(\omega) \in Z_{0\dots0} \\ k & \text{if} \quad T^i(\omega) \in Z_{0\dots010\dots0} \quad (1 \text{ at entry } k \in \{1, \dots, d\}) \end{cases}$$

Clearly $a : \Omega \to a(\Omega)$ is continuous and if σ is the shift, then $a \circ T = \sigma \circ a$. Let us prove that a is injective, and therefore homeomorphic. We have seen before that T is injective on $\Omega \setminus T^{-1}\langle 0 \rangle$. Furthermore, each preimage of $\langle 0 \rangle$ lies in a different element of $\mathcal{Z}_d$. This shows that if $\omega, \omega' \in T^{-n}\langle 0 \rangle$ for some $n \geq 1$ and $\omega \neq \omega'$, then $a(\omega) \neq a(\omega')$. We use the induction hypothesis (H_k): If $\omega, \omega' \in \Omega$ are such that $\omega_i = \omega'_i$ for $i < k$ and $\omega_k \neq \omega'_k$, then $a(\omega) \neq a(\omega')$. For $k \leq d$ this is obviously true, because then ω and ω' lie in different elements of $\mathcal{Z}_d$. So assume that (H_{k-d}) is true, and that $\omega, \omega' \in \Omega$ are such that $\omega_i = \omega'_i$ for $i < k$ and $\omega_k \neq \omega'_k$. By the above remark, we can assume that $T^n(\omega) \neq T^n(\omega')$ for all $n \geq 0$. Assume $\omega_k = 0$, then $N = \sum_{i<k} \omega_i S_i = \sum_{i<k} \omega'_i S_i < S_{k-d}$. It is easy to check that $(T^{S_{k-d}-N}\omega)_i = (T^{S_{k-d}-N}\omega')_i$ for $i < k-d$ and $(T^{S_{k-d}-N}\omega)_{k-d} = 1 \neq (T^{S_{k-d}-N}\omega')_{k-d}$. So (H_k) follows, proving the induction.

Let $\alpha := a\langle 0 \rangle$ denote the itinerary of $\langle 0 \rangle$. It is easy to verify that the string $\chi^k(0)$ has length S_k and coincides with $\alpha_0, \dots, \alpha_{S_k - 1}$. Moreover, $\alpha_{S_{k-1}}, \dots, \alpha_{S_k - 1} = \alpha_0, \dots, \alpha_{S_k - S_{k-1} - 1}$. Since $\chi^k(0)$ converges to the unique fixed point of χ, this fixed point coincides with α. As the orbit of $\langle 0 \rangle$ is dense in Ω, $a(\Omega)$ equals Σ_χ, which by definition is the orbit closure with respect to the shift of the fixed point α. $\square$

Remark 3 The substitution χ on Σ_χ is conjugate to the right shift $\psi := \omega \mapsto 0\omega$ on Ω. Indeed,

$$a \circ \psi\langle n \rangle \;=\; a \circ \psi\langle \textstyle\sum_i \langle n \rangle_i S_i \rangle \;=\; a\langle \textstyle\sum_i \langle n \rangle_i S_{i+1} \rangle \;=\; \sigma^{\sum_i \langle n \rangle_i S_{i+1}}(\alpha) \; .$$

---

2) Observe that Lyubich and Milnor show that X is never an attractor for quadratic Fibonacci maps. However, in [BKNS] it is proved that for Fibonacci maps with a critical point of higher order X is indeed an attractor.

Furthermore $\chi \circ \sigma^n(\alpha) = \sigma^{n+t_n}(\alpha)$, where $t_n = \#\{0 \leq i < n : \alpha_i = 0\}$. Since $\alpha_{S_{k-1}}, \ldots, \alpha_{S_k-1} = \alpha_0, \ldots, \alpha_{S_k-S_{k-1}-1}$, we have $t_{S_k} = t_{S_{k-1}} + t_{S_{Q(k)}}$. By induction we get $t_{S_k} = S_{Q(k+1)}$ and finally $t_n = \sum_i \langle n \rangle_i S_{Q(i+1)}$, whence $t_n + n = \sum_i \langle n \rangle_i S_{i+1}$. Now

$$\chi \circ a\langle n \rangle \; = \; \chi \circ \sigma^n(\alpha) \; = \; \sigma^{\sum_i \langle n \rangle_i S_{i+1}}(\alpha) \; = \; a \circ \psi\langle n \rangle$$

This relation extends immediatly from $\langle \mathbb{N} \rangle$ to Ω. □

Now we investigate the cutting times for $d \geq 1$,

$$\begin{aligned} S_k &= k+1 \quad , \; k = 0, \ldots, d-1 \\ S_{k+1} &= S_k + S_{k-d} \quad , \; k \geq d \; . \end{aligned}$$

According to Theorem 5 in [GLT] these particular S–odometers are uniquely ergodic.

The characteristic polynomial $P_d(x) = x^{d+1} - x^d - 1$ of this recurrence has $d+1$ distinct roots $\lambda_0, \ldots, \lambda_d$, therefore the sequences $(\lambda_0^k)_{k\geq 0}, \ldots, (\lambda_d^k)_{k \geq 0}$ form a fundamental system of solutions. To get a nice formula for S_k it is more convenient to consider first the sequence $(s_k)_{k \geq 0}$ obtained from the initial condition $(s_0, \ldots, s_d) = (0, \ldots, 0, 1)$. The solution is $s_k = \sum_{\mu=0}^d c_\mu \cdot \lambda_\mu^k$, where $c_\mu^{-1} = \prod_{\nu=0,\, \nu \neq \mu}^d (\lambda_\mu - \lambda_\nu) = \lambda_\mu^d + d/\lambda_\mu$. So we have

$$S_k = s_{k+2d} = \sum_{\mu=0}^{d} C_\mu \cdot \lambda_\mu^k \quad , \text{ where } \quad C_\mu = c_\mu \cdot \lambda_\mu^{2d} = \frac{\lambda_\mu^{2d+1}}{\lambda_\mu^{d+1} + d} \tag{5}$$

For the following analysis we need some information about the roots of the characteristic polynomial.

Lemma 7 [Sel, NV]

i) Every polynomial $P_d(x)$ has a unique positive root $\lambda_0^{(d)}$ which is greater than 1.

ii) Let $O_d := \{\mu \in \{0, \ldots, d\} : |\lambda_\mu| \geq 1\}$. Then $O_d = \{0\}$ iff $d \in \{1, 2, 3\}$. In all other cases $\#O_d \geq 3$. (In fact $\#O_d = \min\{2k+1 : k \in \mathbb{N},\, 2k+1 \geq \frac{d}{3}\}$.)

iii) If $d \not\equiv 4 \pmod 6$, then $P_d(x)$ is irreducible. If $d \equiv 4 \pmod 6$, then $P_d(x) = (x^2 - x + 1) \cdot \tilde{P}_d(x)$ for some irreducible $\tilde{P}_d \in \mathbb{Z}[x]$.

iv) If $d \not\equiv 4 \pmod 6$, then the Galois group of P_d acts as the full permutation group on the $d+1$ roots of P_d.

