COMPLEX MAPS WITHOUT INVARIANT DENSITIES

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Abstract. We consider complex polynomials $f(z) = z^\ell + c_1$ for $\ell \in 2\mathbb{N}$ and $c_1 \in \mathbb{R}$, and find some combinatorial types and values of ℓ such that there is no invariant probability measure equivalent to conformal measure on the Julia set. This holds for particular Fibonacci-like and Feigenbaum combinatorial types when ℓ sufficiently large and also for a class of 'long–branched' maps of any critical order.

1. Introduction

The question of finding invariant probability measures that are absolutely continuous with respect to a natural reference measure is central in many areas of dynamical systems. Such invariant probabilities are called *acip*s. In dynamical systems on manifolds, the reference measure is usually taken to be Lebesgue (or some Riemannian volume). In this paper we consider unicritical polynomials $f(z) = z^\ell + c$ on the complex plane with critical order $\ell \in 2\mathbb{N}$ and parameter $c \in \mathbb{R}$, and in this setting, the natural reference measure is δ-conformal measure μ_δ, supported on the Julia set $\mathcal{J} = \mathcal{J}(f)$. This probability measure is defined by the relation

$$\mu_\delta(A) = \int_A |Df(z)|^\delta d\mu_\delta(z),$$

whenever $f : A \to f(A)$ is one-to-one. There is a δ–conformal measure for at least one $\delta \in [0, 2]$, see Sullivan [S]; in fact, a δ_*-conformal measure exists for $\delta_* := \inf\{\delta > 0 : \text{a conformal measure } \mu_\delta \text{ exists}\}$.

Given $\delta > 0$ and $z \in \mathbb{C} \setminus \cup_{n\geqslant 1} f^n(0)$, let the *Poincaré series* be defined as

$$\Xi_\delta(z) := \sum_{n=0}^{\infty} \sum_{f^n(y)=z} \frac{1}{|Df^n(y)|^\delta}.$$

As explained in [GS], a bounded distortion argument implies that this number is independent of $z \in \mathbb{C} \setminus \omega(0)$. We let $\delta_{Poin} := \inf\{\delta > 0 : \Xi_\delta(z) < \infty \text{ for } z \in \mathbb{C} \setminus \omega(0)\}$. For more information on the Poincaré series see, for example, [AL1, GS].

We denote the Hausdorff dimension of a set X by $HD(X)$. The Hausdorff dimension of a measure μ on X is $HD(\mu) := \inf\{HD(A) : A \subset X, \mu(A) = 1\}$. We define the *dynamical dimension* $DD(X) := \sup\{HD(\mu)\}$ where the supremum is taken over all ergodic invariant measures of positive entropy. By [PU], $\delta_* = DD(\mathcal{J})$, so clearly $HD(\mathcal{J}) \geqslant \delta_*$. Also, by [Bi], if the Lebesgue measure of the Julia set is zero then $\delta_{Poin} \geqslant HD(\mathcal{J})$. Whenever $\delta_* = \delta_{Poin}$, μ_{δ_*} is called a *geometric measure*.

We consider Feigenbaum and Fibonacci maps in this paper, which are interesting examples in the measure theoretical context, especially when the critical order ℓ is large. In this paper a Feigenbaum map is an infinitely renormalisable map with periodic combinatorics, i.e., with periodic renormalisation operator, see [dMvS]. In the Fibonacci case, the questions whether $\delta_* < 2$, whether Area$(\mathcal{J}(f)) = 0$ and the conservativity/dissipativity of δ–conformal measure remain unsolved. Avila and Lyubich [AL1] show that in the Feigenbaum case, μ_{δ_*} is dissipative

2000 *Mathematics Subject Classification.* Primary: 37F35; Secondary 37F10, 37C40, 37D35, 37F25.
This research was supported by EPSRC grant GR/S91147/01.

and that $\mathrm{Area}(\mathcal{J}(f)) = 0$ implies $\delta_* < 2$ (as well as further results on a class of maps similar to Feigenbaum maps).

In the following theorem, proved in Section 4, we show that for Feigenbaum and Fibonacci-like (i.e., satisfying (1) below) polynomials admit no acip $\mu \ll \mu_\delta$ provided ℓ is sufficiently large.

Theorem 1. *Suppose that $f : z \mapsto z^\ell + c_1$ is either a complex Fibonacci-like map satisfying (4) or a Feigenbaum map. If ℓ is sufficiently large, then f cannot have an acip w.r.t. any conformal measure, unless the acip is supported on $\omega(0)$.*

Note that $HD(\omega(0)) \leqslant 1$, so if $\mu_{\delta_*}(\omega(0)) > 0$ then $HD(\mathcal{J}) = 1$. By [Z], this only occurs when $\mathcal{J}$ is a Jordan curve. So in particular, the maps in Theorem 1 have no acip w.r.t. μ_{δ_*}.

In fact, in [AL1] it is proved that for quadratic Feigenbaum maps, any conformal measure is dissipative (indeed this is claimed for unicritical Feigenbaum maps of any critical order). Lemma 9 then implies that there cannot be an acip. However, we include the Feigenbaum case in our proofs for interest. Further examples of maps having no acip are presented in the following theorem.

Theorem 2. *There exists a class of maps 'long–branched maps' each of which has no acip not supported on $\omega(0)$, for any even critical order.*

The paper is arranged as follows. We start by outlining some useful combinatorial facts for interval maps in Section 2. In Section 3, we explain the general complex setup, and what conditions we require to show there are no acips. In Section 4, we show that these conditions are satisfied by certain Fibonacci and Feigenbaum maps. In Section 5 we show that these conditions are satisfied by some 'long–branched' maps. Section 6 applies the above results to the question of (non–)existence of equilibrium states. In the appendices we first give some useful results on dissipativity/conservativity, and then we supply some technical results on the kneading map.

Acknowledgements: We would like to thank Juan Rivera–Letelier, Feliks Przytycki and the referees for useful comments.

2. Preliminaries

Let $f(z) = z^\ell + c_1$ for $\ell \in 2\mathbb{N}$ and $c_1 \in \mathbb{R}$, so the critical point is 0 and

$$f(e^{2\pi k i/\ell}\,\mathbb{R}) = \begin{cases} [c_1, \infty) & \text{if } k \text{ is even;} \\ (-\infty, c_1] & \text{if } k \text{ is odd.} \end{cases}$$

Any maximal closed interval $J \subset \mathbb{R}$ such that $f^n|_J$ is monotone is called a *branch* and if $0 \in \partial J$, then the branch is called *central*; there are two central branches one on either side of 0, denoted J_n and $\hat{J}_n$. The numbers $1 = S_0 < S_1 < S_2 < \dots$ such that if $n = S_k$ then

$$D_n := f^n(J_n) \ni 0,$$

are called the *cutting times* of f. It can be shown, see [H, B2], that if there is no periodic attractor, then there are infinitely many cutting times, and there is a function $Q : \mathbb{N} \to \mathbb{N} \cup \{0\}$, called the *kneading map*, such that

$$S_k - S_{k-1} = S_{Q(k)} \text{ for all } k \in \mathbb{N} \text{ (and hence } Q(k) < k).$$

As $D_{S_k} \ni 0$, there are points $\zeta_k < 0 < \hat{\zeta}_k$ such that $f(\zeta_k) = f(\hat{\zeta}_k)$ and $f^{S_k}(\zeta_k) = 0$. Moreover, there is no $z \in (\zeta_k, 0)$ such that $f^n(z) = 0$ for some $n \leqslant S_k$. For this reason, the points ζ_k and $\hat{\zeta}_k$ are called *closest precritical points.*

Lemma 1. *The following properties hold:*

(a) *the central branches of* f^{S_k} *are* $J_{S_k} = [\zeta_k, 0]$ *and* $\hat{J}_{S_k} = [0, \hat{\zeta}_k]$;
(b) *the point* $f^{S_k}(0)$ *lies in* $(\zeta_{Q(k+1)-1}, \zeta_{Q(k+1)}) \cup (\hat{\zeta}_{Q(k+1)}, \hat{\zeta}_{Q(k+1)-1})$.

