# NON-MONOTONICITY OF ENTROPY OF INTERVAL MAPS.

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Abstract. We prove that the topological entropy of a unimodal map (with a unique maximum) can increase if the map is perturbed to a map whose graph lies below the original map. The perturbed map can be symmetric, concave, smooth and/or have negative Schwarzian derivative, if the original map has these properties.

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Let $f: I \to I$, $I = [0,1]$ be a unimodal map. This means that $f$ has a unique turning point, $c$. Assume that $f(c)$ is the global maximum. The question of increasing topological entropy concerns the generally observed phenomenon that unimodal maps becomes more complicated topologically if the map increases. Topological complexity may be measured by the topological entropy $h_{top}$. Then the question reads:

$$\text{Does } f(x) \leq g(x) \text{ for all } x \text{ imply } h_{top}(f) \leq h_{top}(g)\,? \tag{1}$$

Often this question is put in a stronger form:

$$\text{Is the map } a \mapsto h_{top}(f_a) \text{ increasing?} \tag{2}$$

Here $f_a(x) = a \cdot f(x)$ or $f_a(x) = a + f(x)$, where $f$ is some fixed unimodal map. For tent-maps, $f_a = a(\frac{1}{2} - |x - \frac{1}{2}|)$, (2) is true. Indeed $h_{top}(f_a) = \max(0, \log a)$ in this case. If $f_a(x) = a - x^\ell$, $\ell$ an even integer, (2) was answered affirmatively in [DH]. For trapezoidal maps, there are results by Brucks et al. [BML].

In general (2) is not true; several counter-examples have been found [K,NY,Z]. These are not pathological; the scenario described in [NY] is encountered in physical experiments [BB]. Yet these examples involve asymmetric maps or attracting fixed points, which restricts their generality. They do not explain why the entropy decreases in the below example.

*Example:* To the unimodal map $f_a(x) = 1 - ax^2$ we add a symmetric $C^3$ perturbation $\varepsilon g$, where

$$g(x) = \begin{cases} (x - \frac{1}{\sqrt{a}})^3 \cdot (x - (\frac{1}{\sqrt{a}} - \frac{1}{10}))^3 & \text{if } x \in (\frac{1}{\sqrt{a}} - \frac{1}{10}, \frac{1}{\sqrt{a}}), \\ (x + \frac{1}{\sqrt{a}})^3 \cdot (x + (\frac{1}{\sqrt{a}} - \frac{1}{10}))^3 & \text{if } x \in (-\frac{1}{\sqrt{a}}, -\frac{1}{\sqrt{a}} + \frac{1}{10}), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f_a^{-1}(0) = \{-\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{a}}\}$. If $\varepsilon$ is not too large, then $f_a + \varepsilon g$ is concave and has negative Schwarzian derivative. (The Schwarzian derivative of $f$ is defined as $\frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2$. Maps with negative Schwarzian derivative are very frequently studied in the theory of iterated interval maps.) Table 1 shows that for certain values of $a$, the entropy of $f_a + \varepsilon g$ decreases if $\varepsilon$ increases.

**Table 1** $a = 1.72980800$

| $\varepsilon$ | $h_{top}(f_a + \varepsilon g)$ | $\varepsilon$ | $h_{top}(f_a + \varepsilon g)$ |
|---|---|---|---|
| -400 | 0.47755558644103 | -0.04 | 0.47755554758150 |
| -300 | 0.47755558173050 | -0.03 | 0.47755554758036 |
| -200 | 0.47755556806528 | -0.02 | 0.47755554757877 |
| -100 | 0.47755555329306 | -0.01 | 0.47755554757708 |
| 0 | 0.47755554757621 | 0.00 | 0.47755554757621 |
| 100 | 0.47755554565188 | 0.01 | 0.47755554757532 |
| 200 | 0.47755554203778 | 0.02 | 0.47755554757500 |
| 300 | 0.47755553490973 | 0.03 | 0.47755554757491 |
| 400 | 0.47755551882449 | 0.04 | 0.47755554757457 |

Our Main Theorem explains this example, and it shows that the answer to question (1) is 'no' for a very general set of maps.

A unimodal map $f$ is called *long-branched* if there exists $K > 0$ such that for every $n$ and every maximal interval $T$ on which $f^n$ is monotone, $|f^n(T)| > K$. Here $f^n$ denotes the $n$-fold composition.

