Examples of expanding C^1 maps having no σ-finite invariant measure equivalent to Lebesgue

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Abstract

In this paper we construct a C^1 expanding circle map with the property that it has no σ-finite invariant measure equivalent to Lebesgue measure. We extend the construction to interval maps and maps on higher dimensional tori and the Riemann sphere. We also discuss recurrence of Lebesgue measure for the family of tent maps.

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1 Introduction

The problem of constructing smooth maps on manifolds with no equivalent σ-finite invariant measure equivalent to the Riemannian measure has a long history. The map is called *type III* (with respect to the measure) in this case. We describe briefly the history of the problem of constructing type *III* maps on manifolds with respect to Lebesgue measure.

Ornstein was the first to construct a type *III* invertible map [24]; it is a continuous interval map. There are by now many examples of type *III* diffeomorphisms, the earliest constructions being C^∞ circle diffeomorphisms [13, 16, 17]. In [12] a noninvertible C^∞ type *III* map of the torus is given. None of these examples is expanding; indeed there are obstructions to finding smooth expanding type *III* maps. This fact is illustrated by a result by Krzyzewski and Szlenk [20], stating that every expanding C^2 transformation on a compact manifold carries an absolutely continuous invariant probability measure (acip).

Later, several authors obtained the same result while weakening the C^2 assumption for expanding maps, for example [6, 23, 27]. In each of these papers the map was assumed to be C^1 and to satisfy an additional condition on the derivative (Hölder, bounded variation etc.). In the C^1 expanding case examples have been constructed which admit no finite absolutely continuous measures [7]; however the existence of an infinite σ-finite invariant measure is not excluded. In addition, many examples of noninvertible C^∞ maps, for instance quadratic maps on the interval, with infinite σ-finite invariant measures have been found, e.g. [2, 14].

In this paper we construct ergodic type *III* maps of manifolds which are C^1 and expanding. Questions on the Lebesgue ergodicity of C^1 expanding maps have been addressed by Quas [25] and references therein. We prove the following theorems.

Theorem 1.1 *There exist maps f for which there is no σ-finite invariant measure equivalent to Lebesgue. f can be constructed to satisfy one of the following sets of properties.*

- *f is a C^1 expanding circle map of degree $d \geq 2$.*
- *f is an interval map topologically conjugate to the full tent map T_2, and f is either C^1, or C^1 and expanding on both branches separately.*
- *f is an interval map topologically conjugate to the tent map with slope a, for any $a \in (1, 2]$ such that the critical point has a nowhere dense orbit.*

Theorem 1.2 *There exists an ergodic type III Borel measure ν on $\mathbf{C}_\infty$ with respect to the rational map $R(z) = \frac{(z^2+1)^2}{4z(z^2-1)}$.*

We use a construction of Hamachi [9] of a type *III* shift map with a product measure to prove our main result. This gives us in a natural way a type *III* measure for the angle doubling map on the circle which is the basis for our smooth examples.

It is known that for noninvertible maps, equivalent measures can exhibit different recurrence properties. Given a nonsingular ergodic endomorphism $(X, \mathcal{B}, \mu, T)$, $\mu(X) = 1$, the (global) Radon-Nikodým derivative of T, denoted ω_μ, is defined to be the unique $T^{-1}\mathcal{B}$-measurable function satisfying:

$$\int_X f \circ T \cdot \omega_\mu d\mu = \int_X f d\mu \text{ for all } f \in L^1(X, \mu).$$

The higher derivatives are defined for each $n \in \mathbf{N}$ by $\omega_\mu(n, x) = \prod_{i=0}^{n-1} \omega_\mu(T^i x)$. The measure μ is said to be *recurrent for T* if

$$\sum_n \omega_\mu(n, x) = \infty \quad \mu\text{-a.e.}$$

In the invertible case, all measures are recurrent. For noninvertible maps, nonrecurrent measures equivalent to recurrent ones are known to exist [5]. In general it is difficult to determine whether a given ergodic measure is recurrent; an invariant measure μ is always recurrent since $\omega_\mu(x) \equiv 1$ in this case. In a noninvertible system, if the measure is known to be recurrent, all existing invariants for invertible systems (e.g. ratio sets and coboundary Radon-Nikodým derivatives) can be used to detect the existence or absence of an equivalent invariant measure [12]. When the measure is nonrecurrent none of the noninvertible tests are valid. Furthermore, nonrecurrent measures provide obstacles to obtaining a conservative natural extension, even if the original map has an acip [28].

Let T_a be the tent map on $[0, 1]$ with constant slope a. For $1 < a \leq 2$, T_a is known to admit an acip (e.g. [21]). We prove the following theorem.

Theorem 1.3 *There are only two values of $a \in (1, 2]$, namely $\sqrt{2}$ and 2, for which Lebesgue measure is recurrent.*

The paper is organized as follows: after some definitions and notations, we prove the nonrecurrence of Lebesgue measure for tent maps in Section 3. In Section 4, we briefly discuss Hamachi's construction on the shift space. In the next two sections we apply his construction to obtain our results for circle and interval maps. In the last section we focus on higher dimensional generalizations.

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2 Preliminaries

Throughout this paper we will only consider measurable nonsingular transformations T of Lebesgue spaces X, where X is usually a manifold, endowed with the σ-algebra of Borel sets, denoted $\mathcal{B}$, and a Lebesgue measure μ on $\mathcal{B}$ which is σ-finite. These assumptions on T mean that for all $A \in \mathcal{B}$, $T^{-1}(A) \in \mathcal{B}$, and $\mu(A) = 0$ if and only if $\mu T^{-1}(A) = 0$. By replacing T by an isomorphic copy if necessary, we also assume that T is forward measurable and nonsingular; i.e., for all $A \in \mathcal{B}$, $T(A) \in \mathcal{B}$, and $\mu(A) = 0$ if and only if $\mu T(A) = 0$ [26]. Under these assumptions T will always be surjective as well with respect to μ.

Definition 2.1 *Let $(X, \mathcal{B}, \mu)$ be a Lebesgue measure space with μ a finite measure. We say that T is a* bounded-to-one endomorphism *of X is there exists a measurable partition $\mathcal{P} = \{A_1, \dots, A_n\}$ of X such that $\mu(A_i) > 0$, $T|_{A_i} \equiv T_i$ is one-to-one, T_1 is one-to-one and onto X, and each A_i is maximal with respect to μ in $X \setminus \bigcup_{j<i} A_j$.*

Definition 2.2 *T is* conservative *(with respect to μ) if for every $A \in \mathcal{B}$ of positive measure, there exists an $m \in \mathbf{N}$ such that $\mu(T^{-m}(A) \cap A) > 0$. T is* ergodic *(with respect to μ) if for every $A \in \mathcal{B}$ such that $T^{-1}(A) = A$, we have $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.*

Definition 2.3 The Jacobian and the Radon-Nikodým derivative: *Given a nonsingular bounded-to-one endomorphism $(X, \mathcal{B}, \mu; T)$, for each $x \in A_i$ let $J_\mu T(x) = \frac{d\mu T_i}{d\mu}(x)$. We set $J_\mu T(x) = 0$ for all $x \in X \setminus \bigcup_i A_i$. This is called the* Jacobian *of T with respect to μ.*

The (global) Radon-Nikodým derivative *of T, denoted ω_μ, is defined to be the unique $T^{-1}\mathcal{B}$-measurable function satisfying:*

$$\int_X f \circ T \cdot \omega_\mu d\mu = \int_X f d\mu \text{ for all } f \in L^1(X, \mu).$$

The higher derivatives are defined for each $n \in \mathbf{N}$ *by* $\omega_\mu(n,x) = \prod_{i=0}^{n-1} \omega_\mu(T^i x)$. *The measure* μ *is said to be* recurrent for T *if*

$$\sum_n \omega_\mu(n,x) = \infty \quad \mu\text{-}a.e.$$

We have the following easily verified identity linking the two types of derivatives:

$$\omega_\mu(x) = \left(\sum_{y \in T^{-1}Tx} \frac{1}{J_\mu T(y)} \right)^{-1}. \tag{1}$$

When T is invertible, all measures are recurrent and the Jacobian and the Radon-Nikodým derivative are the same function. It is an open problem as to whether every endomorphism admits an equivalent recurrent measure. In general it is difficult to determine whether a given ergodic measure is recurrent; an invariant measure μ is always recurrent since $\omega_\mu(x) \equiv 1$ in this case.

The following theorem represents a compilation of results.