Proof. See [B2]. □

Simple examples of the kneading map are $Q(k) = k-1$, when $S_k = 2^k$ and the corresponding map is a Feigenbaum map; and $Q(k) = \max\{0, k-2\}$, when S_k are the Fibonacci numbers, and f is called the Fibonacci map. More generally, we call a map *Fibonacci–like* if there exists N such that

$$k - Q(k) \leqslant N \text{ for all } k \geqslant 1. \tag{1}$$

A Feigenbaum map is Fibonacci-like according to this definition, but not all infinitely renormalisable maps are. In fact, (real) renormalisability can be expressed in terms of the kneading map as follows: There exists k_0 such that

$$Q(k_0) = k_0 - 1 \text{ and } Q(k) \geqslant k_0 - 1 \text{ for all } k \geqslant k_0.$$

In this case, f has a periodic interval of period S_{k_0}, see [B2]. The infinitely renormalisable maps that we are interested in are those for which the renormalisation operator is periodic. Hence, some iterate of the renormalisation operator will fix the combinatorial structure, and this can be expressed by periodicity in the kneading map: There exists k_0 and period p such that

$$Q(k_0) = k_0 - 1, Q(k) \geqslant k_0 - 1 \text{ and } Q(k+p) = Q(k) + p \text{ for all } k \geqslant k_0. \tag{2}$$

Not every kneading map Q corresponds to a real unimodal polynomial. Hofbauer, in *e.g.*, [H], states the following sufficient condition for a kneading map to be realised by a polynomial.

$$\{Q(k+j)\}_{j\geqslant 1} \succeq \{Q(Q^2(k)+j)\}_{j \geqslant 1} \text{ for all } k \geqslant 1, \tag{3}$$

where $\succeq$ indicates lexicographical order and $Q^2(n) = Q(Q(n))$, and so on. A slightly stronger version of this condition is as follows: there is k_0 such that

$$Q(k+1) > Q(Q^2(k)+1) \quad \text{for all } k > k_0. \tag{4}$$

We will use this condition, which excludes the existence of "almost saddle node" bifurcations, in Section 4 to simplify some of our results.

3. Conditions which preclude acips

3.1. **The first return map to a wake.** We suppose throughout that $f : z \to z^\ell + c_1$ for $c_1 \in \mathbb{R}$ has connected Julia set. We suppose further that 0 is recurrent and $0 \in \mathcal{J}$ (if $0 \notin \mathcal{J}$, then $f|_\mathcal{J}$ is hyperbolic and an acip exists). This implies that the filled Julia set $K(f) := \{z \in \mathbb{C} : f^n(z) \not\to \infty\}$ has no interior. We let $r_\theta := \{re^{i\theta} : r \in [0,\infty)\}$. Then for $\theta \in T_\ell := \left\{\frac{\pi}{\ell}, \frac{3\pi}{\ell}, \dots, \frac{(2\ell-1)\pi}{\ell}\right\}$ we have a connected segment $r_\theta \cap \mathcal{J}$ mapping onto $\mathbb{R}^- \cap \mathcal{J}$. For example, in the quadratic case, $i\mathbb{R}^+ \cap \mathcal{J}$ and $i\mathbb{R}^- \cap \mathcal{J}$ map onto $\mathbb{R}^- \cap \mathcal{J}$.

Let the Green function $G : \mathcal{F} \to \mathbb{R}$ be defined by $G(z) = \lim_{n\to\infty} \frac{\log|f^n(z)|}{\ell^n}$. The *equipotentials* (i.e. level sets of the Green function) form a foliation of the *Fatou set* $\mathcal{F}$ consisting of nested Jordan curves, see [M]. The orthogonal foliation is the foliation of *external rays*. Let $B(\mathcal{J})$ be the bounded connected component of $\mathbb{C} \setminus G^{-1}(\{1\})$. Each external ray R has its *external angle* $\alpha = \lim_{r\to\infty} \arg(R \cap \{|z| = r\})$, so we write $R = R_\alpha$. Given external rays $R_\alpha, R_{\alpha'}$ landing at the same point z, the corresponding *wake* is the set of points in $B(\mathcal{J})$ lying between R_α and $R_{\alpha'}$.

Let $0 \leqslant \alpha < \frac{2\pi}{\ell}$ be such that $R_{\ell\alpha}, R_{\ell\tilde{\alpha}}$ are external rays landing at c_1 where $\tilde{\alpha} = -\alpha$. We pull these rays back by f to get the external rays $R_{\alpha_0}, R_{\tilde{\alpha}_0}, R_{\alpha_1}, R_{\tilde{\alpha}_1}, \dots, R_{\alpha_{\ell-1}}, R_{\tilde{\alpha}_{\ell-1}}$ where $\alpha_k = \alpha + \frac{2\pi k}{\ell}$ and $\tilde{\alpha}_k = -\alpha + \frac{2\pi k}{\ell}$. We denote the wake corresponding to $\tilde{\alpha}_k$ and α_k by W^k, see Figure 1. There are ℓ such wakes. Each $\theta \in T_\ell$ is of the form $\theta = \frac{(2k+1)\pi}{\ell}$, so $r_\theta \cap \mathcal{J} \subset W^k$ and we write $W(\theta) = W^k$.

Fix $\theta \in T_\ell$ for the moment and consider the first return map to a wake $W = W(\theta)$. We may index a domain of this map by the point w, called *root point*, that maps to 0. We denote the set of root points by R.

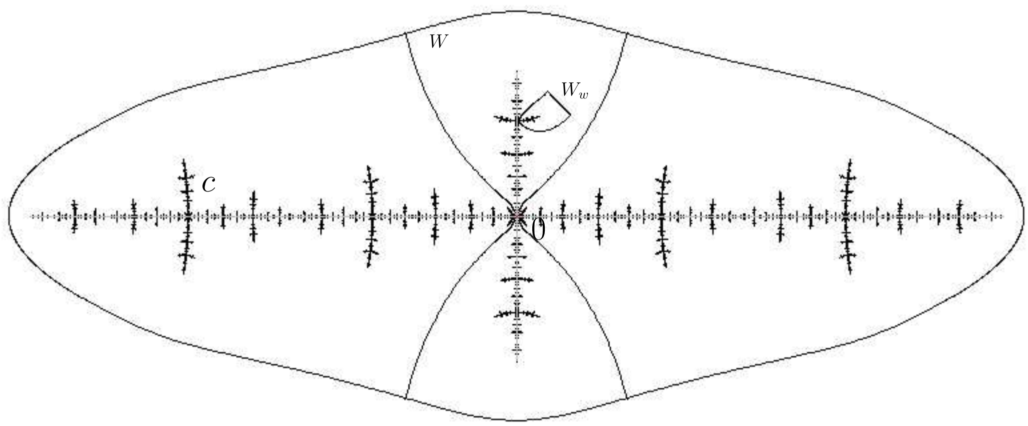

FIGURE 1. The neighbourhood $B(\mathcal{J})$ of the Julia set with the wake $W = W^1 = W(\frac{\pi}{2})$. Here the map f is quadratic, and the first return map to W maps the subwake W_w onto W. The points 0 and c are shown as well.

The following proposition is a tool we use throughout the rest of this paper.

Proposition 1. *Let $W = W(\theta)$ be a wake where $\theta \in T_\ell$ and suppose $\tau(z) := \min\{k \geqslant 1 : f^k(z) \in W(\theta)\}$, wherever this is defined. If*

$$\sum_{w \in R} \tau(w)\mu_\delta(W_w) = \infty,$$

then there is no acip with respect to μ_δ, unless the acip is supported on $\omega(0)$.

We will show that in fact the above sum is infinite when it is taken over the set of domains whose root point w lies on r_θ.

To prove this proposition, we first need the following result, the proof of which is postponed to Appendix A.

Lemma 2. *Suppose that μ is an acip w.r.t. μ_δ and $\mu(\omega(0)) < 1$. Then $\mu(W(\theta)) > 0$ for all $\theta \in T_\ell$.*

Lemma 3 (Koebe Distortion Lemma for complex maps)**.** *Assume that $\mathbb{D}$ is the open unit disk in $\mathbb{C}$ and $g\colon \mathbb{D} \to \mathbb{C}$ is univalent, then for each $z \in \mathbb{D}$,*

$$\frac{1-|z|}{(1+|z|)^3} \leqslant \frac{|Dg(z)|}{|Dg(0)|} \leqslant \frac{1+|z|}{(1-|z|)^3}.$$

and

$$\frac{1-|z|}{1+|z|} \leqslant \frac{|z| \cdot |Dg(z)|}{|g(z)-g(0)|} \leqslant \frac{1+|z|}{1-|z|}.$$

Thus, supposing that $z, w \in r\mathbb{D}$ for some $0 < r < 1$, we have

$$\frac{|Dg(z)|}{|Dg(w)|} \leqslant \frac{(1+r)^4}{(1-r)^4}. \tag{5}$$

Proof. This lemma can be found in [Po]: the first statement is formula (15) of Theorem 1.3, page 9, and the second follows by substituting the Koebe transform $h(z) = (g(\frac{z+w}{1+\bar{w}z}) - g(w))/(1 - |w|^2)Dg(w)$ in formula (14) and then taking $z = -w$, cf. Exercise 3 on page 13 in [Po]. □

Proof of Proposition 1. Suppose that we do have an acip μ and $\mu(\omega(0)) < 1$.

