# On the chaotic behavior of the Primal–Dual Affine–Scaling Algorithm for Linear Optimization

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**Abstract.** Affine scaling (AFS) methods solve linear optimization problems by iteration. These methods all involve a parameter $\theta \in (0,1)$ that controls the maximum step-size during the iteration. It is known that some of these AFS methods display chaotic behavior if $\theta$ is too large. In this paper we study the primal-dual affine-scaling method that was proposed in [8]. We convert this into a dynamical system that is related to the well-known logistic family. We show that our dynamical system enters a chaotic regime once $\theta$ passes a threshold.

**Key words:** interior-point method, affine scaling method, primal–dual method, chaotic behavior.

## 1 Introduction

Interior point methods (IPM) solve linear optimization problems (LO) by iteration. Starting from an initial vector $x$ in the feasible set, one iterates the IPM until it gets sufficiently close to an optimum $x^*$, which may not be unique. In other words, an IPM is a dynamical system (or iterative process) that is defined by a vector-valued map $x \mapsto F(x)$ that depends on the optimization problem. One says that an IPM *converges* if and only if for every optimization problem the iterates $F^k(x)$ converges to some optimal solution for each feasible initial vector $x$.

The earliest IPM is Dikin's affine scaling (AFS) method, which solves a linear optimization problem by iterating $x \mapsto x + \alpha\Delta x$ and $s \mapsto s + \alpha\Delta s$ that is controlled by a parameter $\alpha$. The vectors $\Delta x$ and $\Delta s$ depend on $x$ and on the linear optimization problem. The parameter $0 < \alpha < 1$ controls the step-size. If $\alpha = 1$ then the step is *full*, but in general a smaller value of $\alpha$ is needed to guarantee convergence. Dikin's original method has been adapted over the years and there exist several different versions of the AFS method. A method that has been studied extensively is the AFS that has been developed by Vanderbei *et al.* [13]. It is known [12] that this method converges if $\alpha \leq 2/3$ and that it need not converge if $\alpha > 2/3$, see [7]. It is also known that this AFS behaves chaotically for $\alpha > 2/3$, see [2, 10].

All previous studies of non-convergence or chaotic behavior were carried out for a particular LO problem. Hall and Vanderbei [7] presented an LO problem with four variables and one constraint, for which they demonstrated that $F^k(x)$ converges to the optimal solution if $\alpha \leq 2/3$, while it converges to a set of two elements if $\alpha > 2/3$. Castillo and Barnes extended this result and demonstrated chaotic behavior that arises from an LO problem in five variables under two constraints. They were able to compute Feigenbaum diagrams that resemble those of the logistic family. The studies in [2, 7] consider specific LO problems, so it is natural to ask whether these specific examples represent the general case. That is the principal motivation behind our paper.

In this paper we consider the primal-dual AFS method that was proposed by Jansen *et al.* [8]. It solves the LO problem by reducing the duality gap $s^T x$ under iteration, until the gap gets sufficiently close to zero. In other words, $(s + \alpha\Delta s)^T(x + \alpha\Delta x) < s^T x$, which can be rewritten as a sum of a linear and a quadratic term for the step size: $\alpha\left(\Delta s^T x + s^T\Delta x\right) + \alpha^2\Delta x\Delta s < 0$. The primal-dual AFS is designed to minimize $\Delta s^T x + s^T\Delta x$ and the quadratic term $\Delta x\Delta s$ decreases rapidly under iteration, since $\Delta s$ and $\Delta x$ converge to zero. In our analysis below, we assume that the quadratic term is zero, which enables us to derive a dynamical system that is independent of the original LO problem. This dynamical system is related to the well-known logistic family. We analyze the limit behavior of this system. Our results are very close to the experimental results in [2]. It appears that our dynamical system represents the behavior of AFS methods in general.

Our paper is organized as follows. We first recall the primal-dual AFS method for solving LO problems. We then derive an iterative process to analyze the convergence of the AFS method under variation of the step-size. We analyze this process for increasing values of a parameter $\theta$ and show that it behaves chaotically as $\theta$ increases beyond 2/3. We supplement this analysis by Feigenbaum diagrams. In the final section, we compare this to results of the primal-dual AFS method and show that it also behaves chaotically as the step-size increases beyond 2/3 of the maximal step-size, which illustrates that our process resembles the general behavior of the primal-dual AFS method.

### 1.1 Notation and terminology

We reserve the symbol $e \in \mathbb{R}^n$ for the vector of all ones. For any vector $x$, the capital $X$ denotes the diagonal matrix with the entries of $x$ on the diagonal. Furthermore, if $f : \mathbb{R} \to \mathbb{R}$ is any function and $x \in \mathbb{R}^n$, then we denote by $f(x)$ the vector $(f(x_1), \ldots, f(x_n))$. If $s$ is another vector, then $xs$ will denote the coordinatewise product of $x$ and $s$ and $x/s$ will denote the coordinatewise quotient of $x$ and $s$. In other words, $xs = Xs$ and $x/s = S^{-1}x$. Finally, $\|.\|$ denotes the $l_2-$norm. Our basic assumption is that a primal-dual pair $(x, s)$ of feasible solutions exists and that $A$ is an $m \times n$ matrix of rank $m$ for $m < n$.

We adopt the standard terminology from dynamical systems, see e.g. [3]. A fixed point $x^*$ is *locally stable* if $\lim_{k\to\infty} f^k(x) = x^*$ for any $x$ that is sufficiently close to $x^*$. For differentiable $f$, if the derivative $Df(x^*)$ has all eigenvalues $< 1$, then $x^*$ is locally stable. The fixed point is *globally stable* if $\lim_{k\to\infty} f^k(x) = x^*$ for all $x$. Saying that an IPM converges therefore is equivalent to saying that the optimal solution is globally stable. If $f^n(x^*) = x^*$ for some $n \geq 1$, then we say that $x^*$ is a periodic point. A periodic point is a fixed point of an iterate $f^n$. The periodic point is called locally or globally stable if it has this property as a fixed point of $f^n$. The set that consists of all the iterates $f^k(x)$ is called the *orbit* of $x$. In general, it need not converge to a periodic point, but if the space is compact, then the descending chain of closed sets $K_n = \overline{\{f^k(x) : k \geq n\}}$ has a non-empty intersection which is called the *omega-limit of* $x$. It is equal to the set of all density points of the orbit.

If the LO problem is primal non-degenerate, then the primal problem has a unique optimum, but the dual problem may consists of a polyhedron $P$ of optimal solutions. It turns out that the primal variable converge to the optimum under the AFS algorithm, but the dual variable need not converge. The omega-limit of the dual variable is a subset of $P$.

## 2 Primal-dual affine scaling

In linear optimization, the notion of affine scaling has been introduced by Dikin [4] as a tool for solving the (primal) problem in standard format

$$(P) \qquad \min\{c^T x \,:\, Ax = b,\ x \geq 0\}.$$

The underlying idea is to replace the nonnegativity constraints $x \geq 0$ by the ellipsoidal constraint

$$\|\bar{X}^{-1}(\bar{x} - x)\| \leq 1, \tag{1}$$

where $\bar{x}$ denotes some given interior feasible point, and $\bar{X}$ the diagonal matrix corresponding to $\bar{x}$. The resulting subproblem is easily solved and renders a new interior feasible point with a better objective value. Dikin showed, under the assumption of primal nondegeneracy, that this process converges to an optimal solution of $(P)$. This so–called *affine scaling* (AFS) method of Dikin remained unnoticed until 1985. The work of Karmarkar [9] sparked a large amount of research in polynomial–time methods for LO, and gave rise to many new and efficient interior point methods (IPMs) for LO. For a survey of this development we refer to the books of Wright [15], Ye [16], Vanderbei [14] and Roos *et al.* [11].