Proof: i), ii), and iii) follow from [Sel] with some additional elementary considerations. iv) is from [NV], see also [O, Corollary 2]. □

So for $d = 1, 2, 3$ the root $\lambda_0^{(d)}$ is an algebraic integer greater than 1 whose conjugates have modulus less than 1. Such numbers are called *Pisot–Vijayaraghavan numbers* (cf.

[Pi], [Sal] and [Sie]). Henceforth we fix some d and write λ_μ instead of $\lambda_\mu^{(d)}$. Observe that for each μ

$$S_k\lambda_\mu^p \;=\; C_\mu\lambda_\mu^{k+p}+\sum_{\substack{\nu=0\\ \nu\neq\mu}}^{d}C_\nu\lambda_\mu^p\lambda_\nu^k \;=\; S_{k+p}+\sum_{\substack{\nu=0\\ \nu\neq\mu}}^{d}C_\nu(\lambda_\mu^p-\lambda_\nu^p)\lambda_\nu^k\;. \tag{6}$$

Therefore the factor map $\Pi_p:\Omega\to\mathbb{T}^1$ from the last section is well defined for $\rho=\lambda_0^p$ if λ_0 is a PV–number, *i.e.* in cases $d=1,2,3$. As $P_d(x)$ is the minimal polynomial of λ_0 in these cases, the numbers $\lambda_0,\dots,\lambda_0^d$ are rationally independent. Therefore (Ω,T) factors onto the d–torus equipped with the rotation $R_{\lambda_0}\times\cdots\times R_{\lambda_0^d}$ via the factor map $\Pi_{\lambda_0}\times\cdots\times\Pi_{\lambda_0^d}$.

The next theorem shows that these are essentially all rotations we can get as continuous factors of (Ω,T).

Theorem 3 *Let (Ω,T) be as above.*

1. *Suppose $d\in\{1,2,3\}$. If $\phi:(\Omega,T)\to(\mathbb{T}^1,R_\rho)$ is a continuous factor map, then there are $q_0,\dots,q_d\in\mathbb{Z}$ such that $\rho=q_0+q_1\lambda_0+\cdots+q_d\lambda_0^d$ and ϕ factors over $\Pi_{\lambda_0}\times\cdots\times\Pi_{\lambda_0^d}$ via $\phi=q_1\Pi_{\lambda_0}+\cdots+q_d\Pi_{\lambda_0^d}$. In fact, there are no other L^2–eigenfunctions of T than these functions ϕ.*
2. *If $d\geq4$ the system (Ω,T) is weakly mixing with respect to its unique invariant probability measure, but not mixing.*

Proof: Let m denote the unique invariant probability measure for T (see [GLT, Theorem 5]). Suppose that $\phi:(\Omega,T)\to(\mathbb{T}^1,R_\rho)$ is a measurable homomorphism, *i.e.* $e^{2\pi i\phi}$ is an L_m^2–eigenfunction of T. In view of Lemma 6 we can apply Theorem 1.4 of [Ho] from which it follows that ϕ coincides a.e. with a continuous map. So we may assume that ϕ is continuous.

We first rule out the possibility that $\rho\in\mathbb{Q}\setminus\mathbb{Z}$. To this end assume that $\rho=\frac{p}{q}$, $(p,q)=1$, $q>1$. As T is minimal (Lemma 2), it follows that the restriction of R_ρ to $\phi(\Omega)$ is minimal such that $\phi(\Omega)=\{R_\rho^k(0):k=0,\dots,q-1\}$. Hence $(\mathbb{Z}_q,+1)$ is a factor of (Ω,T) and in view of Remark 2, q must be an eventual divisor of $(S_n)_{n>0}$, a contradiction to the definition of S_n.

From Lemma 3 it follows that $\sum_k\|\rho S_k\|<\infty$. Let $I_d=\{0,\dots,d\}\setminus O_d$. As $S_k=\sum_{\mu\in I_d}C_\mu\lambda_\mu^k+\sum_{\mu\in O_d}C_\mu\lambda_\mu^k$ by (5), we find that

$$\sum_k\left\|\sum_{\mu\in O_d}\rho C_\mu\lambda_\mu^k\right\|\leq\sum_k\|\rho S_k\|+\sum_{\mu\in I_d}\sum_k\rho\cdot|C_\mu|\cdot|\lambda_\mu|^k<\infty\;.$$

It follows from [Ma, Lemma 2] that $\rho C_\mu \in \mathbb{Q}(\lambda_\mu)$ for all $\mu \in O_d$. As $C_\mu \in \mathbb{Q}(\lambda_\mu)$ by (5), we conclude that $\rho \in \mathbb{Q}(\lambda_\mu)$ for all $\mu \in O_d$.

1st case: $d \equiv 4 \pmod 6$
Then $(x^2-x+1)|P_d(x)$ so that $\frac{1}{2}(1\pm i\sqrt{3}) \in \{\lambda_\mu \mid \mu \in O_d\}$. As $\rho \in \mathbb{R}\cap\mathbb{Q}(\frac{1}{2}(1\pm i\sqrt{3}))$, it follows that $\rho \in \mathbb{Q}$, and because of the introductory part of this proof, $\rho \in \mathbb{Z}$. So (T,μ) is weakly mixing. The non mixing assertion follows in both cases from [DK, Theorem 2] together with Lemma 6.

2nd case: $d \not\equiv 4 \pmod 6$
Now P_d is irreducible by Lemma 7. As $\rho \in \mathbb{Q}(\lambda_\mu)$ for all $\mu \in O_d$, there are polynomials $Q_\mu \in \mathbb{Z}[x]$, $Q_\mu(x) = \sum_{i=0}^d q_{\mu,i}x^i$, and $q_\mu \in \mathbb{Z}\setminus\{0\}$ such that $Q_\mu(\lambda_\mu) = q_\mu\rho$. If $\rho \in \mathbb{Q}$, then as before, $\rho \in \mathbb{Z}$ and $\phi = 0$ is constant. Otherwise $\rho \notin \mathbb{Q}$ and the $Q_\mu(x)$ are nonconstant.