Proposition 2.1 *(cf. [28]) Suppose* T *is countable-to-one and preserves a* σ*-finite measure* ν *equivalent to* μ*, then the following are equivalent:*

1. μ *is recurrent;*
2. $\phi = \frac{d\nu}{d\mu}$ *is a* $T^{-1}\mathcal{B}$*-measurable density function;*
3. ω_μ *is a coboundary; i.e.,* $\omega_\mu = \frac{\phi}{\phi \circ T}$ μ*-a.e.*

Proof: (3) $\Rightarrow$ (2): Since by definition ω_μ is $T^{-1}\mathcal{B}$-measurable, so is the quotient $\frac{\phi}{\phi \circ T}$. Obviously $\phi \circ T$ is $T^{-1}\mathcal{B}$-measurable, so the product of ω_μ with $\phi \circ T$, which is ϕ, is $T^{-1}\mathcal{B}$-measurable as well. (2) $\Rightarrow$ (3) follows since $\omega_\nu = 1$ and

$$\frac{\omega_\nu}{\omega_\mu} = \frac{\phi \circ T}{E(\phi | T^{-1}\mathcal{B})},$$

where $E(\cdot|\mathcal{A})$ denotes the usual conditional expectation onto $\mathcal{A} \subseteq \mathcal{B}$. (3) $\Rightarrow$ (1) since coboundaries are easily shown to be recurrent (see [11]). (1) $\Rightarrow$ (3) is proved in [28]. □

Definition 2.4 *Let T be a endomorphism of a Lebesgue space $(X, \mathcal{B}, \mu)$ such that μ is ergodic, conservative, and nonsingular. We call μ a* type II_1 measure for T *if it is absolutely continuous with respect to some invariant probability measure. When X is a Riemannian manifold, and μ is an invariant probability measure which is absolutely continuous with respect to the volume form, then we call μ an* acip. *We call μ a* type *III* measure for T *or we say T is a* type *III endomorphism if T admits no σ-finite invariant measure equivalent to μ.*

In this paper many examples deal with unimodal maps. A map $f : I \to I$, $I = [0, 1]$, is called *unimodal*, if there exists a unique point, c, the *critical point*, such that $f|_{[0,c)}$ is increasing and $f|_{(c,1]}$ is decreasing. We write $c_n := f^n(c)$. The forward orbit of a point x is denoted as $orb(x)$. A unimodal map is onto on the *dynamical core*, i.e. the interval $[c_2, c_1]$. Therefore we will always restrict to this interval. A unimodal map f is bounded-to-one; Definition 2.1 is satisfied by taking $A_1 = (c, c_1]$ and $A_2 = [c_2, c)$. For $x \neq c$, let the *symmetric point* $\hat{x}$ be the point such that $\hat{x} \neq x$ and $f(x) = f(\hat{x})$. Restricted to the dynamical core, $\hat{x}$ is only defined if $x \in [c_2, \hat{c}_2] \setminus \{c\}$. We call a Radon-Nikodým g derivative *symmetric* if $g(x) = g(\hat{x})$ for all $x \in [c_2, \hat{c}_2] \setminus \{c\}$.

It can happen that there exists an interval $J \ni c$, $J \neq [c_2, c_1]$, such that $f^n(J) \subset J$ for some $n \geq 1$. In this case f is called *renormalizable*. Take J maximal and n minimal with these properties. Then J is called a *restrictive* or *periodic* interval of period n. An appropriate affine rescaling of $f^n|_J$, called the *renormalization*, is again a unimodal map, which can be renormalizable or not. Therefore we can distinguish between infinitely renormalizable and finitely renormalizable maps. In the latter case, the deepest, i.e. the last renormalization is itself nonrenormalizable.

3 Non-recurrent measures

In this section we consider the family of tent maps on I, defined by

$$T_a(x) = \begin{cases} ax & \text{if } x \leq \frac{1}{2}, \\ a(1-x) & \text{if } x \geq \frac{1}{2}, \end{cases}$$

and we take the slope $a \in (1, 2]$. T_2 is the *full tent map*. The critical point is $c = \frac{1}{2}$ and the interval $[T_a^2(c), T_a(c)] = [(2-a)\frac{a}{2}, \frac{a}{2}]$ is the dynamical core. It is easily verified that T_a is renormalizable (of period 2) if and only if $a \leq \sqrt{2}$. The renormalization is the tent map with slope a^2. Therefore T_a is at most finitely renormalizable for

$a > 1$. We put the normal Borel structure on the dynamical core, and let m_a be the normalized Lebesgue measure on it. In this setting T_a is bounded-to-one, and the partition $\zeta = \{(c, c_1], [c_2, c)\}$ generates the σ-algebra of Borel sets under T_a. T_a is clearly nonsingular. It is well known that T_a is ergodic with respect to m_a, and also conservative, provided it is nonrenormalizable.

We can compute $\omega_{m_a}(x)$ explicitly from equation (1). At points where the map T_a is one-to-one, (i.e. $x = T_a^{-1} T_a x$) we have $\omega_{m_a}(x) = a$; at the points where T_a is two-to-one, (i.e. $\{x, \hat{x}\} = T_a^{-1} T_a x$, $x \neq \hat{x}$) we have $\omega_{m_a}(x) = \frac{a}{2}$. We do not define $\omega_{m_a}(c)$. We compute the higher derivatives to be:

$$\omega_{m_a}(n, x) = a^n 2^{-r(n,x)},$$

where

$$r(n, x) = \#\{0 \leq i < n; T_a \text{ is two-to-one at } T^i(x)\}.$$

Since T_a admits an ergodic acip $\nu_a \sim m_a$, it follows that

$$\lim_{n \to \infty} \frac{r(n, x)}{n} \text{ exists and is constant } \nu_a\text{-a.e.}$$

Define the sequence $\theta_i \in \{-1, 1\}^{\mathbf{N}}$ as follows:

$$\theta_n = \begin{cases} 1 & \text{if } \#\{2 \leq i \leq n; c_i > 1\} \text{ is even,} \\ -1 & \text{if } \#\{2 \leq i \leq n; c_i > 1\} \text{ is odd.} \end{cases}$$

We have the following theorem describing the density of ν_a:

Theorem 3.1 ([3]) *The Radon-Nikodým derivative*

$$\frac{d\nu_a}{dm_a}(x) = \varphi(x) = \sum_{\substack{n \geq 1 \\ c_{n+1} < x < c_1}} \frac{\theta_n}{a^n}$$

We use this result to prove the following:

Theorem 3.2 *There are only two values of a such that T_a has a symmetric density function for ν_a. Hence Lebesgue measure is not recurrent for tent maps, except for the slopes $a = 2$ and $a = \sqrt{2}$.*

Proof: We divide the proof into two steps. First we claim that the set $orb(c)$ is not $T_a^{-1}\mathcal{B}$-measurable, unless $a = 2$ or $a = \sqrt{2}$. Indeed, suppose it were, then $\hat{c}_m \in orb(c)$ for every $m \in \mathbf{N}$. So there exists n such that $c_n = \hat{c}_m$. In particular, if $a \neq 2$, there exists $n > 2$ such that $c_n = \hat{c}_2$. Then $c_{n+1} = c_3$, whence c_3 is $n-2$-periodic. Setting $n = 3$, we see that this can occur with c_3 a fixed point. In this case $orb(c) = \{c_1, c_2, c_3 = \hat{c}_2\}$ is indeed symmetric. This is met when $a = \sqrt{2}$. Using Theorem 3.1, one can show that $\frac{d\nu_a}{dm_a}$ is constant on $(c_2, \hat{c}_2)$ for both $a = \sqrt{2}$ and $a = 2$. Hence by Proposition 2.1, μ_a is recurrent. We show that nothing else can occur.

If $n > 3$, then $c_n \neq \hat{c}_2$. If $c_3 \notin [c_2, \hat{c}_2]$, then $c_3 = (2-a)\frac{a^2}{2} > \frac{2-2a+a^2}{2} = \hat{c}_2$. This in turn implies that $a^3 - a^2 - 2a + 2 < 0$, which is impossible for $a > \sqrt{2}$. If $a < \sqrt{2}$, then T_a is renormalizable: $T_a^2([c_2, \hat{c}_2]) \subset [c_2, \hat{c}_2]$ and $[c_2, \hat{c}_2] \cap [c_3, c_1] = \emptyset$. Now $c_4 \in (c_2, \hat{c}_2)$, and $\hat{c}_4 = c_{n'}$ for some $n' > 4$. But this is impossible because c_3 is periodic. The remaining possibility is $c_3 \in (c_2, \hat{c}_2)$, but then also $\hat{c}_3 = c_{n'}$ for some $n' > 3$. This again is impossible because c_3 is periodic.