Every known method for solving $(P)$ essentially also solves the dual problem

$$(D) \qquad \max\{b^T y \,:\, A^T y + s = c,\ s \geq 0\}$$

by closing the duality gap $s^T x$ between $c^T x$ and $b^T y$. A pair of vectors $x^*, s^*$ solves $(P)$ and $(D)$ if and only if they are orthogonal. Since $x^* \geq 0$ and $s^* \geq 0$, this means that the coordinatewise product $x^* s^*$ is equal to the all-zero vector. The primal-dual AFS method that we consider in this paper has been proposed by Jansen *et al.* [8]. Dikin's ellipsoidal constraint (1) involves the the primal variable only. In primal-dual AFS this is replaced by a constraint that includes both the primal and the dual variable:

$$\|\bar{X}^{-1}(\bar{x} - x) + \bar{S}^{-1}(\bar{s} - s)\| \leq 1, \tag{2}$$

where similarly as before, $\bar{S}$ denotes the diagonal matrix corresponding to the slack vector $\bar{s}$. In this notation, $(\bar{x}, \bar{s})$ is the original pair of primal vector and slack vector and $(x, s)$ is an updated pair. The differences $\Delta x = x - \bar{x}$ and $\Delta s = s - \bar{s}$ are called the primal-dual AFS directions.

Let $v = (xs)^{1/2}$ be the coordinatewise square root of the coordinatewise product and let $v^k$ be the coordinatewise power of $v$. Jansen *et al.* have shown that the directions $\Delta x$ and $\Delta s$ can be derived from the vector

$$p_v = -\frac{v^3}{\|v\|^2}$$

by first projecting $p_v$ onto the null space (for $\Delta x$) and the row space (for $\Delta s$) of $AD$, and then rescaling the result by a coordinatewise product. More specifically, if $d = (x/s)^{1/2}$ then

$$\Delta x = d P_{AD}(p_v),\ \ \Delta s = d^{-1} Q_{AD}(p_v)$$

where $P_{AD}$ and $Q_{AD}$ denote the orthogonal projections onto the null space of $AD$ and the row space of $AD$, respectively. These projections recombine in the Dikin ellipsoid to

$$X^{-1}\Delta x + S^{-1}\Delta s = -\frac{v^2}{\|v^2\|}. \tag{3}$$

This gives the primal-dual AFS directions but not the size of the step, which is controlled by an additional parameter $\alpha$. It is shown in [8] that if $\alpha < 1/(15\sqrt{n})$, then the iterative process $x \mapsto x + \alpha\Delta x$, $s \mapsto s + \alpha\Delta s$ follows an 'almost centered' path, i.e., $xs$ is close to $e$ in projective space, and hence that process remains in the feasible set and converges to a solution.

## 3 The iterative process under the assumption of exact Dikin steps

Starting with a primal-dual feasible pair $(x, s)$, the next iterated pair is given by

$$x^+ = x + \alpha \Delta x, \quad s^+ = s + \alpha \Delta s,$$

and hence we have

$$x^+ s^+ = (x + \alpha \Delta x)(s + \alpha \Delta s) = xs + \alpha (x \Delta s + s \Delta x) + \alpha^2 \Delta x \Delta s.$$

The directions $\Delta x$ and $\Delta s$ are orthogonal. We call the Dikin step *exact* if the coordinatewise product is equal to zero, so $\Delta x \Delta s = 0$. We emphasize that Dikin steps are exact only under special conditions, for instance if both directions are non-negative. So the assumption of exact steps is rather restrictive. However, if the AFS iterations are close to a solution, then $\Delta x$ and $\Delta s$ will be relatively small, and the product $\Delta x \Delta s$ will be negligible.

If the step is exact then the quadratic term $\alpha^2 \Delta x \Delta s$ vanishes and the reduction of $xs$ is proportional to $x \Delta s + s \Delta x$, which can be rewritten to

$$xs \left( x^{-1} \Delta x + s^{-1} \Delta s \right)$$

Observe that $x^{-1} \Delta x + s^{-1} \Delta s$ is equal to the ellipsoidal constraint (3) of the primal-dual AFS method. So we find under the assumption of exact steps that

$$x^+ s^+ = xs + \alpha (x \Delta s + s \Delta x) = xs - \alpha \frac{x^2 s^2}{\|xs\|} = xs \left( e - \alpha \frac{xs}{\|xs\|} \right),$$

where $e$ denotes the all-one vector. In the original primal-dual AFS method, one needs to choose $\alpha$ such that $x$ and $s$ remain feasible. Now we have arrived at an iterative process for the product vector $xs$. Since we require $x \geq 0$ and $s \geq 0$, we need to require $xs \geq 0$ in the iterative process. From this we deduce that the maximal step size is equal to

$$\alpha_{\max} = \frac{\|xs\|}{\max xs}$$

Defining

$$\theta = \frac{\alpha}{\alpha_{\max}} = \alpha \frac{\max xs}{\|xs\|}$$

we may write

$$x^+ s^+ = xs \left( e - \theta \frac{xs}{\max xs} \right), \quad \theta \in [0, 1].$$

Now if we write $w = xs$, then we get

$$w^+ = w \left( e - \theta \frac{w}{\max w} \right), \quad \theta \in [0, 1].$$

This iterative process depends on a parameter $\theta$ which is related to the original step size $\alpha$ by the equation $\alpha = \theta \alpha_{max}$. If $w$ is close to the central line and has coordinates that are approximately equal, then $\alpha_{max} \approx \sqrt{n}$. In general, $1 \leq \alpha_{max} \leq \sqrt{n}$.

We make one further reduction. If $u = \lambda w$ for a scalar $\lambda$ then $u^+ = \lambda w^+$, so the iterative process preserves projective equivalence. We may therefore reduce our system up to projective equivalence by scaling vectors so that their maximum coordinate is equal to one. So we end up with the iterative process that is the subject of our paper:

$$\bar{w}^k = w^k \left( e - \theta w^k \right), \quad w^{k+1} = \frac{\bar{w}^k}{\max \bar{w}^k}, \quad k = 0,\ 1,\ \ldots \tag{4}$$

This process, which is dependent of the original LO problem, involves two steps: multiplication and scaling. To describe the process more succinctly we use the map $f_\theta(x) = x(1 - \theta x)$. The iterative process is then given by:

$$w^{k+1} = f_\theta(w^k) / \max\{f_\theta(w^k)\}. \tag{5}$$

The map $f_\theta$ is very similar to the logistic map $y \mapsto \theta y(1-y)$ on the unit interval, which is a well studied object and one of the archetypes of chaotic dynamical systems. Indeed, if we replace $x$ by the variable $y$ which is given by $x = \frac{1-y}{\theta}$, then we replace $f_\theta(x)$ by the logistic map.