For $\mu \in O_d$ and $k \in \mathbb{N}$ it follows from (6) that

$$\begin{aligned} S_k q_\mu \rho &= S_k Q_\mu(\lambda_\mu) = \sum_{i=0}^d q_{\mu,i} S_k \lambda_\mu^i \qquad (7)\\ &= \sum_{i=0}^d q_{\mu,i}S_{k+i} + \sum_{i=0}^d q_{\mu,i} \sum_{\nu\in I_d} C_\nu(\lambda_\mu^i - \lambda_\nu^i)\lambda_\nu^k + \sum_{i=0}^d q_{\mu,i} \sum_{\nu\in O_d\setminus\{\mu\}} C_\nu(\lambda_\mu^i - \lambda_\nu^i)\lambda_\nu^k\ , \end{aligned}$$

and since $\sum_k \|q_\mu S_k\rho\| \le |q_\mu|\sum_k \|S_k\rho\| < \infty$, we conclude that for $\mu \in O_d$

$$\sum_k \Big\| \sum_{\nu\in O_d\setminus\{\mu\}} \underbrace{\left(\sum_{i=0}^d q_{\mu,i}C_\nu(\lambda_\mu^i - \lambda_\nu^i)\right)}_{=:A_{\mu,\nu}} \lambda_\nu^k \Big\| < \infty\ .$$

If $A_{\mu,\nu} \neq 0$ for some $\mu,\nu \in O_d$, then $|\lambda_\mu| < 1$ by [Ma, Lemma 2], which contradicts $\mu \in O_d$. Hence $A_{\mu,\nu} = 0$ for all $\mu,\nu \in O_d$, *i.e.* $Q_\mu(\lambda_\mu) = Q_\mu(\lambda_\nu)$ for all $\mu,\nu \in O_d$. (This part of the proof is similar to arguments leading to Theorem 2 of [L2], see also [So4, Theorem 2.2].)

For the rest of the proof we adapt an argument from [So4, Corollary 2.3]. Let F be the splitting field of $P_d(x)$ and denote $\alpha := Q_0(\lambda_0)$. Let $A := \{\nu \in \{0,\dots,d\} : Q_0(\lambda_\nu) = \alpha\}$. Then $O_d \subseteq A$. If τ is any automorphism of F over $\mathbb{Q}$, then $Q_0(\tau\lambda_\nu) = \tau Q_0(\lambda_\nu) = \tau\alpha$ for all $\nu \in A$ such that $\tau A = A$ (if $\tau\alpha = \alpha$) or $\tau A \cap A = \emptyset$ (if $\tau\alpha \neq \alpha$). But as the Galois group of $P_d(x)$ allows all permutations of $\{\lambda_0,\dots,\lambda_d\}$ (Lemma 7), and as $Q_0(x)$ is a nonconstant polynomial of degree at most d, this is possible only if $\#A = 1$. But then $\#O_d = 1$, too, and hence $d \in \{1,2,3\}$ by Lemma 7. This proves the theorem for $d \ge 4$, $d \not\equiv 4 \pmod 6$.

Rewriting (7) for $\mu = 0$ yields

$$S_k\rho = \sum_{i=0}^d \frac{q_{0,i}}{q_0} S_{k+i} + \sum_{\nu\in I_d}\left(\sum_{i=0}^d q_{0,i}C_\nu(\lambda_0^i - \lambda_\nu^i)\right)\lambda_\nu^k\ .$$

As $\|S_k\rho\| \to 0$, it follows that $\sum_{i=0}^{d} \frac{q_{0,i}}{q_0} S_{k+i} \in \mathbb{Z}$ for sufficiently large k. Suppose this is true for $k \geq k_0$. Then it is also true for $k = k_0 - d$ in view of the recurrence relation for the S_k, and by induction it follows that it holds for $k \geq -2d$. Evaluating this relation for $k = -2d, \ldots, -d$ and observing that $S_{-2d} = \cdots = S_{-d-1} = 0$, $S_{-d} = \cdots = S_1 = 1$ yields successively $q_0 | q_{0,d}, q_0 | q_{0,d-1}, \ldots, q_0 | q_{0,0}$, *i.e.* $q_0 = 1$ and $\rho = Q_0(\lambda_\mu)$. Now the conclusions of the theorem follow immediately. □

The isomorphism onto the 2 and 3–torus

Theorem 4 ([VL, GLT]) *For $d = 2, 3$, the factor map ϕ is a metric isomorphism onto the 2– and 3–dimensional torus, respectively.*

Proof: For $d = 2$ we can use [GLT, Theorem 6 and Remark 5], because Hypothesis B of that paper is satisfied. This hypothesis requires a certain uniform control on "backward carries" when adding numbers $\langle S_m \rangle$ to elements $\omega \in \Omega$. In our terminology it reads:

> There is $b \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, all $m \geq k + b + 2$ and all $\omega \in \Omega$ with $\omega_{k+1} = \cdots = \omega_{k+b+1} = 0$ holds: $(T^{S_m}\omega)_j = \omega_j$ for $j = 0, \ldots, k$.

Observing that $2S_m = S_{m+1} + S_{m-2} + S_{m-7}$, $S_m + S_{m+1} = S_{m+2} + S_{m-3}$, and $S_m + S_{m+2} = S_{m+3}$ for $d = 2$, it is not hard to check this with $b = 6$. For $d = 3$ the hypothesis fails however.

We can prove both cases using a result of Vershik and Livshits [VL, Theorem 3]. Recall that for $d = 2, 3$, the space Ω is homeomorphic to the shift space generated by the substitution $0 \to 01$, $1 \to 2$, $2 \to 0$, respectively $0 \to 01$, $1 \to 2$, $2 \to 3$, $3 \to 0$. The assumptions of their theorem are satisfied for $\Delta_C = \{\binom{0}{0}_1, \binom{01}{10}_2, \binom{012}{120}_1\}$ and $\Delta = \{\binom{1}{1}_1, \binom{0120}{2001}_2\}$ when $d = 2$, and for $\Delta_C = \{\binom{0}{0}_1, \binom{01}{10}_2, \binom{012}{120}_2, \binom{0123}{1230}_4\}$, $\Delta = \{\binom{1}{1}_1, \binom{2}{2}_1, \binom{3}{3}_1, \binom{1230}{3012}_3, \binom{01230}{30012}_2, \binom{1230010}{3001012}_2, \binom{01201230}{23001012}_3, \binom{010120123}{120123001}_2, \binom{0101201230}{3001012012}_5, \binom{12300123001}{30010120123}_2\}$, when $d = 3$. Here $\binom{a}{b}_k$ denotes a pair of balanced blocks and k the number of substitutions needed before a decomposition in elements of $\Delta_C \cup \Delta$ is possible. □

