ENTROPY OF HOMEOMORPHISMS ON UNIMODAL INVERSE LIMIT SPACES

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ABSTRACT. We prove that every self-homeomorphism $h : K_s \to K_s$ on the inverse limit space K_s of the tent map T_s with slope $s \in (\sqrt{2}, 2]$ has topological entropy $h_{top}(h) = |R| \log s$, where $R \in \mathbb{Z}$ is such that h and σ^R are isotopic. Conclusions on the possible values of the entropy of homeomorphisms of the inverse limit space of a (renormalizable) quadratic map are drawn as well.

1. INTRODUCTION

Since Williams' work on expanding attractors [20], inverse limit spaces play a key role in constructing and analyzing examples in dynamical systems. For a continuous map $f : X \to X$ of a metric space X, its shift homeomorphism $\sigma_f : X_f \to X_f$, where $X_f := \varprojlim(X, f)$, is a homeomorphism of the inverse limit space X_f, and it is the dynamically minimal extension of f to a homeomorphism. Although the space X_f is defined as a subspace of $X^{\mathbb{N}}$, Whitney's Embedding Theorem allows one to embed the inverse limit in a manifold M (of dimension 3 if $X \subset \mathbb{R}$), and $\sigma_f : X_f \to X_f$ can be extended to a homeomorphism of M for which X_f is a global attractor.

Barge and Martin [5] described a construction to embed the inverse limit of any interval endomorphism as a global attractor of a planar homeomorphism and their construction readily generalizes to graphs other than the interval. Bruin [9] showed that for unimodal maps, this planar homeomorphism can be taken Lipschitz continuous. Very recently, Boyland, de Carvalho and Hall have developed a C^0 parametrized version [8] of the Barge-Martin construction. In the same paper, they apply their results to the one of the most studied family of unimodal maps, the tent family $T_s : [0, 1] \to [0, 1]$ with slope $\pm s$, $s \in [1, 2]$, defined as $T_s(x) = \min\{sx, s(1 - x)\}$. For this family it has been proven that for $1 \le s < t \le 2$, the inverse limits $K_s := \varprojlim([0, 1], T_s)$ and K_t are not homeomorphic, see [13, 19, 18, 2]. By Barge and Martin [5], any map of the interval can be extended to the disk, and their construction gives a family of extensions of any unimodal family. Thus

Boyland, de Carvalho, and Hall provide a continuously varying family $\phi_s : \mathbb{D} \to \mathbb{D}$ of homeomorphisms of the disk $\mathbb{D}$, with ϕ_s having an embedded copy of K_s as a global attracting set; the dynamics of ϕ_s restricted to this attractor is conjugate to the shift homeomorphism $\sigma := \sigma_{T_s} : K_s \to K_s$. Moreover, by [8], these attractors vary continuously in the Hausdorff topology, although no two are homeomorphic.

In this paper, we add to the description of the group of homeomorphisms $h : K_s \to K_s$ for arbitrary s. In [6, 7] and [10] it is proven that for $s \in (\sqrt{2}, 2]$ and a homeomorphism $h : K_s \to K_s$, there exists $R \in \mathbb{Z}$ such that h is isotopic to σ^R. Here we study the topological entropy $h_{top}(h)$ and show:

Theorem 1.1. If $s \in (\sqrt{2}, 2]$, $h : K_s \to K_s$ is a homeomorphism and $R \in \mathbb{Z}$ is such that h is isotopic to σ^R, then the topological entropy $h_{top}(h) = |R| \log s$.

Isotopy does in general not preserve topological entropy, as is easy to see in dimension ≥ 2. The one-dimensional situation of Theorem 1.1 is more rigid, and one is inclined to compare it with skew-products where all fiber maps are monotone interval maps. Indeed [10], the rigidity of isotopy implies that h can differ from σ^R only at *non-folding points,* (see Definition 2.2 below). Using Bowen's entropy formula and the notion of sequence entropy, Kolyada, Misiurewicz and Snoha [14, 15] proved that for such skew-products the entropy is the same as the entropy of the base map. In our case, we cannot easily identify the base map, and at best it would be defined on a non-compact union of Cantor sets. Therefore it is unclear if results from the skew-product setting can be applied and a proof must be given. In this paper we give our own proof.

We call the fact that the homeomorphisms on a space X can only taking specific values *entropy rigidity.* Clearly the circle has entropy rigidity: zero is the only possible entropy. A related example is the pseudo-arc which can be written as inverse limit space of an interval map in various ways. Mouron [17] proved that zero and infinity are the only possible values of entropy for the corresponding shift homeomorphism.

The entropy rigidity result of Theorem 1.1 can be extended to the inverse limit spaces of quadratic maps $q_a(x) = 1 - ax^2$. Each q_a, $a \in [0, 2]$, with positive topological entropy, is semi-conjugate to a tent map T_s with $\log s = h_{top}(q_a)$ [16] and the semi-conjugacy collapses (pre)periodic intervals to points. If q_a has no non-trivial periodic intervals (*i.e.,* q_a is not renormalizable), then Theorem 1.1 applies to its inverse limit space. Otherwise, the first return map to such a periodic interval is a new unimodal map, called the *renormalization* of the previous; thus we get a (possibly infinite) sequence of nested cycles of periodic intervals, with periods $(p_i)_{i \geq 0}$, where p_i divides p_{i+1} and $p_0 = 1$. The effect of renormalization on the

structure of the inverse limit space is well-understood [3]: it produces proper subcontinua that are p_i-periodic under the shift homeomorphism and homeomorphic with the inverse limit space of the renormalized map. Here we show that even in this case an analogous of Theorem 1.1 holds:

Theorem 1.2. Assume that q_a is a quadratic map with positive topological entropy and $\log s = h_{top}(q_a)$. If h is a homeomorphism on the inverse limit space of q_a, then the topological entropy $h_{top}(h) = |R| \log s$ where $R \in \mathbb{Z}$ is such that h is isotopic to σ^R.