## 4 The process converges if $\theta \le 2/3$

The iterative process is easier to understand than the original primal AFS method. We analyze its limit behavior for increasing values of $\theta$. We suppress the subscript $\theta$ in $f_\theta$ and simply write the iterative process as

$$w^{k+1} = f(w^k)/\max\{f(w^k)\}.$$

On the real line, $f$ has a global maximum $f(1/2\theta) = 1/4\theta$. If $\theta \le 1/2$ then $1/2\theta \ge 1$ and in this case $f$ is increasing on the unit interval.

**Lemma 1.** *If $\theta \le 1/2$ then $w^k$ converges to $e$ for every initial $w^0 > 0$.*

*Proof.* Suppose that $w_i$ is the maximum coordinate of the vector $w$. In particular, $w_i = 1$. Since $f$ is increasing for $\theta \le 1/2$, the maximum coordinate of $f(w)$ is $f(w)_i = (1-\theta)$. Since the position of the maximum coordinate remains fixed, we can write $w^{k+1} = g(w^k)$ with $g(x) = \frac{x(1-\theta x)}{1-\theta}$. It is immediately clear that $g(x)$ is monotonically increasing on the unit interval, so the limit of $w^k$ exists. It is necessarily a fixed point of $g$, and the only fixed point of $g$ is $e$. We conclude that $w^k$ converges to the all-one vector $e$. □

In other words, $e$ is a global attractor if $\theta \le 1/2$. We extend this to $\theta \le 2/3$. If $\theta > 1/2$ then $f$ is unimodal with its maximum at $1/2\theta$. It is increasing on $[0, 1/2\theta]$ and decreasing on $[1/2\theta, 1]$. The map is point symmetric with respect to $1/2\theta$:

$$f\left(\frac{1}{2\theta} + z\right) = f\left(\frac{1}{2\theta} - z\right). \tag{6}$$

It follows that $f[1/2\theta, 1] = f[1/\theta - 1, 1/2\theta]$, so if $x < 1 - 1/\theta$ and $y \ge 1 - 1/\theta$ then $f(x) < f(y)$.

**Lemma 2.** *Suppose that $\theta > 1/2$. For every initial $w^0$ there exists a $k_0$ such that $\min w^k \ge 1/\theta - 1$ for all $k \ge k_0$.*

*Proof.* Suppose that $\min w^k < 1/\theta - 1$ and let $w_i^k$ be the minimal coordinate. Since $f$ is increasing on $[0, 1/2\theta$, the $i$-th coordinate of $f(w^k)$ is minimal and has value $f(w_i^k)$. The maximal coordinate has value $\le \max(f) = 1/4\theta$. It follows that

$$\min w^{k+1} \ge 4\theta w_i^k(1-\theta w_i^k) = 4\theta^2 w_i^k(1/\theta - w_i^k) \ge 4\theta^2 w_i^k.$$

Therefore as long as $\min w < 1/\theta - 1$ the minimal coordinate increases by a factor $4\theta^2 > 1$. There must be a $k_0$ such that $\min w^{k_0} \ge 1/\theta - 1$. For this $k_0$, $\min f(w^{k_0}) = f(1) = 1 - \theta$, while $\max f(w^{k_0}) \le f(1/2\theta) = 1/4\theta$. Therefore $\min w^{k_0+1} \ge 4\theta(1-\theta) = 4\theta^2(1/\theta - 1) > 1/\theta - 1$. It follows that all coordinates remain in the interval $[1/\theta - 1, 1]$ from the $k_0$-th iterate onwards. □

We want to show that $w^k$ converges to $e$ if $1/2 \le \theta \le 2/3$. In other words, we want to show that the minimum coordinate of $w^k$ converges to 1. So we only need to keep track of the minimum coordinate under iteration. By the point symmetry in (6), if we replace the coordinates $w_i < 1/2\theta$ in $w^k$ by their reflections $1/\theta - w_i$, then this does not affect $w^{k+1}$. So we may assume that $w^k$ is a vector that has all coordinates $\ge 1/2\theta$. Now let $x = \min w^k \ge 1/2\theta$. Since $f$ is decreasing for $x \ge 1/2\theta$ we have that $f(x) = \max f(w^k)$ and $f(1) = \min f(w^k)$. So the minimum coordinate of $w^{k+1}$ is given by

$$h(x) = \frac{f(1)}{f(x)} = \frac{1-\theta}{x(1-\theta x)}.$$

**Lemma 3.** *If $1/2 < \theta \leq 2/3$ then $h(x) \geq x$ and $\lim_{k\to\infty} h^k(x) = 1$ for every $x \in [1/\theta - 1, 1]$.*

*Proof.* It suffices to prove that $h(x) \geq x$, because this implies that the limit of $h^k(x)$ exists and is equal to the unique fixed point of $h$. Now $h(x) \geq x$ can be rewritten as

$$\theta x^3 - x^2 + 1 - \theta \geq 0. \tag{7}$$

The derivative $(3\theta x - 2)x$ of the cubic $\theta x^3 - x^2 + 1 - \theta$ is negative on the unit interval, by our assumption that $\theta \leq 2/3$. So the cubic has its maximum at 0 and its minimum at $x = 1$, which is a zero of the cubic. Hence the inequality holds. □

**Theorem 1.** *If $\theta \leq 2/3$ then the all-unit vector is globally stable, i.e., $\lim_{k\to\infty} w^k = 1$ for all $w^0$.*

*Proof.* We already proved this result for $\theta \leq 1/2$. If $1/2 < \theta \leq 2/3$ then we may assume that $\min w^0 \geq 1/\theta - 1$ by Lemma 2. We already argued that we may replace such a vector by its point reflection in $1/2\theta$. Lemma 3 now implies that the minimum coordinate of $w^k$ converges to 1. □

## 5 The process converges to a point of period two if $2/3 < \theta \leq \frac{1+\sqrt{5}}{4}$

We continue our analysis of the process for $\theta > 2/3$. It is useful to consider the two-dimensional case $w = (x, 1)$, so the iterative process is given by $(x, 1) \to (1, h(x)) \to (h^2(x), 1) \to \cdots$. Note that a fixed point of $h$ corresponds to a point of period two for the iterative process. If $\theta > 2/3$ then $h$ has a unique fixed point $r \in (0, 1)$, which can be found by solving the cubic equation $h(x) = x$ that we already encountered in equation (7). It factors as $\theta x^3 - x^2 + 1 - \theta = (x - 1)(\theta x^2 + (\theta - 1)x + \theta - 1) = 0$. Therefore $r$ satisfies the quadratic equation

$$\theta r^2 + (\theta - 1)r + (\theta - 1) = 0. \tag{8}$$

The positive solution for $r$ is equal to

$$r = \frac{1 - \theta + \sqrt{(1-\theta)^2 + 4\theta(1-\theta)}}{2\theta}, \tag{9}$$

which is $\leq 1$ if and only if $\theta \geq 2/3$.

**Lemma 4.** *If $\theta > 2/3$ then $\lim_{k\to\infty} h^k(x) = r$ for every $x \in [1/\theta - 1, 1)$, i.e., $r$ is a global attractor in $[1/\theta - 1, 1)$.*

*Proof.* The cubic equation $h(x) = x$ has zeros in $1, r$ and the third zero $s$ is negative. In particular, $h(x) > x$ on $(s, r)$ and $h(x) < x$ on $(r, 1)$ and the result follows. □

Note that $r$ is not a global attractor in the closed interval $[1/\theta - 1, 1]$ since $h(1) = 1$ is a fixed point. As a useful corollary of Lemma 4, we obtain that in the two-dimensional case $(x, 1) \to (1, h(x))$ the process converges to a fixed point if $\theta \leq 2/3$ and it converges to an orbit of period two if $\theta > 2/3$. The exact same limit behavior has been observed by Hall and Vanderbei for the primal AFS method [7]. They considered an LO problem with four variables and one constraint, but due to the symmetry of the problem, it has two degrees of freedom. Hall and Vanderbei prove that the process converges to a fixed point for step-size $\leq 2/3$, while it converges to a point of period two for step-size $> 2/3$.