Additionally we give an elementary geometric proof of Theorem 4 for $d = 2$, which at the same time makes explicit the relation to a certain tiling of the plane. It bears some resemblance with [Ra] where the recurrence $S_{n+1} = S_n + S_{n-1} + S_{n-2}$ is treated. We try to understand the nature of the factor map

$$\begin{array}{rcccl} \Pi_{\lambda_0} \times \Pi_{\lambda_0^2} & : & \Omega & \to & \mathbb{T}^2 \\ & & \omega & \mapsto & \sum_{k\geq 0} \omega_k \,(S_k\lambda_0, S_k\lambda_0^2) \quad (\bmod\ \mathbb{Z}\times\mathbb{Z}\ ) \, . \end{array}$$

We will prove the following:

> The projection $\Pi_{\lambda_0} \times \Pi_{\lambda_0^2}$ is almost everywhere one–to–one with respect to the Lebesgue measure on the 2-torus. More precisely: For $x \in \Omega$ let $Z_{x_0\ldots x_{n-1}}$ denote the cylinder of order n consisting of all $\omega \in \Omega$ such that $\omega_i = x_i$ for $i = 0, \ldots, n-1$. Then for fixed n the images of two different sets $Z_{x_0\ldots x_{n-1}}$ and $Z_{y_0\ldots y_{n-1}}$ are closed subsets of the torus which intersect only in a set of Lebesgue measure zero.

Since according to (6) we have $S_k\lambda_0^p - S_{k+p} = K_1^{(p)}\lambda_1^{k+4} + K_2^{(p)}\lambda_2^{k+4}$, where we defined $K_j^{(p)} := c_j(\lambda_0^p - \lambda_1^p)$, the factor map can be decomposed in the following way:

1. A map
$$\begin{array}{rcccl} \tilde{\Pi} & : & \Omega & \to & \mathbb{C} \\ & & \omega & \mapsto & \sum_{k\geq 0} \omega_k \, \lambda_1^{k+4} \end{array} \tag{8}$$
which creates a "fractal" set $\tilde{\mathrm{P}} := \tilde{\Pi}(\Omega)$ in the shape of a five legged poodle, cf. Figure 1.
2. A linear deformation $\tilde{\mathrm{P}} \mapsto \phi(\tilde{\mathrm{P}}) =: \mathrm{P}$, where
$$\begin{array}{rcccl} \phi & : & \mathbb{C} \simeq \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & & z & \mapsto & (\phi_1(z), \phi_2(z)) \ := \ (K_1^{(1)}z + K_2^{(1)}\bar{z}, K_1^{(2)}z + K_2^{(2)}\bar{z}) \end{array}$$
3. and finally the canonical projection $\pi_{\mathbb{T}^2}$ from $\mathbb{R}^2$ onto $\mathbb{T}^2$.

Lemma 8 *i) Fix $n \geq 2$. Then every set $\tilde{\mathrm{P}}_{x_0\ldots x_{n-1}} := \tilde{\Pi}(Z_{x_0\ldots x_{n-1}})$ is up to a translation identical with one of the three sets $\lambda_1^n\tilde{\mathrm{P}}$, $\lambda_1^{n+1}\tilde{\mathrm{P}}$ or $\lambda_1^{n+2}\tilde{\mathrm{P}}$. More precisely:*

$$\tilde{\mathrm{P}}_{x_0\ldots x_{n-1}} = \sum_{i=0}^{n-1} x_i\lambda_1^{i+4} + \left\{ \begin{array}{l} \lambda_1^n \ \ \tilde{\mathrm{P}} \\ \lambda_1^{n+1}\,\tilde{\mathrm{P}} \\ \lambda_1^{n+2}\,\tilde{\mathrm{P}} \end{array} \right\} \ \textit{if the string } x_0\ldots x_{n-1} \textit{ ends with } \left\{ \begin{array}{l} 00 \\ 10 \\ 01 \end{array} \right\} . \tag{9}$$

ii) The poodle $\tilde{\mathrm{P}}$ is self–similar in the sense of [Hu], i.e. it coincides with the unique nonvoid compact set invariant under the three contractions

$$\psi_{00}: \ z \mapsto \lambda_1^2 z \ , \quad \psi_{100}: \ z \mapsto \lambda_1^4 + \lambda_1^3 z \quad \textit{and} \quad \psi_{0100}: \ z \mapsto \lambda_1^5 + \lambda_1^4 z \ ,$$

which map $\tilde{\mathrm{P}}$ to $\tilde{\mathrm{P}}_{00}$, $\tilde{\mathrm{P}}_{100}$ and $\tilde{\mathrm{P}}_{0100}$ respectively. These three sets form a partition of $\tilde{\mathrm{P}}$ modulo a null set.

iii) The translates $k_0\lambda_1^0 + k_1\lambda_1^1 + \tilde{\mathrm{P}}$, $k_i \in \mathbb{Z}$, of the poodle intersect only in null sets.

Proof: i) If $x_0 \dots x_{n-1}$ ends with 10 or 01 then the set $Z_{x_0 \dots x_{n-1}}$ is identical with $Z_{x_0 \dots x_{n-1}0}$ and $Z_{x_0 \dots x_{n-1}00}$ respectively, since any two ones have to be separated by at least two zeroes. So consider only the case where $x_0 \dots x_{n-1}$ ends with 00. In this case $\sigma^n(Z_{x_0 \dots x_{n-1}})$ equals Ω where σ denotes the left shift on Ω. Now for any $\omega \in Z_{x_0 \dots x_{n-1}}$ we have $\tilde{\Pi}(\omega) = \sum_{i=0}^{n-1} x_i \lambda_1^{i+4} + \lambda_1^n \tilde{\Pi}(\sigma^n \omega)$, which proves (9).

ii) Clearly, $\Omega = Z_{00} \cup Z_{100} \cup Z_{0100}$, so the first assertion follows from (9). To prove the second one we show that the Lebesgue measure $\mu(\tilde{\mathrm{P}})$ equals $\mu(\tilde{\mathrm{P}}_{00}) + \mu(\tilde{\mathrm{P}}_{100}) + \mu(\tilde{\mathrm{P}}_{0100})$: According to i), $\mu(\tilde{\mathrm{P}}_{00}) + \mu(\tilde{\mathrm{P}}_{100}) + \mu(\tilde{\mathrm{P}}_{0100}) = (|\lambda_1^2|^2 + |\lambda_1^3|^2 + |\lambda_1^4|^2) \cdot \mu(\tilde{\mathrm{P}})$, whence it suffices to verify $\lambda_1^2\lambda_2^2 + \lambda_1^3\lambda_2^3 + \lambda_1^4\lambda_2^4 = 1$. But $\lambda_1^2\lambda_2^2 + \lambda_1^3\lambda_2^3 + \lambda_1^4\lambda_2^4 - 1 = \lambda_0^{-2} + \lambda_0^{-3} + \lambda_0^{-4} - 1$, since $\lambda_0\lambda_1\lambda_2 = 1$. The right hand term, multiplied by λ_0^4, is equal to $\lambda_0^2 + \lambda_0 + 1 - \lambda_0^4 = -(\lambda_0 + 1)(\lambda_0^3 - \lambda_0^2 - 1) = 0$.