As unimodal inverse limit spaces mimic to some extent the structure of Hénon attractors, Theorems 1.1 and 1.2 can be seen as a step towards an entropy rigidity result for Hénon attractors. Recall that the Hénon map is $H_{a,b}(x,y) = (1 + y - ax^2, bx)$, and for parameters $a \in (0, 2)$ and small b, there is a bounded open forward invariant set U such that $\cap_{n \geq 0} H^n(U)$ is a continuum, called the *(global) attractor*. Barge and Holte [4] proved that for parameter values a where q_a has an attracting periodic orbit, and small values of b, $H_{a,b}$ restricted to the global attractor is topologically conjugate to shift homeomorphisms on inverse limit of an interval with bonding map q_a. Hence Theorem 1.2 applies in this situation, but due to a result by Barge [1], this may be the only case where Hénon attractors are homeomorphic to unimodal inverse limit spaces with a single bonding map.

Despite this, it would be interesting to see if general Hénon (or Lozi) attractors exhibit rigidity of entropy. Specifically, what is the class of Hénon maps $H_{a,b}$ for which entropy of every self-homeomorphism of the corresponding attractor is an integer multiple of $h_{top}(H_{a,b})$?

Let us mention a final example of entropy rigidity pertaining to certain skew-products of tent maps, see [12]. In this case, the attractor is a projection of the inverse limit space of a tent map, and the entropy rigidity is as described in Theorem 1.1.

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2. Preliminaries

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We consider the family of tent maps $T_s : [0,1] \to [0,1]$ with slope $\pm s$, $s \in [1,2]$, defined as $T_s(x) = \min\{sx, s(1-x)\}$. The critical or turning point is $c := 1/2$. We write $c_k := T_s^k(c)$, so in particular $c_1 = s/2$ and $c_2 = s(1 - s/2)$. The closed T_s-invariant interval $[c_2, c_1] = [s - s^2/2, s/2]$ is called the *core*.

The inverse limit space $K_s := \varprojlim([0, 1], T_s)$ is the collection of all backward orbits

$$\{x = (\ldots, x_{-2}, x_{-1}, x_0) : T_s(x_{-i-1}) = x_{-i} \in [0, s/2] \text{ for all } i \in \mathbb{N}_0\},$$

equipped with metric $d(x, y) = \sum_{n \leq 0} 2^n |x_n - y_n|$ and *induced* (or *shift*) *homeomorphism*

$$\sigma_{T_s}(x) := \sigma(\ldots, x_{-2}, x_{-1}, x_0) = (\ldots, x_{-2}, x_{-1}, x_0, T_s(x_0)).$$

Let $\pi_k : \varprojlim([0, 1], T_s) \to [0, c_1]$, $\pi_k(x) = x_{-k}$ be the k-th projection map. We fix $s \in (\sqrt{2}, 2]$; for these parameters T_s is not renormalizable and $\varprojlim([c_2, c_1], T_s)$ is indecomposable. For any point $x \in K_s$, the composant of x in K_s is the union of all proper subcontinua of K_s containing x, and the arc-component of x in K_s is the union of all arcs in K_s containing x.

We review some of the main tools introduced in [2] and which are necessary here as well. Since $0 \in [0, c_1]$ is a fixed point of T_s, the endpoint $\bar{0} := (\ldots, 0, 0, 0)$ is contained in $\varprojlim([0, 1], T_s)$. The arc-component of K_s which contains $\bar{0}$ will be denoted as C; it is a ray converging to, but disjoint from the inverse limit of the core $\varprojlim([c_2, c_1], T_s)$.

We define *p-points* as those points $x = (\ldots, x_{-2}, x_{-1}, x_0) \in K_s$ such that $x_{-p-k} = c$ for some $k \in \mathbb{N}_0$. The number $L_p(x) := k$ is called the *p-level* of x. By convention, the endpoint $\bar{0}$ of C is also a p-point and $L_p(\bar{0}) := \infty$, for every p.

The ordered set $\{z^1, z^2, \ldots, z^n, \ldots\}$ of all p-points of the arc-component C is denoted by E_p, and the ordered set of all p-points of the arc-component C of p-level k by $E_{p,k}$. The sequence of p-levels $(L_p(x))_{x \in E_p}$ is called the *folding pattern* of C, because it indicates the way how C "folds" back and forth when it compactifies on the core $\varprojlim([c_2, c_1], T_s)$. We denote the folding pattern as

$$FP_p(C) = L_p(z^1), L_p(z^2), \ldots, L_p(z^n), \ldots$$

Let $q \in \mathbb{N}$, $q > p$, and $E_q = \{y^1, y^2, \ldots, y^n, \ldots\}$. Since σ^{q-p} is an order-preserving homeomorphism of C, it is easy to see that $\sigma^{q-p}(z^i) = y^i$ for every $i \in \mathbb{N}$, and $L_p(z^i) = L_q(y^i)$. Therefore, the folding pattern of C does not depend on p. In fact, the graph of $T_s^p|_{[0,c_1]}$ oscillates in the same way as the initial arc of C, and therefore $FP_p(C)$ starts as ∞ 0 1 0 2 0 1 $\ldots$ for every slope $s > 1$ and $p \in \mathbb{N}_0$.

