LARGE DERIVATIVES, BACKWARD CONTRACTION AND INVARIANT DENSITIES FOR INTERVAL MAPS

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Abstract. In this paper, we study the dynamics of a smooth multimodal interval map f with non-flat critical points and all periodic points hyperbolic repelling. Assuming that $|Df^n(f(c))| \to \infty$ as $n \to \infty$ holds for all critical points c, we show that f satisfies the so-called backward contracting property with an arbitrarily large constant, and that f has an invariant probability μ which is absolutely continuous with respect to Lebesgue measure and the density of μ belongs to L^p for all $p < \ell_{\max}/(\ell_{\max} - 1)$, where $\ell_{\max}$ denotes the maximal critical order of f. In the appendix, we prove that various growth conditions on the derivatives along the critical orbits imply stronger backward contraction.

1. Introduction

The concept of absolutely continuous invariant measures plays an important role in studying the chaotic behavior of non-uniformly hyperbolic dynamical systems. In the area of interval dynamics, various conditions have been shown to guarantee the existence of an invariant probability which is absolutely continuous with respect to Lebesgue measure (acip). In [**M**] it was shown that an acip exists for an S-multimodal map (here and below, "S-" stands for negative Schwarzian derivative) without periodic attractors or recurrent critical points. In [**CE**], it was proved that an S-unimodal map f satisfying the following condition (the *Collet-Eckmann condition*) has an acip:

$$(CE) \qquad \liminf_{n\to\infty} \frac{\log |Df^n(f(c))|}{n} > 0,$$

where c denotes the critical point of f. In [**NS2**], the following summability condition (the *Nowicki-van Strien condition*) was shown to imply the existence

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of an acip for an S-unimodal map:

$$(NS) \qquad \sum_{n=0}^{\infty} \frac{1}{|Df^n(f(c))|^{1/\ell}} < \infty,$$

where ℓ is the order of the critical point. Moreover, it was proved that the density of the acip with respect to Lebesgue measure belongs to L^p for all $p < \ell/(\ell-1)$. (Note that this regularity is the best possible since the density is never $L^{\ell/(\ell-1)}$.) Later on, this result was extended to the multimodal setting in [**BS**].

Observe that the Collet-Eckmann condition requires that for every critical point c, the derivatives $|Df^n(f(c))|$ grow exponentially fast with n, which is more restrictive than the Nowicki-van Strien condition that still requires these derivatives to grow at a sufficiently fast rate. In [**BSS**] it was shown that for S-unimodal maps, these conditions are far too restrictive for the existence of an acip. In fact, there exists a constant $C > 0$ depending on the critical order such that

$$\liminf_{n\to\infty} |Df^n(f(c))| \geq C$$

implies the existence of an acip. In this paper we shall extend this last result to the multimodal setting.

Main Theorem. *Let $f : [0,1] \to [0,1]$ be a C^3 multimodal interval map with non-flat critical points and with all periodic points hyperbolic repelling. Assume that for each critical point c, we have*

$$\lim_{n\to\infty} |Df^n(f(c))| = \infty,$$

then f has an acip μ whose density with respect to Lebesgue measure belongs to L^p for all $1 \leq p < \ell_{\max}/(\ell_{\max}-1)$, where $\ell_{\max}$ is the maximum of the orders of the critical points.

Observe that the hypothesis of this result rules out maps for which a critical point is mapped to another critical point under forward iteration.

Here, as usual, by saying that f is *of class C^3 with non-flat critical points*, we mean that f is C^1 everywhere and satisfies the following:

- f is C^3 outside $\mathrm{Crit}(f) = \{x \in [0,1] : Df(x) = 0\}$;
- for each $c \in \mathrm{Crit}(f)$, there exists a number $\ell_c > 1$ (called *the order of* c) and C^3 diffeomorphisms ϕ, ψ of $\mathbb{R}$ with $\phi(c) = \psi(f(c)) = 0$ such that

  $$|\psi \circ f(x)| = |\phi(x)|^{\ell_c}$$

  holds in a neighborhood of c.

We shall use $\mathcal{A}$ to denote the class of all C^3 interval maps with non-flat critical points and with all periodic points hyperbolic repelling.

In order to state a more precise version of our main theorem, we need to introduce more notation. For a fixed positive integer N and a positive number $\ell_{\max} > 1$, denote by $\mathcal{A}(N, \ell_{\max})$ the collection of all maps $f \in \mathcal{A}$ having exactly N critical points, and for which the maximum of the orders of the critical points is $\ell_{\max}$. For $K > 0$ denote by $\mathcal{A}(N, \ell_{\max}, K)$ the class of all maps $f \in \mathcal{A}(N, \ell_{\max})$ for which the following holds:

$$(*)\qquad \begin{array}{c} \textit{There exists a neighborhood } \mathcal{V} \textit{ of } Crit(f) \textit{ such that if} \\ f^n(c) \in \mathcal{V} \textit{ for some critical point } c \in Crit(f), \textit{ then} \\ |Df^n(f(c))| > K. \end{array}$$

We shall actually prove the following Main Theorem', from which we easily deduce the Main Theorem.

Main Theorem'. *Given a positive integer N, real numbers $\ell_{\max} > 1$ and $p \in [1, \ell_{\max}/(\ell_{\max} - 1))$, there exists a constant $K = K(N, \ell_{\max}, p)$ such that if $f \in \mathcal{A}(N, \ell_{\max}, K)$, then f has an acip μ whose density with respect to Lebesgue measure belongs to the space L^p.*

In fact, in the Main Theorem' we show that, if we denote the Lebesgue measure by m, then each accumulation point in the weak* topology of the sequence $(\frac{1}{n}\sum_{i=0}^{n-1} f_*^i m)_{n\geq 0}$ is an acip with an L^p density. Thus the Main Theorem is a direct consequence of the Main Theorem'. Notice however that, although having sufficiently large derivatives at each critical value is enough to get an acip, the statement in the Main Theorem about the density being in L^p for *all* $p < \ell_{\max}/(\ell_{\max}-1)$ requires the derivatives at each critical value to grow to infinity.

The result of the density is new even in the unimodal setting. We note that the proof in this paper is quite different from and significantly simpler than that of [**BSS**]. In fact, the proof follows much closer the proof in [**NS2**] but is simpler; the main reason for this is that we use here the notion of nice intervals which significantly simplifies the proof of formula (7) compared to the corresponding inductive statement in [**NS2**].

It is probably difficult to improve these results. It is not possible to give a topological condition equivalent to the existence of an acip since the last property is not a topological invariant, not even a quasisymmetric invariant, see [**B1**]. Moreover, although no invariant density needs to exist if $(*)$ is satisfied with a small constant K [**BKNS**], $\liminf_{n\to\infty} |Df^n(f(c))| = 0$ does not rule out the existence of an acip [**B2**].

Condition $(*)$ is almost a C^1 invariant in the unimodal case, i.e., if two unimodal maps $f, g \in \mathcal{A}$ are C^1 conjugate, and f satisfies $(*)$ for $K = K_0$, then g satisfies $(*)$ for all $K < K_0$. However, for multimodal maps, this is not the case. In fact, let $K(c, c')$ be the infimum of the set of all numbers α for which there exists a sequence $n_k \to \infty$ with

$$f^{n_k}(c) \to c', \text{ and } |Df^{n_k}(f(c))| \to \alpha,$$

and let

$$K(f) = \inf_{c,c' \in \mathrm{Crit}(f)} K(c, c').$$

Then $(*)$ requires that $K(f) \geq K$. Note that the number $K(c, c)$ is C^1 invariant for any $c \in \mathrm{Crit}(f)$, while $K(c, c')$, $c \neq c'$, is not.