iii) First we have to estimate the shape of the poodle in order to prove that only adjacent poodles can intersect[3)]: For fixed n let $d_n := \max\{|\tilde{\Pi}(\omega)| \bigm| \omega_k = 0, \forall k \geq n\}$. Then $|\sum_{k \geq n} \omega_k \lambda_1^{k+4}| \leq |\lambda_1|^n \sum_{k \geq 0} d_n |\lambda_1|^{nk} = |\lambda_1|^n \cdot d_n/(1 - |\lambda_1|^n) =: r_n$ holds for any $\omega \in \Omega$, so the set $\tilde{\mathrm{P}}$ is contained in the union of S_n balls of radius r_n, centered at the points $\tilde{\Pi}(\langle k \rangle)$, $0 \leq k < S_n$. For $n = 9$ the estimates are good enough to show that this union is contained in the parallelogram $\{\, \alpha + \beta\lambda_1 \mid \alpha \in [-0.79, 0.90],\ \beta \in [-0.84, 1.12] \,\}$. whose sides are strictly smaller than 2 resp. $2 \cdot |\lambda_1|$. So two poodles $k_0 + k_1\lambda_1 + \tilde{\mathrm{P}}$ and $k_0' + k_1'\lambda_1 + \tilde{\mathrm{P}}$ can only intersect if $|k_i - k_i'| \leq 1$, $i = 1, 2$.

In order to prove iii) it remains to show that the sets $\tilde{\mathrm{P}}$, $1 + \tilde{\mathrm{P}}$, $\lambda_1 + \tilde{\mathrm{P}}$ and $1 + \lambda_1 + \tilde{\mathrm{P}}$ are mutually disjoint up to a null set. The clue to the proof is the observation that these four poodles can already be found "en miniature" after a similarity transformation $z \mapsto \lambda_1^7 z$ in different parts $\tilde{\mathrm{P}}_{x_0 \dots x_{n-1}}$ of the original poodle $\tilde{\mathrm{P}}$:

$$\begin{array}{lcl}
\tilde{\mathrm{P}} & \xrightarrow{\cdot\lambda_1^7} & \lambda_1^7\tilde{\mathrm{P}} & = & \tilde{\mathrm{P}}_{0000000} \\
1 + \tilde{\mathrm{P}} & \xrightarrow{\cdot\lambda_1^7} & \lambda_1^7 + \lambda_1^7\tilde{\mathrm{P}} & = & \tilde{\mathrm{P}}_{0001000} \\
\lambda_1 + \tilde{\mathrm{P}} & \xrightarrow{\cdot\lambda_1^7} & \lambda_1^8 + \lambda_1^7\tilde{\mathrm{P}} & = & \tilde{\mathrm{P}}_{0000100} \\
1 + \lambda_1 + \tilde{\mathrm{P}} & \xrightarrow{\cdot\lambda_1^7} & \lambda_1^7 + \lambda_1^8 + \lambda_1^7\tilde{\mathrm{P}} & \subseteq & \tilde{\mathrm{P}}_{10000100} \cup \tilde{\mathrm{P}}_{10000000}\,.
\end{array}$$

Here we made use of the "carry rule" $\lambda_1^{k+1} = \lambda_1^k + \lambda_1^{k-2}$ to convert the "illegal" sums in the middle row into "legal" ones and of equation (9). By ii), these parts intersect only in null sets. □

As an immediate consequence of Lemma 8 we get the following: For any $n \in \mathbb{N}$ the sets $\tilde{\mathrm{P}}_{\langle k \rangle_0 \dots \langle k \rangle_{n-1}}$, $k = 0, \dots, S_n - 1$, form a partition of $\tilde{\mathrm{P}}$ modulo null sets. Since all points $z \in \mathbb{C}$ which have more than one preimage in ω have to lie in at least two distinct sets $\mathrm{P}_{x_0 \dots x_{n-1}}$ and $\mathrm{P}_{y_0 \dots y_{n-1}}$ for some $n \in \mathbb{N}$, the set of all those points has Lebesgue measure zero.

[3)]The calculations are tedious but straightforward, so the reader may prefer to just have a quick glance at Figure 2 in order to be convinced of this assertion.

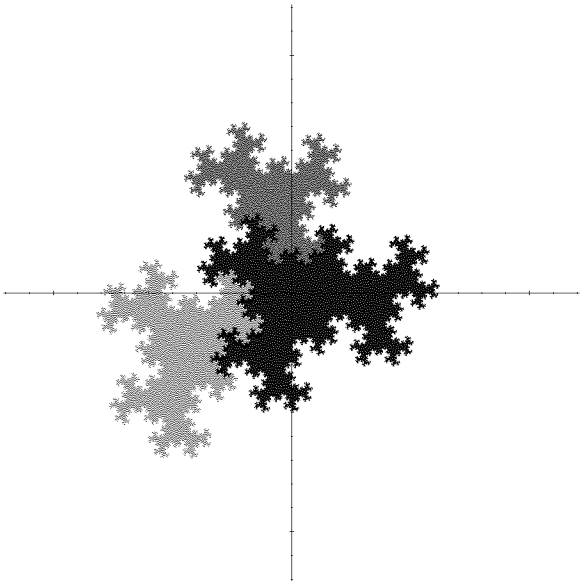


Figure 1: The poodle $\tilde{\mathrm{P}}$ with the parts $\tilde{\mathrm{P}}_{00}$, $\tilde{\mathrm{P}}_{100}$ and $\tilde{\mathrm{P}}_{0100}$ shaded differently. For better comparison between Figures 1 and 2 we plotted $-\Re(z)$ versus $-\Im(z)$.

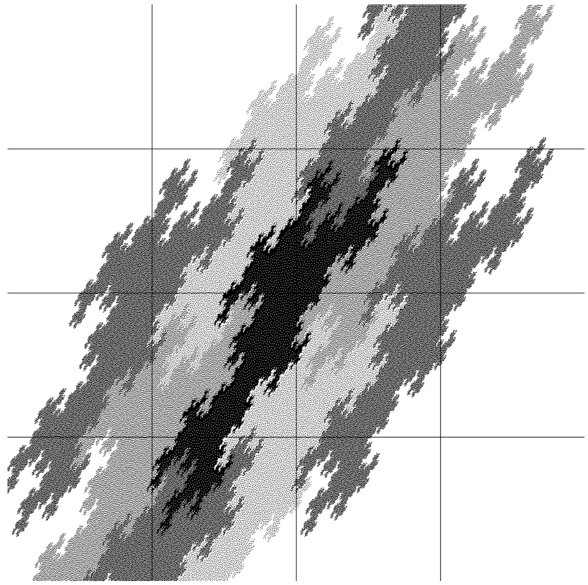


Figure 2: The deformed poodle P and its eight immediate neighbours.