Claim. *There is a lower bound on the density* $b := \inf \frac{d\mu}{d\mu_\delta} > 0$.

Proof. First note that by Lemma 2 we have $\mu(W) > 0$. We consider a domain $Q = W_w$ with $\mu(Q) > 0$. There will be some Koebe space for the map $f^n : Q \to W$ where $\tau|_Q = n$. So there exists K such that $\left|\frac{Df^n(z)}{Df^n(v)}\right| < K$ for all $z, v \in Q$. Fix this $v = v(Q)$ once and for all. We may assume that the first return map $F_Q : \bigcup_i Q_i \to Q$ also has distortion bounded by K. Let $\mu_{\delta,Q} := \frac{\mu_\delta|_Q}{\mu_\delta(Q)}$ and $\mu_Q := \frac{\mu|_Q}{\mu(Q)}$ be the normalised restrictions to Q of μ_δ and μ respectively. Note that since μ is an invariant probability measure, Poincaré recurrence implies that the set of points which return infinitely often to Q has positive μ–measure, and thus positive μ_δ measure. Since the set of points which enter Q infinitely often is an invariant set with positive μ_δ measure, the ergodicity of μ_δ given by [Pr] means that this set has full measure.

We can now apply the Folklore Theorem to $(F_Q, \mu_{\delta,Q})$, see for example [dMvS], and obtain an ergodic invariant measure m_Q. Then there exists some $b' > 0$ such that $\frac{dm_Q}{d\mu_{\delta,Q}} > b'$. But since m_Q is ergodic and invariant and $\mu \ll \mu_\delta$, we have $m_Q = \mu_Q$. Hence $\frac{d\mu_Q}{d\mu_{\delta,Q}} > b'$. Now consider a domain $Q' = W_{w'}$ of f^τ. For any $U' \subset Q'$ there exists a unique $U \subset Q$ such that $f^n(U) = U'$. By the conformality of μ_δ and the invariance of μ,

$$\mu_\delta(U') = \int_U |Df^n|^\delta d\mu_\delta \leqslant K|Df^n(v)|^\delta \mu_\delta(U) \leqslant \frac{K}{b'}|Df^n(v)|^\delta \mu(U)\frac{\mu_\delta(Q)}{\mu(Q)}$$
$$\leqslant \frac{K}{b'}|Df^n(v)|^\delta \mu(U')\frac{\mu_\delta(Q)}{\mu(Q)}.$$

Taking $b := \frac{b'}{K|Df^n(v)|^\delta}\frac{\mu(Q)}{\mu_\delta(Q)}$, the proof of the claim is finished. □

Since μ is invariant, Kac's Lemma implies

$$1 \geqslant \sum \tau(w)\mu(W_w) \geqslant b \sum \tau(w)\mu_\delta(W_w),$$

contradicting the assumptions of our lemma. So there is no acip. □

3.2. **Wake boundaries.** To apply Proposition 1 we will need to estimate $\mu_\delta(W_w)$ in terms of $|Df^{\tau(w)}(w)|$. We do this by establishing distortion bounds for the first return map to a suitably chosen wake W for some domains W_w and then applying Lemma 3.

In order to find the necessary Koebe space, we make use of the fact that any forward iterate of the critical point for our maps $f : z \mapsto z^\ell + c_1$ must lie in the real line and that the wake W is separated from certain iterates of the critical point as explained below. In Section 4 we need W to be separated from $\mathbb{R} \setminus \{0\}$, and in Section 5 for a neighbourhood U of 0 we need $W \setminus U$ to be a bounded distance away from $\mathbb{R}$.

Lemma 4. *Suppose that* $f(z) = z^\ell + c_1$ *for* $c_1 \in \mathbb{R}$ *and* $\ell \geqslant 6$. *Then*

$$W^k \subset \left\{re^{i\theta} : r \geqslant 0,\ \theta \in \left(\frac{2\pi}{\ell}, \frac{(\ell-1)\pi}{\ell}\right)\right\}$$

whenever $2 \leqslant k \leqslant \frac{\ell-1}{2}$.

Proof. Given W^k, let $\mathcal{W}^k$ denote the wake constructed as W^k was, but not bounded by an equipotential. So $W^k \subset \mathcal{W}^k$ and on the Riemann sphere $\mathcal{W}^k \cap \mathcal{W}^j = \{\infty\}$ for $j \neq k$. We prove the lemma for $\mathcal{W}^k$ which implies that it holds for W^k too.

We fix $r > 0$ and let $C_r := \{re^{i\theta} : \theta \in [0, 2\pi)\}$. Let $\alpha_k := \sup\{\theta \in [0, 2\pi) : re^{i\theta} \in \partial\mathcal{W}^k\}$ and $\beta_k := \inf\{\theta \in [0, 2\pi) : re^{i\theta} \in \partial\mathcal{W}^k\}$. Notice that $\beta_k = \beta_{k-1} + \frac{2\pi}{\ell}$ and $\alpha_k = \alpha_{k-1} + \frac{2\pi}{\ell}$.

Claim. *There is no $r > 0$ such that $re^{i\xi_{k+1}} \in \mathcal{W}^{k+1}$ with $\xi_{k+1} < \beta_k$.*

Suppose that the claim is false. Then there exists $r > 0$ such that $re^{i\xi_1} \in \mathcal{W}^1 \cap C_r$ with $\beta_1 \leqslant \xi_1 < \beta_0$. Since $\beta_0 < 0 < \beta_1$ (no ray may cross $\mathbb{R}$), we have a contradiction, proving the claim.

Since $\alpha_k, \beta_k > \frac{2\pi}{\ell} + \beta_1$ for $k \geqslant 2$, the claim implies

$$\mathcal{W}^k \subset \left\{re^{i\xi} : r \geqslant 0,\ \xi \in \left(\frac{2\pi}{\ell}, \frac{(\ell-1)\pi}{\ell}\right)\right\}$$

whenever $\alpha_k \leqslant \frac{(\ell-1)\pi}{\ell}$. Since $\alpha_k \leqslant \frac{2\pi k}{\ell}$, the lemma is proved. □

Lemma 5. *Suppose that U is a neighbourhood of 0. Then for any ray γ landing at 0 there exists $\varepsilon = \varepsilon(\gamma)$ such that $d(\gamma \setminus U, \mathcal{J} \cap \mathbb{R}) > \varepsilon$, where d denotes distance in the Euclidean metric.*

Proof. First notice that $\mathcal{J}$ is symmetric with respect to both $\mathbb{R}$ and $i\mathbb{R}$. So if there is an external ray landing at 0 in one quadrant of $\mathbb{C}$ then there must be one in all other quadrants of $\mathbb{C}$. Therefore, close to $\mathcal{J}$, for example inside the equipotential $\{G(z) = \delta\}$ for small enough $\delta > 0$, a ray landing at 0 must start in a given quadrant of $\{G(z) = \delta\}$ and remain in that quadrant until it lands at 0. Of course, rays cannot intersect $\mathcal{J}$.

Now suppose that the lemma is false. Then there is some sequence of points $y_n \in (\mathcal{J} \cap \mathbb{R}) \setminus U$ such that $d(y_n, \gamma) \to 0$ as $n \to \infty$. But this implies that for rays to land at points in $\mathcal{J} \cap \mathbb{R} \cap U$ they must intersect $\mathcal{J} \cap \mathbb{R}$ which is not possible. Since $\mathcal{J}$ is locally connected, see [LeSt], every point of $\mathcal{J}$ has a ray landing by Carathéodory's Theorem. So we have a contradiction. □

4. Fibonacci-like and Feigenbaum maps

This section is devoted to the proof of Theorem 1.

Proposition 2. *Assume that Q is an admissible kneading map satisfying either* (1) *and* (4) *(Fibonacci-like), or* (2) *(Feigenbaum). Then for each $\ell \in 2\mathbb{N}$, there is a unique c_1 such that $f(z) = z^\ell + c_1$ has kneading map Q. If $\ell > 2$, there exists $\lambda = \lambda(\ell) < 1$ such that*

$$\frac{|f^{S_{k+1}}(0) - 0|}{|f^{S_k}(0) - 0|} \to \lambda, \text{ as } k \to \infty. \text{ Moreover, } \lambda \to 1 \text{ as } \ell \to \infty.$$

Proof. The existence of the parameters $c_1 = c_1(\ell)$ follows from the admissibility of the kneading map Q. Uniqueness of $c_1(\ell)$ follows from rigidity results due to [Ly, GS] for $\ell = 2$ and [KSS] for $\ell > 2$. Finally the scaling properties were proved in [BKNS] for the Fibonacci map, generalised to the Fibonacci–like case in [B3, Proposition 10.6]. The Feigenbaum case follows from [LeSw]. □

Let S_n denote the n–th cutting time (for example, the n–th Fibonacci number for the Fibonacci map). We will need the following technical lemma.

