HOMEOMORPHIC RESTRICTIONS OF UNIMODAL MAPS

HENK BRUIN

ABSTRACT. Examples are given of tent maps $\mathcal{T}$ for which there exist non-trivial sets $B \subset [0,1]$ such that $\mathcal{T} : B \to B$ is a homeomorphism.

1. INTRODUCTION

Let $\mathcal{T} : [0,1] \to [0,1]$ be a unimodal map, *i.e.* $\mathcal{T}$ is a continuous map with a unique turning point $c \in [0,1]$ such that $\mathcal{T}|_{[0,c]}$ is increasing and $\mathcal{T}|_{[c,1]}$ is decreasing. Obviously, $\mathcal{T} : [0,1] \to [0,1]$ is not homeomorphic, but we can ask ourselves if there are sets $B \subset [0,1]$ such that $\mathcal{T} : B \to B$ is homeomorphic. If B is a union of periodic orbits, then this is obviously the case. Also if there is a subinterval $J \subset [0,1]$ such that $|\mathcal{T}^n(J)| \to 0$, it is easy to construct an uncountable set $B \subset \cup_{n\in\mathbb{Z}}\mathcal{T}^n(J)$ such that $\mathcal{T} : B \to B$ is homeomorphic. A third example is $B = \omega(c)$ ($\omega(x) := \cap_i \overline{\cup_{j\geq i}\mathcal{T}^j(x)}$), when $\mathcal{T}$ is infinitely renormalizable. Indeed, in this case $\omega(c)$ is a so-called solenoidal attractor and $\mathcal{T} : \omega(c) \to \omega(c)$ is topological conjugate to an adding machine and therefore a homeomorphism. For these and other general results on unimodal maps, see *e.g.* [1, 10].

In [3] the above question was first raised, and properties of B were discussed. To avoid the mentioned trivial examples let us restrict the question for maps $\mathcal{T}$ that are *locally eventually onto*, *i.e.* every interval $J \subset [0,1]$ iterates to large scale: $\mathcal{T}^n(J) \supset [\mathcal{T}^2(c), \mathcal{T}(c)]$ for n sufficiently large. Because every locally eventually onto unimodal map is topologically conjugate to a some tent map T_a, $T_a(x) = \min(ax, a(1-x))$ with $a > \sqrt{2}$, we can restate the question to

> Are there tent maps T_a, $a > \sqrt{2}$, that admit an infinite compact set B such that $T_a : B \to B$ is a homeomorphism?

This turns out to be the case. To be precise, we prove

Theorem 1. *There exists a locally uncountable dense set $A \subset [\sqrt{2}, 2]$ such that $T_a : \omega(c) \to \omega(c)$ is homeomorphic for every $a \in A$.*

We remark that because $\omega(c)$ is nowhere dense for each $a \in A$, A is a first category set of zero Lebesgue measure, see [4, 2].

This paper is organized as follows. In the next section, we discuss, following [3], some properties that B has to satisfy. In section 3 we recall some facts from kneading theory. Theorem 1 is proven in section 4 and in the last section we give a different construction to solve the main question.

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2. Properties of B

Throughout the paper we assume that $\mathcal{T} = T_a$ is a tent map with slope $a > 1$ and that B is a compact infinite set such that $\mathcal{T} : B \to B$ is a homeomorphism.

Proposition 1. *Under the above assumptions, $B = \omega(c)$ modulo a countable set, and $\omega(c)$ is minimal.*

Let us first recall a result of Gottschalk and Hedlund [8]. A self map f on a compact metric space is called *locally expanding* if there exist $\varepsilon_0 > 0$ and $\lambda > 1$ such that $d(f(x), f(y)) > \lambda d(x, y)$ whenever $d(x, y) < \varepsilon_0$.

Lemma 1 ([8])**.** *If X is a compact metric space and $f : X \to X$ is a locally expanding homeomorphism, then X is finite.*

Proof: Because f^{-1} is continuous and X is compact, there exists $\delta > 0$ such that $d(x, y) < \varepsilon$ implies $d(f^{-1}(x), f^{-1}(y)) < \delta$. Obviously δ can be taken small as $\varepsilon \to 0$. In particular, if $\varepsilon \ll \varepsilon_0$, local expandingness gives that we can take $\delta = \frac{1}{\lambda}\varepsilon$. Let $\cup_i U_i$ be an open cover of X such that $\mathrm{diam}(U_i) < \varepsilon$ for each i. As X is compact we can take a finite subcover $\cup_{i=1}^N U_i$. By definition of δ, $\mathrm{diam}(f^{-1}U_i) < \delta$ and as f is locally expanding, $\mathrm{diam}(f^{-1}U_i) < \frac{1}{\lambda}\mathrm{diam}(U_i) < \frac{\varepsilon}{\lambda}$. Repeating this argument, we obtain for each n a finite cover $\cup_{i=1}^N f^{-n}U_i$ of X and $\mathrm{diam}(f^{-n}U_i) < \lambda^{-n}\varepsilon \to 0$ uniformly. Hence X must be finite. $\square$

The proof of Proposition 1 uses ideas from [3]:

Proof: The map $\mathcal{T}$ is locally expanding on every compact set that excludes c. Therefore, if $c \notin B$, the previous lemma shows that B is finite. Assume therefore that $c \in B$, and hence $\omega(c) \subset B$. If $\omega(c)$ is not minimal, then there exists $x \in \omega(c)$ such that $c \notin \omega(x)$. Then $\mathcal{T} : \omega(x) \to \omega(x)$ is a homeomorphism. Hence by Lemma 1, x must be a periodic orbit, say with period N. Take $U \ni x$ so small that for each $0 \leq i < N$, $\mathcal{T}^{-1}(\mathcal{T}^i(U))$ has only one component that intersects B. This is possible because $\mathcal{T} : B \to B$ is one-to-one and $c \notin \mathrm{orb}(x)$. Let m be minimal such that $c_m \in \cup_{i=0}^{N-1}\mathcal{T}^i(U)$, say $c_m \in \mathcal{T}^i(U)$. But then $c_{m-1} \in B$ belongs to a component of $\mathcal{T}^{-1}(\mathcal{T}^i(U))$ that does not intersect B. This contradiction shows that $\omega(c)$ is minimal.

Now assume that $x \in B \setminus \omega(c)$ is such that $c \in \omega(x)$. Let $0 < \varepsilon < d(x, \omega(c))$, and let $U_1 = B(c; \frac{\varepsilon}{2})$ be the open $\frac{\varepsilon}{2}$-ball centered at c. By taking ε smaller if necessary we can assume that if U is any interval disjoint from $\mathcal{T}(U_1)$ and with $\mathrm{diam}(U) < \varepsilon$, at most one component of $\mathcal{T}^{-1}(U)$ intersects B. Finally assume that $\mathcal{T}^n(\partial U_1) \cap U_1 = \emptyset$ for all $n \geq 1$. This happens if *e.g.* ∂U_1 contains the point in a periodic orbit which is closest to the critical point. Because c is an accumulation point of such periodic points (for tent maps T_a with $a > 1$), this last assumption can be realized.

For $i \geq 1$ define U_{i+1} to be the component of $\mathcal{T}^{-1}(U_i)$ that intersects B. As $\mathrm{diam}(U_{i-1}) < \varepsilon$ and $\mathcal{T}$ has slope > 1, $\mathrm{diam}(U_{i+1}) < \varepsilon$ and we can continue the construction, at least as long as $U_i \cap \mathcal{T}(U_1) = \emptyset$. Let N be minimal such that $U_N \cap U_1 \neq \emptyset$. Because c is recurrent, N exists. Then $U_N \subset U_1$, because otherwise $\partial U_1 \subset U_N$ and $\mathcal{T}^N(\partial U_1) \subset U_1$. This would contradict the assumption on ∂U_1.

Because $c \in \omega(x)$, there exists m minimal such that $\mathcal{T}^m(x) \in \cup_{i=1}^N U_i$, say $\mathcal{T}^m(x) \in U_i$. Then, as before, $\mathcal{T}^{m-1}(x) \in B$ lies in a component of $\mathcal{T}^{-1}(U_1)$ that is disjoint from B. This contradiction shows $c \notin \omega(x)$ and using the above arguments x must be a periodic point. Therefore $B \subset \omega(c)$ up to a countable set. □

3. Preliminaries about Kneading Theory

Let us start with some combinatorics of unimodal maps. Write $c_n := \mathcal{T}^n(c)$. We define *cutting times* and the *kneading map* of a unimodal map. These ideas were introduced by Hofbauer, see *e.g.* [9]. A survey can be found in [5].

If J is a maximal (closed) interval on which $\mathcal{T}^n$ is monotone, then $\mathcal{T}^n : J \to \mathcal{T}^n(J)$ is called a *branch.* If $c \in \partial J$, $\mathcal{T}^n : J \to \mathcal{T}^n(J)$ is a *central branch.* Obviously $\mathcal{T}^n$ has two central branches, and they have the same image. Denote this image by D_n.

If $D_n \ni c$, then n is called a *cutting time.* Denote the cutting times by $\{S_i\}_{i\geq 0}$, $S_0 < S_1 < S_2 < \dots$ For interesting unimodal maps (tent maps with slope > 1) $S_0 = 1$ and $S_1 = 2$. The sequence of cutting times completely determines the tent map and vice versa. It can be shown that $S_k \leq 2S_{k-1}$ for all k. Furthermore, the difference between two consecutive cutting times is again a cutting time. Therefore we can write

$$S_k = S_{k-1} + S_{Q(k)}, \tag{1}$$

for some integer function Q, called the *kneading map.* Each unimodal map therefore is characterized by its kneading map. Conversely, each map $Q : \mathbb{N} \to \mathbb{N} \cup \{0\}$ satisfying $Q(k) < k$ and the *admissibility condition*

$$\{Q(k+j)\}_{j\geq 1} \succeq \{Q(Q^2(k)+j)\}_{j\geq 1} \tag{2}$$

(where $\succeq$ denotes the lexicographical ordering) is the kneading map of some unimodal map. Using cutting times and kneading map, the following properties of the intervals D_n are easy to derive:

$$D_{n+1} = \begin{cases} \mathcal{T}(D_n) & \text{if } c \notin D_n, \\ [c_{n+1}, c_1] & \text{if } c \in D_n. \end{cases}$$