**Lemma 5.** *Suppose that $\frac{1}{2} < \theta \leq \frac{1+\sqrt{5}}{4}$. For any initial $w^0$ there exists a $k_0$ such that $\min w^k \geq 1/2\theta$ for all $k \geq k_0$.*

*Proof.* By Lemma 2 we may assume that $\min w^0 \geq 1/\theta - 1$. In this case, the minimal coordinate of $f(w^0)$ has value $f(1) = 1 - \theta$ and the maximal coordinate is $\leq \max f = 1/4\theta$. Therefore, $\min w^1 \geq 4\theta(1 - \theta)$ which is greater than or equal to $1/2\theta$ if and only if $8\theta^2(1 - \theta) \geq 1$. We solve the cubic equation

$$8\theta^2(1 - \theta) - 1 = 0 \iff (2\theta - 1)(1 - 2\theta - 4\theta^2) = 0.$$

The two roots of the quadratic equation are $\frac{1\pm\sqrt{5}}{4}$. We conclude that $8\theta^2(1-\theta) \geq 1$ for $\frac{1}{2} < \theta \leq \frac{1+\sqrt{5}}{4}$. □

Suppose $\theta \le \frac{1+\sqrt{5}}{4} \approx 0.809$. Consider an initial condition $w^0 = (w_1, \dots, w_n)$ with increasing coordinates $w_1 \le w_2 \le \cdots \le w_n = 1$ and such that $w_1 \ge 1/2\theta$. Then $f(w)$ has decreasing coordinates and, like before, we conclude that we can keep track of the iterative process by its extreme coordinates $w_1$ and $w_n$. For an intermediate coordinate $w_i$ the process is given by $w_i \to \frac{w_i(1-\theta w_i)}{w_1(1-\theta w_1)}$. We know that the minimum coordinate converges to $r$ so we may as well put $w_1 = r$, in which case we get that $w_i \to g(w_i)$ for the map

$$g(x) = \frac{x(1-\theta x)}{r(1-\theta r)}.$$

Note that $g(r) = 1$ and that $g(1) = h(r) = r$ so that $g$ has a fixed point $s \in (r, 1)$ as is illustrated by the graph of $g^2$ in the figure below. Indeed, we leave it to the reader to verify that $g$ has a unique fixed point in $(r, 1)$ at $s = (r + \theta - 1)/r\theta$. The derivative of $g$ is given by

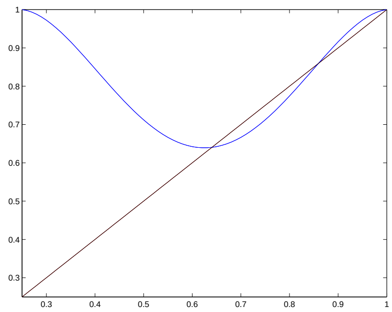


**Fig. 1.** The function $g^2$ on the interval [1/4,1] for $\theta = 0.8$ plotted against the diagonal.

$$g'(x) = \frac{1-2\theta x}{r(1-\theta r)} = \frac{r(1-2\theta x)}{1-\theta}, \tag{10}$$

where we use that $1 - \theta = r^2 - \theta r^3$.

**Lemma 6.** *Suppose that $2/3 < \theta \le \frac{1+\sqrt{5}}{4}$ and that $x \in (r, 1)$. Then the $\omega$-limit set of $g^k(x)$ is either equal to $\{r, 1\}$ or $x$ is equal to the fixed point $s$.*

*Proof.* Note that $g(x) = x$ is a quadratic equation with solutions $x = 0$ and $x = s$. The equation $g^2(x) = x$ is an equation of degree four with solutions $0, r, s, 1$. It follows that $g^2(x) \neq x$ on the two subintervals $(r, s) \cup (s, 1)$ and that $g^2(x) > x$ on the one interval while $g^2(x) < x$ on the other interval. Using equation (10), and using that $rs\theta = r + \theta - 1$, we find that the derivative at $s$ is

$$g'(s) = \frac{r\,(1-2\theta s)}{1-\theta} = \frac{2-r-2\theta}{1-\theta}.$$

To prove that $s$ is unstable, we need to verify that $\frac{2-r-2\theta}{1-\theta} < -1$, or equivalently, that $r > 3 - 3\theta$. Substituting (9) for $r$ and simplifying equations we end up with $(1-\theta)+\sqrt{(1-\theta)^2 + 4\theta(1-\theta)} > 6\theta(1-\theta)$. Taking squares to remove the root gives $(1-\theta)^2 + 4\theta(1-\theta) > (6\theta-1)^2(1-\theta)^2$. which simplifies to $1 + 3\theta > (6\theta - 1)^2(1-\theta)$. Collecting all terms and dividing by $\theta$ we finally arrive at the inequality $9\theta^2 - 12\theta + 4 > 0$, or equivalently, $(3\theta - 2)^2 > 0$. This obviously holds if $\theta > 2/3$. It follows that $g^2(x) < x$ on $(r, s)$ and that $g^2(x) > x$ on $(s, 1)$. □

We note that $s$ is weakly unstable. Its derivative is only marginally smaller than $-1$, and so it is a weak repellor. This property of $s$ will be illustrated by the Feigenbaum diagram of the primal-dual AFS method in the final section.

We introduced the map $g$ to keep track of the iterative process $w^k$ on a fixed coordinate. So we now know that the $\omega$-limit of $w^k$ on each coordinate is equal periodic with values $\{r, 1\}$ or it is equal to a fixed point $s$. Since $s$ is unstable, the $\omega$-limit is almost surely equal to $\{r, 1\}$.

**Theorem 2.** *If $2/3 < \theta \leq \frac{1+\sqrt{5}}{4}$ then the $\omega$-limit is almost surely equal to an orbit of period two. The coordinates of the two elements of such an $\omega$-limit are equal to $r$ or $1$.*

*Proof.* Denote the $\omega$-limit by $V$. This set is invariant under $v \mapsto f(v)/\max f(v)$. By the previous lemma, the minimum coordinate of $v \in V$ is equal to $r$. So $\max f(v) = f(r)$, which implies that $f(v)/\max f(v) = g(v)$ and we conclude that $V$ is invariant under $g$ and the $\omega$-limit of $v \in V$ has coordinates that are equal to $r$ and 1, unless $v$ has a coordinate that happens to be equal to $s$. The periodic points with coordinates $r$ and 1 only are locally stable. Therefore, the $\omega$-limit almost surely consists of such a periodic point. □