Lemma 6. *Suppose that the critical point of f has order ℓ. Assume that the kneading map Q of f either satisfies* (1) *and* (4) *(Fibonacci-like) or* (2) *(Feigenbaum). Then there exists $B = B(\ell) > 0$ such that given $\theta \in T_\ell$ such that $W(\theta) \subset \left\{re^{i\xi} : r \geqslant 0,\ \xi \in \left(\frac{2\pi}{\ell}, \frac{(\ell-1)\pi}{\ell}\right)\right\}$, for each n there is $w_n \in r_\theta$ such that $f^{S_n}(w_n) = 0$ and $|Df^{S_n}(w_n)| < B$.*

Proof. Let (a,b) denote the open interval with endpoints a and b, even if $b<a$.

Recall that $\zeta_n < 0 < \hat{\zeta}_n$ are such that $f^{S_n}(\zeta_n) = f^{S_n}(\hat{\zeta}_n) = c$, and $f^{S_n} : (\zeta_n, 0) \to (0, f^{S_n}(0))$, $f^{S_n} : (0, \hat{\zeta}_n) \to (0, f^{S_n}(0))$ are monotone. Let $w_n \in r_\theta$ be the point which has $f(w_n) = f(\zeta_n)$.

We give bounds for derivatives $|Df^{S_n}(c_1)|$, $|Df^{S_n}(\zeta_n)|$ and $|Df^{S_n-1}(c_1)|$. By Lemma 12 in Appendix B there is some interval V containing $c_1, f(w_n), f(\zeta_n)$ such that the map $f^{S_n-1} : V \to (f^{S_{Q(n)}}(0), f^{S_{Q^2(n)}}(0))$ is monotone, as in Figure 2.

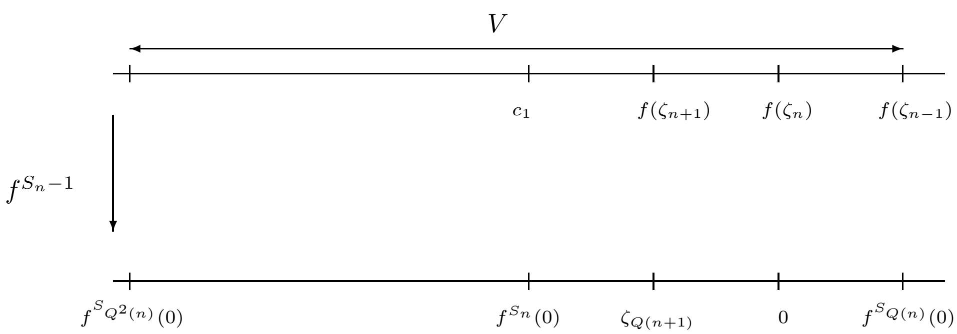

FIGURE 2. Koebe space for f^{S_n-1} to show that $|Df^{S_n-1}(f(\zeta_n))| \approx |Df^{S_n-1}(c_1)|$.

Moreover, $(0, f^{S_n}(0))$ is well inside $(f^{S_{Q^2(n)}}(0), f^{S_{Q(n)}}(0))$; indeed by Proposition 2 and assumption (4) it is a $(1-\lambda)$–scaled neighbourhood of $(0, f^{S_n}(0))$ (note that λ depends on ℓ). Thus we can use the Koebe lemma to get

$$|Df^{S_n-1}(c_1)| \approx \frac{|f^{S_n}(0) - 0|}{|f(\zeta_n) - c_1|} \approx \frac{|f^{S_n}(0) - 0|}{|\zeta_n - 0|^\ell}.$$

Hence

$$|Df^{S_n}(c_1)| \approx \frac{|f^{S_n}(0) - 0|}{|\zeta_n - 0|^\ell} |f^{S_n}(0) - 0|^{\ell-1} = \left(\frac{|f^{S_n}(0) - 0|}{|\zeta_n - 0|} \right)^\ell.$$

Since, by Lemma 1(b), $f^{S_n}(0) \in (\zeta_{Q(n+1)-1}, \zeta_{Q(n+1)}) \subset (\zeta_{Q(n+1)-1}, \zeta_n)$, if Q satisfies (1) then the right hand side above is bounded above by $\left(\frac{|\zeta_{n-N-1}-0|}{|\zeta_n - 0|}\right)^\ell = O(1)$. Similarly if Q satisfies (2) then letting $M := Q(k_0) - k_0$, the right hand side above is bounded above by $\left(\frac{|\zeta_{n-M-1}-0|}{|\zeta_n-0|}\right)^\ell = O(1)$ for all $n \geqslant k_0$.

So in either case, for any large n we have

$$\begin{aligned} |Df^{S_n}(\zeta_n)| &= |Df^{S_n-1}(f(\zeta_n)) Df(\zeta_n)| \approx |Df^{S_n-1}(c_1) Df(\zeta_n)| \\ &\approx \frac{|f^{S_n}(0) - 0|}{|\zeta_n - 0|^\ell} |\zeta_n - 0|^{\ell-1} = O(1) \end{aligned}$$

as required. □

Proof of Theorem 1. Let *Pre* be the set of points constructed in Lemma 6. By Lemma 4 for $\ell \geqslant 6$ we can choose θ so that the wake $W = W(\theta)$ described in Section 3.2 has $r_\theta \cap \mathcal{J} \subset W$ and $W \subset \left\{ re^{i\xi} : r \geqslant 0,\ \xi \in \left(\frac{2\pi}{\ell}, \frac{(\ell-1)\pi}{\ell} \right) \right\}$. We now wish to estimate $\mu_\delta(W_{w_n})$ using the derivative $Df^{S_n}(w_n)$ and distortion properties of $f^{S_n} : W_{w_n} \to W$.

As in Figure 2, there is some interval $V \ni c_1, f(w_n)$ such that $f^{S_n-1} : V \to (f^{S_{Q^2(n)}}(0), f^{S_{Q(n)}}(0))$ is monotone. Since $|f^{S_n}(0) - 0| \geqslant C\lambda^n$ we have some Koebe space for $f^{S_n} : W_{w_n} \to W$, which we use below.

We can trace out Koebe space for $f^{S_n} : W_{w_n} \to W$ using an enlarged wake W' whose boundary is parallel to this boundary curve of W. By Proposition 2 and since we chose $W \subset \left\{re^{i\xi} : r \geqslant 0,\ \xi \in \left(\frac{2\pi}{\ell}, \frac{(\ell-1)\pi}{\ell}\right)\right\}$, the Koebe space is greater than $C\lambda^n$ where $C > 0$ depends only on ℓ. According to Lemma 3 applied to $f^{-S_n} : W' \to W'_{w_n}$, we have $\mu_\delta(W_{w_n}) \geqslant K|Df^{S_n}(w_n)|^{-\delta}\lambda^{4n}$. To compare with Proposition 1 we sum over the points $w_n \in Pre$ obtained in Lemma 6 to get

$$\sum_{w_n \in Pre} \tau(w)\mu_\delta(W_{w_n}) \geqslant K \sum_{w_n \in Pre} S_n|Df^{S_n}(w_n)|^{-\delta}\lambda^{4n} > KB^{-\delta} \sum_{w_n \in Pre} S_n\lambda^{4n}. \tag{6}$$

For Fibonacci–like maps, S_n increases exponentially, say $S_n \asymp \sigma^n$ for some $\sigma > 1$ depending only on the combinatorial type of f. For example, if f is the Fibonacci map, $S_n \asymp \gamma^n$ where $\gamma = (1 + \sqrt{5})/2$ is the golden ratio. If $\sigma\lambda^4 > 1$, then the proof is completed by Proposition 1. Recalling, from Proposition 2, that $\lambda \to 1$ as $\ell \to \infty$, we can increase the order of the critical point to reach this conclusion. For Feigenbaum maps with periodic combinatorics period p we note that $S_n \asymp p^n$, so we may apply the same argument as for the Fibonacci–like case. □

5. Long-branched maps

A real map f is called *long-branched* if there is $\kappa > 0$ such that for all n and all maximal intervals $J \subset \mathbb{R}$ such that $f^n|_J$ is monotone, $|f^n(J)| > \kappa$. In [B1], these maps were studied in connection to cascade of saddle-node bifurcations, and it was shown that under appropriate combinatorial conditions, no invariant probability measure can be absolutely continuous with respect to one-dimensional Lebesgue measure. In this section, we prove a similar non-existence result for probability measures that are absolutely continuous with respect to the δ-conformal measure on the Julia set. As before, let $(S_k)_{k\geqslant 0}$ be the cutting times, and let ζ_k and $\hat{\zeta}_k$ stand for the closest precritical points.