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2. Ideas and organisation of the proof

To prove the existence of an absolutely continuous invariant measure one has somehow to control the density of $(f^n)_*m$ for n large. One approach is by means of the Perron-Frobenius operator acting on densities. Here, as in [**NS2**], we estimate the measure $(f^n)_*m$ more directly by bounding the Lebesgue measure of $f^{-n}(A)$ for small Borel sets A. The proof of the Main Theorem can be divided into two steps.

Step 1: Show that if $\lim_{n\to\infty} |Df^n(f(c))| \to \infty$ for each critical point c, then f satisfies the backward contracting property $BC(r)$, with an arbitrarily large constant r. This property says that for ϵ sufficiently small, if for some $c, c' \in \mathrm{Crit}(f)$ the pullback W of an $r\epsilon$-neighborhood of $f(c)$ is ϵ-close to $f(c')$, then $|W| < \epsilon$, see Definition 1 below. A more precise version of this step is stated in Theorem 1.

Step 2: Show that if f satisfies the backward contracting property with an arbitrarily large constant, then for each $\kappa \in (0,1)$ there exists M so that for each Borel set A, we have the key inequality,

$$|f^{-n}(A)| \le M|f(A)|^{\kappa/\ell_{\max}}. \tag{1}$$

Here $|A|$ denotes the Lebesgue measure of A. A more precise version of this step is stated in Theorem 2.

Step 1 is fairly easy, and relies on controlling repeated pullbacks using the growth assumption on the derivative and the one-sided Koebe Principle, see part (ii) of Proposition 1. This is done in the first part of Section 3. In these pullbacks we are only guaranteed Koebe space on one side, and therefore we cannot apply the usual Koebe Principle. We should emphasise that there is no analogue of the one-sided Koebe Principle in the complex case, and this is one reason why there is no analogue of our theorem for non-real holomorphic maps.

Step 2 is more involved. The backward contracting property only gives information on *individual* components J of preimages $f^{-n}(A)$ when A and J are intervals near critical values of f. To extend this to information on the entire set $f^{-n}(A)$ (for arbitrary Borel sets A) we consider first return maps to so-called nice intervals (the real analogue of Yoccoz puzzle pieces). To do this, we establish in Subsection 3.2 some implications of the backward contracting property:
(i) There are nice intervals of essentially any given size, see Proposition 3.
(ii) Return maps to nice intervals are λ-nice: the first return domains to nice intervals are very deep inside, see Lemma 3.
(iii) A nested sequence of pullbacks of a nice interval (called 'children', i.e., intervals which contain a critical point and which are unicritical pullbacks of the original interval) shrinks exponentially fast, see Lemma 4.

Using this information and the one-sided Koebe Principle, we prove in Section 4 the inductive inequality (7) which fairly easily implies the key inequality (1) for any interval A near a critical point. For more general sets A we cannot immediately use information about first return maps. Therefore in Section 5 we first assume that A is a Borel set which is contained in some nice neighborhood V of the critical set. If we subdivide A into subpieces and then compare the pullback of A with those of certain intervals, we get an estimate which depends on the order of some pullback, see inequality (14). On the other hand, for pullbacks of arbitrarily high order we get an exponential estimate. Summing over all combinatorial possibilities we get (1) for any set $A \subset V$ in Lemma 9. For arbitrary sets A we then finally use hyperbolicity of the dynamics away from the critical points (i.e., Mañé's Theorem).

One of the main innovations compared to the proof given in [**NS2**] is to use first return maps to well-chosen nice intervals and nested chains of children, see Subsection 3.2.

We will now give a more precise description of the proof of the Main Theorem', which implies the Main Theorem as was described above. We shall use the following terminology, which was first introduced in [**R**]. For any $c \in \mathrm{Crit}(f)$ and $\delta > 0$, let $\hat{B}_\delta(c)$ be the component of $f^{-1}(B_\delta(f(c)))$ which contains c. Let $CV = CV(f) = f(\mathrm{Crit}(f))$ be the set of critical values of f.

Definition 1. For a constant $r > 1$, that will be usually large, we say that f satisfies *the backward contracting property with constant* r ($BC(r)$ in short) if the following holds: there exists $\varepsilon_0 > 0$ such that for each $\varepsilon < \varepsilon_0$, each $s \geq 1$ and each component W of $f^{-s}(\hat{B}_{r\varepsilon}(c))$ for some $c \in \mathrm{Crit}(f)$,

$$\mathrm{dist}(W, CV) < \varepsilon \text{ implies } |W| < \varepsilon. \tag{2}$$

We say that f satisfies $BC(\infty)$ if it satisfies $BC(r)$ for all $r > 1$.

Clearly, for any $r > 1$, property $BC(r)$ implies that f has no critical relation, i.e., no critical point is mapped into the critical set under forward iteration.

The Main Theorem' follows easily from the following two theorems.

Theorem 1. *For real numbers $\ell_{\max} > 1$ and $r > 1$, there exists $K = K(r, \ell_{\max})$ such that if f is a map in the class $\mathcal{A}(N, \ell_{\max}, K)$ for some $N \geq 1$, then it satisfies property $BC(r)$.*

We shall provide two proofs of this theorem. The first one is given in Section 3 as a consequence of Lemma 2 and Proposition 2. The second one is given in the appendix, see Theorem 3. The reason that K depends only on f through$\ell_{\max}$, is that all constants related to f (such as those related to non-flatness) vanish when looking at sufficiently small scales.

In the appendix we state and prove a result which is somewhat related to the proof of Theorem 1. It shows that various growth conditions of the derivatives along the critical orbits imply stronger backward contraction.

Theorem 2. *For any positive integer N, real numbers $\ell_{\max} > 1$ and $\kappa \in (0, 1)$, there exists $r = r(N, \ell_{\max}, \kappa)$ such that if $f \in \mathcal{A}(N, \ell_{\max})$ satisfies the $BC(r)$ property then there exists a constant M such that for every Borel set A we have*

$$|f^{-n}(A)| \leq M|f(A)|^{\kappa/\ell_{\max}}. \tag{3}$$

The proof of Theorem 2 is given in Sections 4 and 5.

We will now explain how to obtain the Main Theorem' from Theorems 1 and 2. We use the argument given above Theorem A in [**NS2**], see also [**MS**, p. 378]. Let N be a positive integer, $\ell_{\max} > 1$ and $p \in [1, \frac{\ell_{\max}}{\ell_{\max}-1})$ be given. Then choose $\kappa \in (0,1)$ sufficiently close to 1 so that $1 - \frac{\kappa}{\ell_{\max}} < 1/p$, let $r = r(N, \ell_{\max}, \kappa)$ be given by Theorem 2, and let $K = K(r, \ell_{\max})$ be given by Theorem 1. Given $f \in \mathcal{A}(N, \ell_{\max}, K)$, let $C > 0$ be such that for every measurable set A we have $|f(A)| \le C|A|$, and put $M' = MC^{\frac{\kappa}{\ell_{\max}}}$. It follows from the inequality (1) that, if we denote Lebesgue measure by m, then for each $n \ge 1$ the measure

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} (f^i)_* m$$

is such that for every measurable set A we have $\mu_n(A) \le M'|A|^{\frac{\kappa}{\ell_{\max}}}$. Fix an accumulation point μ in the weak* topology of the sequence $(\mu_n)_{n\ge1}$. Then μ is f-invariant and for every measurable set A we have $\mu(A) \le M'|A|^{\frac{\kappa}{\ell_{\max}}}$. In particular μ is absolutely continuous with respect to the Lebesgue measure; we denote by ρ its density. For each $k \ge 0$ put $D_k = \{\rho^p \ge k\}$ and observe that

$$\int \rho^p dx \le \sum_{k\ge0} (k+1)|D_k \setminus D_{k+1}| = \sum_{k\ge0} |D_k|.$$

For each $k \ge 1$ we have

$$k^{\frac{1}{p}}|D_k| \le \mu(D_k) \le M'|D_k|^{\frac{\kappa}{\ell_{\max}}},$$

and $|D_k| \le (M'k^{-\frac{1}{p}})^{(1-\frac{\kappa}{\ell_{\max}})^{-1}}$. By the choice of κ we have $\frac{1}{p}(1 - \frac{\kappa}{\ell_{\max}})^{-1} > 1$, so $\sum_{k\ge0} |D_k| < \infty$. This shows that $\rho \in L^p$.