In the second step of the proof we show that φ is not a symmetric density if $orb(c)$ is not symmetric. Let c_m be such that $\hat{c}_m \notin orb(c)$, but exists. The function $\varphi(x) = \sum_{\substack{n \geq 1 \\ c_{n+1} < x < c_1}} \frac{\theta_n}{a^n}$ clearly has a discontinuity at c_m, with a jump of size $\frac{1}{a^{m-1}}$. Choose k satisfying $\frac{1}{a^k}\frac{a}{a-1} < \frac{1}{a^m} =: \varepsilon$, and choose neighbourhoods U of c_m and $\hat{U}$ of $\hat{c}_m$ such that $c_i \notin U \cup \hat{U}$ for all $i \leq k$. Then it follows that $\sup_{x,y \in U} |\varphi(x) - \varphi(y)| > \varepsilon$, while $\sup_{x,y \in \hat{U}} |\varphi(x) - \varphi(y)| \leq \varepsilon$. Hence φ cannot be symmetric. This concludes the proof of the theorem. □

Remark: A similar situation may hold for differentiable families, and in particular the quadratic family $f_b(x) = bx(1-x)$. For the full quadratic map, i.e. $b = 4$, the invariant density is known to be $\frac{d\mu_b}{dm}(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}$, which is clearly symmetric. For $b = 3.67857\ldots$, the parameter value corresponding to the tent map with slope $\sqrt{2}$, the density is not symmetric, see [8]. Therefore Lebesgue measure is nonrecurrent for this value.

For all other values of $b \in [0, 4]$ such that f_b has an acip, we expect Lebesgue measure to be nonrecurrent. We outline why this should be true. Using the same proof as for the tent maps, we can show that the critical orbit is not symmetric. If an acip exists, its density function will have a pole at every forward image of c, cf. [18]. For example, if f_b additionally satisfies the Collet-Eckmann condition (i.e. $\liminf_n \frac{1}{n} \log Df^n(c_1) > 0$), the density has the form $\frac{d\mu_b}{dm}(x) = g(x) \sum_{i \geq 1} \lambda^r |x - c_r|^{\frac{1}{2}}$

for some $\lambda \in (0,1)$ and a function g of bounded variation (see [19]). Whenever the closure of the critical orbit is nonsymmetric, this density is clearly nonsymmetric.

4 Hamachi measure

The starting point for all the constructions in this paper is an example of Hamachi from 1981 [9]. He constructed a measure μ for the two-sided shift space $X = \{0,1\}^{\mathbf{Z}}$ with the usual Borel structure with the following properties.

Theorem 4.1 ([9]) *There exists a Borel measure α for the shift σ on $X = \{0,1\}^{\mathbf{Z}}$ such that:*

1. *α is a product measure on X;*

2. *α is nonsingular, conservative, and ergodic for σ;*

3. *α is a type III measure for σ.*

Later it was shown by Dajani and Hawkins that a one-sided version of Hamachi measure gives a type *III* one-sided shift [4]. This observation about the Hamachi measure was also made independently by Silva and Thieullen [28]. We state the precise result since it is the basic measure on which further constructions are based.

Corollary 4.1 ([4]) *There exists a Borel measure for the shift on $X^+ = \{0,1\}^{\mathbf{N}}$ such that:*

1. *α is a product measure on X^+;*

2. *α is nonsingular, conservative, and ergodic for σ;*

3. *α is recurrent for σ;*

4. *α is a type III measure for σ.*

We give a brief description of the type *III* measure of Hamachi here.

4.1 A description of the Hamachi product measure

We define the measure α to be of the form $\alpha = \prod_{n\geq 1} \alpha_n$ by specifying each factor measure α_n. We begin by defining some measures on the 2-point set $\{0, 1\}$:

1. We denote by β the equally distributed measure $\beta(0) = \beta(1) = \frac{1}{2}$. If we put the measure β on each factor of X^+, we will denote this Bernoulli measure by $\hat{\beta}$.

2. For a sequence λ_k such that $\lambda_1 > \lambda_2 \cdots > 1$ (to be chosen by induction later), we define a measure $\gamma_k(0) = \frac{1}{1+\lambda_k}$ and $\gamma_k(1) = \frac{\lambda_k}{1+\lambda_k}$.

We will define sequences of integers $\{M_k\}_{k\geq 1}$, $\{N_k\}_{k\geq 1}$, satisfying: $M_1 > 1$ is arbitrary, also $N_1 > M_1$ is arbitrary, and $N_k = M_k + n_k$, $M_{k+1} = N_k + m_k$. Here, n_k and m_k are positive integers chosen inductively, with the inductive step outlined below.

We then define:

$$\alpha_n = \begin{cases} \beta & \text{if } 0 < n \leq M_1, \\ \gamma_k & \text{if } M_k < n \leq N_k, \\ \beta & \text{if } N_k < n \leq M_{k+1}. \end{cases}$$

As was shown in [15], the measure μ is nonsingular for the shift if and only if

$$\sum_{k=1}^{\infty} (\log \lambda_k)^2 < \infty. \tag{2}$$

For later purposes we will choose the λ_k's such that $\prod_{k=1}^{\infty} \lambda_k < \frac{11}{10}$. Clearly formula (2) is still satisfied. We add another condition on the choice of λ_k in the inductive step.

Hamachi gives an inductive algorithm for choosing the sequence $(\lambda_k, n_k, m_k)_{k\geq 1}$ so that the shift is nonsingular, ergodic, and type *III* with respect to α. Since we will modify the measure later, we will outline the inductive step, omitting the details since they are carefully written out in [9].

A. Starting the inductive argument.

We choose $\lambda_1 > 1$ to be arbitrary; also $n_1 \in \mathbf{N}$ is arbitrary (greater than 1) and $m_1 \in \mathbf{N}$ is any number such that $m_1 > n_1 + 1$. Choose any decreasing sequences of positive numbers $\{p_k\}$ and $\{\varepsilon_k\}$, $k \geq 1$, such that

1. As $k \to \infty$, $p_k \to 0$ and $\varepsilon \to 0$;

2. $\sum_{k=1}^{\infty} p_k = \infty$;

3. $\sum_{k=1}^{\infty} \varepsilon_k < \infty$;

4. Define $\eta_k = \sum_{t=k}^{\infty} \varepsilon_t$ (the tail of the ε_k series).

B. The inductive choice of λ_k in order to introduce the distortion from the $(\frac{1}{2}, \frac{1}{2})$ measure $\hat{\beta}$.

Keeping in mind that if $\lambda_k = 1$ for all k, then we have the β measure on each factor, and preserving the nonsingularity condition given by formula (2), we choose $1 < \lambda_k < \lambda_{k-1}$ so that

$$(2\lambda_k/1+\lambda_k)^{M_{k-1}} < \lambda_k^{M_{k-1}} < e^{\varepsilon_k}$$

We also choose $\rho_k > 0$ such that

$$1 < (\lambda_1)^{2M_{k-1}} < (\lambda_k)^{\rho_k}$$

C. The inductive choice of n_k, the integer which determines how long we must distort the measure by λ_k.

We consider for the moment all possible cylinders of length $N_k = M_k + n_k$; and we note that the measure of all such cylinders with exactly t 1's occurring somewhere between M_k and N_k is given by the binomial distribution formula:

$$f_k(t) = \binom{n_k}{t} (\frac{\lambda_k}{1+\lambda_k})^t (\frac{1}{1+\lambda_k})^{n_k - t},$$

$t = 0, 1, \ldots, n_k$. We choose n_k large enough so that $f_k(t)$ is the "correct size for enough of the t's enough of the time" in order to reflect the fact that we have changed from the $(\frac{1}{2}, \frac{1}{2})$ measure to the $(\frac{1}{1+\lambda_k}, \frac{\lambda_k}{1+\lambda_k})$ measure. In particular, if we solve for $c_k > 0$ so that

$$\frac{1}{\sqrt{2\pi}} \int_{-c_k}^{c_k} e^{(-s^2/2)} ds = p_k,$$

then we apply the Central Limit Theorem and choose n_k large enough so that

$$\sum_{|t - \frac{n_k \lambda_k}{1+\lambda_k}| < \frac{\sqrt{n_k \lambda_k} c_k}{1+\lambda_k} - 2\rho_k} f_k(t) > \frac{p_k}{4}.$$

D. The inductive choice of m_k, the integer which determines how long we must spend back at the β measure for conservativity.