Now the only thing left is to show that by the map $\pi_{\mathbb{T}^2} \circ \phi$ the set $\tilde{\mathrm{P}}$ is squeezed nicely into the torus, in such a way that only boundary points[4)] of $\tilde{\mathrm{P}}$ become identified (compare fig. 2). To do this one just has to transform the corresponding result for $\tilde{\mathrm{P}}$ in Lemma 8 iii). Since

$$\phi_k(\lambda_1^p) = K_1^{(p)}\lambda_1^p + K_2^{(p)}\lambda_2^p = S_{k-4}\lambda_0^p - S_{k+p-4} = \begin{cases} 0, & 0 \le k+p < 2 \\ -1, & 2 \le k+p < 4 \end{cases},$$

the images of λ_1^1 and λ_1^0 are $\phi(\lambda_1^1) = -\binom{1}{1}$ and $\phi(\lambda_1^0) = -\binom{0}{1}$. This shows that the projection $\pi_{\mathbb{T}^2} : \mathrm{P} \to \mathbb{T}^2$ is one–to–one except possibly on the boundary of P. Since the factor map $\Pi_{\lambda_0} \times \Pi_{\lambda_0^2}$ induces a semi-conjugacy between T and an irrational rotation on the torus, the image $\pi_{\mathbb{T}^2}(\mathrm{P})$ must be the whole torus.

Fractal tilings of the plane by sets like the poodle and their connection to Pisot numbers and finite state automata are investigated in numerous papers, see *e.g.* [Th, Ba, BG]. The poodle itself was studied also by [Ge]. For tilings in dimension $d > 2$ see [So2].

[4)] By "boundary" we mean the intersection of two adjacent poodles.

Factor maps onto d–dimensional tori

A generalization of the previous construction leads to

Theorem 5 *For every $d \in \mathbb{N}$ there exist rationally independent irrationals $\rho_1, \dots, \rho_d$ and a unimodal map f such that the torus rotation $(x_1, \dots, x_d) \mapsto (x_1+\rho_1, \dots, x_d+\rho_d)$ is a continuous factor of $f|_X$. The factor map is also a measure theoretic isomorphism. Furthermore, the map f is not renormalizable, and can be chosen to have a wild attractor.*

Proof: Let $a_0 \geq a_1 \geq \dots \geq a_d \geq 1$ be integers and let $\lambda_0, ..., \lambda_d$ be the roots of the equation

$$x^{d+1} - a_0 x^d - \dots - a_d = 0.$$

According to [Bra], the largest root λ_0 is an irrational PV–number. Hence the numbers $\rho_i := \lambda_0^i$ for $i = 1, ..., d$ are rationally independent irrationals. Let $(R_n)_{n \geq 0}$ be the sequence of integers satisfying

$$R_{k+1} = \begin{cases} k+1 & \text{if} \quad 0 \leq k \leq d. \\ a_0 R_k + \dots + a_d R_{k-d} & \text{if} \quad k > d. \end{cases}$$

According to Solomyak [So3], the odometer based on the numbers R_i is measure theoretic isomorphic to the torus rotation $(x_1, \dots, x_d) \to (x_1 + \rho_1, \dots, x_d + \rho_d)$.

Next we construct a sequence $(m_i)_{i \geq 0}$ and cutting times $(S_i)_{i \geq 0}$ with the property that $S_{m_i} = R_i$. Let $A = (\sum_{i=0}^d a_i) - 1$ and

$$m_i = \begin{cases} i & \text{if} \quad 0 \leq i \leq d. \\ m_{i-1} + A = (i-d)A + d & \text{if} \quad i > d. \end{cases}$$

Let $h : \{1, \cdots, A\} \to \{0, \cdots, d-1\}$ be defined as

$$\begin{array}{c} h(1) = h(2) = \dots = h(a_0 - 1) = 0. \\ h(a_0) = h(a_0 + 1) = \dots = h(a_0 + a_1 - 1) = 1. \\ \vdots \qquad \vdots \qquad \vdots \\ h(a_0 + \dots + a_{d-1}) = h(a_0 + \dots + a_{d-1} + 1) = \dots = h(A) = d. \end{array}$$

Next let the kneading map

$$\begin{cases} Q(m) = 0 & \text{if} \quad m \leq d. \\ Q(m_i + j) = m_{i-h(j)} & \text{if} \quad i \geq d \quad \text{and} \quad 1 \leq j \leq A. \end{cases}$$

A straightforward calculation shows that Q satisfies the admissibility condition (1), as well as the conditions of Theorem 0. Moreover, $S_{m_i} = R_i$. It is shown in e.g. [Bru, Lemma 2.3] that f is renormalizable if and only if there exists $k \geq 1$ such that

$$Q(k+j) \geq k \quad \text{for all} \quad j \geq 1. \tag{10}$$

This is clearly not the case, so f is not renormalizable. By the construction of $(R_i)_i$ and the fact that λ_0 is a PV–number it follows that $|\rho_r R_k - R_{k+r}| \to 0$ exponentially. Therefore $\sum_m \|\rho_r S_m\| < \infty$. Indeed, for $1 \leq j \leq A$ and $i \geq d$,

$$S_{m_i+j} = \sum_{k=0}^{d} e_{k,j} S_{m_{i-k}},$$

for integers $0 \leq e_{k,j} \leq B := \max_i a_i$. Hence

$$\begin{aligned}\sum_m \|\rho_r S_m\| &\leq d+1+\sum_{i\geq d}\sum_{j=1}^{A} \|\rho_r S_{m_i+j}\| \leq d+1 \sum_{i\geq d}\sum_{j=1}^{A}\sum_{k=0}^{d} e_{k,j}\|\rho_r S_{m_{i-k}}\| \\ &\leq d + A\,B\,(d+1)\sum_{i\geq 0}\|\rho_r R_i\| < \infty.\end{aligned}$$