**Numerical results for $\theta > \frac{1+\sqrt{5}}{4}$.** Numerical experiments show that the period two limit cycle persists beyond $\frac{1+\sqrt{5}}{4}$. The analysis gets involved and we limit ourselves to the case that $w$ has three coordinates. Assuming that the coordinates are ordered $x < y < 1$ we can write

$$
\begin{aligned}
(x, y, 1) &\mapsto \left(1, \frac{y(1-\theta y)}{x(1-\theta x)}, \frac{1-\theta}{x(1-\theta x)}\right) \\
&\mapsto \left(\frac{x(1-\theta x)}{1-\theta\frac{1-\theta}{x(1-\theta x)}}, \frac{y(1-\theta y)}{1-\theta}\, \frac{1-\theta\frac{y(1-\theta y)}{x(1-\theta x)}}{1-\theta\frac{1-\theta}{x(1-\theta x)}}, 1\right)
\end{aligned}
$$

so we can describe the second iterate by the function

$$
F(x, y) = \left(\frac{x(1-\theta x)}{1-\theta\frac{1-\theta}{x(1-\theta x)}}, \frac{y(1-\theta y)}{1-\theta}\, \frac{1-\theta\frac{y(1-\theta y)}{x(1-\theta x)}}{1-\theta\frac{1-\theta}{x(1-\theta x)}}\right).
$$

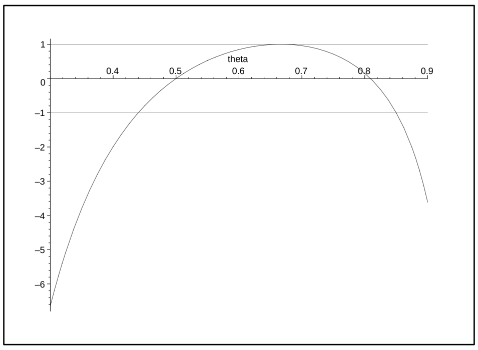


**Fig. 2.** Value of the second 'transversal' eigenvalue $\frac{\partial F_2}{\partial y}(r, r)$ as function of $\theta$.

This function preserves the diagonal, on which we have the two-dimensional process, which as we have seen already has a period two global attractor for $\theta > 2/3$. So, the instability has to occur in the direction transversal to the diagonal. We can study this stability by taking the derivative

$$
DF(x, y) = \begin{bmatrix} \frac{\partial F_1(x,y)}{\partial x} & \frac{\partial F_2(x,y)}{\partial x} \\ 0 & \frac{\partial F_2(x,y)}{\partial y} \end{bmatrix}
$$

where

$$\frac{\partial F^1(x,y)}{\partial x} = \frac{(x - 3\theta x^2 - 2\theta + 2\theta^2 + 2\theta^2 x^3 + 4\theta^2 x - 4\theta^3 x)x(1-\theta x)}{(-x + \theta x^2 + \theta - \theta^2)^2}$$

and

$$\frac{\partial F_2(x,y)}{\partial y} = \frac{x - \theta x^2 - 2y\theta + 6y^2\theta^2 - 2x^2\theta + 2y\theta^2 x^2 - 4y^3\theta^3}{(-x + \theta x^2 + \theta - \theta^2)\cdot(-1+\theta)}.$$

Maple computations show that fixed point becomes unstable at $\theta = 0.8499377796$, where the eigenvalue $\frac{\partial F_2(r,r)}{\partial y}$ becomes equal to $-1$. At this value of $\theta$ we expect $(r, r, 1)$ to become unstable, splitting off a stable period 4 point in a period doubling bifurcation. This is confirmed by the Feigenbaum diagrams in the final section of the paper.

**Remark:** For generic period doubling bifurcations in smooth dynamical systems, the parameter curve of the periodic points of period $2n$ is parabolic and intersects the curve of the periodic point of period $n$ transversally. At the point of intersection, the period $n$ point changes from stable to unstable, or vice versa. Curiously, this scenario fails in at least the first two periodic doublings in the Feigenbaum diagram of Figure 3. In this section we have described precisely these two bifurcations, and they are different from the usual bifurcations in dynamical systems.

## 6 The process converges to a periodic point for $\theta$ near 1.

The process $w^k$ is hard to analyze for values beyond $\theta \geq \frac{1+\sqrt{5}}{4}$, since the degree of the algebraic equations increases and periodic points cannot be found in closed form. Some structure of period orbits of the logistic family is retained, though.

**Proposition 1.** *Let $\theta \in [0,1]$ be such that $c = \frac{1}{2}$ is $m$-periodic for the logistic map $Q_\theta : x \mapsto 4\theta x(1-x)$. Then the process $w^{k+1} = \frac{f_\theta(w^k)}{\max\{f_\theta(w^k)\}}$ from (4) has a locally stable $m$-periodic orbit too, provided the number of coordinates $n \geq m$.*

*Proof.* Assume that the first $m$ coordinates $w_i^k$ of the vector $w^k$ are equal to $Q_\theta^i(c)/\theta$ (so the coordinates are not put in increasing order here). In particular, $w_m^k = c/\theta = 1/2\theta$, and $f_\theta(w_m^k) = 1/4\theta = \max\{f_\theta(x)\}$. Then $w_i^{k+1} = 4\theta f_\theta(w_i^k) = 4\theta w_i^k(1-\theta w_i^k)$. The linear scaling $h(x) = \theta x$ conjugates this to $Q_\theta$, since $h^{-1} \circ Q_\theta \circ h(x) = 4\theta x(1-\theta x)$. So the critical point of $f_\theta$ is periodic too: $w_i^{k+1} = w^k_{(i \bmod m)+1}$ for $i = 1, \dots, m$, and in particular $w_{m-1}^{k+1} = 1/2\theta$. Therefore the scaling remains the same at all iterates.

This periodic orbit attracts the coordinates $w_i$ for $m < i \leq n$ and Lebesgue-a.e. initial choice of $w_i$. Let us now verify that the orbit is also stable under small changes in the coordinates $w_i$ for $1 \leq i \leq m$. Renaming these $w_i$ to $y_i$, $i = 1, \dots, m$, where $y_{m-1} = 1/2\theta$, $y_m = 1$, $y_1 = f(1)/f(y_{m-1})$ and $y_{i+1} = f(y_i)/f(y_{m-1})$ for $1 \leq i < m$, we can described them by the map

$$F(y_1, \dots, y_{m-1}, 1) = \left( \frac{f(1)}{f(y_{m-1})}, \frac{f(y_1)}{f(y_{m-1})}, \dots, \frac{f(y_{m-2})}{f(y_{m-1})}, 1 \right). \tag{11}$$

The final coordinate is redundant, so $DF$ is an $(m-1) \times (m-1)$ matrix. Recall that $f'(x) = 1 - 2\theta x$. Therefore

$$DF(y) = \begin{pmatrix} 0 & 0 & \dots & 0 & -(1-2\theta y_{m-1})\frac{f(1)}{f(y_{m-1})^2} \\ \frac{1-2\theta y_1}{f(y_{m-1})} & 0 & & \vdots & -(1-2\theta y_{m-1})\frac{f(y_1)}{f(y_{m-1})^2} \\ 0 & \frac{1-2\theta y_2}{f(y_{m-1})} & \ddots & & -(1-2\theta y_{m-1})\frac{f(y_2)}{f(y_{m-1})^2} \\ \vdots & & \ddots & & \vdots \\ 0 & & & \frac{1-2\theta y_{m-2}}{f(y_{m-1})} & -(1-2\theta y_{m-1})\frac{f(y_{m-2})}{f(y_{m-1})^2} \end{pmatrix} \tag{12}$$

and since $y_{m-1} = 1/2\theta$, the right-most column is zero. Therefore all eigenvalues are zero, and $DF^n$ is a contraction. □

This shows that the structure of the Feigenbaum map of the logistic family should in principle be present in the Feigenbaum diagrams for (4), see Section 7. However, the basins of the periodic orbits of Proposition 1 and also the parameter windows in which these orbits persist may be very small, and therefore they may be hard to detect.