Equivalently:

$$D_n = [c_n, c_{n-S_k}], \text{ where } k = \max\{i; S_i < n\}, \tag{3}$$

and in particular

$$D_{S_k} = [c_{S_k}, c_{S_{Q(k)}}].$$

Let $z_k < c < \hat{z}_k$ be the boundary points of the domains the two central branches of $\mathcal{T}^{S_{k+1}}$. Then z_k and $\hat{z}_k$ lie in the interiors of the domains of the central branches of $\mathcal{T}^{S_k}$ and $\mathcal{T}^{S_k}(z_k) = \mathcal{T}^{S_k}(\hat{z}_k) = c$. Furthermore, $\mathcal{T}^j$ is monotone on (z_k, c) and $(c, \hat{z}_k)$ for all $0 \leq j \leq S_k$. These points are called *closest precritical points*, and the relation (1) implies

$$\mathcal{T}^{S_{k-1}}(c) \in (z_{Q(k)-1}, z_{Q(k)}] \cup [\hat{z}_{Q(k)}, \hat{z}_{Q(k)-1}). \tag{4}$$

We will use these relations repeatedly without specific reference.

Let us also mention some relations with the standard kneading theory. The *kneading invariant* $\kappa = \{\kappa_n\}_{n\geq 1}$ is defined as

$$\kappa_n = \begin{cases} 0 & \text{if } \mathcal{T}^n(c) < c, \\ * & \text{if } \mathcal{T}^n(c) = c, \\ 1 & \text{if } \mathcal{T}^n(c) > c. \end{cases}$$

If we define

$$\tau : \mathbb{N} \to \mathbb{N}, \quad \tau(n) = \min\{m > 0; \kappa_m \neq \kappa_{m+n}\}, \tag{5}$$

then we retrieve the cutting times as follows:

$$S_0 = 1 \text{ and } S_{k+1} = S_k + \tau(S_k) = S_k + S_{Q(k+1)} \text{ for } k \geq 0.$$

In other words (writing $\kappa'_i = 0$ if $\kappa_i = 1$ and vice versa),

$$\kappa_1 \dots \kappa_{S_k} = \kappa_1 \dots \kappa_{S_{k-1}} \kappa_1 \dots \kappa'_{S_{Q(k)}}. \tag{6}$$

For the proofs of these statements, we refer to [5].

4. Proof of the Theorem

Let us introduce an adding machine-like number system that factorizes over the action $\mathcal{T} : \omega(c) \to \omega(c)$: Let $\{S_k\}$ be the cutting times of a unimodal map and assume that the corresponding kneading map Q tends to infinity. Any non-negative integer n can be written in a canonical way as a sum of cutting times: $n = \sum_i e_i S_i$, where

$$e_i = \begin{cases} 1 & \text{if } i = \max\{j; S_j \leq n - \sum_{k>i} e_k S_k\}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular $e_i = 0$ if $S_i > n$. In this way we can code the non-negative integers $\mathbb{N}$ as zero-one sequences with a finite number of ones: $n \mapsto \langle n \rangle \in \{0,1\}^{\mathbb{N}}$. Let $E_0 = \langle \mathbb{N} \rangle$ be the set such sequence, and let E be the closure of E_0 in the product topology. This results in

$$E = \{e \in \{0,1\}^{\mathbb{N}}; e_i = 1 \Rightarrow e_j = 0 \text{ for } Q(i+1) \leq j < i\},$$

because if $e_i = e_{Q(i+1)} = 1$, then this should be rewritten to $e_i = e_{Q(i+1)} = 0$ and $e_{i+1} = 1$. It follows immediately that for each $e \in E$ and $j \geq 0$,

$$e_0 S_0 + e_1 S_1 + \dots + e_j S_j < S_{j+1}. \tag{7}$$

There exists the standard addition of 1 by means of 'add and carry'. Denote this action by P. Obviously $\mathrm{P}(\langle n \rangle) = \langle n+1 \rangle$. It is known (see *e.g.* [6, 7]) that $\mathrm{P} : E \to E$ is continuous if and only if $Q(k) \to \infty$, and that P is invertible on $E \setminus \{\langle 0 \rangle\}$. The next lemma describes the inverses of $\langle 0 \rangle$ precisely.

Lemma 2. *For a sequence $e \in E$, let $\{q_j\}_{j \geq 0}$ be the index set (in increasing order) such that $e_{q_j} = 1$. We have $P(e) = \langle 0 \rangle$ if and only if $e \notin E_0$, $Q(q_0 + 1) = 0$ and $Q(q_j + 1) = q_{j-1} + 1$ for $j \geq 1$.*

Proof: This follows immediately from the add and carry construction, because the condition on $\{q_j\}$ is the only way the addition of 1 carries 'to infinity'. □

The next lemma gives conditions under which P is invertible on the whole of E.

Lemma 3. *Let Q be a kneading map such that $Q(k) \to \infty$. Suppose that there is an infinite sequence $\{k_i\}$ such that for all i and $k > k_i$*

- *either $Q(k) \geq k_i$,*
- *or $Q(k) < k_i$ and there are only finitely many $l > k$ such that $Q^n(l) = k$ for some $n \in \mathbb{N}$,*

then P is a homeomorphism of E.