This shows that there is a continuous factor map. Finally, in order to apply the result in [So3], let us show that the R–odometer and the S–odometer are homeomorphic. Any integer $n \geq 0$ can be represented in both number systems: $n = \sum_j b_j R_j = \sum_i a_i S_i$. The sequences $(a_i)_i$ and $(b_j)_j$ can be expressed in terms of each other; in fact $b_j = b_j(a_{m_j}, a_{m_j+1}, \ldots)$ and $a_i = a_i(b_k, b_{k+1}, \ldots)$, where k is such that $m_k \leq i < m_{k+1}$. So the transformation relating $(a_i)_i$ to $(b_j)_j$ is uniformly continuous in both directions, and therefore extends to a homeomorphism between the whole R–odometer and S–odometer. □

4 Examples based on continued fractions

Let $\rho \in [0,1] \setminus \mathbb{Q}$ and $[0, a_1, a_2, \ldots]$ be its continued fraction. The convergents will be written as $\frac{p_i}{q_i}$. Hence $\frac{p_0}{q_0} = \frac{0}{1}$, $\frac{p_1}{q_1} = \frac{1}{a_1}$ and in general $p_i = a_i p_{i-1} + p_{i-2}$ and $q_i = a_i q_{i-1} + q_{i-2}$. It follows by induction that $p_i q_{i+1} - p_{i+1} q_i = (-1)^{i+1}$. Furthermore $\frac{p_{2i}}{q_{2i}} < \rho < \frac{p_{2i+1}}{q_{2i+1}}$, whence $|\rho - \frac{p_j}{q_j}| \leq \frac{1}{q_j q_{j+1}}$.

Theorem 6 *For every $\rho \in [0,1] \setminus \mathbb{Q}$ there exists a sequence of cutting times $(S_k)_k$ such that $\Pi_\rho : \Omega \to \mathbb{S}^1$ is continuous. The sequence (S_k) is admissible, the corresponding map f is non–renormalizable and its kneading map $Q(k)$ tends to infinity. Moreover, the map Π_ρ is one–to–one, except in the preimages of 0, where it is two–to–one.*

The construction of this kind of examples is related to so–called almost restricted intervals (cf. [J]). In [GS] they are called rotation–like mappings. During our research we learned that Paul Glendinning was working on these examples, too. The Fibonacci map is delayed, at every closest return, by a_{n+1} loops in an almost restrictive interval of period q_n.

Remark 4 If ρ is of bounded type, i.e. (a_i) is bounded, then the kneading map Q constructed in the proof satisfies $n - Q(n) \leq 2 \cdot \max_k a_k$. So it follows from [Bru] that this combinatorial type can coexist with a wild attractor. For rotation numbers of unbounded type we think that the same is true.

Proof: Let again $(q_k)_{k \geq 0}$ be the denominators of the convergents of ρ. We will construct a sequence of cutting times, and a sequence $(n_k)_{k \geq 0}$ with the property that $S_{n_k} = q_k$. Let the kneading map Q be as follows:

$$Q(n) = 0 \quad \text{for} \quad n < q_1 \ .$$

Then $S_0 = 1$ and $S_{q_1 - 1} = q_1$, so $n_0 = 0$ and $n_1 = q_1 - 1$. Inductively, set $n_{k+1} = n_k + a_{k+1}$ and

$$Q(n) = \begin{cases} n_k & \text{if} \quad n_k < n < n_{k+1} \ , \\ n_{k-1} & \text{if} \quad n = n_{k+1} \ . \end{cases}$$

Then indeed $S_{n_k} = q_k$ and $S_{n_k + j} = (j+1) q_k$ for $1 \leq j < a_{k+1}$. Q is admissible, see (1), because for $n_k \leq n < n_{k+1}$,

$$Q(Q^2(n) + 1) \leq n_{k-2} < n_{k-1} \leq Q(n+1) \ .$$

It follows from (10) that f is not renormalizable.

Let $\omega = \omega_0 \omega_1 \ldots \in \Omega$. As $Q(n+1) \leq n_k$ for every $n_k \leq n < n_{k+1}$, there can be only one $n_k \leq n < n_{k+1}$ such that $\omega_n = 1$. As $\|\rho j q_k\| \leq j |\rho q_k - p_k| \leq j q_k |\rho - \frac{p_k}{q_k}| \leq \frac{j}{q_{k+1}}$ and $S_{n_k + j} = (j+1) S_{n_k}$ for $0 \leq j < a_{k+1}$, it follows that

$$\sum_k \omega_k \|\rho S_k\| \leq \sum_k \max\{\|\rho j q_k\| : 0 < j \leq a_{k+1}\} \leq \sum_k \frac{1}{q_k} < \infty \ .$$

Let us now prove the last statement of the theorem. The choice of kneading map shows that for each $\omega \in \Omega$, there is at most one n, $n_k \leq n < n_{k+1}$ such that $\omega_n = 1$, and if $\omega_n = 1$ for some $n_k \leq n < n_{k+1}$, then $\omega_{n_{k+1} + a_{k+2} - 1} = \omega_{n_{k+2} - 1} = 0$. Therefore there is a one–to–one correspondence between Ω and Σ, where

$$\Sigma = \{(b_i)_{i \geq 0} : 0 \leq b_i \leq a_{i+1}, \quad b_i \neq 0 \Rightarrow b_{i+1} < a_{i+2}\}.$$

Indeed, one assigns $b \in \Sigma$ to $\omega \in \Omega$, where $b_i = 0$ if $\omega_n = 0$ for $n_i \leq n < n_{i+1}$, else $b_i > 0$ is chosen such that $\omega_{n_i + b_i - 1} = 1$. A sequence ω is *eventually maximal* if there exists K such that either $\omega_{n_{2k} - 1} = 1$ for all $k \geq K$, or $\omega_{n_{2k+1} - 1} = 1$ for all $k \geq K$. It follows that $T^m(\omega) = \langle 0 \rangle$ for some $m \geq 1$ if and only if ω is eventually maximal. The eventually maximal sequences correspond to the following sequences in Σ: There exists K such that either $b_{2k} = a_{2k+1}$ (and therefore $b_{2k-1} = 0$) for all $k \geq K$, or $b_{2k+1} = a_{2k+2}$ (and $b_{2k} = 0$) for all $k \geq K$. It is known that the number system Σ is isomorphic to the circle rotation over ρ [VS, Theorem 2.1]. The isomorphism is given by

$$\Pi'_\rho : \Sigma \to \mathbb{S}^1, \quad \Pi'_\rho(b) = \sum_i \|b_i \rho q_i\| \pmod 1.$$

In fact, Π' is one–to-one, except in the eventually maximal sequences (which correspond to the preimages of the critical point), where it is two–to–one. □