There are two conditions that determine our choice at this step. We choose m_k large enough so that

$$\frac{N_k e^{2\eta_{k+1}} \lambda_1^{3N_k}}{m_k - N_k} < \frac{\varepsilon_k}{2}.$$

This maintains conservativity of α; but in addition, in order to ensure that the product measure α is of type *III* (and not just equivalent to an infinite invariant measure), we need to choose m_k large enough so that we can obtain some correct Birkhoff Ergodic Theorem averages with respect to the measure $\hat{\beta}$ on most of the space X^+ (on a set of $\hat{\beta}$ measure $> 1 - \varepsilon_k$) for certain real-valued functions. We refer the reader to [9] for the details of this inductive step.

For the invertible shift, the measure is extended to the negative indices by setting $\alpha_n = \beta$ if $n \leq 0$. We will call the original two-sided measure constructed by Hamachi $\hat{\alpha}$, and it is easy to show that the noninvertible measure α constructed above satisfies:

$$\frac{d\hat{\alpha}\sigma^i}{d\hat{\alpha}}(\ldots, x_{-1}, x_0, x) = \omega_\alpha(i, x) \text{ for all } x \in X^+, \text{ and } x_j \in \{0,1\},\ j \leq 0.$$

The choice of m_k will insure that:

$$\sum_{i=0}^{\infty} \frac{d\hat{\alpha}\sigma^i}{d\hat{\alpha}}(x) = \infty$$

for the invertible shift, giving conservativity, and for the noninvertible one we have

$$\sum_{i=0}^{\infty} \omega_\alpha(i, x) = \infty.$$

Therefore α constructed in this way is a recurrent measure.

Definition 4.1 *We will call the measure α constructed in this way, using [9], the* Hamachi (*type III*) measure.

4.2 Some variations on the Hamachi measure.

Given $X_d^+ = \prod_{n\in\mathbf{N}}\{0, 1, \ldots, 2d-1\}_n$, we define a product measure α_d closely related to the Hamachi measure on X^+ as follows.

Choose any positive numbers $p_1, \dots, p_d$ and $q_1, \dots, q_d$ such that

$$\sum_{k=1}^{d} p_k = \sum_{k=1}^{d} q_k = \frac{1}{2}.$$

Since the Borel structure on X_d^+ is generated by cylinders, we define α_d by specifying its values on cylinder sets. We note first that the p_i's and q_i's determine a measure on X_d^+ by:

$$P = \prod_{n \in \mathbf{N}} P_n,$$

with

$$P_n(0) = p_1, P_n(1) = p_2, \dots, P_n(d-1) = p_d,$$
$$P_n(d) = q_1, \dots, P_n(2d-1) = q_d$$

for each $n \in \mathbf{N}$. The measure P is a Bernoulli measure preserved by the shift σ, so $\omega_P(x) = 1$ for all $x \in X_d^+$. Each cylinder $C_{e_1 \dots e_n}$ of length n lies in a dyadic cylinder of length n, by recoding each e_k into a 0 if $0 \leq e_k < d$ or as a 1 if $d \leq e_k < 2d$. We denote the coding map from $2d$ symbols to 2 symbols by π; π is defined pointwise in the obvious way. We now define

$$\alpha_d(C_{e_1 \dots e_n}) = P(C_{e_1 \dots e_n}) \frac{\alpha(\pi C_{e_1 \dots e_n})}{\hat{\beta}(\pi C_{e_1 \dots e_n})},$$

where α is the Hamachi measure constructed above, and $\hat{\beta}$ the $(\frac{1}{2}, \frac{1}{2})$ Bernoulli measure. With this definition, we have linearly rescaled the Hamachi measure to any even number of states, so that $\alpha_d = P \cdot \frac{\alpha}{\hat{\beta}} \circ \pi$.

Letting σ_k denote the shift on the k-symbol space, we have the following lemma.

Lemma 4.1 *The Radon-Nikodým derivative $\omega_{\alpha_d}(x)$ for σ_{2d} equals $\omega_\alpha(\pi x)$ for σ_2.*

Proof: We note that since $J_{\alpha_d}\sigma_{2d}(x) = \frac{1}{2} J_P \sigma_{2d}(x) J_\alpha \sigma_2(\pi x)$, then it is easy to compute that

$$\omega_{\alpha_d} = \left(\sum_{y \in \sigma_{2d}^{-1} \sigma_{2d} x} \frac{1}{J_{\alpha_d} \sigma_{2d}(y)} \right)^{-1} = \omega_\alpha(\pi x).$$

□

Remark. Choose any two sequences of positive numbers $\{p_k\}$ and $\{q_k\}$ such that

$$\sum_{k=1}^{\infty} p_k = \sum_{j=1}^{\infty} q_j = \frac{1}{2}.$$

The p_k's and q_j's determine a probability measure on: $X_{\mathbf{N}}^+ = \prod_{n=0}^{\infty}\{0, 1, \ldots, 2k, 2k+1, \ldots\}_n$ by: $P = \prod_{n\geq 1} P_n$, with $P_n(2k) = p_k$, $P_n(2k+1) = q_k$, for each $k \geq 0$ for each n. We define a factor map π from $X_{\mathbf{N}}^+$ onto the dyadic X^+ above by

$$\pi(x) = (x_1(\bmod 2), x_2(\bmod 2), \ldots,).$$

We can consider the usual one-sided shift map $\sigma_{\mathbf{N}}$ on $X_{\mathbf{N}}^+$. If we define the measure

$$\alpha_{\mathbf{N}}(C_{e_1\ldots e_n}) = P(C_{e_1\ldots e_n})\frac{\alpha(\pi C_{e_1\ldots e_n})}{\hat{\beta}(\pi C_{e_1\ldots e_n})},$$

where α is Hamachi measure and $C_{e_1\ldots e_n}$ is any cylinder of length n, then $\omega_{\alpha_{\mathbf{N}}}(x) = \omega_{\alpha}(\pi x)$.

5 Hamachi measure for circle maps

In this section we construct the basic differentiable example, on which the other examples are built. A map f on a metric space X (endowed with metric ρ) is called *expanding* if there exists $C > 1$ such that $\rho(f(x), f(y)) \geq C\rho(x, y)$ for all $x, y \in X$.

Theorem 5.1 *There exists a C^1 expanding circle map which is type III with respect to Lebesgue measure m. Furthermore m is recurrent for this map.*

Proof: Let $S : \mathbf{S}^1 \to \mathbf{S}^1$, $\mathbf{S}^1 \simeq \mathbf{R}/\mathbf{Z}$, be the ordinary angle doubling map: $x \mapsto 2x(\bmod 1)$. S preserves Lebesgue measure m. Let α be Hamachi measure on $\Sigma = \{0, 1\}^{\mathbf{N}}$. Because (Σ, σ) is measure-theoretically isomorphic to $(\mathbf{S}^1, S)$, S is type *III* with respect to the measure μ which is induced by α. We fix an orientation on $\mathbf{S}^1$, and define $h : \mathbf{S}^1 \to \mathbf{S}^1$ as $h(x) = \mu([0, x))$. As μ is nonatomic and its support is the whole circle, h is indeed a homeomorphism, and the measure $\mu \circ h^{-1}$ is Lebesgue measure. Therefore $f := h \circ S \circ h^{-1}$ is a type *III* circle map with respect to Lebesgue measure. We will analyze h in detail to show that f can be a Lipschitz map. Using a slight perturbation of h, we can obtain an expanding C^1 map which is type *III* with respect to Lebesgue measure.