Proof: Because P is continuous and invertible outside $\langle 0\rangle$ and E is compact, it suffices to show that $\#\mathrm{P}^{-1}(\langle 0\rangle) = 1$. Let $e \in \mathrm{P}^{-1}(\langle 0\rangle)$ and let $\{q_j\}_j$ be the index sequence of the non-zero entries of e. By the previous lemma and our assumption, we see that $\{k_i - 1\}$ must be a subsequence of $\{q_j\}_j$. But because any q_j determines $q_{j'}$ for $j' < j$, there can only be one such sequence $\{q_j\}_j$ and one preimage e. □

Example 1 Take any sequence $\{k_i\}$ with $k_i > k_{i-1} + 10$ and define Q as

$$Q(k) = \begin{cases} k-4 & \text{if } k > K \text{ and } k-4 \in \{k_i\}, \\ k-3 & \text{if } k > K \text{ and } k-5 \in \{k_i\}, \\ \text{arbitrary} & \text{if } k \leq K, \\ k-2 & \text{otherwise,} \end{cases}$$

provided Q is admissible. It is shown in [5] that Q belongs to a renormalizable map of period S_k if and only if

$$Q(k+1) = k \text{ and } Q(k+j) \geq k \text{ for all } j \geq 1. \tag{8}$$

Because $Q(k) \leq k - 2$ for $k \geq K$, we can avoid renormalizable maps. This shows that there is a locally uncountable dense set of tent maps, whose kneading maps Q satisfy Lemma 3 and $Q(k) \to \infty$. In fact, this example also satisfies conditions (11) and (12) of Theorem 2 below.

Remark: Lemmas 2 and 3 also indicate how to construct a number systems (E, P) where for $d \in \mathbb{N} \cup \{\aleph_0\}$, $\langle 0\rangle$ has exactly d preimages. For example, if $Q(k) = \max(0, k-d)$, then $\langle 0\rangle$ is d preimages. The rest of this section makes clear that this yields examples of maps where $\mathcal{T} : \omega(c) \to \omega(c)$ is one-to-one except for d points in $\cup_{n\geq 0} \mathcal{T}^{-n}(c)$.

Given $n \in (S_{k-1}, S_k]$, define $\beta(n) = n - S_{k-1}$. It is easy to check that $\langle \beta(n)\rangle$ is $\langle n\rangle$ with the last non-zero entry changed to 0. The map β also has a geometric interpretation in the Hofbauer tower: It was shown in [6, Lemma 5] that for all $n \geq 2$,

$$D_n \subset D_{\beta(n)}. \tag{9}$$

In fact, D_n and $D_{\beta(n)}$ have the boundary point $c_{\beta(n)}$ in common. Recall that for $e \in E$, $\{q_j\}_j$ is the index sequence of the non-zero entries of e. Define

$$b(i) := \sum_{j \leq q_i} e_j S_j.$$

We have $b(i) \geq S_{q_i}$ by definition of q_i and $b(i) < S_{q_i+1}$ by (7). It follows that $\beta(b(i)) = b(i-1)$. By a *nest* of levels will be meant a sequence of levels $D_{b(i)}$. By (9) and the fact that $\beta(b(i)) = b(i-1)$, these levels lie indeed nested, and because $Q(k) \to \infty$ implies that $|D_n| \to 0$ (see [5]), each nest defines a unique point $x = \cap_i D_{b(i)} \in \omega(c)$. Therefore the following projection (see [6]) makes sense:

$$\pi(\langle n\rangle) = c_n$$

and

$$\pi(e \notin E_0) = \cap_i D_{b(i)}. \tag{10}$$

Obviously $\mathcal{T} \circ \pi = \pi \circ \mathrm{P}$ and it can be shown (see [6, Theorem 1]) that $\pi : E \to E$ is uniformly continuous and onto.

Note that a nest contains exactly one cutting level $D_{b(0)}$. If $\{D_{b(i)}\}_i$ is some nest converging to x, then $\{\mathcal{T}(D_{b(i)})\}_i$ is a nested sequence of levels converging to $\mathcal{T}(x)$.

To obtain a nest of $\mathcal{T}(x)$, we may have to add or delete some levels, but $\{\mathcal{T}(D_{b(i)})\}$ asymptotically coincides with a nest converging to $\mathcal{T}(x)$.

Theorem 2. *If Q is a kneading map that satisfies Lemma 3 as well as*

$$Q(k+1) > Q(Q^2(k)+1)+1 \tag{11}$$

for all k sufficiently large, and

$$Q(s+1) = Q(\tilde{s}+1) \text{ for } s \neq \tilde{s} \text{ implies } Q^{n+1}(s) \neq Q^{\tilde{n}+1}(\tilde{s}), \tag{12}$$

for any $n, \tilde{n} \geq 0$ such that $Q^n(s) \neq Q^{\tilde{n}}(\tilde{s})$, then any map $\mathcal{T}$ with kneading map Q is homeomorphic on $\omega(c)$.