Theorem 3. *If $f(z) = z^\ell + c$, $c \in \mathbb{R}$, $\ell \in 2\mathbb{N}$, is long-branched and*

$$\sum_k S_k|\zeta_k - \zeta_{k+1}|^{-\delta} = \infty, \tag{7}$$

then f cannot have an acip with respect to any δ–conformal measure, unless the acip is supported on $\omega(0)$.

Remark 1. *If f is long–branched, then it is straightforward to show that the kneading map Q is bounded, say $Q(k) \leqslant B$, (in fact, this is equivalent to long–branchedness) and that $k - 1 \leqslant S_k \leqslant S_B k$ for all $k \geqslant 0$, see* [B1].

Lemma 7. *If f is long–branched, then there exists $\kappa' > 0$ such that $\kappa' \leqslant |f^{S_k}(0) - \zeta_{Q(k+1)}| \leqslant |f^{S_k}(0) - 0|$ for all $k \geqslant 0$.*

Proof. Suppose by contradiction that there is a sequence $\{k_i\}_i$ such that $|f^{S_{k_i}}(0) - \zeta_{Q(k_i+1)}| \to 0$. Since $Q(k_i + 1) \leqslant B$, it follows that

$$|f^{1+S_{Q(k_i+1)}} \circ f^{S_{k_i}}(0) - f^{1+S_{Q(k_i+1)}}(\zeta_{Q(k_i+1)})| = |f^{1+S_{k_i+1}}(0) - f(0)| \to 0$$

as well. But this contradicts that f is long-branched. □

Proof of Theorem 3. Fix some odd $m < \ell$. Let $r > 1$ be minimal such that $f(0) < f^r(0) < 0$. Then $f^r : e^{\frac{\pi m i}{\ell}}\mathbb{R} \to [f^r(0), \infty)$ is an onto 2-to-1 map with a single branch point at 0. Hence for each $\zeta_k \in (f^r(0), 0)$, there exists $w_k \in e^{\frac{\pi m i}{\ell}}\mathbb{R}$ such that $f^r(w_k) = \zeta_k$, and the corresponding domains W_{w_k} have $f^r(W_{w_k}) = W_{\zeta_k}$, see Section 3.2.

The map $f^{S_k} : (\zeta_{k-1}, 0) \to (f^{S_{Q(k)}}(0), f^{S_k}(0))$ is monotone onto, and by Lemma 7, its image $(f^{S_{Q(k)}}(0), f^{S_k}(0))$ contains a κ'-neighbourhood of $f^{S_k}([\zeta_k, \zeta_{k+1}]) = [0, \zeta_{Q(k+1)}]$. As f has negative Schwarzian derivative, the distortion of $f^{S_k}|_{[\zeta_k,\zeta_{k+1}]}$ is bounded, say by $K = K(\kappa')$, independently of k. We find

$$|Df^{S_k}(\zeta_k)| \leqslant K \ \frac{|\zeta_{Q(k+1)} - 0|}{|\zeta_k - \zeta_{k+1}|}.$$

Since $|w_k - 0| \approx |\zeta_k - f^r(0)| > 0$ for large k, we get that $Df^r(w_k)$ is bounded and bounded away from 0, uniformly in k. Therefore, by Lemma 7,

$$|Df^{r+S_k}(w_k)|^{-\delta} \approx |\zeta_k - \zeta_{k+1}|^{\delta}.$$

Recall that by Lemma 10(a) f^{S_k} maps $(\zeta_{k-1}, 0)$ diffeomorphically onto $(f^{S_k}(0), f^{S_{Q(k)}}(0))$. Hence, by Lemma 5, there exists a neighbourhood W' of W, with $\partial W'$ intersecting $\mathbb{R}$ at $f^{S_k}(0)$ and $f^{S_{Q(k)}}(0)$, and neighbourhoods $W'_{w_k} \supset W_{w_k}$ such that $f^{r+S_k} : W'_{w_k} \to W'$ is univalent. By Lemma 7, the Koebe space for this map is of order κ', whence

$$\mu_\delta(W_{w_k^\pm}) \approx |\zeta_k - \zeta_{k+1}|^{\delta}.$$

Now since

$$\sum_k \tau(w_k)\ \mu(W_{w_k}) \approx \sum_k (r + S_k)|\zeta_k - \zeta_{k+1}|^{\delta} = \infty,$$

the theorem follows by Proposition 1. □

To show that there are indeed maps satisfying the conditions of Theorem 3, we are going to specify combinatorial conditions that imply condition (7). We suppose that $f^N : J \to f^N(J)$ is the central branch and $(f^{kN}(0))_{k=0}^d$ is a monotone sequence of points in J. When d is large (so $f^N : J \to f^N(J)$ is "very close to the diagonal"), we call this an *almost tangency*. The largest $d \in \mathbb{N}$ such that $f^{dN}(0) \in J$ is called the *depth* of the almost-tangency. We speak of a *cascade of almost saddle-node bifurcations* if there is a sequence $(N_j)_{j\in\mathbb{N}}$ of iterates such that the graphs of the central branches of f^{N_j} are close to tangency for all j. Let $(d_j)_{j\in\mathbb{N}}$ be the corresponding depths. Note that given N_j, one can adjust the parameter c such that d_j can be arbitrarily large, see [B1].

Proposition 3. *If f is long-branched, and has an almost saddle node cascade satisfying*

$$\sum_j \frac{d_j}{N_j L^{2N_j/(\ell-1)}} = \infty,$$

where $L := \sup\{|Df(x)| \ : \ x \in [f(0), f^2(0)]\} \leqslant 2\ell$, then condition (7) holds for $\delta \leqslant 2$.

Proof. Since the central branch of f^{N_j} near 0 consists of $x \mapsto x^\ell$ composed with a diffeomorphism of bounded distortion uniformly in i, we can write

$$f^{N_j}(x) \approx g(x) := \alpha x^\ell + \beta,$$

for x close to 0.

Without loss of generality we can assume that f^{N_j} has a local maximum at 0. Let $[z, 0]$ be the domain of this central branch, so $f^n(z) = 0$ for some $n < N_j$. Let $a \in \mathbb{N}$ be such that $f^{N_j}(0) \in (\zeta_{a-1}, \zeta_a)$ and $f^{N_j}(\zeta_a) = \zeta_b$ is a closest precritical point too, so $N_j = S_a - S_b$, and $f^{2N_j}(0) \in (\zeta_{b-1}, \zeta_b)$. Repeating this iteration, we have $f^{d_jN_j}(\zeta_a) = \zeta_k$ and $f^{N_j}(\zeta_k)$ is still a closest precritical point, but it lies outside $[z, 0]$, so $S_k \leqslant 2N_j$. It follows that $d_jN_j < S_a < (d_j + 2)N_j$.

Let x_0 be such that $Dg(x_0) = 1$, i.e., $x_0 = (\alpha\ell)^{-1/(\ell-1)}$. Because g has no fixed point,

$$x_0 < g(x_0) = \alpha\,(\alpha\ell)^{-\ell/(\ell-1)} + \beta = \alpha^{-1/(\ell-1)}\ell^{-\ell/(\ell-1)} + \beta.$$

This gives

$$\beta > \alpha^{-1/(\ell-1)}\left(\ell^{-1/(\ell-1)} - \ell^{-\ell/(\ell-1)}\right) \geqslant \frac{1}{4}\alpha^{-1/(\ell-1)},$$

and using $\alpha \leqslant L^{N_j}$,

$$g^2(0) - g(0) = \alpha\beta^\ell + \beta - \beta \geqslant 4^{-\ell}\alpha^{-1/(\ell-1)} \geqslant 4^{-\ell}L^{-N_j/(\ell-1)}.$$

There are at most N_j integers t such that $\zeta_t \in (f^{N_j}(0), f^{2N_j}(0))$, so for at least one such t,

$$|\zeta_t - \zeta_{t+1}| \geqslant \frac{4^{-\ell}}{N_j - 1}L^{-N_j/(\ell-1)}.$$

For this t, $S_t \geqslant (d_j - 1)N_j$. Therefore we find

$$\sum_k S_k|\zeta_k - \zeta_{k+1}|^\delta \geqslant \sum_j (d_j - 1)N_j\left(\frac{4^{-\ell}}{N_j - 1}L^{-N_j/(\ell-1)}\right)^\delta > 17^{-\ell}\sum_j d_j L^{-2N_j/(\ell-1)}/N_j.$$

By the assumption of the proposition, this sum diverges (and hence no acip exists). □

6. Equilibrium States

We show the implications of the results in the previous section to the question of the existence of equilibrium states for the potential $-\delta \log|Df|$, i.e., whether or not there exists a measure which achieves the supremum (called *pressure*)

$$P := \sup_\mu \left(h_\mu - \delta\int_{\mathcal{J}} \log|Df|\ d\mu\right)$$

taken over set of ergodic invariant Borel probability measures. This question is answered by Proposition 4. We need the following result from [Led] to prove our proposition.