More generally, given an arc $A \subset K_s$ with successive p-points $x^0, \ldots, x^n$, the sequence of their p-levels is denoted as

$$FP_p(A) := L_p(x^0), \ldots, L_p(x^n).$$

Note that every arc of C has only finitely many p-points, but an arc A of the core of K_s can have infinitely many p-points. In this case, if $(u^i)_{i \in \mathcal{I}}$ is the set of p-points of A, then $FP_p(A) = (L_p(u^i))_{i \in \mathcal{I}}$, for some countable index set $\mathcal{I}$ (not necessarily of the same ordinal type as $\mathbb{N}$ or $\mathbb{Z}$).

In [2] it is shown that the asymptotic structure of the folding pattern of C is a topological invariant, but it takes a long argument to reach to that conclusion. For instance, p-points have no topological characterization and a homeomorphism will in general not map p-points to q-points for any integers p, q. However, among the p-points there are special ones, which we call *salient*, which are center points of symmetries in C. Homeomorphisms preserve these symmetries to such an extent that it is possible to prove that salient points map close to salient points.

Definition 2.1. We call a p-point $y \in C$ *salient* if $0 \leq L_p(x) < L_p(y)$ for every p-point $x \in (\bar{0}, y)$. Let $(s_p^i)_{i \in \mathbb{N}}$ be the sequence of all salient p-points of C, ordered such that $s_p^i \in (\bar{0}, s_p^{i+1})$ for all $i \geq 1$.

Since by definition $L_p(s_p^i) > 0$, for all $i \geq 1$, we have $L_p(s_p^1) = 1$. Also, since $s_p^i = \sigma^{i-1}(s_p^1)$, we have $L_p(s_p^i) = i$, for every $i \in \mathbb{N}$. Therefore, for every p-point x of K_s with $L_p(x) \neq 0$, there exists a unique salient p-point s_p^k such that $L_p(x) = L_p(s_p^k) = k$. Also, for every $k \in \mathbb{N}$, among all p-points $E_{p,k}$ of C with p-level k there exists precisely one p-point s_p^k which is salient and has p-level k. Note that the salient p-points depend on p: if $p \geq q$, then the salient p-point s_p^i equals the salient q-point s_q^{i+p-q}.

Definition 2.2. A *folding point* is any point x in the core of K_s such that no neighborhood of x in the core of K_s is homeomorphic to the product of a Cantor set and an arc.

In [10] it was shown that $x \in K_s$ is a folding point if and only if for some $p \in \mathbb{N}$ there is a sequence of p-points $(x^k)_{k \in \mathbb{N}}$ such that $x^k \to x$ and $L_p(x^k) \to \infty$.

3. Construction of chains $\mathcal{C}_p$ and $\widetilde{\mathcal{C}}_{p+M}$

A continuum is *chainable* if for every $\varepsilon > 0$, there is a cover $\{\ell^1, \ldots, \ell^n\}$ of open sets (called *links*) of diameter $< \varepsilon$ such that $\ell^i \cap \ell^j \neq \emptyset$ if and only if $|i - j| \leq 1$. Such a cover is called a *chain*. Clearly the interval $[0, s/2]$ is chainable. Throughout, we will use sequence of chains $\mathcal{C}_p$ of $\varprojlim([0, 1], T_s)$ satisfying the following properties:

(1) there is a chain $\{I_p^1, I_p^2, \ldots, I_p^n\}$ of $[0, s/2]$ such that $\ell_p^j := \pi_p^{-1}(I_p^j)$ are the links of $\mathcal{C}_p$;

(2) each point $x \in \cup_{i=0}^p T_s^{-i}(c)$ is a boundary point of some link I_p^j;

(3) for each i there is j such that $T_s(I_{p+1}^i) \subset I_p^j$.

If $\max_j |I_p^j| < \varepsilon s^{-p}/2$ then $\mathrm{mesh}(\mathcal{C}_p) := \max\{\mathrm{diam}(\ell_p) : \ell_p \in \mathcal{C}_p\} < \varepsilon$, which shows that $\varprojlim([0, 1], T_s)$ is indeed chainable. Condition (3) ensures that $\mathcal{C}_{p+1}$ *refines* $\mathcal{C}_p$ (written $\mathcal{C}_{p+1} \preceq \mathcal{C}_p$).

Note that all p-point $E_{p,k}$ of p-level k belong to the same link of $\mathcal{C}_p$. (This follows by property (1) of $\mathcal{C}_p$, because $L_p(x) = L_p(y)$ implies $\pi_p(x) = \pi_p(y)$.) Therefore, every link of $\mathcal{C}_p$ which contains a p-point of p-level k, contains also the salient p-point s_p^k.

Let $h : \varprojlim([0,1], T_s) \to \varprojlim([0,1], T_s)$ be a homeomorphism. Take $q, p \in \mathbb{N}_0$ such that

$$h(\mathcal{C}_q) \preceq \mathcal{C}_p.$$

This means that it becomes natural to measure the p-level of p-points close to the images $h(s_q^i)$ of salient q-points. Let us denote by ℓ_p^x a link of $\mathcal{C}_p$ which contains the point x. From [19] and [2] (for the finite and infinite critical orbit case, respectively) we have the following proposition:

Proposition 3.1. There exists $M \in \mathbb{Z}$ such that the following holds:

(i) Let $k \in \mathbb{N}$ and let s_q^k be a salient q-point. Then $u := h(s_q^k) \in \ell_p^{s_p^{k+M}}$ and the arc component $A_u \subset \ell_p^{s_p^{k+M}}$ containing u, also contains the salient p-point s_p^{k+M}.