Remark. If f is unimodal, then the acip is ergodic and unique. A general multimodal interval map may have more than one acip. For a map satisfying $BC(r)$ for a large r, the estimate provided by Theorem 2 rules out the existence of Cantor attractors. Thus any compact forward invariant set of positive measure contains a cycle of periodic intervals [**SV**], each supporting an acip.

Notation. Unless otherwise stated, $X = [0,1]$ and $f : X \to X$ is a map in the class $\mathcal{A}$. We will assume, without loss of generality, that $f(\partial X) \subset \partial X$ and that $Df(x) \neq 0$ for $x \in \partial X$.

If J is an interval and $\lambda > 0$, we use λJ to denote the concentric open interval which has length $\lambda|J|$. We say that J is λ-*well inside* another interval I or that I *contains the* λ-*scaled neighborhood of* J, if $I \supset (1 + 2\lambda)J$.

3. Real bounds

We shall use the following result throughout our analysis.

Proposition 1. *For any $f \in \mathcal{A}$, there exists $\eta(f) > 0$ such that the following holds. Let $s \ge 1$ be an integer and let $T = (a, b)$ be an interval. Assume that $f^s|T$ is a diffeomorphism onto its image and that $|f^s(T)| < \eta(f)$. Then*

(i) *(the Minimum Principle) for every* $x \in T$,
$$|Df^s(x)| \geq 0.9 \min\left(|Df^s(a)|, |Df^s(b)|\right);$$
(ii) *(the one-sided Koebe Principle) Let* $x \in T$ *be such that* $|f^s(a) - f^s(x)| \geq \tau |f^s(x) - f^s(b)|$. *Then*
$$|Df^s(x)| \geq 0.9 \left(\frac{\tau}{1+\tau}\right)^2 |Df^s(b)|;$$
(iii) *(the Koebe Principle) If* J *is a subinterval of* T *such that* $f^s(J)$ *is* τ*-well inside* $f^s(T)$, *then for any* $x, y \in J$,
$$0.9 \left(\frac{\tau}{1+\tau}\right)^2 \leq \frac{|Df^s(x)|}{|Df^s(y)|} \leq \frac{1}{0.9}\left(\frac{1+\tau}{\tau}\right)^2;$$
(iv) *(the Macroscopic Koebe Principle) If* J *is a subinterval of* T *such that* $f^s(J)$ *is* τ*-well inside* $f^s(T)$, *then* J *is* τ'*-well inside* T, *where* $\tau' = 0.9\tau^2/(1+2\tau)$.

Proof. If f has negative Schwarzian derivative, then so does $f^s|T$. In this case, it is well-known that the statements hold with 0.9 being replaced by 1 (and without the assumption that $f^s(T)$ has a small length), see for example [**MS**]. For the general case, we first note that $|f^s(T)|$ small implies that $|T|$ is small as well since f has no wandering interval, see [**MS**]. Then we apply a theorem of Graczyk and Sands [**GS**] which states that any $f \in \mathcal{A}$ is real-analytically conjugate to a map with negative Schwarzian derivative. □

An alternative proof. Let us say that a diffeomorphism φ between intervals is almost linear if for any x, y in its domain, we have $|D\varphi(x)| \geq 0.9|D\varphi(y)|$. It suffices to prove that there exists $0 \leq s_0 < s$ such that

- $f^{s-s_0} : f^{s_0}(T) \to f^s(T)$ is almost linear;
- either $s_0 = 0$ or $f^{s_0} : T \to f^{s_0}(T)$ has negative Schwarzian.

By the third statement of Theorem C in [**SV**], there exists a neighborhood U of $\mathrm{Crit}(f)$ such that for any $x \in X$ and $n \geq 0$ with $f^n(x) \in U$, we have $Sf^{n+1}(x) < 0$, where $S\phi$ denotes the Schwarzian derivative of ϕ. Let $V \Subset U$ be a smaller neighborhood of $\mathrm{Crit}(f)$. Provided that $|f^s(T)|$ is small enough, $\max_{i=0}^s |f^i(T)| < d(\partial U, \partial V)$, so that for each $i \in \{0, 1, \ldots, s\}$, either $f^i(T) \subset U$ or $f^i(T) \cap V = \emptyset$. Let $s_0 \in [0, s)$ be minimal such that $f^i(T) \cap V = \emptyset$ for all $s_0 \leq i < s$. Then either $s_0 = 0$ or $f^{s_0 - 1}(T) \subset U$ so that $f^{s_0} : T \to f^{s_0}(T)$ has negative Schwarzian derivative. Moreover, by Mañé's theorem, f is uniformly expanding outside V. Thus provided that $|f^s(T)|$ is small enough, $f^{s-s_0} : f^{s_0}(T) \to f^s(T)$ is almost linear. □

3.1. **Backward contraction.** A sequence of open intervals $\{G_j\}_{j=0}^s$ is called *a chain* if for each $0 \leq j < s$, G_j is a component of $f^{-1}(G_{j+1})$. The *order* of the chain is defined to be the number of j's with $0 \leq j < s$ and such that G_j contains a critical point.

For each critical point c and $\varepsilon > 0$, let $\hat{B}_\varepsilon(c)$ be the connected component of $f^{-1}((f(c)-\varepsilon, f(c)+\varepsilon))$ containing c. Moreover, let

$$\hat{B}_\varepsilon = \bigcup_{c \in \mathrm{Crit}(f)} \hat{B}_\varepsilon(c).$$

Note that provided that ε is small enough,

$$\hat{B}_{r\varepsilon}(c) \approx r^{1/\ell_c} \hat{B}_\varepsilon(c).$$

Lemma 1. *For any $\rho > 0$ and $\ell_{\max} > 1$, there exists $K > 1$, and for each $f \in \bigcup_{N=1}^\infty \mathcal{A}(N, \ell_{\max}, K)$ there exists $\varepsilon_0 > 0$ with the following property. Let $c, c' \in Crit(f)$ and $\varepsilon \in (0, \varepsilon_0)$. If $f^s(c) \in \hat{B}_\varepsilon(c')$ for some $s \geq 1$, and if J is the component of $f^{-s}(\hat{B}_\varepsilon(c'))$ containing c, then*

$$J \subset \hat{B}_{\rho\varepsilon}(c).$$

Proof. We may assume that $\rho \in (0,1)$. Put $r = 2^{\ell_{c'}}$ and consider the chains $\{G_j\}_{j=0}^s$ and $\{H_j\}_{j=0}^s$ with $G_s = \hat{B}_{r\varepsilon}(c') \supset H_s = \hat{B}_\varepsilon(c')$ and $G_0 \supset H_0 = J$. Let $s_1 < s$ be maximal such that G_{s_1} contains a critical point c_1. Let H'_{s_1+1} be the convex hull of $H_{s_1+1} \cup \{f(c_1)\}$, and observe that $H'_{s_1+1} \subset G_{s_1+1}$.