Let A be the dyadic adding machine, i.e. $A = \{0,1\}^{\mathbb{N}}$ endowed with the metric $d_A(x,y) = \sum \frac{|x_i - y_i|}{2^i}$. The natural numbers are represented by their dyadic expansion. Let B be the dyadic solenoid, i.e. $B = \mathbb{S}^1 \times A \,/ \sim$, where $(x,y) \sim (x',y')$ if $(x,y) = (x',y')$ or $x = x' = 0$ and $y' = y+1$ or vice versa. We only need the local topology of B, which has the structure of a Cantor set cross an arc. Therefore assume that $arc(x,x')$, the shorter arc between x and $x' \in \mathbb{S}^1$, has length $< \frac{1}{4}$. Then

$$d_B((x,y),(x',y')) = \|x - x'\| + \sum_i \frac{|z_i - z_i'|}{2^i},$$

where

$$\begin{array}{llll} z = y, z' = y' & \text{if} \quad 0 \notin arc(x,x') & \text{or} & x = x' = 0, \\ z = y+1, z' = y' & \text{if} \quad 0 \in arc(x,x') & \text{and} & x' \in [0,\frac{1}{4}], \\ z = y, z' = y'+1 & \text{if} \quad 0 \in arc(x,x') & \text{and} & x \in [0,\frac{1}{4}]. \end{array}$$

Theorem 7 *There exists a unimodal map f such that $B \times A$ is a continuous factor of (X,f). Moreover, f is not renormalizable, and the kneading map Q satisfies the conditions of Theorem 0.*

Proof: Let $(F_n)_{n \geq 0} = \{1,2,3,5,...\}$ be the Fibonacci numbers. We will construct the cutting times of f in such a way that

$$S_n = 2^{k(n)} F_{r(n)} \ ,$$

where $r(n) \gg k(n)$ for n large, and $k(n) \to \infty$ as $n \to \infty$. Hence, for every $k \in \mathbb{N}$, 2^k is an eventual divisor of $(S_n)_n$ We start taking $S_n = F_n$, so $Q(n) = n - 2$ for $n > 1$. Then we modify Q and (S_n) recursively according to the following algorithm, starting with $i = 1$:

(i) Choose $n_i > 5^i$ such that 2^i divides S_{n_i}.

(ii) Put $Q(n_i+1) = n_i - 3$, $Q(n_i+2) = n_i - 2$, $Q(n_i+3) = n_i - 2$, $Q(n_i+4) = n_i + 1$, and for $n > n_i + 4$, we again take $Q(n) = n - 2$.

(iii) Increase i by 1 and continue with step (i).

For example, the sequence (S_n) with $n_1 = 7$ starts with

$$1,2,3,5,8,13,21,34,42,55,68 = 2 \times 34, 110 = 2 \times 55, 178 = 2 \times 89, ...$$

It is easy to show that Q defined this way satisfies the admissibility condition. Moreover $n-5 \leq Q(n) \leq n-2$ for all $n \geq 2$. Hence by (10), f is not renormalizable.

It is a straightforward computation that $S_{n_i+1} = 2S_{n_i-1}$, $S_{n_i+2} = S_{n_i} + S_{n_i-1}$, and for $n_i + 3 \leq n \leq n_{i+1}$, $S_n = 2^i F_{r-3+(n-n_i)}$, where r is such that $2^{i-1}F_r = S_{n_i}$. Hence $S_n = 2^i F_r$ for some $r \gg i$, whenever $n \in \{n_i+1, n_i+3, n_i+4, n_i+5, ..., n_{i+1}, n_{i+1}+2\}$.

Next take $\gamma = \frac{\sqrt{5}-1}{2}$ and define the factor map $\pi : \langle \mathbb{N} \rangle \to B \times A$ as $\pi(\langle n \rangle) = (n\gamma, [n\gamma], n)$. Here the first component is taken in $\mathbb{S}^1$, the second is the integer part of $n\gamma$, taken in A, and the third component is again taken in A. It is clear that on $\langle \mathbb{N} \rangle$, $\pi \circ T = T' \circ \pi$, where T' denotes the addition of $(\gamma, 0, 1)$ in $B \times A$. We claim that π is uniformly continuous, so that it extends continuously to the whole of Ω.

Take $\epsilon > 0$ arbitrarily small and find n such that $C \sum_{i \geq n} 2^{k(i)} \gamma^{r(i)} \leq \frac{\epsilon}{3}$ and $4 \cdot 2^{-k(n)} \leq \epsilon/3$. Here C is such that $\gamma |F_n - F_{n-1}| \leq C\gamma^n$ for all n. Take any $\langle a \rangle \in \langle \mathbb{N} \rangle$. We will show that $\pi(Z_n(\langle a \rangle) \cap \langle \mathbb{N} \rangle)$ is contained in an ϵ–neighbourhood of $\pi(\langle a \rangle)$. Indeed, take $\langle b \rangle \in Z_n(\langle a \rangle)$. All cutting times S_i with $i \geq n$ are multiples of $2^{k(n)-1}$. So $d_A(a,b) \leq 2 \cdot 2^{1-k(n)} \leq \frac{\epsilon}{3}$. Also $\gamma |b-a| \bmod 1 \leq \sum_{i \geq n} \|\gamma S_i\| = \sum_{i \geq n} \|\gamma 2^{k(i)} F_{r(i)}\| \leq C \sum_{i \geq n} 2^{k(i)} \gamma^{r(i)} \leq \frac{\epsilon}{3}$. So $arc(\gamma a, \gamma b)$ is indeed short. Note that

$$|\gamma S_i - 2^{k(i)} F_{r(i)-1}| = 2^{k(i)} |\gamma F_{r(i)} - F_{r(i)-1}| \leq 2^{k(i)} C \gamma^{r(i)} \ .$$

So if $arc(\gamma a, \gamma b) \not\ni 0$, then $[\gamma a] - [\gamma b] = \sum_{i > n} \sigma_i 2^{k(i)} F_{r(i)-1}$, for some sequence $(\sigma_i)_i \in \{-1,0,1\}^{\mathbb{N}}$. This is a multiple of $2^{k(n)-1}$, hence

$$d_B((\gamma a, [\gamma a]), (\gamma b, [\gamma b])) \leq \|\gamma a - \gamma b\| + \frac{2}{2^{k(n)-1}} \leq \frac{2}{3}\epsilon.$$

If $arc(\gamma a, \gamma b) \ni 0$ and $[\gamma b] \in [0, \frac{1}{4}]$, then $[\gamma a] + 1 - [\gamma b] = \sum_{i > n} \sigma_i 2^{k(i)} F_{r(i)-1}$, so again

$$d_B((\gamma a, [\gamma a]), (\gamma b, [\gamma b])) \leq \|\gamma a - \gamma b\| + \frac{2}{2^{k(n)-1}} \leq \frac{2}{3}\epsilon.$$

The third possibility goes likewise. Hence $\pi(\langle a \rangle)$ and $\pi(\langle b \rangle)$ are at most ϵ apart. □

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