Let $\{M_k\}_{k\geq 1}$, $\{N_k\}_{k\geq 1}$, $1 < M_k < N_k < M_{k+1} < N_{k+1} \ldots$, and $\{\lambda_k\}_{k\geq 1}$, $\lambda_1 > \lambda_2 > \ldots > 1$, be the sequences appearing in the construction of the Hamachi product measure. Recall that $\prod_k \lambda_k < \frac{11}{10}$. Let $\varepsilon = \{\varepsilon_k\}$ be a sequence of nonnegative reals. If $\varepsilon_k \equiv 0$, then the construction below yields $h(x) = \mu([0,x))$. Otherwise we will choose $\varepsilon_k > 0$ inductively (with $\varepsilon_1 = 0.1$ and $\varepsilon_k \searrow 0$) to obtain a C^1 degree 2 circle map $f_\varepsilon = h_\varepsilon \circ S \circ h_\varepsilon^{-1}$. For each $k \in \mathbf{N}$ and $\eta \in (-\frac{1}{2}, \frac{1}{2})$ let $\psi_{k,\eta}$ (satisfying $\psi_{k,\eta}(0) = 0$) be the map with a piecewise linear derivative:

$$D\psi_{k,\eta}(x) = \begin{cases} 1 + \frac{x}{\varepsilon_k}\frac{1-\lambda_k}{1+\lambda_k} & \text{if } x \in [0, \varepsilon_k), \\ \frac{2}{1+\lambda_k} & \text{if } x \in [\varepsilon_k, \frac{1}{2} + \eta - \varepsilon_k], \\ 1 - \frac{x-\frac{1}{2}-\eta}{\varepsilon_k}\frac{1-\lambda_k}{1+\lambda_k} & \text{if } x \in (\frac{1}{2} + \eta - \varepsilon_k, \frac{1}{2} + \eta + \varepsilon_k), \\ \frac{2\lambda_k}{1+\lambda_k} & \text{if } x \in [\frac{1}{2} + \eta + \varepsilon_k, 1 - \varepsilon_k], \\ 1 + \frac{x-1}{\varepsilon_k}\frac{1-\lambda_k}{1+\lambda_k} & \text{if } x \in (1 - \varepsilon_k, 1]. \end{cases}$$

Then $\int_0^1 D\psi_{k,\eta}(x)dx = 1 + 2\eta\frac{1-\lambda_k}{1+\lambda_k}$, and $\psi_{k,\eta}(I) = I$ if $\eta = 0$. Moreover, $D\psi_{k,\eta}(0) = D\psi_{k,\eta}(1) = 1$. This is necessary to glue these maps together and still have a C^1 diffeomorphism.

Let $C_{e_1 e_2 \ldots e_n} \subset \mathbf{S}^1$ be an n-cylinder, labelled by the first n coordinates of its itinerary: $C_0 = [0, \frac{1}{2})$, $C_1 = [\frac{1}{2}, 1)$, $C_{01} = [\frac{1}{4}, \frac{1}{2})$, and so on. Therefore $\mathbf{S}^1 = \bigcup_{e_1 e_2 \ldots e_n \in \{0,1\}^n} C_{e_1 e_2 \ldots e_n}$. Define h_n as follows:

- If $n \leq M_1$ or $N_k < n \leq M_{k+1}$ for some k, then h_n is the identity.
- If $M_k < n \leq N_k$ for some k, then h_n is made up of 2^{n-1} scalings of $\psi_{k,\eta}$. Let $\xi = h_{n-1} \circ \ldots h_1(\zeta)$, where ζ is the midpoint of $C_{e_1 \ldots e_{n-1}}$. Let $[x, y) = h_{n-1} \circ \ldots \circ h_1(C_{e_1 \ldots e_{n-1}})$ and $\eta_n = \frac{\xi - x}{y - x} - \frac{1}{2}$. (Note that if $\varepsilon_k \equiv 0$, $h_{n-1} \circ \ldots \circ h_1$ is linear on $C_{e_1 \ldots e_{n-1}}$ and $\eta_n \equiv 0$.) Let $\tilde{h}_n(0) = 0$, and assuming inductively that $\tilde{h}_n$ is defined on the cylinder $[x', y')$ to the left of $[x, y)$, we define for $z \in [x, y)$.

$$\tilde{h}_n(z) = \lim_{z' \nearrow y'} \tilde{h}_n(z') + (y - x)\psi_{k,\eta_n}(\frac{z - x}{y - x}), \tag{3}$$

  Finally let $h_n(z) = t_n \tilde{h}_n(z)$, where $t_n > 0$ is such that $t_n \int_0^1 D\tilde{h}_n(z)dz = 1$.

Let $H_n = h_n \circ \ldots \circ h_1$ and $h_\varepsilon = \lim_n H_n$. We will see below that $t_n \to 1$ exponentially. This will imply that $\lim_n H_n$ exists pointwise. Define also $f_n = H_n \circ S \circ H_n^{-1}$. We need to show that for $\{\varepsilon_k\}$ well chosen, $\{f_n\}$ is a convergent sequence in the C^1 topology.

Suppose that we have chosen ε_l for $l < k$ and take $M_k < n \leq N_k$. Let us first derive some estimates for Df_{N_k} which hold if $\varepsilon_k = 0$. Then we choose upper bounds $\bar{\varepsilon}_k$ for ε_k for the inductive step in formula (4) using a continuity argument.

By the *relative displacement* of a point $c \in (a,b)$ *due to* a homeomorphism h we mean $\frac{h(c)-h(a)}{h(b)-h(a)} - \frac{c-a}{b-a}$. Each number η_n measures the relative displacement of the midpoint of (one of the) $C_{e_1\ldots e_{n-1}}$ due to H_{n-1}. We estimate η_n under the temporary assumption $\varepsilon_k = 0$. We only have to consider h_m for $m \leq N_{k-1}$, since for all $N_{k-1} < m < n$, the homeomorphisms h_m are piecewise affine on $H_{m-1}(C_{e_1\ldots e_{n-1}})$. For $m \leq N_{k-1}$, the interval $H_{m-1}(C_{e_1\ldots e_{n-1}})$ is exponentially small compared to $H_{m-1}(C_{e_1\ldots e_{m-1}})$. Indeed, because the distortions $\frac{\sup Dh_l}{\inf Dh_l} \leq \frac{6}{5}$ for all $l \in \mathbf{N}$ and $\frac{M_k}{N_{k-1}} \gg 1$, $|H_{m-1}(C_{e_1\ldots e_{n-1}})| \leq 1.9^{-(n-m)}|H_{m-1}(C_{e_1\ldots e_{m-1}})|$. Hence the relative displacement of any $c \in H_{m-1}(C_{e_1\ldots e_{n-1}})$ due to h_m is $\leq \mathcal{O}((1.9)^{-(n-m)})$. This gives a relative displacement due to H_{n-1} (as $\frac{M_k}{N_{k-1}} \gg 1$) $\eta_n \leq \sum_{m=1}^{N_{k-1}} \mathcal{O}((1.9)^{-(n-m)}) \leq (1.8)^{-n}$. By the same argument $\eta_n \geq -(1.8)^{-n}$. Because this is true for every cylinder $C_{e_1\ldots e_{n-1}}$ and $\int_0^1 D\psi_{k,\eta_n}(z)dz = 1 + 2\eta_n\frac{1-\lambda_k}{1+\lambda_k}$, it follows also that $\sup_z|\tilde{h}_n(z) - z| \leq \mathcal{O}((1.8)^{-n})$ and also $1 - \mathcal{O}((1.8)^{-n}) \leq t_n \leq 1 + \mathcal{O}((1.8)^{-n})$. Therefore the normalized homeomorphism h_n satisfies $\sup_z|h_n(z) - z| \leq \mathcal{O}((1.8)^{-n})$ as well.

Clearly $Df_n(x) = \frac{Dh_n(f_{n-1}(y))}{Dh_n(y)}Df_{n-1}(y)$, where $y = h_n^{-1}(x)$. Now setting $n = N_k$ and $y_i = h_i^{-1} \circ \cdots \circ h_{N_k}^{-1}(x)$, we get (because h_m is the identity for $N_{k-1} < m \leq M_k$)

$$\begin{aligned} Df_{N_k}(x) &= Df_{N_{k-1}}(y_{N_{k-1}+1})\frac{Dh_{N_{k-1}+1}(f_{N_{k-1}}(y_{N_{k-1}+1}))}{Dh_{N_{k-1}+1}(y_{N_{k-1}+1})}\cdots\frac{Dh_{N_k}(f_{N_k-1}(y_{N_k}))}{Dh_{N_k}(y_{N_k})} \\ &= Df_{N_{k-1}}(y_{N_{k-1}+1})\frac{Dh_{M_k+1}(f_{M_k}(y_{M_k+1}))}{Dh_{M_k+1}(y_{M_k+1})}\cdots\frac{Dh_{N_k}(f_{N_k-1}(y_{N_k}))}{Dh_{N_k}(y_{N_k})}. \end{aligned}$$