Claim. Provided that ε is small enough and that K is large enough, we have

$$H_{s_1} \subset \hat{B}_{\rho\varepsilon}(c_1). \tag{4}$$

In fact, since

$$f^{s-s_1-1} : G_{s_1+1} \to G_s$$

is a diffeomorphism with $|G_s|$ small, it follows from the one-sided Koebe Principle (Proposition 1 (ii)), applied to each of the connected components of $G_{s_1+1} \setminus \{f(c_1)\}$ intersecting H'_{s_1+1}, that for each $x \in H'_{s_1+1}$, we have

$$|Df^{s-s_1-1}(x)| \geq C|Df^{s-s_1-1}(f(c_1))|, \tag{5}$$

where $C > 0$ is a universal constant. Provided that ε is small enough, we have

$$|Df^{s-s_1}(f(c_1))| \geq K$$

by the hypothesis. Moreover, by non-flatness of the critical points there is a constant $C_1 > 0$ such that

$$|Df(f^{s-s_1}(c_1))| \leq C_1 \frac{|fG_s|}{|G_s|}.$$

Thus

$$|Df^{s-s_1-1}(f(c_1))| = \frac{|Df^{s-s_1}(f(c_1))|}{|Df(f^{s-s_1}(c_1))|} \geq KC_1^{-1} \frac{|G_s|}{|fG_s|}.$$

This, equation (5) and the mean value theorem imply

$$\frac{|G_s|}{|H'_{s_1+1}|} \geq CC_1^{-1} K \frac{|G_s|}{|fG_s|},$$

which implies that

$$|H'_{s_1+1}| \leq \rho\varepsilon$$

provided that K is sufficiently large. The claim follows.

If $s_1 = 0$ then the proof of the lemma is completed. For the general case, the lemma follows by an easy induction on s. $\square$

Let us say that f satisfies *property $BC^*(r)$* if the following holds: for any $\varepsilon > 0$ small enough, $c, c' \in \mathrm{Crit}(f)$, and $s \geq 1$, if $f^s(c) \in \hat{B}_{r\varepsilon}(c')$ and J is the component of $f^{-s}(\hat{B}_{r\varepsilon}(c'))$ which contains c, then $J \subset \hat{B}_\varepsilon(c)$. (So the difference with property $BC(r)$ is that in equation (2), the assumption $\mathrm{dist}(W, CV) < \varepsilon$ is replaced by $W \cap CV \neq \emptyset$.)

The above lemma can be reformulated as

Proposition 2. *For any $\ell_{\max} > 1$ and $r > 1$ there exists $K \geq 1$ such that each $f \in \bigcup_{N=1}^{\infty} \mathcal{A}(N, \ell_{\max}, K)$ satisfies property $BC^*(r)$.*

Property $BC^*(r)$ is closely related to $BC(r)$. Clearly the latter implies the former. The other direction is shown in the following proposition.

Lemma 2. *For any $f \in \mathcal{A}$, $BC^*(8^{\ell_{\max}} r)$ implies $BC(r)$, where $\ell_{\max}$ is the maximum of the order of the critical points of f.*

Proof. Let $\varepsilon > 0$ be a small constant. Then for all $c \in \mathrm{Crit}(f)$, $\hat{B}_{8^{\ell_{\max}} r\varepsilon}(c)$ contains the 3-scaled neighborhood of $\hat{B}_{r\varepsilon}(c)$.

Let $c, c' \in \mathrm{Crit}(f)$ and $x \in \hat{B}_\varepsilon(c)$. Let $s \geq 1$ be such that $f^s(x) \in \hat{B}_{r\varepsilon}(c')$ and let J_k be the component of $f^{-(s-k)}(\hat{B}_{r\varepsilon}(c'))$ which contains $f^k(x)$. We want to show that $|J_1| < \varepsilon$.

Let us prove this by induction on s. For $s = 1$ the statement is trivially true. Fix s_0 and assume that the statement holds if $s < s_0$. To prove the statement for $s = s_0$, consider the chain $\{G_j\}_{j=0}^{s}$ with $G_s = \hat{B}_{8^{\ell_{\max}} r\varepsilon}(c')$ and $G_0 \ni x$. We distinguish two cases:

Case 1. There exists $0 \leq s_1 < s$ such that G_{s_1} contains a critical point c_1. By the definition of the BC^* property, it follows that $G_{s_1} \subset \hat{B}_\varepsilon(c_1)$. If $s_1 = 0$, then $c_1 = c$ and $J_0 \subset G_0 \subset \hat{B}_\varepsilon(c)$. Otherwise, the statement follows by the induction hypothesis.

Case 2. For any $0 \leq k < s$, G_k contains no critical point. Then $f^{s-1} : G_1 \to G_s$ is a diffeomorphism. By the Macroscopic Koebe Principle (Proposition 1 (iii)), we obtain that G_1 contains the 1-scaled neighborhood of J_1. Since $c \notin G_0$, $f(c) \notin G_1$ and $f(x) \in B_\varepsilon(f(c))$, it follows that $|J_1| < \varepsilon$. $\square$

Proof of Theorem 1. Combine Lemma 2 and Proposition 2. $\square$

3.2. **Nice sets.** An open set $V \subset [0, 1]$ is called *nice* if for each $x \in \partial V$ and for any $k \geq 1$, $f^k(x) \notin V$.

Proposition 3. *For any $f \in \mathcal{A}$ satisfying $BC(2)$ and any $\varepsilon > 0$ sufficiently small, the following holds: for each $c \in Crit(f)$, there exists an open interval V_c*

such that $\bigcup_{c\in Crit(f)} V_c$ *is nice and such that*

$$\hat{B}_\varepsilon(c) \subset V_c \subset \hat{B}_{2\varepsilon}(c).$$

Proof. The proof follows from the following argument due to Rivera-Letelier, see Lemma 6.2 in [**R**]. For $\varepsilon > 0$ small, define the open set $V^n = \bigcup_{i=0}^n f^{-i}(\hat{B}_\varepsilon)$. Clearly V^∞ is nice. Take V_c^n to be the connected component of V^n which contains c, and let $V_c = V_c^\infty$. It remains to show that $V_c^n \subset \hat{B}_{2\varepsilon}(c)$ for each n. We do this by induction. For $n = 0$ this holds by definition, so assume it holds for n. Consider $Z = f(V_c^{n+1}) \setminus B(f(c), \varepsilon)$. For $z \in Z$, there exists $m(z) \in \{0, 1, \ldots, n\}$ and $c_0(z) \in \text{Crit}(f)$ so that $f^{m(z)}(z) \in \hat{B}_\varepsilon(c_0(z))$. Now choose $z_0 \in Z$ so that $m_0 = m(z_0)$ is minimal among $m(z)$ for points $z \in Z$ and let $\hat{c}_0 = c_0(z)$. Since $f^{m_0}(z_0) \in \hat{B}_\varepsilon(\hat{c}_0)$, and since $f^{m_0}(Z) \subset V_{\hat{c}_0}^{n-m_0} \subset V_{\hat{c}_0}^n$, the induction hypothesis implies

$$f^{m_0}(Z) \subset \hat{B}_{2\varepsilon}(\hat{c}_0).$$

Since f satisfies $BC(2)$ and Z has distance ε to $f(c)$, it follows that $|f(Z)| < \varepsilon$ and $Z \subset \hat{B}_{2\varepsilon}(c)$. This completes the induction step. □

For $\lambda > 0$ we say that a nice open set V is λ-*nice* if for each return domain J of V, we have $(1 + 2\lambda)J \subset V$.

A nice open set $V \supset \text{Crit}(f)$ will be called *a puzzle neighborhood of* Crit(f) if each component of V contains exactly one critical point of f.