Theorem 4. *Suppose that f has a δ_*–conformal measure and μ is an invariant measure with $\lambda(\mu) > 0$. Then*

$$h_\mu = \delta_* \int \log|Df|\ d\mu \quad \text{if and only if} \quad \mu \ll \mu_{\delta_*}.$$

Proposition 4. *Suppose that $h_\mu = \delta_* \int \log|Df|\ d\mu$ for a unicritical map $z \mapsto z^\ell + c$ for $c \in \mathbb{R}$ and $\ell \in 2\mathbb{N}$. If $\lambda(\mu) > 0$ then $\mu \ll \mu_{\delta_*}$. Otherwise μ is in the convex hull of weak accumulation points of $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(0)}$.*

Proof. By Theorem A of [Prz1], for any invariant measure μ supported on $\mathcal{J}$, $\lambda(\mu) \geqslant 0$. First suppose that $\lambda(\mu) > 0$. According to for example [DU] or Theorem A2.9 of [Prz2], the pressure is zero. So we must have $h_\mu = \delta_* \int \log|Df|\ d\mu$. If $\lambda(\mu) > 0$ then Theorem 4 implies $\mu \ll \mu_{\delta_*}$.

If $\lambda(\mu) = 0$ and $\mu(\mathcal{J} \setminus \mathbb{R}) = 0$, then the proposition is completed by the result of Hofbauer and Keller [HK] for interval maps which states that μ belongs to the convex hull of the weak accumulation points of $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(0)}$. Otherwise, we can choose a set A such that $\mathrm{dist}(A, \mathbb{R}) > 0$ and $\mu(A) > 0$.

Given $\rho > 0$ and $M > 0$ we say that z *reaches large scale* at time i if there are neighbourhoods $\mathbb{C} \supset V_0 \supset V_1 \ni z$ such that $f^n : V_0 \to f^n(V_0)$ is univalent, $f^i(V_1)$ contains a round ball of radius ρ (measured in Euclidean distance) and the modulus of $\mathrm{mod}(V_0, V_1) > M$.

Since μ is ergodic and invariant, this means that μ–a.e. z visits A with positive frequency, and hence goes to large scale (with bounded distortion) with positive frequency. The paper [BT] then implies that we can 'lift' μ, which implies that μ has positive Lyapunov exponent, which is a contradiction. □

Corollary 1. *The maps considered in Theorems 1 and 2 may only have an equilibrium state in the convex hull of weak accumulation points of* $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(0)}$.

For Fibonacci–like maps exactly one equilibrium state exists and is supported on the minimal Cantor set $\omega(0)$, see [B2] and [B4] for the unique ergodicity of $f|_{\omega(0)}$.

Proof. For $\delta=\delta_*$, this follows directly from the non–existence of acips, Theorem 4 and Proposition 4. In the case $\delta>\delta_*$ we have the following argument. Suppose that μ is an an equilibrium state. First suppose that $\mu(\mathbb{R})>0$. By ergodicity $\mu(\mathbb{R})=1$. Then the argument of [B2] implies that μ is in the convex hull of weak accumulation points of $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(0)}$.

Now suppose that $\mu(\mathbb{R})=0$. We can use the argument in the proof of Proposition 4 to show that $\lambda(\mu)>0$. Next we use a result of [Led, M], see also [PU, Theorem 9.4.1], which states that $HD(\mu)=\frac{h_\mu}{\lambda(\mu)}$. Since by [DU], $DD(\mathcal{J})=\delta_*$, we have $h_\mu-\delta\int\log|Df|\,d\mu<0$. However, [BK] implies that the pressure of $-\delta\log|Df|$ is greater than or equal to zero: a contradiction. □

Corollary 2. *There exists a quadratic long–branched map with no equilibrium state in its Julia set for the potential* $z\mapsto-\delta\log|Df(z)|$ *for each* $\delta\geqslant\delta_*$.

Proof. From the proof of Corollary 1 we only need to exclude the case that the equilibrium state is supported on $\mathbb{R}$.

In [HK] an example of a real quadratic map was presented such that $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(x)}$ converges to the Dirac measure at the (repelling) fixed point $p\in[f(0),0]$ for Lebesgue–a.e. $x\in[f(0),f^2(0)]$. In [BK, Example 5.4] this example was put in the context of long–branched maps, and it was shown that it did not have an equilibrium state. The same example works in the complex case. Corollary 1 implies that the only equilibrium state is the Dirac measure at $\{p\}$, so its free energy is $-\log|f'(p)|<0$. On the other hand, there is a sequence $\{p_i\}_{i\geqslant1}$ of periodic points whose periods $\text{per}(p_i)\to\infty$ but whose multipliers $|Df^{\text{per}(p_i)}(p_i)|$ remain bounded. So the pressure is $P\geqslant0$. This shows that no equilibrium state exists. □

Appendix A

We follow [AL1] in proving some facts about dissipativity/conservativity. While the ideas presented here are very similar to those in [AL1], we do not make any assumption on the combinatorics of our maps. Notice that we restrict ourselves to unicritical polynomials for simplicity, but all the results below extend to general complex polynomials. Recall that (X,T,μ) is dissipative if there exists $A\subset X$ such that $\mu(A)>0$ and $\mu(T^{-n}A\cap A)=0$ for all $n\geqslant1$.

Lemma 8. *Suppose that f is a unicritical polynomial with $\omega(0)\neq\mathcal{J}$. Then the following are equivalent:*

(a) μ_δ *is dissipative;*
(b) $\Xi_\delta(z)$ *is convergent for all* $z\notin\omega(0)$;
(c) $f^n(z)\to\omega(0)$ *for* μ_δ*–a.e.* $z\in\mathcal{J}$.

Moreover, if μ_δ is conservative then almost all orbits are dense in $\mathcal{J}$.

Proof. Notice that $\omega(0)\neq\mathcal{J}$ implies that $\mu_\delta(\omega(0))<1$.

(a)⇒ (b): Let Y be a wandering set of positive measure. Choose $r>0$ such that $B_{2r}(z)\cap\omega(0)=\emptyset$. Let $D=B_r(z)$. Since there exists $N\geqslant0$ such that $f^N(D)\supset\mathcal{J}$, there exists $Y'\subset D$ such that $f^N(Y') \subset Y$ and $\mu_\delta(Y') > 0$. By the Koebe Lemma, and the fact that preimages of Y' are pairwise disjoint, we have

$$\Xi_\delta(z) \approx \sum_{n \geqslant 0} \mu_\delta(f^{-n}(Y')) < \infty$$

as required.

(b) ⇒ (c): Take any $z \in \mathcal{J}$ and $r > 0$ such that $B_{2r} \cap \omega(0) = \emptyset$. Letting $D = B_r(z)$, we have

$$\sum_{n \geqslant 0} \mu_\delta(f^{-n}(D)) \approx \Xi_\delta(z) < \infty.$$

By the Borel–Cantelli Lemma, for μ_δ–a.e. $y \in \mathcal{J}$, the orbit of y visits D only finitely often. Since this is true of any open set bounded away from $\omega(0)$ this implies (c).

(c) ⇒ (a): Let U be a neighbourhood of $\omega(0)$ with $0 < \mu_\delta(U) < 1$. Define $Y_n := \{z \in \mathbb{C} : f^j(z) \in U \text{ for all } j \geqslant n\}$. By (c), $\mu_\delta\left(\bigcup_{n \geqslant 0} Y_n\right) = 1$. Therefore there exists some n_0 such that $\mu_\delta(Y_{n_0}) > 0$. Since $Y_n \subset Y_{n+1}$ for all n, we must have $\mu_\delta(Y_n) > 0$ for $n \geqslant n_0$.

Now let n be such that $\mu_\delta(Y_n \cap U^c) > 0$ and let $Y := Y_n \cap U^c$. Then $f^j(Y) \cap Y = \emptyset$ for all $j \geqslant n$. Now let $k := \max\{j : \mu_\delta(f^j(Y) \cap Y) > 0\} < n$ and $Y' := f^k(Y) \cap Y$. Then $f^{n-k}(Y') \subset U$, so certainly $\mu_\delta(f^j(Y') \cap Y') = 0$ for $j \geqslant n - k$. Moreover, by definition of k, $\mu_\delta(f^j(Y') \cap Y') = 0$ for $1 \leqslant j < n - k$. Clearly Y' is a wandering set of positive measure, so μ_δ is dissipative.