(ii) Let $k \in \mathbb{N}$ and let x' be a q-point with $L_q(x') = k$. Then $u := h(x') \in \ell_p^{s_p^{k+M}}$ and the arc component $A_u \subset \ell_p^{s_p^{k+M}}$ containing u, also contains a p-point x such that $L_p(x) = k + M$. Moreover, the number of q-points in $[s_q^i, s_q^{i+1}]$ with q-level k, $k \le i$, is the same as the number of p-points in $[s_p^{M+i}, s_p^{M+i+1}]$ with p-level $M + k$, for every $i \in \mathbb{N}$.

From [6, 7] and [10] we can derive

Proposition 3.2. The integer $R = M + p - q$ does not depend on M, p, q, and is such that h and σ^R are isotopic.

Let us write $x \approx_p y$ if x and y belong to the same arc component of the same link of the chain $\mathcal{C}_p$. Using this notation we can write Proposition 3.1 in the following way: $h(x) \approx_p \sigma^R(x)$ for every $x \in E_q$. A fortiori, it was shown in [10] that $\sigma^{-R} \circ h$ is the identity on the set of folding points of $\varprojlim([c_2, c_1], T_s)$. At non-folding points, there is flexibility that a priori might allow a difference in entropies of h and σ^R. For an arbitrary $m \in \mathbb{N}$, the h^m-images of links of $\mathcal{C}_q$ are hard to control; for a link $\ell \in \mathcal{C}_q$, images $h^m(\ell)$ and $\sigma^{Rm}(\ell)$ need not even be close to "parallel" in $\mathcal{C}_p$, where "parallel" in $\mathcal{C}_p$ means $\pi_g(h^m(\ell)) = \pi_g(\sigma^{Rm}(\ell))$ for $g < p$. To overcome this problem, we need to introduce new chains $\widetilde{\mathcal{C}}_{p+M}$ with very elongated links (being the concatenation of links of $\mathcal{C}_{p+M}$) such that h^m-image of $\mathcal{C}_q$ can be shown to be "parallel" to $\widetilde{\mathcal{C}}_{q+Rm} = \widetilde{\mathcal{C}}_{p+M}$ (we define the notion 'basically collinear' in Remark 3.5 to make this precise). The rest of this section is devoted to the properties of these additional chains.

Take $p < q \in \mathbb{N}_0$ such that $h^m(\mathcal{C}_q) \prec \mathcal{C}_p$ and $\sigma^{Rm}(\mathcal{C}_q) \prec \mathcal{C}_p$ and let $M = Rm + q - p$. For each $j \geq 1$, $E_{q,j}$ is contained in a single link $\ell_q \in \mathcal{C}_q$ and by Proposition 3.1, for $\ell_p \supseteq h^m(\ell_q)$, every point of $h^m(E_{q,j})$ is contained in an arc component of ℓ_p which contains a p-point of $E_{p,M+j} = E_{p+M,j}$.

By using h^{-1} instead of h if necessary, we can assume that $R \geq 0$. Then $M > 0$ and $\mathcal{C}_{p+M} \prec \mathcal{C}_p$. Let $\widetilde{\mathcal{C}}_{p+M}$ be a chain with $\mathcal{C}_{p+M} \preceq \widetilde{\mathcal{C}}_{p+M} \preceq \mathcal{C}_p$ satisfying the following property: Each link $\widetilde{\ell}_{p+M} \in \widetilde{\mathcal{C}}_{p+M}$ is much wider in the 'tangential' direction than in the 'transversal' direction; in fact, the 'tangential' length of $\widetilde{\ell}_{p+M}$ is the same as the 'tangential' length of $\ell_p \in \mathcal{C}_p$, and the 'transversal' length of $\widetilde{\ell}_{p+M} = \widetilde{\ell}_{q+Rm}$ (recall that $p + M = q + Rm$) is the same as the 'transversal' length of $\ell_{q+Rm} \in \mathcal{C}_{q+Rm}$, see Figure 1.
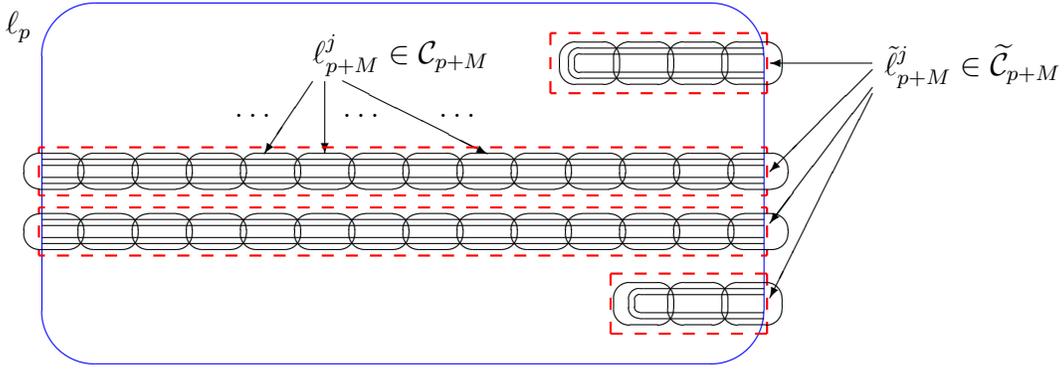
FIGURE 1. Impression of the links ℓ_p (blue), $\widetilde{\ell}^j_{p+M} \in \widetilde{\mathcal{C}}_{p+M}$ (red dashed lines) and $\ell^j_{p+M} \in \mathcal{C}_{p+M}$ (black).