Lemma 3. *For any* $\lambda > 0$ *and real number* $\ell_{\max} > 1$*, there exists* $r > 1$ *such that for any* N *and any* $f \in \mathcal{A}(N, \ell_{\max})$ *which satisfies* $BC(r)$ *the following holds. There exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *there exists a puzzle neighborhood* $V = \bigcup_c V_c$ *with the following properties:*

- V *is* λ*-nice;*
- *for each* $c \in Crit(f)$ *we have*

$$\hat{B}_\varepsilon(c) \subset V_c \subset \hat{B}_{2\varepsilon}(c).$$

Proof. Let $r \geq 2$ and let W be a puzzle neighborhood of Crit(f) such that

$$\hat{B}_{r\varepsilon/2}(c) \subset W_c \subset \hat{B}_{r\varepsilon}(c),$$

given by Proposition 3.

For each $c \in \text{Crit}(f)$, let V_c be the union of $\hat{B}_\varepsilon(c)$ and the return domains of W which intersect the boundary of $\hat{B}_\varepsilon(c)$. Clearly, $V = \bigcup_c V_c$ is a puzzle neighborhood of Crit(f) and moreover, for each $x \in \partial V$ and $k \geq 1$, $f^k(x) \notin W$. Provided r is large enough, each component of V is deep inside a component of W. It follows that each return domain of V is deep inside V, see for example Theorem B(2) in [**SV**]. □

If I is an interval which contains a critical point and J (with $J \cap \mathrm{Crit} \neq \emptyset$) is a unicritical pull back of I then we say that J is a *child* of I. So there exists $c \in \mathrm{Crit}(f)$, $s \geq 1$ (called the *transition time from J to I*) and an interval $\tilde{J} \ni f(c)$ such that

- f^{s-1} maps $\tilde{J}$ diffeomorphically onto I;
- J is the component of $f^{-1}(\tilde{J})$ which contains c.

We shall use the following lemma in the next section:

Lemma 4. *Let $c, c' \in Crit(f)$, let $I \ni c$ be a λ-nice interval with $|I|$ small and let*

$$J_1 \supsetneq J_2 \supsetneq \cdots \supsetneq J_m$$

be children of I which contain c'. Then

$$|f(J_i)| \leq \rho^{i-1} |f(J_1)|$$

holds for all i, where $\rho = \rho(\lambda) > 0$ is such that $\rho \to 0$ as $\lambda \to \infty$.

Proof. By definition, for each $i \geq 1$, there exists a positive integer s_i and an interval $T_i \supset f(J_i)$ such that $f^{s_i - 1}$ maps T_i diffeomorphically onto I and J_i is the component of $f^{-1}(T_i)$ containing c'. Note that $f^{s_i}(T_{i+1})$ is contained in a return domain of I, hence λ-well inside I. By the Koebe Principle, it follows that each T_{i+1} is λ_1-well inside T_i, where $\lambda_1 \to \infty$ as $\lambda \to \infty$. The conclusion follows. □

4. Pull back of intervals

The goal of this section is to prove the following proposition, which is as Theorem 2, but in the special case when the set A is an interval contained in a small neighborhood of the set of critical points. We will complete the proof of Theorem 2 in the next section.

Proposition 4. *For $\kappa \in (0,1)$, $N \in \mathbb{N}$ and $\ell_{\max} > 1$, there exists $r > 1$ such that if $f \in \mathcal{A}(N, \ell_{\max})$ satisfies $BC(r)$, then there exists a neighborhood U of $Crit(f)$ such that for any interval $A \subset U$ and any $n \geq 0$, the following holds:*

$$|f^{-n}(A)| \leq M |f(A)|^{\kappa/\ell_{\max}}, \tag{6}$$

where M is a constant depending on f.

We start with the following easy general lemma.

Lemma 5. *Let f be a C^3 multimodal map with non-flat critical points. Then there is $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, every $r > 1$, every critical point c of f, and every pair of intervals A and I such that $A \subset I \subset \hat{B}_{2\varepsilon}(c)$ and $\overline{A} \not\subset \hat{B}_{\varepsilon/r}(c)$, we have*

$$\frac{|f(I)|}{|I|} \frac{|A|}{|f(A)|} \leq 4 r^{1 - 1/\ell_c}.$$

Proof. By definition of multimodal maps with non-flat critical points, there are diffeomorphisms ψ and ϕ of class C^3 such that $\phi(c) = \psi(f(c)) = 0$ and such that $g = \psi \circ f \circ \phi^{-1}$ is of the form $g(x) = \pm|x|^{\ell_c}$ near 0. It is thus enough to prove the lemma with f replaced by g and with $4r^{1-1/\ell_c}$ replaced by $2^{2-1/\ell_c} r^{1-1/\ell_c}$. To prove this, just observe that if J is an interval and a is a point in $\overline{J}$ which is farthest from 0, then we have $|f(J)|/|J| \geq \frac{1}{2}|a|^{\ell_c - 1}$ and, if in addition $0 \in J$, then we have $|f(J)|/|J| \leq |a|^{\ell_c-1}$. □

Let r be a large constant, and assume that f satisfies $BC(r)$. By Lemma 3, there exist $\lambda = \lambda(r, \ell_{\max})$ and $\varepsilon_0 = \varepsilon_0(f) > 0$ small such that for any $\varepsilon < \varepsilon_0$ there exists a λ-nice puzzle neighborhood of $\mathrm{Crit}(f)$ which lies in-between $\hat{B}_\varepsilon$ and $\hat{B}_{2\varepsilon}$. Moreover, $\lambda \to \infty$ as $r \to \infty$.

For each $n \geq 0$ and $\delta > 0$, define

$$L_n(\delta) = \sup\{|f^{-m}(A)| : 0 \leq m \leq n, A \subset \hat{B}_{\varepsilon_0/r} \text{ is an interval}, |f(A)| \leq \delta\}.$$

Lemma 6. *Let I be a λ-nice interval such that $\hat{B}_\varepsilon(c) \subset I \subset \hat{B}_{2\varepsilon}(c)$, where $c \in Crit(f)$ and $\varepsilon < \varepsilon_0$, and let A be an interval such that*

$$A \subset \hat{B}_{\varepsilon/2}(c) \text{ and } \overline{A} \not\subset \hat{B}_{\varepsilon/r}(c).$$

Then for all $n \geq 1$,

$$|f^{-n}(A)| \leq C\frac{|A|}{|I|}|f^{-n}(I)| + 2N\sum_{i=1}^{\infty} L_{n-1}(\rho^i|f(A)|), \tag{7}$$

where $N = \#Crit(f)$, the constant $C > 0$ only depends on $\ell_{\max}$, and the constant $\rho \in (0,1)$ depends only on r and $\ell_{\max}$, and for a fixed $\ell_{\max}$ we have $\rho \to 0$ as $r \to \infty$.

Proof. Let $\mathcal{J}_0$ be the collection of all components J of $f^{-n}(I)$ such that $f^n : J \to I$ is a diffeomorphism, and let $\mathcal{J}_1$ be the collection of all other components of $f^{-n}(I)$. The collection $\mathcal{J}_0$ contributes to the first part of inequality (7) (it is the easy part of the argument), while $\mathcal{J}_1$ contributes to the second part.

As $A \subset \hat{B}_{\varepsilon/2}(c)$ is well inside I, it follows by the Koebe Principle that there is a constant $C > 0$ only depending on $\ell_{\max}$, such that for all $J \in \mathcal{J}_0$ we have

$$\frac{|f^{-n}(A) \cap J|}{|J|} \leq C\frac{|A|}{|I|}. \tag{8}$$

For each $J \in \mathcal{J}_1$, there exist $n_1 = n_1(J) \in \{0, 1, \ldots, n-1\}$ and an interval $J' \supset f^{n-n_1}(J)$ such that J' has a common endpoint with $f^{n-n_1}(J)$ and such that J' is a child of I.