For example, $f$ is long-branched if $c$ is not recurrent, i.e. if the forward orbit of $c$ does not accumulate on $c$. But there are also long-branched maps with a recurrent turning point, see [B].

**Main Theorem.** *Let $f$ a long-branched unimodal map with a recurrent non-periodic turning point. Then for each $\varepsilon > 0$ and $r \in \mathbb{N}$, there exists a map $g$ such that*

*i) $g(x) < f(x)$ for all $x \in I$,*
*ii) $h_{top}(g) > h_{top}(f)$,*
*iii) $\| g - f \|_r < \varepsilon$, where $\| \, \|_r$ denotes the $C^r$ norm,*
*iv) if $f$ is symmetric, concave and/or has negative Schwarzian derivative, $g$ has the same properties.*

*Remarks:*

- The set of long-branched unimodal maps is densely uncountable in the following sense: Let $\{f_a\}_{a\in\mathbb{R}}$ be a one-parameter family of unimodal maps, and let $(a_0, a_1)$ be an interval on which $a \mapsto h_{top}(f_a)$ is not constant. Then $f_a$ is long-branched and has a recurrent non-periodic turning point for uncountable many parameters $a \in (a_0, a_1)$.
- The Main Theorem can be seen as a partial answer to the $C^r$-closing lemma, as it asserts that $f$ cannot be $C^r$-structurally stable. The theorem extends a result of Gambaudo and Tresser [GT], indicating an uncountable set of parameters for which the map $f_a$ is not $C^r$ structurally stable.

**Lemma 1.** *Let $f_a$ be a family of unimodal maps such that $f_a$ exhibits a saddle node bifurcation at $a_0$. If $a$ is such that this saddle node bifurcation has not taken place yet, then $h_{top}(f_a) < h_{top}(f_{a_0})$.*

*Sketch of proof.* The proof relies on the fact that each unimodal map is semi-conjugate to a tent-map with the same topological entropy [ALM,MT]. $f_{a_0}$ is semi-conjugate to a tent-map with a periodic turning point. The topological type of this periodic point is not exhibited by the tent-map to which $f_a$ is semi-conjugate. The latter tent-map therefore has smaller entropy than the first one. □

As we said before, there are long-branched maps with a recurrent turning point. Yet the returns of $c$ are somewhat special:

**Lemma 2.** *If $f$ is long-branched, then there exist a neighbourhood $U \ni c$ such that whenever $f^n(c) \in U$, then $|f^n(x) - c|$ has a local minimum in $c$.*

*Proof.* Let $U \ni c$ be such that $|U|, |f(U)| < K$. Clearly $c$ is a turning point of $t \mapsto |f^n(t) - c|$. Suppose $f^n(c) \in U$. Let $y$ be maximal such that $f^n|(c,y)$ is monotone. By long-branchedness, $f^n(y) \notin U$. If $f^n(c)$ is a local maximum of $t \mapsto |f^n(t) - c|$, then $f^n(x) = c$ for some $x \in (c, y)$. So $(c, x)$ is a maximal interval such that $f^{n+1}|(c,x)$ is monotone. But then $|f^{n+1}((c,x))| \leq |f(U)| < K$. □

**Lemma 3.** *Let $f$ be long-branched and have a recurrent non-periodic turning point. Then for every $\varepsilon > 0$ there exists $\tilde{f}$ such that*
- $\tilde{f}(x) \geq f(x)$ *for all* $x \in I$,
- $h_{top}(\tilde{f}) > h_{top}(f)$ *and*
- $\| \tilde{f} - f \|_r < \varepsilon$.

*Proof.* Choose $z$ such that $f^N(z) = c$ and $f^i(z) \notin (z, \hat{z})$ for $i < N$. Here $\hat{z} \neq z$ is the point such that $f(\hat{z}) = f(z)$. Let $U = (z, \hat{z})$ and assume that $z$ is so close to $c$ that $U$ satisfies the properties of Lemma 2. We construct $\tilde{f}$ by perturbing $f$ in $U$ in the following way:
- $\tilde{f}(x) \geq f(x)$ for $x \in U$ and $\tilde{f}(x) = f(x)$ for $x \notin U$, but $\tilde{f} \neq f$.
- $\| \tilde{f} - f \|_r \leq \varepsilon$,
- $c$ is also the unique turning point of $\tilde{f}$.