We now prove the last statement of the lemma. Suppose that $\{D_n\}_n$ is the countable basis of the topology of $\mathbb{C}$. Define $E_n := \{z \in \mathbb{C} : f^j(z) \notin D_n \text{ for all } j \geqslant 0\}$. If the statement is not true then $\mu_\delta\left(\bigcup_{n, D_n \cap \mathcal{J} \neq \emptyset} E_n\right) = 1$. So there exists n such that $\mu_\delta(E_n) > 0$. Hence there is a disk D intersecting $\mathcal{J}$ and a forward invariant set X of positive measure such that $D \cap X = \emptyset$. Since $f^N(D) \supset \mathcal{J}$ for some N, there exists $Y \subset D$ such that $f^N(Y) = X$. Since X has positive measure, Y must also have positive measure. Moreover, $f^n(Y) \cap Y = \emptyset$ for $n \geqslant N$.

As above, let $k = \max\{j : \mu(f^j(Y) \cap D) > 0\} < N$ and $Y' := f^k(Y) \cap D$. By definition, $\mu_\delta(Y') > 0$ and $Y' \subset D$. Then $f^{N-k}(Y') \subset X$, so certainly $\mu_\delta(f^j(Y') \cap Y') = 0$ for $j \geqslant N - k$. Moreover, by definition of k, $\mu_\delta(f^j(Y') \cap Y') = 0$ for $1 \leqslant j < N - k$. Clearly Y' is a wandering set of positive measure, so μ_δ is dissipative. □

When discussing whether (X, T, m) is dissipative or not, it is important to distinguish between totally dissipative (i.e., there is no invariant subset $Y \subset X$ of positive measure on which (Y, T, m) is conservative), or only "partially" dissipative, when such a proper subset exists.

It is easy to see that given a dynamical system (X, T), an invariant probability measure μ must be conservative, since, as in the Poincaré recurrence Theorem, if $U \subset X$ is a wandering set of positive measure then $\mu\left(\bigcup_{n \geqslant 0} f^{-n}(U)\right) = \sum_{n \geqslant 0} \mu(f^{-n}(U)) = \sum_{n \geqslant 0} \mu(U) = \infty$: a contradiction. However, there are systems where $\mu \ll m$ and μ is an acip, but m is dissipative (although not totally dissipative). An example of this is a renormalisable interval map such that the renormalisation has an acip (supported on a proper subset of the interval), while Lebesgue measure is dissipative. The following lemma shows that this does not occur in our setting.

Lemma 9. *If μ_δ is dissipative, then there is no acip, unless the acip is supported on $\omega(0)$.*

Proof. We suppose that μ is an acip for μ_δ such that $\mu(\omega(0)) < 1$. We show that μ is dissipative if and only if μ_δ is dissipative. We use the method of Lemma 8, the slight difference being that we are concerned with μ, which is not conformal. But since an invariant probability measure cannot be dissipative, the lemma follows.

By Lemma 8, we know that μ_δ is dissipative if and only if $f^n(z) \to \omega(0)$ for μ_δ–a.e. $z \in \mathcal{J}$. This condition immediately implies that $f^n(z) \to \omega(0)$ for μ–a.e. $z \in \mathcal{J}$. We now show that $f^n(z) \to \omega(0)$ for μ–a.e. $z \in \mathcal{J}$ implies that μ is also dissipative.

Let U be a neighbourhood of $\omega(0)$ with $0 < \mu(U) < 1$. Define $Y_n := \{z \in \mathbb{C} : f^j(z) \in U \text{ for all } j \geqslant n\}$. By assumption, $\mu\left(\bigcup_{n\geqslant 0} Y_n\right) = 1$. Therefore there exists some n_0 such that $\mu(Y_{n_0}) > 0$. Since $Y_n \subset Y_{n+1}$ for all n, we must have $\mu(Y_n) > 0$ for $n \geqslant n_0$.

Now let n be such that $\mu(Y_n \cap U^c) > 0$ and let $Y := Y_n \cap U^c$. Then $f^j(Y) \cap Y = \emptyset$ for all $j \geqslant n$. Let $k := \max\{j : \mu(f^j(Y) \cap Y) > 0\} < n$. Let $Y' := f^k(Y) \cap Y$. Then $f^{n-k}(Y') \subset U$, so certainly $\mu(f^j(Y') \cap Y') = 0$ for $j \geqslant n-k$. Moreover, by definition of k, $\mu(f^j(Y') \cap Y') = 0$ for $1 \leqslant j < n-k$. Clearly Y' is a wandering set of positive μ–measure, so μ is dissipative, as required. □

Proof of Lemma 2. Recall that $(\mathcal{J}, f, \mu)$ is dissipative if there is a set A with $\mu(A) > 0$ such that $\mu(f^{-n}(A) \cap A) = 0$ for all $n \geqslant 1$. Otherwise $(\mathcal{J}, f, \mu)$ is conservative. Lemma 9 implies that μ_δ is conservative. Hence Lemma 8 implies that μ_δ–a.e. point has a dense orbit in $\mathcal{J}$. Therefore, for any open set U intersecting $\mathcal{J}$ has $\mu_\delta\left(\bigcup_{n\geqslant 0} f^{-n}(U)\right) = 1$. Suppose that $\mu(\bigcup_{n\geqslant 0} f^{-n}(W(\theta))) = 0$, so the complement has positive μ-measure. Then by absolute continuity $\mu_\delta\left(\left(\bigcup_{n\geqslant 0} f^{-n}(W(\theta))\right)^c\right) > 0$ which is a contradiction to the previous line. Therefore, $\mu(W(\theta)) > 0$. □

Appendix B

Here we present some lemmas concerning the combinatorial properties of unimodal maps $f : \mathbb{R} \to \mathbb{R}$ and resulting Koebe space.

Lemma 10. *The following properties hold:*

(a) $D_{S_k} = f^{S_k}([\zeta_{k-1}, 0]) = [f^{S_{Q(k)}}(0), f^{S_k}(0)]$;
(b) *if J is a branch of f^n and $f^n(J) \ni 0$ then $f^n(J) = D_{S_k}$ for some k;*
(c) $D_n = [f^n(0), f^{\beta(n)}(0)]$ *for $\beta(n) = n - \max\{S_j : S_j < n\}$ and if n is not a cutting time, then $D_n \subset D_{\beta(n)}$;*
(d) *if $Q(k) \to \infty$, then the lengths $|D_n| \to 0$ as $n \to \infty$.*

Proof. See [B2]. □

Lemma 11. *Assume that $Q(k) \to \infty$ satisfies* (4). *Then there exists $m_0 \in \mathbb{N}$ such that for each non-cutting time n, D_n contains at most one closest precritical point ζ_m or $\hat{\zeta}_m$ for $m \geqslant m_0$.*

Proof. From Lemma 10, part (c), we can build, for any n, a nested sequence

$$D_n \subset D_{\beta(n)} \subset D_{\beta^2(n)} \subset \cdots \subset D_{\beta^{r-1}(n)} \subset D_{\beta^r(n)},$$

where $\beta^r(n)$ is a cutting time (say $S_{a(n)-1}$) and $\beta^{r-1}(n) = S_{a(n)-1} + S_{b(n)-1}$. See Figure 3. We let $a = a(n),\ b = b(n)$.

By Lemma 10(d), $|D_{S_{b-1}+S_{a-1}}| \to 0$ as $b \to \infty$, so there is a neighbourhood U_a of 0 such that $D_{S_{b-1}+S_{a-1}} \cap U_a = \emptyset$ for all $b \in \mathbb{N}$ such that $Q(b) > a-1$. Take $U = \cap_{n,\ a(n)<k_0} U_{a(n)}$ with k_0 as in (4) and let m_0 be minimal such that $\zeta_m, \hat{\zeta}_m \in U$ for all $m \geqslant m_0$.

Now take n a non-cutting time, and assume by contradiction that D_n contains ζ_m and ζ_{m+1} (or $\hat{\zeta}_m$ and $\hat{\zeta}_{m+1}$) for $m \geqslant m_0$.

Build the nested sequence $(D_{\beta^i(n)})_{i=0}^r$ as above and let $\beta^r = S_{a-1}$ and $\beta^{r-1}(n) = S_{a-1} + S_{b-1}$. Since D_n intersects U, so does $D_{S_{a-1}+S_{b-1}}$, and hence $a \geqslant k_0$.