Proof: In view of Lemma 3 we only have to show that $\pi : E \to \omega(c)$ is one-to-one. First note (see also [6, Theorem 1]) that $\pi^{-1}(c) = \langle 0 \rangle$. Indeed, if $e \neq \langle 0 \rangle$ and $\pi(e) = c$, then, taking $k < l$ the first non-zero entries of e, $c \in D_{S_k+S_l}$. Then S_k+S_l is a cutting time S_m and we have $m = l+1$ and $k = Q(l+1)$. This would trigger a carry to $e_k = e_l = 0$ and $e_m = 1$. Because P is invertible, also $\#\mathcal{T}^{-n}(c) \cap \omega(c) = 1$ for each $n \geq 0$. Assume from now on that $x \in \omega(c) \setminus \cup_{n \geq 0} \mathcal{T}^{-n}(c)$. We need one more lemma:

Lemma 4. *Let $\mathcal{T}$ be a unimodal map whose kneading map satisfies (11) and tends to infinity. Then there exists K such that for any $n \notin \{S_i\}_i$ such that $\beta(n)$ is a cutting time (*i.e. *$n = S_r + S_t$ for some $r < t$ with $r < Q(t+1)$), and every $k \geq K$, intD_n does not contain both c_{S_k} and a point from $\{z_{Q(k+1)-1}, \hat{z}_{Q(k+1)-1}\}$.*

Proof: Assume the contrary. Write $n = S_r + S_t$ with $r < Q(t+1)$ and let k but such that $z_{Q(k+1)-1}$ or $\hat{z}_{Q(k+1)-1} \in D_n \subset D_{S_r}$. Formula (4) implies that $Q(r+1) < Q(k+1)$, see figure 1.

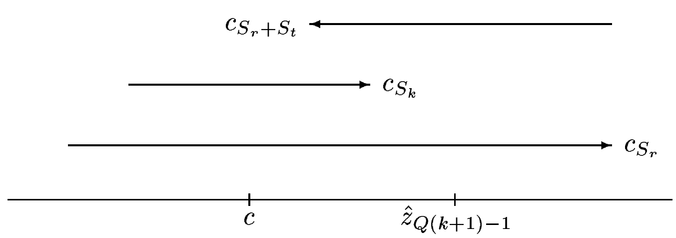

Figure 1: The levels D_{S_k} and $D_{S_r+S_t}$

It follows that also $z_{Q(r+1)}$ or $\hat{z}_{Q(r+1)} \in D_{S_r+S_t}$ and therefore $S_r + S_t + S_{Q(r+1)} = S_{t+1}$. This gives

$$r+1 = Q(t+1), \tag{13}$$

and $S_r + S_t = S_{t+1} - S_{Q^2(t+1)}$. If also $z_{Q(r+1)+1}$ or $\hat{z}_{Q(r+1)+1} \in D_{S_r+S_t}$, then $S_{Q(r+1)+1} - S_{Q(r+1)} + S_{t+1} = S_{t+2}$, which yields $Q(Q(r+1)+1) = Q(t+2)$. Using (13) for $t+1$, this gives $Q(t+1+1) = Q(Q^2(t+1)+1)$. This contradicts (11), if t is sufficiently large. For smaller t, there are only finitely many pairs $r < t$. For k sufficiently large (recall that $Q(k+1) \to \infty$, so $c_{S_k} \to c$), $D_{S_r+S_t} \not\ni c_{S_k}$. Hence

$$D_{S_r+S_t} \text{ contains at most one closest precritical point.} \tag{14}$$

Therefore, as $c_{S_k} \in D_{S_r+S_t}$, $Q(r+1) = Q(k+1) - 1$. Take the $S_{Q(r+1)}$-th iterate of $D_{S_r+S_t}$ and $[c, c_{S_k}]$ to obtain $D_{S_k+S_{Q(r+1)}} \cap D_{S_{t+1}} \neq \emptyset$, see figure 2.

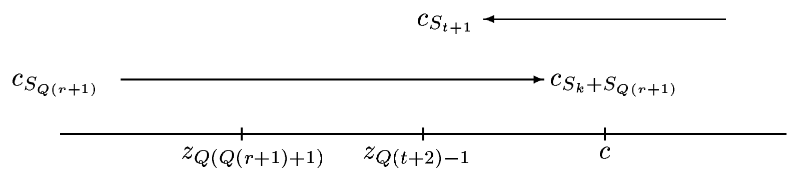

Figure 2: The levels $D_{S_{Q(r+1)}+S_k}$ and $D_{S_{t+1}}$

By (13) we have $Q(Q(r+1)+1) = Q(Q^2(t+1)+1)$, and using (11) on $t+1$, we obtain

$$Q(Q^2(t+1)+1) < Q(t+1+1) - 1 = Q(t+2) - 1.$$

Hence there are at least two closest precritical points contained in $D_{S_k+S_{Q(r+1)}}$. This contradicts the arguments leading to (14). □

We continue the proof of Theorem 2. Observe that if $\pi(e) = \pi(\tilde{e}) = x$ for some $e \neq \tilde{e}$, then the corresponding nests $\{D_{b(i)}\}$ and $\{D_{\tilde{b}(i)}\}$ are different, but both nests converge to x. Because $x \notin \cup_{n \geq 0} \mathcal{T}^{-n}(c)$, $\#\pi^{-1}(\mathcal{T}^n(x)) > 1$ for all $n \geq 0$. We will derive a contradiction.

Claim 1: We can assume that $b(0) \neq \tilde{b}(0)$.