Claim. There exists a constant $C > 0$ depending only on $\ell_{\max}$ such that for any $J \in \mathcal{J}_1$, and each component A' of $f^{-n_1}(A) \cap J'$, we have

$$|f(A')| \leq Cr^{1-1/\ell_c}\frac{|f(J')|}{|f(I)|}|f(A)|. \tag{9}$$

In fact, by definition of a child, there exists $\tilde{J}' \supset f(J')$ such that f^{n_1-1} maps $\tilde{J}'$ diffeomorphically onto I. By the one-sided Koebe Principle, there exists a constant C such that

$$\frac{|f(A')|}{|f(J')|} \leq C\frac{|A|}{|f^{n_1}(J')|}.$$

Since $A \subset I \subset \hat{B}_{2\varepsilon}(c)$ and $\overline{A} \not\subset \hat{B}_{\varepsilon/r}(c)$, we have

$$\frac{|f(I)|}{|I|}\frac{|A|}{|f(A)|} \leq 4r^{1-1/\ell_c}.$$

Since $A \subset \hat{B}_{\varepsilon/2}(c)$, $I \supset \hat{B}_{\varepsilon}(c)$ and since $f^{n_1}(J')$ contains a component of $I \setminus A$, $|f^{n_1}(J')|/|I|$ is bounded away from zero. Inequality (9) follows by redefining the constant C.

For any child P of I, let $s(P)$ be the transition time from P to I and let $\mathcal{J}_1(P)$ be the collection of all elements $J \in \mathcal{J}_1$ with $J' = P$. Clearly,

$$\sum_{J\in\mathcal{J}_1(P)} |f^{-n}(A) \cap J| = |f^{-(n-s(P))}(f^{-s(P)}(A) \cap P)|.$$

Since $f^{-s(P)}(A) \cap P$ has at most two components, applying (9) we obtain that

$$\sum_{J\in\mathcal{J}_1(P)} |f^{-n}(A) \cap J| \leq 2L_{n-1}\left(Cr^{1-1/\ell_c}\frac{|f(P)|}{|f(I)|}|f(A)|\right). \tag{10}$$

For each $c' \in \mathrm{Crit}(f)$, let

$$P_1(c') \supsetneq P_2(c') \supsetneq \cdots$$

be all the children of I which contain c'. By the $BC(r)$ property,

$$|f(P_1(c'))| \leq 2\varepsilon/r.$$

By Lemma 4,

$$|f(P_i(c'))| \subset \rho_1^{i-1}|f(P_1(c'))|,$$

where ρ_1 is a constant depending on λ, and $\rho_1 \to 0$ as $\lambda \to \infty$. So

$$f(P_i(c')) \subset B_{2\rho_1^{i-1}\varepsilon/r}(f(c')).$$

Note that since r is large

$$\rho := \max(\rho_1, 2Cr^{-1/\ell_{\max}})$$

is close to zero. By (10),

$$\sum_{J\in\mathcal{J}_1(P_i(c'))} |f^{-n}(A) \cap J| \leq 2L_{n-1}(\rho^i|f(A)|).$$

Thus

$$\begin{aligned}\sum_{J\in\mathcal{J}_1} |f^{-n}(A) \cap J| &= \sum_{c'}\sum_{i=1}^{\infty}\sum_{J\in\mathcal{J}_1(P_i(c'))} |f^{-n}(A) \cap J| \\ &\leq 2N\sum_{i=1}^{\infty} L_{n-1}(\rho^i|f(A)|).\end{aligned}$$

Combining this with (8), we obtain inequality (7). □

Proof of Proposition 4. Fix $\kappa \in (0,1)$ and write $\alpha = \kappa/\ell_{\max}$. Let $M = (r/\varepsilon_0)^\alpha$. Recall that the constant $C > 0$ given by Lemma 6 does not depend on r. We can thus assume that r is so large that

$$4Cr^{(\kappa-1)/\ell_{\max}} \leq \frac{1}{2}.$$

We shall prove by induction on n that the following inequality (11) holds:

$$L_n(\delta) \leq M\delta^\alpha. \tag{11}$$

The case $n = 0$ is trivial. So assume that the inequality holds for all $n < n_0$ and we will prove it for $n = n_0$. To do this, we shall prove by induction on $m \geq 0$ that the inequality holds for every $\delta \in [\frac{\varepsilon_0}{2^{m+1}}, \frac{\varepsilon_0}{2^m}]$. The choice of M clearly guaranteed that this holds for $m = 0$. Assuming that it holds for all m less than some positive integer m_0, let us prove it for $m = m_0$.

We need to prove that for each open interval $A \subset \hat{B}_{\varepsilon_0/r}$ such that $\varepsilon_0/2^{m_0+1} \leq |f(A)| \leq \varepsilon_0/2^{m_0}$ we have

$$|f^{-n}(A)| \leq M|f(A)|^\alpha. \tag{12}$$

Let $\varepsilon \in (0, \varepsilon_0)$ be minimal such that $A \subset \hat{B}_{\varepsilon/r}(c)$ for some $c \in \mathrm{Crit}(f)$. Let $I \ni c$ be a λ-nice interval such that

$$\hat{B}_\varepsilon \subset I \subset \hat{B}_{2\varepsilon}.$$

Then by Lemma 6, we have

$$|f^{-n}(A)| \leq C\frac{|A|}{|I|}|f^{-n}(I)| + 2N\sum_{i\geq 1} L_{n-1}(\rho^i|f(A)|).$$

The second term in the right hand side is bounded from above by

$$2NM\sum_{i\geq 1}\left(\rho^i|f(A)|\right)^\alpha = 2NM|f(A)|^\alpha\frac{\rho^\alpha}{1-\rho^\alpha} \leq |f(A)|^\alpha M/2,$$

provided that ρ is small enough. If $I \subset \hat{B}_{\varepsilon_0/r}$, then by the induction hypothesis on m, and using the fact that

$$|f(I)| \geq r|f(A)| \geq \varepsilon_0/2^{m_0},$$

we have

$$|f^{-n}(I)| \leq M|f(I)|^\alpha.$$

The same estimate holds in the case $I \supset \hat{B}_{\varepsilon_0/r}$ by the choice of M since $|f^{-n}(I)| \leq 1$. Thus

$$|f^{-n}(A)| \leq CM\frac{|A|}{|I|}|f(I)|^\alpha + \frac{M}{2}|f(A)|^\alpha.$$

Since

$$\frac{|A|}{|I|} \leq \frac{|f(A)|}{|f(I)|}\cdot 4r^{1-1/\ell_c} \leq \frac{|f(A)|^\alpha}{|f(I)|^\alpha}\cdot 4r^{\alpha-1/\ell_c},$$

and $r^{\alpha-1/\ell_c} \leq r^{(\kappa-1)/\ell_{\max}}$, it follows that

$$|f^{-n}(A)| \leq M\left(4Cr^{(\kappa-1)/\ell_{\max}} + \frac{1}{2}\right)|f(A)|^\alpha \leq M|f(A)|^\alpha.$$

This completes the induction step on m, hence the induction step in (11) and the proposition. □

5. Invariant measure and the density

In this section we prove Theorem 2. Using Mañé's theorem we reduce to the case when the set A is contained in a small neighborhood of the set of critical points. This case, which is stated as Lemma 9 below, is obtained from the case of intervals (Proposition 4), using the Minimum Principle (Proposition 1 (i)) to relate the size of the preimage of a general set to the size of preimages of intervals "at the end of branches", see Lemma 7 below. This corresponds to the "sliding argument" used in [**NS2**], see also [**MS**].

So let f be a multimodal interval map as in the proposition. Take $\kappa_1 \in (\kappa, 1)$. Assume that f satisfies $BC(r)$ for a large r. Then by Proposition 4, there exists $\varepsilon_0 > 0$ (small) such that for any interval $Q \subset \hat{B}_{\varepsilon_0}$ and any $n \geq 0$,

$$|f^{-n}(Q)| \leq M_0|f(Q)|^{\kappa_1/\ell_{\max}}, \tag{13}$$

where $M_0 > 0$ is a constant.