To further understand the chains $\mathcal{C}_q$, $\mathcal{C}_{q+Rm}$, $\widetilde{\mathcal{C}}_{q+Rm}$ and $\mathcal{C}_p$, note that for every link $\widetilde{\ell}_{q+Rm} \in \widetilde{\mathcal{C}}_{q+Rm}$, there is a link $\ell_p \in \mathcal{C}_p$ such that every arc component $G \in \widetilde{\mathcal{C}}_{q+Rm}$ is an arc component of ℓ_p as well. In other words, if G_a and G_b are two arc components of ℓ_p, then they can belong to different links of $\widetilde{\mathcal{C}}_{q+Rm}$, say $G_a \subset \widetilde{\ell}^a_{q+Rm}$ and $G_b \subset \widetilde{\ell}^b_{q+Rm}$, but if arc components $G_a \subset \ell_p$ and $\widetilde{G}_a \subset \widetilde{\ell}^a_{q+Rm}$ are such that $G_a \cap \widetilde{G}_a \neq \emptyset$, then $G_a = \widetilde{G}_a$. Therefore, all points of $h^m(E_{q,j})$ are contained in the link $\widetilde{\ell}_{q+Rm} \in \widetilde{\mathcal{C}}_{q+Rm}$ which contains $E_{p+M,j} = E_{q+Rm,j}$. Moreover, by construction of $\widetilde{\mathcal{C}}_{q+Rm}$, we have that $h^m(\mathcal{C}_q) \prec \widetilde{\mathcal{C}}_{p+M} = \widetilde{\mathcal{C}}_{q+Rm}$.

Let $\mathcal{C}_q = (\ell^i_q)^{n_q}_{i=1}$ and $\widetilde{\mathcal{C}}_{q+Rm} = (\widetilde{\ell}^i_{q+Rm})^{n_{q+Rm}}_{i=1}$. Let $A_q = [\bar{0}, s^1_q]$ and $A_{q+mR} = [\bar{0}, s^1_{q+mR}]$, where s^1_q and s^1_{q+mR} are the first salient q-point and salient $(q + mR)$-point respectively. Then $\pi_q(A_q) = [0, c_1] = \pi_{q+mR}(A_{q+mR})$. Also $\bar{0} \in \ell^1_q$, $\bar{0} \in \widetilde{\ell}^1_{q+Rm}$, $s^1_q \in \ell^{n_q}_q$ and $s^1_{q+mR} \in \widetilde{\ell}^{n_{q+Rm}}_{q+Rm}$. Therefore, $\mathcal{C}_{q+Rm}$ and $\widetilde{\mathcal{C}}_{q+Rm}$ coil through $\mathcal{C}_q$ in the same way with the same number of laps as T^{Rm}_s maps $[0, c_1]$ to itself. Also, $\sigma^{Rm}(\mathcal{C}_q)$ is *collinear* with $\widetilde{\mathcal{C}}_{q+Rm}$, i.e., $\sigma^{Rm}(\mathcal{C}_q)$

goes straight through $\widetilde{\mathcal{C}}_{q+Rm}$. More precisely, for every $t \in [c_2, c_1]$ there exists a unique j such that either $\pi^{-1}_{q+Rm}(t) \subset \sigma^{Rm}(\ell^j_q)$, or $\pi^{-1}_{q+Rm}(t) \subset \sigma^{Rm}(\ell^j_q) \cap \sigma^{Rm}(\ell^{j+1}_q)$.

Remark 3.3. Note that if $a, b \in \mathbb{N}$ are such that all of $h^m(\ell^i_q)$, $a \le i \le b$, are mapped into the same link of $\widetilde{\mathcal{C}}_{q+Rm}$, i.e., if $\cup^b_{i=a} h^m(\ell^i_q) \subseteq \tilde{\ell}^j_{q+Rm}$ for some j, it is possible that there exists $t \in [c_2, c_1]$ such that $\pi^{-1}_{q+Rm}(t) \cap h^m(\ell^i_q) \neq \emptyset$ for all $a \le i \le b$ even if $b - a \ge 2$. To avoid this, let us consider the chain $V = (v^j)^N_{j=1}$ such that

- $v^j = \cup^{a_{j+1}-1}_{k=a_j} \ell^k_q$ for an increasing sequence $(a_j)^{N+1}_{j=1}$;
- $a_1 = 1$, $a_{N+1} = n_q + 1$ with $\ell^{a_{N+1}}_q = \emptyset$ by convention;
- for every $j \in \{1, \ldots, N\}$ we have $\cup^{a_{j+1}-1}_{k=a_j} h^m(\ell^k_q) \subseteq \tilde{\ell}^i_{q+Rm}$, for some $i \in \{1, \ldots, n_{q+Rm}\}$;
- $h^m(\ell^{a_{j+1}}_q) \not\subseteq \tilde{\ell}^i_{q+Rm}$ except for a_{N+1} (since $\emptyset = h^m(\ell^{a_{N+1}}_q) \subset \tilde{\ell}^i_{q+Rm}$).