By construction, $f_{n-2}(h_{n-1}^{-1}(z))$ lies in the left part of $H_{n-2}(C_{e_2\ldots e_{n-1}})$ if and only if z lies in the left part of $H_{n-1}(C_{e_1\ldots e_{n-1}})$, for all $M_k < n \leq N_k$. Hence $\frac{|Dh_{n-1}(f_{n-2}(y_{n-1}))|}{|Dh_n(y_n)|} = \frac{t_{n-1}}{t_n}$ and

$$\begin{aligned} Df_{N_k}(x) &= Df_{N_{k-1}}(y_{N_{k-1}+1})\frac{Dh_{N_k}(f_{N_k-1}(y_{N_k}))}{Dh_{M_k+1}(y_{M_k+1})}\prod_{n=M_k+2}^{N_k}\frac{t_{n-1}}{t_n} \\ &\leq Df_{N_{k-1}}(x)\frac{Df_{N_{k-1}}(y_{N_{k-1}+1})}{Df_{N_{k-1}}(x)}\frac{D\tilde{h}_{N_k}(f_{N_k-1}(y_{N_k}))}{D\tilde{h}_{M_k+1}(y_{M_k+1})} \\ &\leq Df_{N_{k-1}}(x)(1+\mathcal{O}((1.8)^{-M_k}))\lambda_k, \end{aligned}$$

where the estimate of $\frac{Df_{N_{k-1}}(y_{N_{k-1}+1})}{Df_{N_{k-1}}(x)}$ follows because

$$\begin{aligned}|y_{N_{k-1}+1} - x| &\leq \sup_z |h^{-1}_{N_{k-1}+1} \circ \ldots \circ h^{-1}_{N_k}(z) - z| \\ &= \sup_z |h^{-1}_{M_k+1} \circ \ldots \circ h^{-1}_{N_k}(z) - z| \\ &\leq \sum_{i=M_k+1}^{N_k} \mathcal{O}((1.8)^{-i}) \leq \mathcal{O}((1.8)^{-M_k}).\end{aligned}$$

A similar argument gives $Df_{N_k}(x) \geq Df_{N_{k-1}}(x)\frac{1}{\lambda_k}(1 - \mathcal{O}((1.8)^{-M_k}))$. These derivatives depend continuously on ε_k. We choose $\bar{\varepsilon}_k > 0$ so small that

$$Df_{N_{k-1}}(x)\frac{1}{\lambda_k}(1 - (1.7)^{-M_k}) \leq Df_{N_k}(x) \leq Df_{N_{k-1}}(x)\lambda_k(1 + (1.7)^{-M_k}), \qquad (4)$$

for all $0 < \varepsilon_k < \bar{\varepsilon}_k$ and $x \in \mathbf{S}^1$. Then both f_{N_k} and Df_{N_k} converge uniformly on $\mathbf{S}^1$. The limit f_ε satisfies $2\prod_k \frac{1-(1.7)^{-M_k}}{\lambda_k} \leq Df(x) \leq 2\prod_k \lambda_k(1 + (1.7)^{-M_k})$ and therefore is an expanding C^1 circle map.

We now check that f_ε is indeed type *III* with respect to Lebesgue measure. Define $\varphi_{\varepsilon,n}(x) = Dh_n(H_{n-1}(x))$, and if $\varepsilon_k \equiv 0$, we write $\varphi_n(x)$. If $\varepsilon_k \equiv 0$, then f_ε is Lipschitz continuous, and μ corresponds to the original Hamachi product measure. For $\varepsilon_k \not\equiv 0$, h_ε defines a measure by $\mu_\varepsilon := m \circ h_\varepsilon$. We will show that for ε_k small enough, μ and μ_ε are equivalent. The Radon-Nikodým derivative is $\Phi_\varepsilon(x) := \frac{d\mu_\varepsilon}{d\mu}(x) = \prod_{n=1}^\infty \frac{\varphi_{\varepsilon,n}(x)}{\varphi_n(x)}$. We can make sure that Φ_ε is bounded and bounded away from 0 μ-a.e. Indeed, let

$$A_n = \{x \mid \frac{\varphi_{\varepsilon,n}(x)}{\varphi_n(x)} < 1 - \frac{1}{n^2} \text{ or } \frac{\varphi_{\varepsilon,n}(x)}{\varphi_n(x)} > 1 + \frac{1}{n^2}\}.$$

Obviously, $\prod_n \frac{\varphi_{\varepsilon,n}(x)}{\varphi_n(x)}$ is finite and positive if x is not too often contained in A_n. But because the numbers λ_k are the same for φ_n and $\varphi_{\varepsilon,n}$, $\mu(A_n) \to 0$ as $\varepsilon_k \to 0$. Take $\varepsilon_k < \bar{\varepsilon}_k$ so small that $\mu(A_n) \leq \frac{1}{n^2}$ for all $M_k < n \leq N_k$. The Borel-Cantelli Lemma gives $\mu(\bigcap_l \bigcup_{n>l} A_n) = 0$, so the set of points visiting an A_n infinitely often has zero measure. Hence μ is equivalent to μ_ε.

It is a property of Hamachi measure that μ is a recurrent measure for S. We claim that μ_ε is also recurrent for S, which is equivalent to showing that the measure m is recurrent for f_ε. Since $\mu \sim \mu_\varepsilon$, it suffices to show that Φ_ε is constant on symmetric points of S ([12]). Suppose that x and y are such that $S(x) = S(y)$. Then $x = y + \frac{1}{2}$ and by the symmetry in the construction of h_ε, $\Phi_\varepsilon(x) = \Phi_\varepsilon(y)$. □

Remark: Using a Hamachi measure on the one-sided shift on d symbols, and the techniques developed in this section, we can construct C^1 expanding type *III* circle maps of any degree $d \geq 2$.

6 Hamachi measure for interval maps

In this section we modify the previous construction for tent maps.

Theorem 6.1 *There exists a unimodal map, conjugate to the full tent map which is type III with respect to Lebesgue measure m. This map can be chosen to be C^1, or piecewise C^1 and expanding.*

Proof: It is well-known that (Σ, σ) is measure-theoretically isomorphic to (I, T), where $T = T_2$ is the full tent map. Indeed, take $a = a_1 a_2 \ldots \in \Sigma$, and let $\vartheta(a_1 \ldots a_n)$ be the number of ones in $a_1 \ldots a_n$. Let $\tilde{a}$ be defined as

$$\tilde{a}_i = \begin{cases} a_i & \text{if } \vartheta(a_1 \ldots a_{i-1}) \text{ is even,} \\ 1 - a_i & \text{if } \vartheta(a_1 \ldots a_{i-1}) \text{ is odd.} \end{cases}$$

If $x(\tilde{a})$ is the point in $I \setminus \bigcup_{n \leq 0} T^n(\frac{1}{2})$ whose itinerary is $\tilde{a}$, then $a \mapsto x(\tilde{a})$ is the required isomorphism. So we can again pull back Hamachi measure α, obtaining a measure with respect to which T is type *III*. We proceed as in the circle case; the only adjustment to be made is to change formula (3) into

$$h_n(z) = \lim_{z' \nearrow y'} h_n(z') + \begin{cases} (y-x)\psi_{k,\eta_n}(\frac{z-x}{y-x}) & \text{if } \vartheta(e_1 \ldots e_{n-1}) \text{ is even,} \\ (y-x)(\psi_{k,\eta_n}(1) - \psi_{k,\eta_n}(\frac{y-z}{y-x})) & \text{if } \vartheta(e_1 \ldots e_{n-1}) \text{ is odd.} \end{cases}$$

This will give us the required piecewise C^1 and expanding map f.

Note that $f'(0) = -f'(1)$. In order to obtain a C^1 (but no longer expanding) example, we can do the following. It can be easily checked that T and the quadratic map $Q(x) = 4x(1-x)$ are smoothly conjugate: If $g(x) = \frac{1-\cos \pi x}{2}$, then $g \circ T = Q \circ g$. Applying the same conjugacy on f, we get a C^1 map $\tilde{f} = g \circ f \circ g^{-1}$, which is type *III* with respect to the measures $m \circ g^{-1}$ and m. □

These results allow the following generalization:

Theorem 6.2 *For every $a \in (1, 2]$ such that the tent map T_a has a nowhere dense critical orbit, there exists a map f, topologically conjugate to T_a, which is type III with respect to Lebesgue measure.*