We say that a sequence of open intervals $\{G_i\}_{i=0}^s$ is a *quasi-chain* if for each $0 \leq i < s$, G_i contains a component of $f^{-1}(G_{i+1})$. The *order* of the quasi-chain is defined to be the number of $i \in \{0, 1, \ldots, s-1\}$ such that G_i contains a critical point. We say that such a quasi-chain is λ-*admissible* if for each $0 \leq i \leq s$, either $G_i \cap \mathrm{Crit}(f) = \emptyset$, or G_i is λ-nice.

By the argument in Sect. 2, choosing ε_0 smaller if necessary, we may assume that for each $j \geq 0$, there exists a λ-nice interval $V_{j,c}$ with the following property:

$$\hat{B}_{\varepsilon_0/2^{j+1}}(c) \subset V_{j,c} \subset \hat{B}_{\varepsilon_0/2^j}(c),$$

where $\lambda \to \infty$ as $r \to \infty$. For each interval $\tilde{G}$ which contains a critical point c and with $\tilde{G} \subset V_{0,c}$, let j be maximal such that $\tilde{G} \subset V_{j,c}$. We call that $V_{j,c}$ is the *enlargement* of $\tilde{G}$. Clearly, the enlargement of any critical interval is uniformly comparable to itself in size.

Let $V = \bigcup_{c\in\mathrm{Crit}(f)} V_{0,c}$. Fix a positive integer n throughout the rest of this section. For each component J of $f^{-n}(V)$, we define a λ-admissible quasi-chain $\{G_i\}_{i=0}^n$ as follows:

- $G_n = V$;
- Assume that $G_{i+1} \supset f^{i+1}(J)$ is defined, and let $\tilde{G}_i$ be the component of $f^{-1}(G_{i+1})$ which contains $f^i(J)$. Then
  - if $\tilde{G}_i$ is not critical, then $G_i := \tilde{G}_i$;
  - if $\tilde{G}_i$ is critical, then G_i is the enlargement of $\tilde{G}_i$ defined as above.

Let $\mathcal{J}_m$ be the collection of all components of $f^{-n}(V)$ such that the order of the corresponding quasi-chain $\{G_i\}_{i=0}^n$ is equal to m. Each $J \in \mathcal{J}_m$ is associated with a sequence of positive integers $k_1, k_2, \ldots, k_m$ and critical points $c_0, c_1, \ldots, c_m$ as follows.

- $s_0 = s > s_1 > \ldots > s_m \geq 0$ are all the integers such that G_{s_j} is critical, and c_j is the critical point contained in G_{s_j};
- $\tilde{G}_{s_j}$ is the k_j-th child of $G_{s_{j-1}}$ which contains c_j.

For any $\mathbf{c} = c_0 c_1 \ldots c_m \in \mathrm{Crit}(f)^{m+1}$ and $\mathbf{k} = k_1 k_2 \ldots k_m \in \mathbb{Z}_+^m$, we use $\mathcal{J}_{\mathbf{c}}^{\mathbf{k}}$ to denote the collection of all components J of $f^{-n}(V)$ with the parameters $\mathbf{c}$ and $\mathbf{k}$. Note that for each $J, J' \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}$ and $\{G_i\}_{i=0}^n$, $\{G'_i\}_{i=0}^n$ the corresponding quasi-chains and for all $0 \leq j \leq m$, we have $s_j = s'_j$ and $G_{s_j} = G'_{s_j}$. Define

$$H_{\mathbf{c}}^{\mathbf{k}} = G_{s_m},$$

and

$$CV_{\mathbf{c}}^{\mathbf{k}} = \{f^{s_0 - s_j}(c_j) : 0 \leq j \leq m\},$$

which has at most $m+1$ elements. Notice that for each $J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}$, the critical values of $f^n|_J$ are contained in $CV_{\mathbf{c}}^{\mathbf{k}}$.

Lemma 7. *For any* $\mathbf{c}, \mathbf{k}$, *and any Borel set* $A \subset V$, *the following holds:*

$$\sum_{J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} |f^{-n}(A) \cap J| \leq 3M_0(m+2)|f(A)|^{\kappa_1/\ell_{\max}}, \tag{14}$$

where M_0 *is as in* (13).

Proof. Let us label the set $CV_{\mathbf{c}}^{\mathbf{k}}$ as $v_1 < v_2 < \ldots < v_{m'+1}$, for some $0 \leq m' \leq m$, and let v_0 and $v_{m'+2}$ be the left and right endpoints of V_{c_0}. These points divide V_{c_0} into subintervals $Q_k = (v_k, v_{k+1})$, $0 \leq k \leq m'+1$. Let $A_k = A \cap Q_k$. It suffices to prove that for each k, the following holds:

$$\sum_{J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} |f^{-n}(A_k) \cap J| \leq 2M_0 \rho^{\kappa_1/\ell_{\max}} \tag{15}$$

where $\rho = |f(A)|$. If $|f(Q_k)| \leq \rho$, then by Proposition 4,

$$\sum_{J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} |f^{-n}(A_k) \cap J| \leq |f^{-n} Q_k| \leq M_0 \rho^{\kappa_1/\ell_{\max}}.$$

So assume that $|f(Q_k)| > \rho$. To show inequality (15), choose $a_k, b_k \in Q_k$ such that

$$L_k = (v_k, a_k), \text{ and } R_k = (b_k, v_{k+1})$$

satisfies

$$|f(L_k)| = |f(R_k)| = |f(A_k)|.$$

For each $J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}$ and each component J' of $f^{-n}Q_k \cap J$, $f^{n+1} : J' \to f(Q_k)$ is a diffeomorphism. By the Minimum Principle it follows that

$$\begin{aligned} |(f^n|J')^{-1}(A_k)| &= |(f^{n+1}|J')^{-1}(f(A_k))| \\ &\le 0.9^{-1}\left(|(f^{n+1}|J')^{-1}(f(L_k))| + |(f^{n+1}|J')^{-1}(f(R_k))|\right) \\ &= 0.9^{-1}\left(|f^{-n}(L_k) \cap J'| + |f^{-n}(R_k) \cap J'|\right). \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} |f^{-n}(A_k) \cap J| &\le 0.9^{-1} \sum_{J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} \left(|f^{-n}(L_k) \cap J| + |f^{-n}(R_k) \cap J|\right) \\ &\le 0.9^{-1}\left(|f^{-n}(L_k)| + |f^{-n}(R_k)|\right) \\ &\le 3M_0 \rho^{\kappa_1/\ell_{\max}}, \end{aligned}$$

where in the last step we used (13). This proves (15), and hence completes the proof of the lemma. □

Applied to the case $\mathbf{c} = \{c\}$, $c \in \mathrm{Crit}(f)$, the lemma gives

(16) $$\sum_{J \in \mathcal{J}_0} |f^{-n}(A) \cap J| \le 6NM_0 |f(A)|^{\kappa_1/\ell_{\max}}.$$

Lemma 8. *There is $\sigma > 0$ depending only on r, such that $\sigma \to 0$ as $r \to \infty$, and such that for each $\mathbf{c} = c_0 c_1 \dots c_m$ and $\mathbf{k} = k_1 \dots k_m$ the following hold:*

(i)
$$\bigcup_{J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} J \subset f^{-(n-s_m)}(H_{\mathbf{c}}^{\mathbf{k}}),$$

(ii)
$$|f(H_{\mathbf{c}}^{\mathbf{k}})| \le \sigma^{k_1+k_2+\dots+k_m} |f(V)|,$$

(iii) *if we put $\sigma_1 = \sigma^{\kappa_1/\ell_{\max}}$, then*
$$\sum_{J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} |J| \le M_1 \sigma_1^{k_1+k_2+\dots+k_m},$$

where $M_1 = M_0 |f(V)|^{\kappa_1/\ell_{\max}}$, and M_0 is as in (13).