Such a chain exists since $h^m(\mathcal{C}_q) \prec \widetilde{\mathcal{C}}_{q+Rm}$, and in fact, by construction we have $h^m(V) \prec \widetilde{\mathcal{C}}_{q+Rm}$.

The following lemma is needed for the proof of the main theorem. In that proof we need the property that $h^m(\mathcal{C}_q)$ coils through $\mathcal{C}_q$ in roughly the same way with the same number of laps as T^{mR}_s maps $[0, c_1]$ into itself.

Lemma 3.4. The image chain $h^m(\mathcal{C}_q)$ is basically collinear with $\widetilde{\mathcal{C}}_{q+Rm}$, i.e., $h^m(\mathcal{C}_q)$ goes straight through $\widetilde{\mathcal{C}}_{q+Rm}$.

Remark 3.5. In the above lemma "basically collinear" means exactly that there exists a chain $V = (v^j)^N_{j=1}$ as in Remark 3.3 for which every link v^j is a union of consecutive links of $\mathcal{C}_q$, such that for every $t \in [c_2, c_1]$, there exists a unique j with either $\pi^{-1}_{q+Rm}(t) \subset h^m(v^j)$, or $\pi^{-1}_{q+Rm}(t) \subset h^m(v^j) \cap h^m(v^{j+1})$.

Proof of Lemma 3.4. Let V be as in Remark 3.3. By Remarks 3.3 and 3.5 we only need to prove that the number of links in V is the same as the number of links in $\widetilde{\mathcal{C}}_{q+Rm}$, i.e., $N = n_{q+Rm}$. In this case, for every link $v^j \in V$ there exist a unique link $\tilde{\ell}^j_{q+Rm} \in \widetilde{\mathcal{C}}_{q+Rm}$ such that $h^m(v^j) \subseteq \tilde{\ell}^j_{q+Rm}$, implying that $h^m(V)$ is collinear with $\widetilde{\mathcal{C}}_{q+Rm}$.

Note firstly that by Theorem 1.1 of [10], we have $h(x) = \sigma^R(x)$ for every folding point x in K_s. Therefore, if the critical point c is dense in the core $[c_2, c_1]$, then $h \equiv \sigma^R$ on the inverse limit of the core $\varprojlim([c_2, c_1], T_s)$. Since $\sigma^{Rm}(\mathcal{C}_q)$ is collinear with $\widetilde{\mathcal{C}}_{q+Rm}$, also $h^m(\mathcal{C}_q)$ is collinear with $\widetilde{\mathcal{C}}_{q+Rm}$.

(1) Let us assume that the critical point c is not dense in the core $[c_2, c_1]$. The construction of $\widetilde{\mathcal{C}}_{p+M}$ and Proposition 3.1 imply that $h^m(x) \approx_{q+Rm} \sigma^{Rm}(x)$ for every $x \in E_q$,

meaning that $h^m(\mathcal{C}_q)|_C$ is basically collinear with $\widetilde{\mathcal{C}}_{q+Rm}|_C$, *i.e.*, that $h^m(\mathcal{C}_q)|_C$ goes straight through $\widetilde{\mathcal{C}}_{q+Rm}|_C$.

(2) Let v^j be any link of V, and let A_j be an arc-component of v^j which is contained in the core of K_s, and does not contain any folding point. Since we want to prove that $N = n_{q+Rm}$, we need not consider those arcs B such that $h^m(B)$ is contained in only one link of $\widetilde{\mathcal{C}}_{q+Rm}$. Hence, by extending the arc A_j to $A = [x,y] \supset A_j$ if necessary, we can assume without loss of generality that x and y are q-points (and/or folding point), but A contains no folding point in its interior. Since C is dense in the core, for every $\delta > 0$, there exists an arc $D_\delta \subset C$ such that A and D_δ are δ-close in the Hausdorff metric, and that $FR_q(A) = FP_q(D_\delta)$. (In the case that one of the boundary points is a folding point $FR_q(\text{Int } A) = FP_q(\text{Int } D_\delta)$.) By continuity of h, this implies that there exists an arc $D \subset C$ such that $h^m(A)$ and $h^m(D)$ go through $\widetilde{\mathcal{C}}_{q+Rm}$ in the same way. Therefore, in this case it suffices to consider only arcs contained in C.

(3) Let us suppose that the arc-component A_j of v^j contains a folding point z (if A_j contains more than one folding point, let z and z' be such that the arc $[z, z'] \subset A_j$ contains all folding points of A_j). Let $A_{j-1} \subset v^{j-1}$ and $A_{j+1} \subset v^{j+1}$ be those arc-components for which $A_j \cap A_{j-1} \neq \emptyset$ and $A_j \cap A_{j+1} \neq \emptyset$ (in the case that A_j contains also a q-point, we have either $A_{j-1}, A_{j+1} \subset v^{j-1}$, or $A_{j-1}, A_{j+1} \subset v^{j+1}$). Let us assume that A_{j-1} and A_{j+1} do not contain any folding point. Since $h(u) = \sigma^R(u)$ for every folding point u in K_s, it suffices to consider arcs $A = [x, z] \supset A_{j-1}$ and $A' = [z', y] \supset A_{j+1}$ such that A and A' do not contain any folding point in its interiors (if A_j contains only one folding point, then $z = z'$). Now, as in (2), we can find arcs $D, D' \subset C$ such that $h^m(A)$ and $h^m(D)$, as well as $h^m(A')$ and $h^m(D')$, go through $\widetilde{\mathcal{C}}_{q+Rm}$ in the same way.