Surprisingly, it is possible to determine the limit of $w^k$ for $\theta$ arbitrarily close to 1. It turns out that $w^k$ converges almost surely to a point $y$ of period $m = n$, and although this orbit is of the same type as those in Proposition 1, the parameter window of stability is larger than for the corresponding orbit for the logistic map: it persists for $\theta$ up to 1, whereas in the logistic family such $m$-periodic orbit is no longer stable for $\theta$ very close to 1.

**Theorem 3.** *For $\theta$ sufficiently close to* 1*, the iterative process* (4) *has a locally stable point of period $n$.*

*Proof.* In this case, the period $m$ coincides with the number of coordinates $n$. Let $w^0$ be any point with maximal coordinate 1 and all other coordinates $\leq \frac{1}{2\theta}$. As before, we assume $\min w \geq 1/\theta - 1$ and this implies that $f(1)$ is the minimal coordinate of $w^1$. We arrange the coordinates of $w^0$ in non-decreasing order. Then $f(y_{m-1})$ is the largest coordinate among all the $f(y_k)$, so we scale by this number and we arrange the coordinates of $w^1$ in non-decreasing order. As in (11), the dynamic process can be described by

$$F(y_1, \ldots, y_{m-1}, 1) = \left( \frac{f(1)}{f(y_{m-1})}, \frac{f(y_1)}{f(y_{m-1})}, \ldots, \frac{f(y_{m-2})}{f(y_{m-1})}, 1 \right).$$

The vector $w^0$ has the required cyclic periodicity if $f(y_k)/f(y_{m-1}) = y_{k+1}$ and $f(1)/f(y_{m-1}) = y_1$. Fix $y_{m-1} < 1/2\theta$ and define a map $g(x) = f(x)/f(y_{m-1})$. Note that $w^0$ has the required periodicity if

$$y_{m-1} = g(y_{m-2}) = g^2(y_{m-3}) = \cdots = g^{m-2}(y_1) = g^{m-1}(1).$$

By the point symmetry of $f$ in (6), we may replace $g^{m-1}(1)$ by $g^{m-1}(1/\theta - 1)$. If we take $y_{m-1} = 1/2\theta$ then a sufficient condition for the cyclic periodic point to exist is

$$g^{m-1}(1/\theta - 1) \leq 1/2\theta. \tag{13}$$

This inequality is satisfied if $\theta$ is sufficiently close to 1. Now $g$ increases as $y_{m-1}$ decreases, so once the condition is satisfied, there exists an $y_{m-1}$ such that $g^{m-1}(1/\theta - 1) = y_{m-1}$.

To show stability of this orbit, we cannot use anymore that the right-most column of the derivative $DF$ vanishes, because $y_{m-1} < 1/2\theta$. Fortunately, $DF$ is of the form

$$DF(y) = \begin{pmatrix} 0 & 0 & \ldots & 0 & -c_1 \\ d_1 & 0 & & \vdots & -c_2 \\ 0 & d_2 & \ddots & & -c_3 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & d_{m-1} & -c_m \end{pmatrix}$$

where $d_1 > d_2 > \ldots > d_{m-1} > 0$ and $0 < c_1 < c_2 < \ldots < c_m < 1$. This follows from (12) and the fact that $y_1 < y_2 < \cdots < y_{m-1} \leq 1/2\theta$ and that $f$ is increasing on $[y_1, y_{m-1}] \subset [0, \frac{1}{2\theta}]$.

In order to estimate the eigenvalue of this matrix we need the following classical result.

**Theorem 4 (Eneström-Kakeya, [5]).** *Let $p(z) = \sum_{k=0}^{m} a_k z^k$ be a polynomial such that all coefficients are positive. Define*

$$\alpha = \min \left\{ \frac{a_k}{a_{k+1}} \right\}, \quad \beta = \max \left\{ \frac{a_k}{a_{k+1}} \right\}.$$

*Then all zeros of $p(z)$ lie in the annulus $\alpha \leq |z| \leq \beta$.*

**Claim:** If $c_i d_i < c_{i+1}$ for all $i \le m-1$, then all eigenvalues of $DF$ are in the open unit disc.

Abbreviate $A = DF$ and let $p(\lambda) = \det(\lambda I_m - A) = \sum_{k=0}^{m} a_k \lambda^k$ be the characteristic polynomial of $A$. We will show by induction that the coefficients are decreasing. More precisely $1 = a_m > \cdots > a_0 > 0$ and $a_0 = c_1 d_1 \cdots d_{m-1}$. By the The proof of the claim is by induction. The claim is obvious for $m = 1$. Assume that the claim is true for $m-1$. The characteristic polynomial is equal to

$$\det(\lambda I_m - A) = \lambda \det(\lambda I_{m-1} - A_{11}) + (-1)^{m-1} c_1 \cdot -d_1 \cdots - d_{m-1},$$

where $A_{11}$ is the $(1,1)$-minor matrix of $A$. By the inductive hypothesis, $\det(\lambda I_{m-1} - A)$ has decreasing coefficients and constant coefficient $c_2 \cdot d_2 \cdots d_{m-1}$. Since $a_0 = \frac{c_1}{c_2} \cdot d_1 \cdot a_1 < a_1$, the claim follows.
We compute

$$\frac{c_i}{c_{i+1}} \cdot d_i = \frac{f(y_{i-1})}{f(y_i)} \frac{1 - 2\theta y_i}{f(y_{m-1})} = \frac{y_i}{f(y_i)}(1 - 2\theta y_i) = \frac{1 - 2\theta y_i}{1 - \theta y_i} < 1.$$

This verifies the condition $c_i d_i < c_{i+1}$ of the claim. By the Eneström-Kakeya Theorem, the roots of $p(\lambda)$ are all in the open unit disc. Hence $DF^m$ is a contraction at $(y_1, \ldots, y_m)$ for $\theta$ sufficiently close to 1. □

Our numerical simulations suggest that the set of initial values $w^0$ that converge to this periodic point is large and has (nearly) full measure, as can be observed in the Feigenbaum diagrams in the next section.

## 7 Feigenbaum diagrams

One of the best known illustrations of a chaotic dynamical system is the Feigenbaum diagram of the quadratic map $t \mapsto \lambda t(1-t)$ on the unit interval. The $x$-axis of the Feigenbaum diagram contains the parameter $\lambda$ and the $y$-axis contains the $\omega$-limit set of a randomly chosen initial value. This Feigenbaum diagram branches nicely like a binary tree, splitting into two until it reaches chaotic behavior. The Feigenbaum diagram of the Dikin iterative process $w^k$ is a bit similar, though it does not branch as nicely as the diagram for the quadratic family. This is because the higher-dimensionality of the process poses some technical problems since now the $\omega$-limit is $n$-dimensional, and cannot easily be depicted. The standard solution is to project the $\omega$-limit set onto the line in some way. In our diagrams we have plotted one single coordinate of the $\omega$-limit set.