Since $D_{S_{a-1}+S_{b-1}}$ contains two closest precritical points and, by Lemma 1(b), $f^{S_{a-1}}(0) \in (\zeta_{Q(a)-1}, \zeta_{Q(a)}) \cup (\hat{\zeta}_{Q(a)}, \hat{\zeta}_{Q(a)-1})$, at least $\zeta_{Q(a)}$ and $\zeta_{Q(a)+1}$ (or $\hat{\zeta}_{Q(a)}$ and $\hat{\zeta}_{Q(a)+1}$) belong to

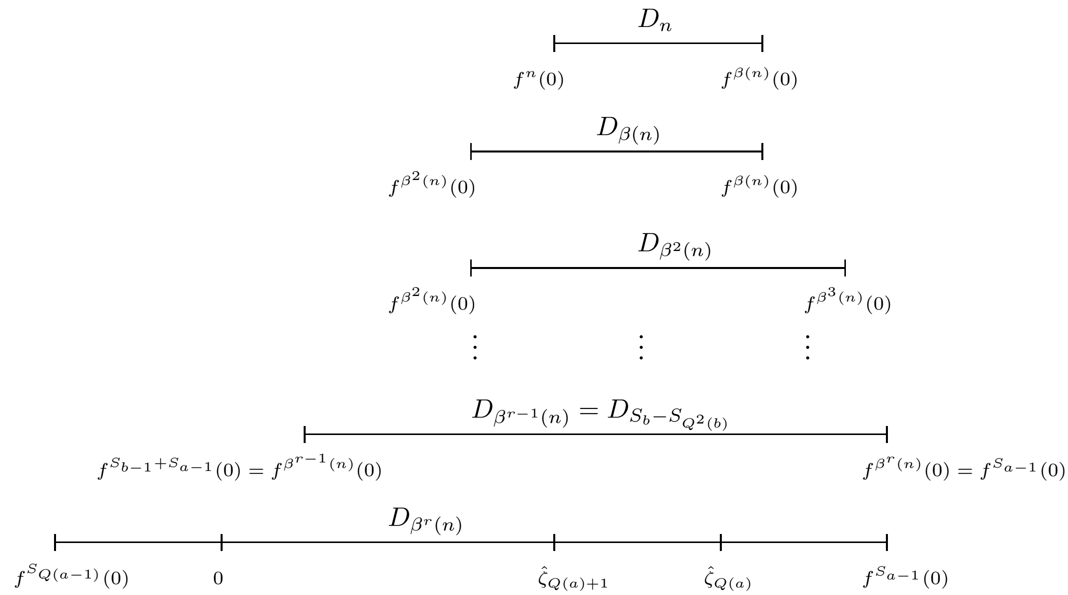

FIGURE 3. The nested sequence of intervals $\left(D_{\beta^i(n)}\right)_{i=0}^r$.

$D_{S_{a-1}+S_{b-1}}$. It follows that the first cutting time beyond $S_{a-1}+S_{b-1}$ is $S_b = S_{b-1}+S_{a-1}+S_{Q(a)}$, so $a = Q(b)$ and

$$S_{b-1} + S_{a-1} = S_b - S_{Q(a)} = S_b - S_{Q^2(b)}.$$

Now apply $f^{S_{Q^2(b)}}$ to the interval $D_{S_b - S_{Q^2(b)}}$; it maps $f^{S_{b-1}+S_{a-1}}(0) = f^{S_b - S_{Q^2(b)}}(0)$ to $f^{S_b}(0)$, $\zeta_{Q^2(b)}$ to 0 and $\zeta_{Q^2(b)+1}$ to a point $\zeta \in f^{S_{Q^2(b)} - S_{Q^2(b)+1}}(0) = f^{-S_{Q(Q^2(b)+1)}}(0)$. In addition, ζ has to be a closest precritical point; otherwise there is a point $\zeta' \in (\zeta, 0) \cap f^{-i}(0)$ for some $i \leqslant S_{Q^2(b)}$ that pulls back under $f^{S_{Q^2(b)}}$ to a closest precritical point strictly between $\hat\zeta_{Q^2(b)}$ and $\hat\zeta_{Q^2(b)+1}$. Thus $\zeta = \zeta_{Q(Q^2(b)+1)} \in [f^{S_{Q^2(b)}}(0), 0]$. The closest precritical point of lowest index in $[f^{S_b}(0), 0]$ is $\zeta_{Q(b+1)}$ by Lemma 1(b), so $Q(b+1) \leqslant Q(Q^2(b)+1)$. But this contradicts assumption (4). □

Lemma 12. *Assume that $Q(k) \to \infty$ and that Condition* (4) *holds. Let m_0 be as in Lemma 11. Then if $V \subset \mathbb{R}$ is the largest neighbourhood of c_1 on which $f^{S_k - 1}$ is monotone, then $f^{S_k-1}(V) = [f^{S_{Q^2(k)}}(0), f^{S_{Q(k)}}(0)]$.*

Proof. Let $V = [f(w), f(z)] \ni c_1$ be the largest interval on which f^{S_k-1} is monotone. By definition of closest precritical point, $z = \zeta_{k-1}$, and $f^{S_k-1}(f(\zeta_{k-1})) = f^{S_k - S_{k-1}}(0) = f^{S_{Q(k)}}(0)$. The other endpoint $f(w)$ is an image of $w \in i\mathbb{R}$. Decompose $f^{S_k-1} = f^{S_{Q(k)}} \circ f^{S_{k-1}-1}$. By Lemma 1(c), $f^{S_{k-1}-1}(c_1) \in (\zeta_{Q(k)-1}, \zeta_{Q(k)}) \cup (\hat\zeta_{Q(k)}, \hat\zeta_{Q(k)-1})$; for simplicity assume that $f^{S_{k-1}-1}(c_1) \in (\zeta_{Q(k)-1}, \zeta_{Q(k)})$, as in Figure 4. There are two possibilities:

- $\zeta_{Q(k)-1} \in f^{S_{k-1}-1}(V)$. In this case, $f^{S_{k-1}-1}(f(w)) = \zeta_{Q(k)-1}$, because otherwise the interval $f^{S_{k-1}-1}(V)$ is not mapped monotonically for another $S_{Q(k)}$ iterates, a contradiction. Therefore,
$$f^{S_k-1}(f(w)) = f^{S_{Q(k)}}(\zeta_{Q(k)-1}) = f^{S_{Q(k)} - S_{Q(k)-1}}(0) = f^{S_{Q^2(k)}}(0).$$
- $\zeta_{Q(k)-1} \notin f^{S_{k-1}-1}(V)$, and then there is $n < S_{k-1}$ such that $f^{S_{k-1}-1}(f(w)) = f^n(0)$, and $f^{S_{k-1}-n-1}(f(w)) = 0$. It follows that $f^{S_{k-1}-n}(\zeta_{k-1})$ is a precritical point. If it is not a closest precritical point, then pulling back the first closest precritical point in $(0, f^{S_{k-1}-n}(\zeta_{k-1}))$ would give another closest precritical point between ζ_{k-1} and 0. Therefore $f^{S_{k-1}-n}(\zeta_{k-1}) = \zeta_j$ or $\hat\zeta_j$ for some $j < k-1$ (in Figure 4, this point is $\hat\zeta_j$),

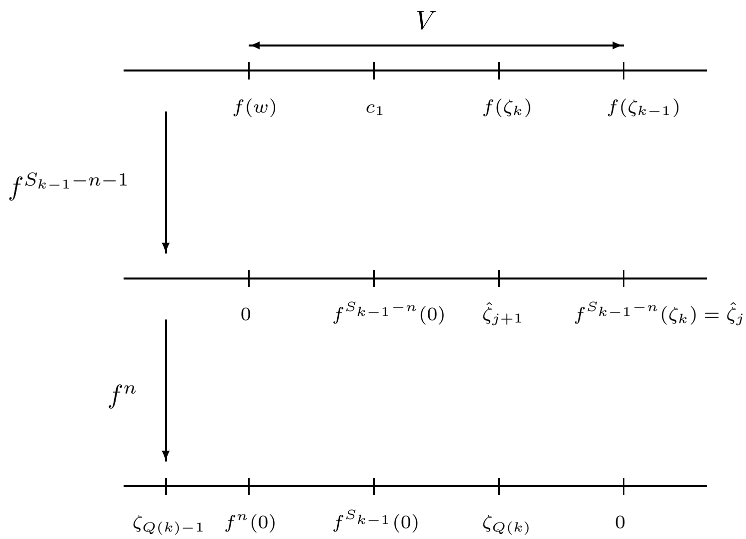

FIGURE 4. The interval V and its images under $f^{S_{k-1}-n-1}$ and $f^{S_{k-1}-1}$.

and $n = S_j$. But pulling back $\zeta_{Q(k)}$ for n iterates gives another closest precritical point in $(f^{S_{k-1}-n}(0), \hat{\zeta}_j)$. (This point must be $\hat{\zeta}_{j+1}$, which also shows that $j+1 > Q(k)$, so $j \geqslant Q(k) \geqslant m_0$.) Therefore $D_{S_{k-1}-n}$ contains two closest precritical points $\hat{\zeta}_j, \hat{\zeta}_{j+1}$ for $j+1 > j \geqslant m_0$, and as $S_{k-1} - n$ is not a cutting time, this violates Lemma 11.

□

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