(4) If A_{j-1}, or A_{j+1}, contains folding points, then either at least one of $A_{j-1} \setminus A_j$, $A_{j+1} \setminus A_j$ is contained in the single link of V, or we can find $j_1 \leq j-1$ (and/or $j_2 \geq j+1$) such that A_{j_1} (and/or A_{j_2}) does not contain any folding point. In the first case, let us suppose that $A_{j-1} \setminus A_j$ is contained in a single link of V. Then $h^m(A_{j-1} \setminus A_j)$ is contained in the single link of $\widetilde{\mathcal{C}}_{q+Rm}$, implying that we need not consider this arc (to prove that $N = n_{q+Rm}$, it suffices to consider those arcs B such that $h^m(B)$ goes through several link of $\widetilde{\mathcal{C}}_{q+Rm}$). The same conclusion follows if $A_{j+1} \setminus A_j$ is contained in a single link of V. In the second case we can proceed as in (3).

From the above steps we conclude that it is sufficient to consider only arcs contained in C. Since by (1), $h^m(\mathcal{C}_q)|_C$ is basically collinear with $\widetilde{\mathcal{C}}_{q+Rm}|_C$, this implies that $N = n_{q+Rm}$. This completes the proof. $\qquad\square$

4. PROOFS OF THEOREMS 1.1 AND 1.2

We will use Bowen's definition of (n, ε)-separated sets to compute topological entropy of h. For interval maps T_s, the entropy is also the exponential growth-rate of the lap-number of T^n, i.e., $h_{top}(T_s) = \lim_{n \to \infty} \frac{1}{n} \log \mathfrak{l}(T_s^n)$, where

$$\mathfrak{l}(T_s^n) := \#\{\text{maximal intervals on which } T_s^n \text{ is monotone}\}.$$

Proof of Theorem 1.1. Fix homeomorphism h and let R be such that h and σ^R are isotopic. By using h^{-1} instead of h if necessary, and noting that $h_{top}(h) = h_{top}(h^{-1})$, we can assume that $R \geq 0$.

We start with the upper bound. Take $\eta > 0$ arbitrary, and find $m \in \mathbb{N}$ such that the lap-number $\mathfrak{l}(T_s^{Rm}) \leq s^{m(R+\eta)}$ and also $(\log 2)/m < \eta$.

Fix $\varepsilon > 0$ and let $p \in \mathbb{N}$ be such that $\text{mesh}(\mathcal{C}_p) < \varepsilon/2$. Find $\varepsilon_0, \varepsilon_1 > 0$ such that $\varepsilon_1 < \varepsilon_0/200 < \varepsilon_0 < \text{mesh}(\mathcal{C}_p)$. Take $q \in \mathbb{N}$ such that $\text{mesh}(\mathcal{C}_q) < \varepsilon_1$, $h^m(\mathcal{C}_q) \prec \mathcal{C}_p$ and $\sigma^{Rm}(\mathcal{C}_q) \prec \mathcal{C}_p$.

Let $G = (g^j)_{j=1}^{N'}$ be a 'chain' composed of 'half-open links' $g^j = (\cup_{k=a_j}^{a_{j+1}} \ell_q^k) \setminus \ell_q^{a_{j+1}}$ for an increasing sequence $(a_j)_{j=1}^{N'}$ such that the following two conditions hold:

(a) $a_1 = 1$, $a_{N'+1} = n_q + 1$ and $\ell_q^{a_{N'+1}} = \emptyset$ by convention,
(b) $\cup_{k=a_j}^{a_{j+1}-1} \ell_q^k \subseteq \ell_p^j$, for every $j \in \{1, \dots N'\}$.

Note that the sequence $(a_j)_{j=1}^{N'}$ is chosen so that

$$\varepsilon_0 < \text{diam}(g^j) < \varepsilon/2.$$

Since $\text{diam}(\ell_q^k) < \varepsilon_1$ there must be at least $\varepsilon_0/\varepsilon_1 \geq 200$ links ℓ_q^k inside each g^j, and g^j is much wider in the 'tangential' direction than in the 'transversal' direction. Although G is not a chain since $g^i \cap g^j = \emptyset$ for every $i \neq j$, we have the intersection of closures $\text{Cl}(g^i) \cap \text{Cl}(g^j) \neq \emptyset$ if and only if $|i - j| \leq 1$. Clearly, $\bar{0} \in g^1$ and $s_q^1 \in g^{N'}$, where s_q^1 is the first salient q-point. Note that G is coarser than V from Remark 3.3, but sufficiently close to it that Lemma 3.4 applies, so $h^m(G)$ is collinear with $\mathcal{C}_{q+mR}$ as well as $h^m(V)$. In fact, G is a partition of K_s and we can use G to code n-orbits of h^m unambiguously.