Let i be the smallest integer such that $b(i) \neq \tilde{b}(i)$, say $b(i) < \tilde{b}(i)$. Then also $q_i < \tilde{q}_i$. Let $l = S_{\tilde{q}_i+1} - \tilde{b}(i)$. By (7), l is non-negative. By the choice of l, $(P^l(\tilde{e}))_j = 0$ for all $j \leq \tilde{q}_i$, but because $b(i) < \tilde{b}(i)$ and $l + b(i) < S_{\tilde{q}_i+1}$, there is some $j \leq \tilde{q}_i$ such that $(P^l(e))_j = 1$.

Replace x by $\mathcal{T}^l(x)$, and the corresponding sequences e and $\tilde{e}$ by $P^l(e)$ and $P^l(\tilde{e})$. Then for this new point, $q_0 < \tilde{q}_0$ and $b(0) < \tilde{b}(0)$. This proves Claim 1.

Claim 2: We can assume that $Q(q_0+1) \neq Q(\tilde{q}_0+1)$

Assume that $Q(q_0+1) = Q(\tilde{q}_0+1) =: r$. We apply P^{S_r} to e and $\tilde{e}$. Write $s = \min\{j; (P^{S_r}(e))_j = 1\}$ and $\tilde{s} = \min\{j; (P^{S_r}(\tilde{e}))_j = 1\}$. To make sure that Claim 1 still holds, assume by contradiction that $s = \tilde{s}$. Then (using (7)), $\sum_{j=0}^{s-1} e_j S_j = \sum_{j=0}^{s-1} \tilde{e}_j S_j = S_s - S_r$. But this would imply that $e_j = \tilde{e}_j$ for all $j < s$, which is not the case. Hence $s \neq \tilde{s}$.

From the add and carry procedure it follows that $e_{j-1} = 1$ for $j = Q(s), Q^2(s), \dots, q_0 + 1 = Q^n(s)$ for some n, and similarly $\tilde{e}_{j-1} = 1$ for $j = Q(\tilde{s}), Q^2(\tilde{s}), \dots, \tilde{q}_0 + 1 = Q^{\tilde{n}}(\tilde{s})$ for some $\tilde{n}$. As $Q(q_0+1) = Q(\tilde{q}_0+1)$, hypothesis (12) implies $Q(s+1) \neq Q(\tilde{s}+1)$. This proves Claim 2. Replace x by $\mathcal{T}^{S_r}(x)$ and the corresponding sequences e and $\tilde{e}$ by $P^{S_r}(e)$ and $P^{S_r}(\tilde{e})$.

Note that we can take $q_0 \neq \tilde{q}_0$ arbitrarily large, with say $Q(q_0+1) < Q(\tilde{q}_0+1)$. Then we are in the situation of Lemma 4, which tells us that $D_{b(1)} \cap D_{\tilde{b}(0)} = \emptyset$. Therefore the two nests cannot converge to the same point. This contradiction concludes the proof. □

Proof of Theorem 1: Combine the previous theorem with Example 1. □

5. A Different Construction

In this section we give a different construction which does not involve the assumption $Q(k) \to \infty$. Let k_1 be arbitrary and $k_2 = k_1 + 1$. Put recursively for $i \geq 3$,

$$k_i = 2k_{i-1} - k_{i-2} + 1, \ \textit{i.e.}\ k_i - k_{i-1} = k_{i-1} + k_{i-2} + 1.$$

Define the kneading map as

$$Q(k_i) = k_i - 1 \text{ for } i \geq 3, \tag{15}$$

and choose $Q(k)$ arbitrary for $k \leq k_2$ so that (2) and (11) below are not violated for $k \leq k_2 + 2$. To finish the definition, let

$$Q(k_i + j) = Q(k_{i-1} + j - 1) \text{ for } i \geq 2, 1 \leq j < k_{i+1} - k_i. \tag{16}$$

A direct computation shows that (15) and (16) imply (11) for all $k > k_2$. Therefore the construction is compatible with the admissibility condition (2). Moreover, (15) and (16) show that condition (8) is not met for $k \geq k_2$. Therefore Q does not belong to a renormalizable map of period $\geq S_{k_2}$.

Theorem 3. *If $\mathcal{T}$ is a unimodal map with the kneading map constructed above, then $\mathcal{T} : \omega(c) \to \omega(c)$ is a homeomorphism.*

Proof: Write $B = \omega(c)$. Using the levels D_n of the Hofbauer tower, we will construct covers of B to show that $\mathcal{T} : B \to B$ is a homeomorphism. Let $\Delta_i = \cup_{n=S_{k_{i-1}}+1}^{S_{k_i}} D_n$. We will use the following claims:

$$c_{S_{k_i-1}} \in [c_{S_k}, 1 - c_{S_k}] \text{ for every } k \leq k_i. \tag{17}$$

$$c_n \notin \operatorname{int} D_{S_{k_i}} \text{ for } 0 < n \leq S_{k_i}. \tag{18}$$

$$\Delta_i \text{ consists of disjoint intervals.} \tag{19}$$

$$c_n \in \Delta_i \text{ for } S_{k_i-1} \leq n < S_{k_{i+1}-1} \tag{20}$$

$$\Delta_{i+1} \subset \Delta_i \tag{21}$$

Proof: Claim (17): Recall the function τ from (5). Obviously $\tau(n) > \tau(m)$ implies that $c_n \in (c_m, 1 - c_m)$. By construction and equation (6), $\tau(S_{k_i-1}) = S_{Q(k_i)} = S_{k_i-1} \geq S_{Q(k+1)}$ for all $k \leq k_i$. Hence $c_{S_{k_i-1}} \in [c_{S_k}, 1 - c_{S_k}]$ for every $k \leq k_i$,