Proof: Assume $a > \sqrt{2}$, because otherwise T_a is renormalizable, and we can consider the deepest renormalization of T_a instead. Let $V \subset [c_2, c_1]$ be an open interval such that $orb(c) \cap V = \emptyset$. Due to the expansion properties of T_a, there exists n minimal such that $c \in T_a^n(V)$. Take any $p \in T_a^n(V)$, $p \neq c$, such that also $\hat{p} \in T_a^n(V)$ and $orb(p) \cap (p, \hat{p}) = \emptyset$. Let $U = T_a^{-n}((p, \hat{p})) \cap V$. Then $orb(c)$ and $orb(\partial U)$ are disjoint from U, and $q = T_a^{-n}(c) \cap U$ is the middle point of U. It follows that the first return map $F : U \to U$ has countably many branches $F : J \to U$, $F|_J = T_a^{s(J)}$, all of which are onto. Also $F(q)$ is not defined and $F(q + \varepsilon) = F(q - \varepsilon)$ for all $\varepsilon < \frac{1}{2}|U|$. By the techniques discussed previously, there exists an ergodic nonsingular nonatomic measure μ with respect to which F is type *III.* Also $\mu(U') > 0$ for every nondegenerate subinterval $U' \subset U$. We can pull back μ to a measure $\bar{\mu}$ of the original map T_a as

$$\bar{\mu}(B) = \sum_J \sum_{i=0}^{s(J)-1} \mu(T_a^{-i}(B) \cap J),$$

where the first sum is taken over all branch domains of F. Due to a result of Silva and Thieullen [28], T_a is type *III* with respect to $\bar{\mu}$. It is easy to show that $\bar{\mu}$ is again ergodic nonatomic and nonsingular, and that $\bar{\mu}(I') > 0$ for every nondegenerate subinterval $I' \subset [c_2, c_1]$. Define again the homeomorphism $h : [c_2, c_1] \to [0, 1]$ as $h(x) = \bar{\mu}([c_2, x))$. Then $f = h \circ T_a \circ h^{-1}$ is conjugate to $T_a|_{[c_2,c_1]}$, and f is type *III* with respect to Lebesgue measure. □

7 Hamachi measure for maps on the sphere

In this section we extend our construction to the Riemann sphere, $\mathbf{C} \cup \infty \equiv \mathbf{C}_\infty$. Our example is motivated by a classical construction given by Lattès to construct a rational map of the sphere whose Julia set is the whole sphere [22].

7.1 Analytic type *III* maps of the Torus

We begin by extending our construction above to the torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ as follows. Let us first remark that the angle doubling map $Sx = 2x \pmod 1$ gives rise to a measure preserving Bernoulli four-to-one map of $\mathbf{T}^2$ by $S \times S(x,y) = Q(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}(x,y) \pmod 1$. With respect to two-dimensional Lebesgue measure on $\mathbf{T}^2$, denoted m_2, this is a one-sided Bernoulli shift of entropy $\log 4$. We can identify $\mathbf{T}^2$ with $\mathbf{S}^1 \times \mathbf{S}^1$ and put μ, the Hamachi measure we constructed in Section 6, on one copy of $\mathbf{S}^1$. The measure we now have on $\mathbf{S}^1 \times \mathbf{S}^1$ is $m \times \mu$, which we denote by μ_2. We have the following result. We will write $\mathcal{B}_2 = \mathcal{B} \times \mathcal{B}$.

Theorem 7.1 *The map $Q = S \times S$ on $(\mathbf{S}^1 \times \mathbf{S}^1, \mathcal{B}_2, \mu_2)$ satisfies:*

1. *Q is a 4-to-1 map with respect to μ_2 ;*
2. *Q is continuous, nonsingular, conservative, and ergodic with respect to μ_2;*
3. *μ_2 is recurrent for Q;*
4. *Q admits no σ-finite invariant measure absolutely continuous with respect to μ_2.*

Proof: It is clear that Q is a nonsingular, 4-to-1 continuous endomorphism on $\mathbf{T}^2$. (It is enough to show there exists a partition $\mathcal{P} = \{P_0, P_1, P_2, P_3\}$ of $\mathbf{T}^2$ into 4 sets such that the restriction of Q to each set is one-to-one and onto with respect to μ_2. The partition into four sets with endpoint coordinates $0, 1$, and $\frac{1}{2}$ obviously works.) Since S is exact with respect to m, and since S is ergodic with respect to μ, it follows by [1] that Q is ergodic with respect to μ_2.

Let ω_m denote the Radon-Nikodým derivative of S with respect to m. Then clearly $\omega_m \equiv 1$. Let ω_μ denote the Radon-Nikodým derivative of S with respect to Hamachi measure. If $\omega_{m_2} \equiv \omega_2$ denotes the Radon-Nikodým derivative of Q with respect to m_2, since ω_μ is $Q^{-1}(\mathcal{B} \times \mathcal{B})$-measurable, it follows that $\omega_2(x,y) = \omega_\mu(y)$ for every (x,y). The fact that Q is type *III* for m_2 now follows exactly as in [12]. $\square$

The following corollary can be proven using the same methods as in the circle case.

Corollary 7.1 *On the d-dimensional torus, there exist maps f with one of the following sets of properties:*

- *f is a toral group endomorphism and f type III with respect to some ergodic and conservative Borel measure.*

- *f is type III ergodic and conservative with respect to Lebesgue measure, and f is C^1 expanding.*

Regarding Q as a map of $\mathbf{R}^2/\mathbf{Z}^2$, we note that $Q(-(x,y)) = -Q(x,y)$ for all $(x,y) \in \mathbf{T}^2$. We now review some basic facts about the Weierstraß elliptic $\wp$ function in conjunction with a classical method of constructing analytic maps of the sphere given by Lattès [22].

7.2 Lattès examples on the Sphere

We consider the Weierstraß elliptic function of the complex plane $\mathbf{C}$; i.e., a meromorphic function on $\mathbf{C}$ which is periodic with respect to a given lattice and is even. In our case we are primarily interested in the lattice $L = \{m+in : m, n \in \mathbf{Z}\}$. We recall that

$$\wp(z) = \frac{1}{z^2} + \sum_{w\in L,\, w\neq 0}\left[\frac{1}{(z-w)^2} - \frac{1}{w^2}\right]$$

satisfies the definition of an even elliptic function and can be regarded as a map from the period parallelogram $\mathbf{C}/L$ which is homeomorphic to $\mathbf{T}^2$. Furthermore $\wp : \mathbf{T}^2 \to \mathbf{S}^2 \cong \mathbf{T}^2/z \sim -z$ is a two-fold branched covering of the sphere by the torus.

Using the identification $z = x + iy = (x,y) \in \mathbf{C}$, when no confusion arises, Q defines a complex endomorphism on $\mathbf{T}^2$ such that $Q(-z) = -Q(z)$. We can pass to the quotient space to obtain an analytic (rational) map of the sphere such that $\wp \circ Q = R \circ \wp$. In fact, using a classical "angle doubling" formula for $\wp$, and the fact that $Q(z) = 2z \,(\bmod L)$, we obtain the following explicit formula for R in this case:

$$R(z) = \frac{(z^2+1)^2}{4z(z^2-1)}.$$

If we put Lebesgue measure m_2 on the torus, then Q, and hence R (using the obvious factor measure) will be isomorphic to the one-sided Bernoulli shift of entropy $\log 4$. By varying the lattice and the integer in the original endomorphism on $\mathbf{C}$, we obtain the earliest examples known of rational maps of the sphere with Julia set the whole sphere.

By varying the measure on $\mathbf{T}^2$, we obtain different factor measures on $\mathbf{C}_\infty$ in the obvious way; i.e., by using the measure $\alpha \circ \wp^{-1}$ on the sphere if α is the measure on the torus. The map $\wp$ gives a 2-set partition (not unique) of $\mathbf{T}^2$ minus exactly 4 branch points with the property that the restriction of $\wp$ to each atom of the partition maps injectively onto the sphere minus 4 points. We will fix from now on a choice of partition and call the 2 disjoint sets A_0 and A_1; they can be chosen to be connected, but we will not use that here. We choose A_0 to be the union of 2 atoms from the partition $\mathcal{P}$ defined earlier for Q. We define $\wp_0 = \wp|_{A_0}$ and $\wp_1 = \wp|_{A_1}$. Using this partition, any nonatomic Borel σ-finite measure ρ on $\mathbf{C}_\infty$ gives rise to a measure on the torus via the map $\wp$ as follows. For each set $C \in \mathcal{B}_2$, define $-C = \{(-x,-y) \,(\bmod L) : (x,y) \in C\}$. Recall that $Q(-C) = -Q(C)$ and $Q^{-1}(-C) = -[Q^{-1}(C)]$.

If $C = \wp^{-1} \circ \wp(C)$, we call C a saturated set (under $\wp$). We can form the saturation of any set $B \in \mathcal{B}_2$ by $B_* \equiv \wp^{-1} \circ \wp(B) \supseteq B$. It is clear that $B_* = B \cup -B$, where the union may or may not be disjoint.