Since by Lemma 3.4, $h^m(\mathcal{C}_q)$ is basically collinear with $\mathcal{C}_{q+mR}$ (it goes straight through $\mathcal{C}_{q+mR}$), and therefore it coils through $\mathcal{C}_q$ in roughly the same way with the same number of laps as T_s^{mR} maps $[0, c_1]$ into itself, each g^i intersects $h^m(g^j)$ for at most $2\mathfrak{l}(T_s^{mR}) \leq 2s^{m(R+\eta)}$ values of j. Continuing by induction, we see that each g^i intersects $h^{nm}(g^j)$ for at most $2^n s^{n(mR+\eta)}$ values of j, counted 'with multiplicity', since the h^m-image of g^j can go through g^i several times. Let us use the partition $(g^i)_{i=0}^{N'}$ to code the n-orbits, i.e., for $x \in K_s$, we define an itinerary $e(x) = e_k(x)_{k \geq 0}$ by setting $e_k(x) = i$ if $h^{mk}(x) \in g^i$. Then there are at

most $N'2^n s^{nm(R+\eta)}$ different itineraries, and h^m-orbits that are (n, ε)-separated must have different length n itineraries. Hence

$$
\begin{aligned}
h_{top}(h) &= \frac{1}{m} h_{top}(h^m) \leq \frac{1}{m} \lim_{n \to \infty} \frac{1}{n} \log N'2^n s^{nm(R+\eta)} \\
&= (R+\eta) \log s + \frac{\log 2}{m} \leq R \log s + 2\eta.
\end{aligned}
$$

Here $\eta > 0$ is arbitrary, so the upper bound $h_{top}(h) \leq R \log s$ follows.

Now for the lower bound, let $X_\delta = \{x \in [0, s/2] : d(\mathrm{orb}(x), \frac{1}{2}) > \delta\}$. If $x \in X_\delta$, then for any $n \in \mathbb{N}$ and maximal interval $J \ni x$ such that $T_s^n|_J$ is monotone, we have $d(T_s^n(x), \partial T_s^n(J)) > \delta$. It is well-known that $h_{top}(T_s|_{X_\delta}) \to \log s$ as $\delta \to 0$. Take $\eta > 0$ arbitrary and fix $\delta > 0$ so that $h_{top}(T_s|_{X_\delta}) > \log(s - \eta)$. Let m be so large that for any interval J of length $\geq 2\delta$, $T_s^{mR}(J)$ has at least $(s - 2\eta)^{mR}$ branches intersecting X_δ.

Take $\varepsilon \in (0, 2\delta/s^m)$ and take q so large that $\mathrm{mesh}(\mathcal{C}_q) < \varepsilon$. For each q-point $x \in C$, let $D_x \subset C$ be the arc of arc-length 2δ centered at x, and set $\mathcal{I}_\delta := \bigcup_{x \in E_q} D_x$. Let $W_1 = h^{-m}(h^m([\bar{0}, s_q^1]) \setminus \mathcal{I}_\delta)$; then W_1 has at least $(s - 2\eta)^{mR}$ connected components. Continue to define inductively $W_j = h^{-jm}(h^{jm}(W_{j-1}) \setminus \mathcal{I}_\delta)$ for $2 \leq j \leq n$. The choice of m guarantees that W_j has at least $(s - 2\eta)^{jRm}$ connected components. If x and y belong to different connected components, then there is $0 \leq j < n$ and $0 \leq k < m$ such that $\pi_0(h^{jm+k}(x))$ and $\pi_0(h^{jm+k}(y))$ are separated by a δ-neighborhood of $\frac{1}{2}$, so x and y are $(n, 2\delta/s^m)$-separated for h^m. Since $\varepsilon < 2\delta/s^m$, we can select an (n, ε)-separated set of cardinality $\geq (s - 2\eta)^{nRm}$, and therefore

$$
h_{top}(h) = \frac{1}{m} h_{top}(h^m) \geq \inf_{\varepsilon > 0} \lim_{n \to \infty} \frac{1}{nm} \log(s - 2\eta)^{nRm} = R \log(s - 2\eta).
$$

As η was arbitrary, the lower bound $h_{top}(h) \geq R \log s$ follows. $\qquad \square$

We finish with the

Proof of Theorem 1.2. Let $q_a : x \mapsto 1 - ax^2$ be a quadratic map with critical point 0, $c_i = q_a^i(0)$ and left fixed point $u = \frac{1}{2a}(-1 - \sqrt{1 + 4a})$. Assume that q_a is renormalizable, say it has a period p cycle of intervals J_k, $k = 0, \ldots, p - 1$ and $u_k \in \partial J_k$ is an orientation preserving p-periodic point. The structure that emerges from [3] of the inverse limit space $\varprojlim([u, c_1], q_a)$ of q_a is as follows: The core of $\varprojlim([u, c_1], q_a)$ has p proper subcontinua G_k, $k = 0, \ldots, p - 1$, which are permuted cyclically by the shift homeomorphism, and each of them is homeomorphic to the inverse limit space of the renormalization $q' := q_a^p|_{J_k}$. One side of the composant of the point $(\ldots, u_{k-2 \bmod p}, u_{k-1 \bmod p}, u_k)$ compactifies onto the core of G_k, playing the role of the zero composant of $\varprojlim(J_k, q')$ itself. Outside $\cup_k G_k$, the core inverse limit space has no other subcontinua than points and arcs. Hence, each

homeomorphism h of $\varprojlim([u, c_1], q_a)$ can at most permute the G_k in some way, isotopically to σ^R, for some $R \in \mathbb{N}_0$. However, one might hope that within the G_k, h needs not act isotopically to σ^R. For example, if h acts as σ^R outside $\{G_k\}$, one might hope that it can still be arranged that h maps G_k to $G_{k+R \bmod p}$ isotopically to some other(!) power, say $R' \neq R$, of σ. In [11] we proved that this cannot happen and that every homeomorphism $h : \varprojlim([u, c_1], q_a) \to \varprojlim([u, c_1], q_a)$ is isotopic to σ^R for some $R \in \mathbb{Z}$. $\square$

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