Claim (18): By construction $D_{S_{k_i}} = [c_{S_{k_i}}, c_{S_{Q(k_i)}}] = [c_{S_{k_i}}, c_{S_{k_i-1}}]$. We have

$$G_i := \min\{\tau(S_{k_i}), \tau(S_{k_i-1})\} = S_{k_{i-1}-1}.$$

Because $\tau(S_k) < G_i$ for all $k < k_i - 1$, $k \neq k_{i-1} - 1$, we obtain $c_{S_k} \notin D_{S_{k_i}}$ for these values of k. If $k = k_i - 1$, then $c_{S_k} \in \partial D_{S_{k_i}}$ and not in the interior. With respect to $k_{i-1} - 1$, note that by (16), $Q(k_{i-1} - 1) = Q(k_i - 1)$, so $\kappa_{S_{k_{i-1}-1}} = \kappa_{S_{k_i-1}}$ and $c_{S_{k_{i-1}-1}}$ and $c_{S_{k_i-1}}$ lie on the same side of c. Because also $\tau(S_{k_{i-1}-1}) = S_{Q(k_{i-1})} < S_{Q(k_i)} = \tau(S_{k_i-1})$, $c_{S_{k_{i-1}-1}} \notin D_{S_{k_i}}$.

It remains to consider non-cutting times $n < S_{k_i}$. Assume by contradiction that $c_n \in \operatorname{int} D_{S_{k_i}}$, *i.e.* D_n intersects $D_{S_{k_i}}$ in a non-trivial interval. Then also $D_{\beta(n)}$ intersects $D_{S_{k_i}}$ where β is as in (9). By taking $\beta^j(n)$ instead of n for some $j \geq 0$, we may assume that $\beta(n)$ is a cutting time. In particular, $c_n \in \operatorname{int} D_{S_{k_i}}$ and $n = S_k + S_t < S_{k_i}$, where $Q(t+1) > k$. If k is such that $\tau(S_k) < G_i$, then by (11) and Lemma 4, $D_n \cap D_{S_{k_i}} = \emptyset$. If $k = k_i - 1$, then $Q(t+1) > k$ implies $t \geq k_i - 1$, contradicting that $S_k + S_t < S_{k_i}$. The last possibility is that $k = k_{i-1} - 1$ and $t = k_{i-1} - 1$. The above arguments showed that $c_{S_{k_i-1}}$ and $c_{S_{k_{i-1}-1}}$ lie on the same side of c. Because $\tau(S_{k_{i-1}-1}) < \tau(S_{k_i-1})$, Lemma 4 applies after all. This proves Claim (18).

Claim (19): Suppose by contradiction that $D_m \cap D_n \neq \emptyset$ for some $S_{k_i-1} < m < n \leq S_{k_i}$. Then also $\mathcal{T}^{S_{k_i}-n}(D_m) \cap \mathcal{T}^{S_{k_i}-n}(D_n) = D_{m+S_{k_i}-n} \cap D_{S_{k_i}} \neq \emptyset$. Because $S_{k_i} + m - n$ is not a cutting time, at least one endpoint of $D_{m+S_{k_i}-n}$ is contained in $D_{S_{k_i}}$. This contradicts the previous claim.

Claim (20): Clearly $c_{S_{k_i-1}} \in [c_{S_{k_i}}, c_{S_{k_i-1}}] = D_{S_{k_i}} \subset \Delta_i$ and for $S_{k_i-1} < n \leq S_{k_i}$, $c_n \in D_n \subset \Delta_i$ by definition. So let us consider $n = S_{k_i} + 1$. By construction of Q and (6) we obtain

$$
\begin{array}{rcl}
m & := & S_{k_{i+1}-1} - S_{k_i} \\
 & = & S_{Q(k_i+1)} + S_{Q(k_i+2)} + \cdots + S_{Q(k_{i+1}-1)} \\
 & = & S_{Q(k_{i-1})} + S_{Q(k_{i-1}+1)} + \cdots + S_{Q(k_i-1)} \\
 & = & S_{k_i-1} - S_{k_{i-1}-1} = S_{Q(k_i)} - S_{k_{i-1}-1},
\end{array} \tag{22}
$$

and

$$\kappa_{S_{k_i}+1} \dots \kappa_{S_{k_{i+1}-1}} = \kappa_{S_{k_{i-1}-1}+1} \dots \kappa_{S_{Q(k_i)}} = \kappa_{S_{k_i-1}+S_{k_{i-1}-1}+1} \dots \kappa'_{S_{k_i}}.$$