Lemma 7.1 *Let ρ be any nonatomic, nonsingular Borel measure for R. Then there exists an associated lifted measure ρ_2 on $\mathbf{T}^2$ such that:*

1. *ρ_2 is nonsingular for Q;*
2. *if ρ is ergodic for R, then ρ_2 has at most 2 ergodic components with respect to Q;*
3. *if ρ is (σ-)finite, then so is ρ_2;*
4. *if ρ is invariant for R, then ρ_2 is invariant for Q.*

Proof: Given any $C \in \mathcal{B}_2$, we can write $C = C_0 \cup C_1 \cup C_{br}$, where $C_0 = C \cap A_0$, $C_1 = C \cap A_1$, and $C_{br} = C \cap$(branch points of $\wp$). Clearly the union is disjoint, and any of these sets in the union could be empty. Define

$$\begin{aligned} \rho_2(C) &= \frac{1}{2}\rho(\wp_0(C_0)) + \frac{1}{2}\rho(\wp_1(C_1)) \\ &= \frac{1}{2}\rho(\wp_1(-C_0)) + \frac{1}{2}\rho(\wp_1(C_1)) \\ &= \frac{1}{2}\rho(\wp_0(C_0)) + \frac{1}{2}\rho(\wp_0(-C_1)), \end{aligned}$$

since $\wp_i(C_i) = \wp_j(-C_i), i,j = 1,2, i \neq j$. We remark that if C is a saturated set, then $\rho_2(C) = \rho(\wp C)$, and for any measurable B, $\rho_2(B_*) = 0$ if and only if $\rho_2(B) = 0$.

To show 1, we suppose $\rho_2(C) = 0$. By the above formulas, this implies that $\rho_2(\wp^{-1}\wp(C)) = \rho_2(C_*) = \rho(\wp C_*) = 0$. By nonsingularity, $\rho(R^{-1}\wp(C_*)) = 0$, and it is easy to see that for any saturated sets $C_* \in \mathcal{B}_2$, $\wp \circ Q^{-1}C_* = R^{-1} \circ \wp C_*$, so it follows that $\rho_2(Q^{-1}C_*) = 0 = \rho_2(Q^{-1}C)$. Similarly we can show that if $\rho_2(Q^{-1}C) = 0$, $\rho_2(C) = 0$ as well, by saturating $Q^{-1}C$.

We will now prove 2. We assume that ρ is ergodic for R and let B be a positive measure Q-invariant set in $\mathcal{B}_2$. If B is saturated, then by the ergodicity of ρ for R, we have that B has full ρ_2 measure. Furthermore, we can show that $B \cap -B$, which is saturated, is also Q invariant. Therefore it has full or zero measure. Suppose then that $\rho_2(B \cap -B) = 0$ (or we are done because if not, then B is saturated hence has full measure). Then since the set $B \cup -B$ is invariant and saturated, we now have 2 disjoint sets, B and $-B$, each of positive measure, disjoint, and invariant. Their union has full measure; this follows from the ergodicity of R with respect to ρ and the fact that $B \cup -B$ is saturated. Then from the discussion it follows that $\wp$ maps B injectively onto $\mathbf{C}_\infty$. From this it follows that no smaller set can be invariant; i.e., the ergodic decomposition can have at most 2 atoms in it, each of which completely covers the sphere. Therefore there are at most two distinct ergodic components.

3. follows easily. Finally we establish the invariance of ρ_2 for Q when R preserves ρ. Let $\mathcal{A}$ denote the σ-algebra of Borel sets on the sphere. If C is a saturated set in $\mathcal{B}_2$, then $\rho_2(Q^{-1}C) = \rho_2(C)$ by the hypothesis on R and the above discussion. So it is enough to check invariance for invariance for $C \subset A_0$ (or A_1); assume $C = C \cap A_0 \in \mathcal{A}$. Then $C \cap (-C) = \emptyset$ and $(Q^{-1}C) \cap Q^{-1}(-C) = (Q^{-1}C) \cap -(Q^{-1}C) = \emptyset$. Then

$$\rho_2(C) = \frac{1}{2}\rho(\wp_0 C) = \frac{1}{2}\rho(\wp C) = \frac{1}{2}\rho(R^{-1}(\wp_0 C)) = \frac{1}{2}\rho(\wp(Q^{-1}C)).$$

Writing $Q^{-1}C = (Q^{-1}C)_0 \cup (Q^{-1}C)_1$, this equals

$$\frac{1}{2}[\rho(\wp_0(Q^{-1}C)_0) + \rho(\wp_1(Q^{-1}C)_1)] = \rho_2(Q^{-1}C).$$

This concludes the proof. □

7.3 Type *III* Rational Maps of the Sphere

We construct the type *III* map on $\mathbf{T}^2$ as above, using the measure μ_2, which is the product of one-dimensional Lebesgue measure m with Hamachi measure μ. On the sphere we use the natural factor measure defined by the factor map $\wp$; i.e., define $\nu(A) = \mu_2(\wp^{-1}(A))$ for every Borel set A on $\mathbf{S}^2 \cong \mathbf{C}_\infty$. It is clear that ν is ergodic and conservative for the factor map R defined above, so the following holds.

Theorem 7.2 *There exists a Borel measure ν on $\mathbf{C}_\infty$ such that, with respect to the rational map $R(z) = \frac{(z^2+1)^2}{4z(z^2-1)}$:*

1. *ν is supported on the Julia set of R (which is the whole sphere);*
2. *R is a 4-to-1 map with respect to ν ;*
3. *R is continuous, nonsingular, conservative, and ergodic with respect to ν;*
4. *R admits no σ-finite invariant measure absolutely continuous with respect to ν.*

Proof: From the discussion above and Lemma 7.1, it is clear that we only need to check 4. We suppose that R admits an invariant measure $\rho \sim \nu$, and that ρ is σ-finite. Then ρ is ergodic since ν is, so we lift ρ to an invariant measure for Q on $\mathbf{T}^2$ as in the preceding lemma. We denote by ρ_2 the lifted measure on $\mathbf{T}^2$, and it has all the properties listed in Lemma 7.1. It remains to show that ρ_2 is equivalent to μ_2 which would contradict the hypothesis on μ_2.

One can easily establish that $\mu_2 \ll \rho_2$ since:

$$\begin{aligned}
\rho_2(C) = 0 \;\; &\Rightarrow \;\; \rho(\wp_0(C_0)) + \rho(\wp_1(C_1)) + \rho(\wp_0(-C_1)) + \rho(\wp_0(-C_1)) = 0 \\
&\Rightarrow \;\; \rho(\wp C) = 0 \\
&\Rightarrow \;\; \nu(\wp_0(C_0)) + \nu(\wp_1(C_1)) + \nu(\wp_0(-C_1)) + \nu(\wp_0(-C_1)) = 0 \\
&\Rightarrow \;\; \mu_2(C) = 0.
\end{aligned}$$

If the measure ρ_2 is ergodic for Q, then we suppose there exists a measurable set C such that $\mu_2(C) = 0$ and $\rho_2(C) > 0$. We can generate an invariant set for Q by C which then has full ρ_2 measure by ergodicity of ρ_2; its complement will have ρ_2-, hence μ_2-measure 0. It cannot happen that μ_2 gives measure 0 to a set and its complement. Therefore we assume, using Lemma 7.1, that ρ_2 has two ergodic components with respect to Q; then we will write the measure as the sum of two ergodics: $\rho_2 = \frac{1}{2}\rho_2^0 + \frac{1}{2}\rho_2^1$. We repeat the above argument on each component separately; i.e., any set C such that $\mu_2(C) = 0$ but $\rho_2(C) > 0$ must lie completely in one ergodic component of ρ_2 (otherwise we repeat the above argument verbatim). Therefore $\mu_2 \sim \rho_2^0$ say. The set C generates an invariant set of full ρ_2^1 measure whose complement has measure 0. Hence μ_2 gives both C and its complement measure 0 and the contradiction establishes the result. □

Remarks. 1. The Lattès examples are constructed for any endomorphism of the form $Qz = nz$ for any $n \geq 2$. In this way we can construct type *III* examples of degree n^2.

2. We can also vary the lattice used in the definition of the Weierstraß elliptic function to obtain different conformal equivalence classes of maps. The measure theoretic properties will remain the same however, as changing the lattice is a measure theoretic isomorphism.

3. Because of averaging that occurs in the Weierstraß factor map, our method does not immediately lead to a C^1 type *III* version of the example.

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