Assume that there exists a conjugacy $h$ such that $h \circ \tilde{f} = f \circ h$. We write $\tilde{y}$ for $h(y)$ for any point $y$ and we use the same notation for sets. By definition of $z$, $z = \tilde{z}$, and therefore $\tilde{x} \in \tilde{U} = U$ whenever $x \in U$. Let $n_0 \geq 0$ be the smallest integer such that $f^{n_0+1}(c) > \tilde{f}^{n_0+1}(c)$. Clearly $f^{n_0} \in U$. Let $\delta = \tilde{f}^{n_0+1}(c) - f^{n_0+1}(c)$. Let $\{n_i\}_{i \geq 0}$ be the sequence of subsequent return iterates of $c$ to $U$. So $n_{i+1} = \min\{n > n_i \mid f^n(c) \in U\}$.

Let $V_i \ni f^{n_i+1}(c)$ be the maximal neighbourhood such that $f^{n_{i+1}-n_i}|V_i$ is monotone, $f^j(V_i) \cap U = \emptyset$ for $j = 0, .., n_{i+1} - n_i - 2$ and $f^{n_{i+1}-n_i-1}(V_i) \subset U$. The boundary points of $V_i$ are preimages of the turning point, possibly preimages of $z$ or $\hat{z}$. Therefore $\tilde{f}^j(\tilde{V}_i) = f^j(V_i)$ for $j = 0, ..., n_{i+1} - n_i - 1$.

By Lemma 2, $f^{n_{i+1}-n_i}|V_i$ preserves orientation. Therefore if $x \in V_i$ and $\tilde{x} \geq x$, then also

$$\tilde{f}^{n_{i+1}-n_i}(\tilde{x}) \geq f^{n_{i+1}-n_i}(x). \tag{3}$$

Let $V = (f^{n_0+1}(c), y)$ be such that
- $f^{n_0+1}(c) < y$,
- $|V| \leq \delta$ and
- There exists $M$ such that $f^M(y) = c$ and $f^M|V$ is monotone.

By definition of $V_i$, $f^{n_i-n_0}(V) \in V_i$ whenever $n_i - n_0 < M$. Applying (3) repeatedly, we obtain that $\tilde{y} \leq y$. On the other hand, $\tilde{f}^{n_0+1}(c) = f^{n_0+1}(c) + \delta$, but then $\tilde{f}^{n_0+1}(c) \geq \tilde{y}$, a contradiction.

So $\tilde{f}$ and $f$ are not conjugate. By continuity, we can choose $\tilde{f}$ such that $\tilde{f}^{M+n_0+1}(c) = c$, while $t \mapsto |f^{M+n_0+1}(t) - c|$ has a local minimum in $c$. Without loss of generality, this new $M + n_0 + 1$-periodic point can be taken to emerge in a saddle node bifurcation. Hence by Lemma 1, $h_{top}(\tilde{f}) > h_{top}(f)$. $\square$

*Proof of the Main Theorem.* Let $f$ be long-branched and have a recurrent non-periodic turning point. By Lemma 3, there exists $\tilde{f}$ arbitrarily close to $f$ such that $h_{top}(\tilde{f}) > h_{top}(f)$. Without loss of generality we may assume that $\tilde{f}(x) = f(x)$ for all $x \leq c$. Let $U$ as in Lemma 3, and let

$$g(x) = \begin{cases} f(x) & \text{if } x \notin f^{-1}(U \cap (c,1)), \\ f^{-1} \circ \tilde{f}^2(x) & \text{if } x \in f^{-1}(U \cap (c,1)). \end{cases}$$

By construction either $g(x) = \tilde{f}(x)$ or $g^2(x) = \tilde{f}^2(x)$. So $g$ and $\tilde{f}$ have the same topological entropy. Because $\tilde{f}(f(x)) \geq f^2(x)$ and $f|(c,1)$ reverses orientation,

$g(x) \leq f(x)$. Finally, because topological entropy is lower semi-continuous [MS], there exists $\eta > 0$ such that $h_{top}(g - \eta) > h_{top}(f)$. Clearly $g(x) - \eta < f(x)$ for all $x$, and it is not hard to see that $g - \eta$ can be symmetric, concave and have negative Schwarzian derivative, if $f$ has these properties. $\square$

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