Here we 'shifted' the word $\kappa_{S_{k_{i-1}-1}+1} \dots \kappa_{S_{Q(k_i)}}$ over S_{k_i-1} entries and used (6) to obtain the second equality. Therefore $c_{S_{k_i}+1}$ lies in the same interval of monotonicity of $\mathcal{T}^{m-1}$ as the level $D_{S_{k_i-1}+S_{k_{i-1}-1}+1} = [c_{S_{k_i-1}+S_{k_{i-1}-1}+1}, c_{S_{k_{i-1}-1}+1}]$. Furthermore $\mathcal{T}^{m-1}(c_{S_{k_i}+1}) = c_{S_{k_{i+1}-1}}$ and

$$\mathcal{T}^{m-1}(D_{S_{k_i-1}+S_{k_{i-1}-1}+1}) = \mathcal{T}^{S_{Q(k_i)}-1}(D_{S_{k_{i-1}-1}+1}) = D_{S_{k_i}}. \tag{23}$$

Claim (17) gives $c_{S_{k_{i+1}-1}} \in D_{S_{k_i}}$. Therefore

$$c_{S_{k_i}+1} \in D_{S_{k_i-1}+S_{k_{i-1}-1}+1}, \tag{24}$$

and $c_n \in D_{S_{k_i-1}+S_{k_{i-1}-1}+n-S_{k_i}} \subset \Delta_i$ for all $S_{k_i} < n < S_{k_{i+1}-1}$.

Claim (21): We need to show that

$$D_{S_{k_{i+1}-1}+j} \subset \Delta_i \text{ for } 1 \leq j \leq S_{k_{i+1}} - S_{k_{i+1}-1} = S_{k_{i+1}-1}.$$

Because $\tau(S_{k_i-1}) < \tau(S_{k_{i+1}-1})$, $c_{S_{k_{i+1}-1}} \in [c_{S_{k_i-1}}, 1 - c_{S_{k_i-1}}]$. Hence $D_{S_{k_{i+1}-1}+j} \subset D_{S_{k_i-1}+j}$ for $0 < j \leq S_{Q(k_i)} = S_{k_i-1}$.

For $j = S_{Q(k_i)}$, $D_{S_{k_i-1}+j} = D_{S_{k_i}}$, and the above line shows that $D_{S_{k_i}} \supset D_{S_{k_{i+1}-1}+j}$ and these two intervals have the boundary point $c_{S_{Q(k_i)}}$ in common. Because also $Q(k_i) = k_i - 1$, we get $D_{S_{k_{i+1}-1}+j} \subset D_{S_{k_i-1}+(j-S_{k_i-1})}$ for $S_{k_i-1} < j \leq S_{k_i-1} + S_{Q(k_i)} = S_{k_i}$.

By formula (24), one boundary point $c_{S_{k_i}+1} \in \partial D_{S_{k_{i+1}-1}+S_{k_i}+1}$ belongs to $D_{S_{k_i-1}+S_{k_{i-1}-1}+1}$. A fortiori, $D_{S_{k_{i+1}-1}+j} \cap D_{S_{k_i-1}+S_{k_{i-1}-1}+(j-S_{k_i})} \neq \emptyset$ for $S_{k_i} < j \leq S_{k_i} + (S_{k_i} - (S_{k_i-1} + S_{k_{i-1}-1}))$. In particular (cf. (23)), for $j = S_{k_i} + (S_{k_i} - (S_{k_i-1} + S_{k_{i-1}-1})) = S_{k_i} + S_{k_i-1} - S_{k_{i-1}-1}$, $D_{S_{k_{i+1}-1}+j}$ intersects the level $D_{S_{k_i-1}+S_{k_{i-1}-1}+(j-S_{k_i})} = D_{2S_{k_i-1}} = D_{S_{k_i}}$. (Here we used $Q(k_i) = k_i - 1$, *i.e.* $S_{k_i} = 2S_{k_i-1}$). At the same time, by (22), $j = S_{k_i} + S_{k_i-1} - S_{k_{i-1}-1} = S_{k_{i+1}-1}$ and therefore $D_{S_{k_{i-1}-1}+j} = D_{S_{k_{i+1}}}$. Thus the intersection is actually an inclusion: $D_{S_{k_{i+1}}} \subset D_{k_i}$ and $D_{S_{k_{i+1}-1}+j} \subset D_{S_{k_i-1}+S_{k_{i-1}-1}+(j-S_{k_i})}$ for all j, $S_{k_i} < j \leq S_{k_{i+1}-1}$. This proves Claim (21).

Let i be arbitrary. By construction, $\Delta_i \supset \{c, c_1, \dots, c_{S_{k_i}}\}$. Claim (21) used repeatedly gives $\mathrm{orb}(c) \subset \Delta_i$, and because Δ_i is closed, $B \subset \cap_i \Delta_i$. Finally, to prove that $\mathcal{T} : B \to B$ is homeomorphic, it suffices to show that $\mathcal{T} : B \to B$ is one-to-one. Suppose by contradiction that there exist $y, y' \in B$, $y \neq y'$, such that $\mathcal{T}(y) = \mathcal{T}(y')$.

Take i so large that y and y' lie in different intervals of Δ_i. Say $y \in D_n$ and $y' \in D_m$. Because $y \neq c \neq y'$, we can assume that $S_{k_i-1} < m < n < S_{k_i}$. But then $\mathcal{T}(D_m) \cap \mathcal{T}(D_n) = D_{m+1} \cap D_{n+1} \neq \emptyset$, contradicting Claim (19). This concludes the proof. □

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Department of Mathematics, California Institute of Technology, Pasadena CA 91125 USA

E-mail address: `bruin@cco.caltech.edu`