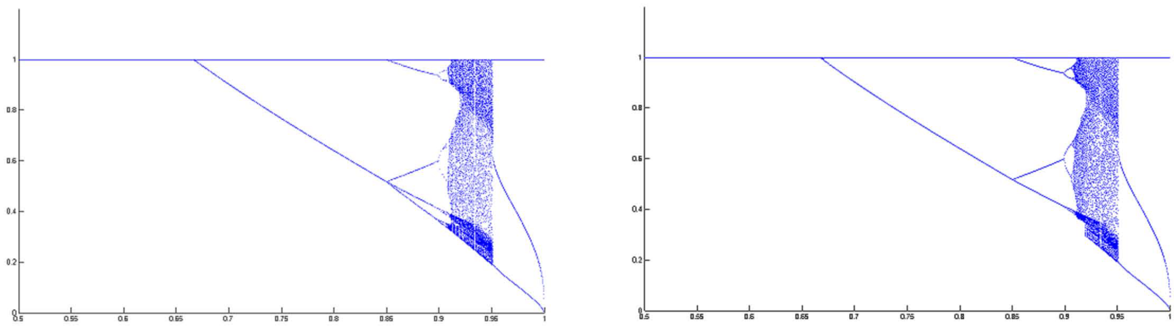

**Fig. 3.** Feigenbaum diagram for the process on three coordinates. Left figure: $\omega$-limit of a random coordinate. Right figure: $\omega$-limit of the middle coordinate. The right figure shows that the bifurcation at 0.8499377796 gives a period four point.

The Feigenbaum diagrams seem to exhibit the usual structure of period doubling cascades of the logistic family $Q_\theta : x \mapsto 4\theta x(1-x)$, $\theta \in [0,1]$. It is well-known that for $Q_\theta$, between two period doubling bifurcations, there is a parameter where the critical point $c = \frac{1}{2}$ is periodic, so by Proposition 1, this periodic point should also exist here, provided the dimension exceeds the period. There are additional grounds for such orbits to be locally stable, but the current resolution and relative small dimension, we cannot expect to see much of the period doubling cascade for higher periods.

To illustrate the consequence of the choice of the projection, compare the Feigenbaum diagrams in Figure 7. On the left we plot the $\omega$-limit set of a random coordinate. On the right we choose the middle coordinate of the ordered vector. We see that the process bifurcates at $\theta = 2/3$, when a point of order two appears, and then at $\theta = 0.849...$, when a point of order four appears. In the left figure, the diagram splits into five lines at $\theta = 0.849...$, in the right figure it splits into four lines. The reason for this is that the point of order four is of the type

$$(r, s_2, 1) \to (1, s_3, s_1) \to (r, 1, s_2) \to (1, s_1, s_3) \to (r, s_2, 1)$$

for values $s_1, s_2$ close to $r$ and $s_3$ close to 1. We will plot the diagrams in the same way as the figure on the left, so the reader should keep in mind that, contrary to standard Feigenbaum diagrams, the period of a point may be smaller than the number of lines.

The diagram indicates that $\omega$-limit set gets positive measure at around $\theta \approx 0.91$ and that the cyclic point of period three appears at around $\theta \approx 0.95$. The coordinates of the period three, for $\theta \approx 0.95$ point are approximately $(0.2, 0.6, 1)$. In Section 6 we found that the period three point exists as soon as inequality (13) is satisfied. If $n = 3$ and $\theta = 0.95$ then $g(1/\theta - 1) \approx 0.1900$, $g^2(1/\theta - 1) \approx 0.5917$ and $1/2\theta \approx 0.5263$. So the appearance of the period three point occurs a little before at the threshold value of $\theta$ predicted by inequality (13), but it is of the required form $(g(1/\theta - 1), g^2(1/\theta - 1), 1)$. This is not surprising. We showed that a cyclic point of that form is stable as soon as the inequality is satisfied. The eigenvalues vary continuously with $\theta$ so the point cannot suddenly become unstable once $\theta$ decreases below the threshold given in inequality (13).

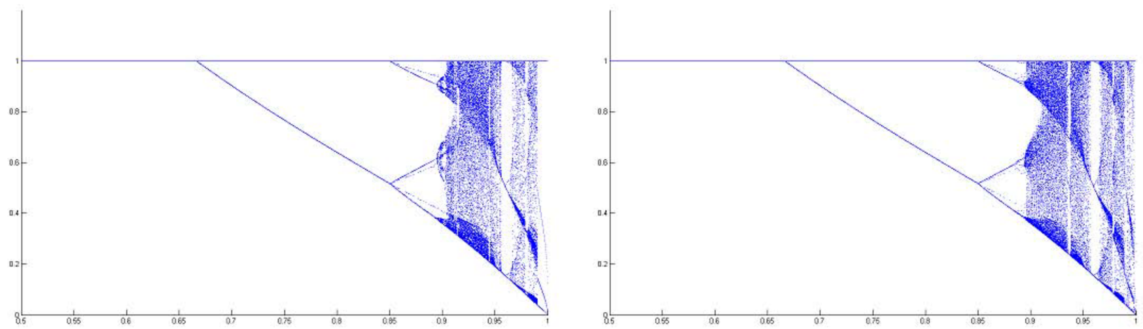

**Fig. 4.** Feigenbaum diagram for the process on four coordinates (left) and five coordinates (right).

The Feigenbaum diagrams for $n = 4$ and $n = 5$ are similar to the diagram for $n = 3$, and as it turns out that this holds in general for all $n > 3$. The main difference between $n = 3$ and $n > 3$ is the appearance of a chaotic region for $0.95 < \theta < 1$. It is remarkable that a stable point of period three reappears around $\theta \approx 0.95$. For $n = 4$ the stable cyclic point of period four appears at $\theta \approx 0.99$ and is still visible in this figure. For $n = 5$ it appears only at $\theta \approx 0.999$ and it is not visible in this picture. To show that our analysis in the previous section holds true and that the periodic point does exist, we zoom in on step sizes in $(0.95, 1)$ in the next figure.

The diagram on the left clearly shows the cyclic period five for $\theta$ near 1. It also shows that the period three orbit appears at $\theta \approx 0.957$. Remarkably, this phenomenon is independent of the dimension, as the diagram for $n = 100$ at the right illustrates. This diagram also shows that the window in which the stable point of period $n$ occurs decreases with $n$. If $n$ is large, then inequality (13) is only satisfied for $\theta \approx 1$.

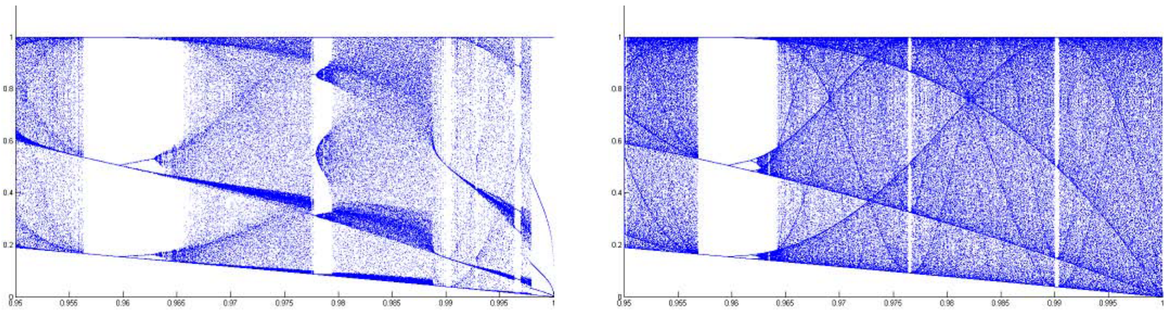

**Fig. 5.** Feigenbaum diagram for the process on five coordinates (left) and hundred coordinates (right) for $0.95 < \theta < 1$.

This concludes our analysis of the iterative process $w^k$. Now to prove that this analysis makes sense, we need to check that the primal-dual AFS method displays the same type of chaotic behavior as $w^k$. We will do that in the next and final section.

## 8 Comparison to the primal dual AFS method

The iterative process $w^k$ has been derived from primal-dual AFS under the restriction that the Dikin step is exact. In general, this step will not be exact, and we need to verify that chaos occurs in the original system. Our iterative process has a parameter $\theta$ that defines the step-size with respect to the maximum $\alpha_{max} = \frac{\|xs\|}{\max xs}$. So if we consider the primal-dual AFS method, then we should set our step size accordingly. This means that $\alpha$ should not be constant as in the original primal-dual AFS method, but we should take it to be equal to $\theta\alpha_{\max}$. We modify the method in this way and we put $\alpha = \theta\alpha_{max}$.

We take the same example as considered by Castillo and Barnes in [2]:

$$\begin{gathered}\min 10x_1 + 10x_2 + 5x_3 + x_4 - x_5 \\ \text{under the constraints} \\ x_1 + 2x_2 - 3x_3 - 2x_4 - x_5 = 0 \\ -x_1 + 2x_2 - x_3 - x_4 - x_5 = 0 \\ x \geq 0\end{gathered} \tag{14}$$

We take the same initial vectors $x_0$ and $y_0$ as Castillo and Barnes and run the modified primal-dual AFS method that we describe in pseudo-code below. The numerical task of computing the limit of the AFS process is not trivial, especially for a larger values of the step size, because $x^k$ rapidly converges to zero which leads to numerical problems, caused by inverting matrices that are ill conditioned. Castillo and Barnes developed analytic formulas that enabled them to still compute Feigenbaum diagrams with high precision. Such an analytic exercise is beyond the scope of our paper. We stop the computation once the duality gap reaches $10^{-10}$.

We have computed the Feigenbaum diagram for the scaled process $\frac{w^k}{\max w^k}$ that is given in Figure 7. The diagram on the right, which depicts the limit of the fourth coordinate. There is a bifurcation for $\theta = 2/3$ and another bifurcation close to $\theta = 0.86$, followed by a chaotic regime. At the end of the diagram, for values of $\theta$ close to 1, we find a stable periodic point. This is similar to the diagrams that we computed earlier for our process $w^k$, although the periodic point at the end of the diagram is period three instead of period five. The Feigenbaum diagram on the left, which depicts the second coordinate, shows a different picture. The diagram bifurcates at $\theta = 2/3$ but the two branches of the graph intersect twice between 2/3 and 0.86: once at $\theta \approx 0.69$ and once at $\theta \approx 0.78$. At these values of $\theta$, the limit lands exactly on the

## Modified Primal–Dual Affine Scaling Algorithm

**Parameters**
  $\varepsilon$ is the accuracy parameter;
  $\theta$ is the scaled step size;
**Input**
  $(x^0, s^0)$: the initial pair of interior feasible solutions;
**begin**
  $x := x^0$; $s := s^0$;
  **while** $x^T s > \varepsilon$ **do**
    $w = xs$;
    $\alpha_{max} = \frac{\|w\|}{\max w}$;
    $\alpha = \theta \alpha_{max}$;
    $x := x + \alpha \Delta x$;
    $y := y + \alpha \Delta y$;
    $s := s + \alpha \Delta s$;
  **end**
**end.**

**Fig. 6.** Primal–dual affine scaling algorithm with modified step size $\alpha$. In our computations we put $\varepsilon = 10^{-10}$ and we plot results as soon as the duality gap reaches values $\leq 0.001$

unstable fixed point $s$ that we found in Lemma 6. We already noticed that this point is weakly repelling, which is why the second coordinate has not yet fully converged to its $\omega$-limit yet, even when the duality gap is $10^{-10}$.

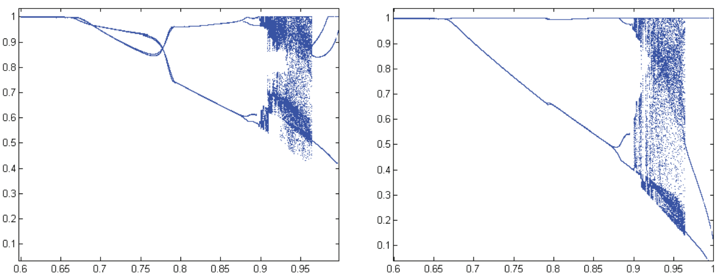

**Fig. 7.** Feigenbaum diagrams for the Castillo-Barnes LO problem. Horizontal coordinate represents $\theta$. Vertical axis contains the $\omega$-limit of a coordinate of the scaled vector $w$. Second coordinate on the left. Fourth coordinate on the right.

The dual problem is degenerate

$$\begin{gathered} \max 0 \\ \text{under the constraints} \\ y_1 - y_2 \leq 10 \\ 2y_1 + 2y_2 \leq 10 \\ -3y_1 - y_2 \leq 5 \\ -2y_1 - y_2 \leq 1 \\ -y_1 - y_2 \leq -1 \end{gathered} \tag{15}$$

All feasible points solve the dual problem. If $\theta \leq 2/3$ then the process $y^k$ converges to $(3.0513, 0.5522)$ but if $\theta$ increases beyond $2/3$ then the process no longer converges to a single point. However, $y^k$ remains within the feasible set even for large values of $\theta$. Figure 8 contains the limit set that we computed for $\theta = 0.94$. It has the contours of a Hénon-like strange attractor, which is an object that is often encountered in chaotic dynamical systems, see [3]. The image of the attractor is slightly blurred since the orbit has not fully converged yet.

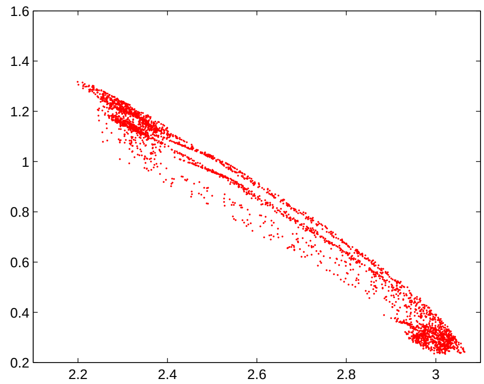


**Fig. 8.** The omega-limit set of the vector $y$ in the dual problem for $\theta = 0.94$ forms a strange attractor in the feasible set.

It seems that the process $w^k$ that we have considered in this paper represents the iterations of primal-dual AFS rather well. We have tested other LO problems as well and we find similar Feigenbaum diagrams for the vector $w$, regardless whether the dual problem is degenerate or not. The algorithm converges to an optimal solution for relatively high values of $\theta$, so for a step-size that is close to $\alpha_{max}$. This may indicate that a step-size that is larger than $1/(15\sqrt{n})$ is possible, if $\alpha$ is not taken to be constant but is allowed to vary with $xs$. To see if this is true, it is worthwhile to try and repeat the stability analysis that we carried out for $w^k$ in this paper, without the restriction that the Dikin step is exact. Such an analysis will require a substantial effort.

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