Proof. The first item is clear. The second follows from Lemma 4, and $|\tilde{G}_{s_i}| \asymp |G_{s_i}|$. For the third item, notice that by item (i), the left hand side of the inequality does not exceed $|f^{-(n-s_m)}(H_{\mathbf{c}}^{\mathbf{k}})|$. So the inequality follows from the item (ii) of this lemma by (13). □

Lemma 9. *Provided that r is large enough, the following holds: for any Borel set $A \subset V$ and any $n \ge 1$,*

$$|f^{-n}(A)| \le M |f(A)|^{\kappa/\ell_{\max}},$$

where M is a constant.

Proof. Pick D_0 to be the maximal integer such that

$$6(1+N)^{D_0+1}D_0 \leq \left(\frac{1}{\rho}\right)^{(\kappa_1-\kappa)/\ell_{\max}}, \tag{17}$$

where $\rho = |f(A)|$ and $N = \#\mathrm{Crit}(f)$. For any $m \geq 1$ and $D \geq 1$, let

$$\mathcal{J}_m(D) = \{J \in \mathcal{J}_m : k_1(J) + k_2(J) + \ldots + k_m(J) = D\},$$

$$\mathcal{J}(D) = \bigcup_{m=1}^{\infty} \mathcal{J}_m(D).$$

For each $1 \leq m \leq D$, the number of tuples $\mathbf{k} = k_1k_2\ldots k_m$ satisfying

$$k_1 + k_2 + \ldots + k_m = D$$

is given by the binomial coefficient $C_{D-1}^{m-1} := \binom{D-1}{m-1}$. Associated to such a $\mathbf{k} = k_1k_2\ldots k_m$, there could be at most N^{m+1} corresponding choices of $\mathbf{c} = c_0c_1\ldots c_m$. From this and Lemma 8 it follows that

$$\sum_{J\in\mathcal{J}(D)} |J| \leq \sum_{m=1}^{D} \sum_{J\in\mathcal{J}_m(D)} |J| \leq M_1 \sum_{m=1}^{D} N^{m+1}C_{D-1}^{m-1}\sigma_1^D \leq M_1N\sigma_2^D,$$

where $\sigma_2 = (N+1)\sigma_1$ and M_1 is as in Lemma 8. Thus

$$\sum_{D=D_0+1}^{\infty} \sum_{J\in\mathcal{J}(D)} |f^{-n}(A)\cap J| \leq \sum_{D=D_0+1}^{\infty} \sum_{J\in\mathcal{J}(D)} |J| \leq M_1\frac{N\sigma_2}{1-\sigma_2}\sigma_2^{D_0}.$$

Provided that r is large enough, σ_2 is close to 0 so that

$$\sigma_2^{\hat{D}} \leq [6(1+N)^{\hat{D}+2}(\hat{D}+1)]^{-\tau}$$

for any $\hat{D} \geq 1$ and where τ is so that $\tau(\kappa_1 - \kappa) = \kappa$. This and (17) imply

$$\sigma_2^{D_0+1} \leq \rho^{\tau(\kappa_1-\kappa)/\ell_{\max}} = \rho^{\kappa/\ell_{\max}}.$$

Since $\rho = |f(A)|$ we get therefore

$$\sum_{D=D_0+1}^{\infty} \sum_{J\in\mathcal{J}(D)} |f^{-n}(A)\cap J| \leq M_2|f(A)|^{\kappa/\ell_{\max}},$$

where M_2 is a constant.

On the other hand, by (14), we have

$$\begin{aligned}\sum_{D=1}^{D_0} \sum_{J\in\mathcal{J}(D)} |f^{-n}(A)\cap J| &\leq \sum_{m=1}^{D_0} \sum_{J\in\mathcal{J}_m} |f^{-n}(A)\cap J| \\ &\leq \sum_{m=1}^{D_0} C_{D_0-1}^{m-1}N^{m+1}3M_0(m+2)|f(A)|^{\kappa_1/\ell_{\max}} \\ &\leq 6M_0(1+N)^{D_0+1}D_0|f(A)|^{\kappa_1/\ell_{\max}} \\ &\leq M_0|f(A)|^{\kappa/\ell_{\max}},\end{aligned}$$

where in the last inequality we used (17) again. Combining with (16), these estimates imply the lemma. □

Proof of Theorem 2. Fix $\kappa \in (0,1)$. Assume that f satisfies $BC(r)$ for a large r. We want to show that there exists $M > 0$ such that inequality (1) holds for any Borel set A and any $n \geq 0$.

By Lemma 9, there exists a neighborhood V of $\mathrm{Crit}(f)$ such that the inequality holds when $A \subset V$.

The general case follows by Mãñé's theorem, which asserts that $f|(X \setminus V)$ is uniformly expanding, i.e., there exists $C > 0$ and $\gamma \in (0,1)$ such that for any $x \in X$ and $k \geq 1$, if $x, f(x), \ldots, f^{k-1}(x) \notin V$ then $|Df^k(x)| \geq C\gamma^{-k}$. It follows that for any Borel set A and $m \geq 1$,

$$A_m := \{x \in X : x, f(x), \ldots, f^{m-1}(x) \notin V, f^m(x) \in A\}$$

has length $C_1\gamma_1^m|A|$, where $C_1 > 0$ and $\gamma_1 \in (0,1)$ are constants.

Set $Q_n = A_n$ and for $0 \leq m < n$, set

$$Q_m := \{x : f^m(x) \in V \text{ and } f^{m+1}(x) \in A_{n-m-1}\}.$$

Clearly,

$$f^{-n}(A) = \bigcup_{m=0}^{n} Q_m.$$

By the argument in the previous paragraph,

$$|Q_n| = |A_n| \leq C_1\gamma_1^n|A|,$$

and for $0 \leq m < n$, since $f^m(Q_m) \subset V \cap f^{-1}(A_{n-m-1})$, we have

$$\begin{aligned}|Q_m| &\leq |f^{-m}(f^m(Q_m))| \leq |f^{-m}(V \cap f^{-1}(A_{n-m-1}))| \\ &\leq M|A_{n-m-1}|^\alpha \leq M'\gamma_2^{n-m}|A|^\alpha,\end{aligned}$$

where the third inequality follows by Lemma 9, and $M' > 0$ and $\gamma_2 \in (0,1)$ are constants. Inequality (1) follows by redefining M. □

Appendix: Growth of derivatives and backward contraction

Let $f : X \to X$ be a map in the class $\mathcal{A}$, i.e., f is C^3 with non-flat critical points and all periodic points hyperbolic repelling. Let $CV = CV(f) = f(\mathrm{Crit}(f))$. Given $\delta' > \delta > 0$ we will say that f is (δ, δ')-*backward contracting* if for every critical point c of f, every $n \geq 1$, and every connected component W of $f^{-n}(\hat{B}_{\delta'}(c))$ we have that

$$\mathrm{dist}(W, CV) < \delta \text{ implies } |W| < \delta.$$

For a given constant $r > 1$, the map f satisfies the backward contracting property with constant r, as defined in the introduction, if for every $\delta > 0$ sufficiently small the map f is $(\delta, \delta r)$-backward contracting. Given $\delta_0 > 0$ and a function $r : (0, \delta_0) \to (1, +\infty)$, we will say that f is *backward contracting with growth function* r, if for every $\delta \in (0, \delta_0)$ sufficiently small the map f is $(\delta, \delta r(\delta))$-backward contracting.

The purpose of this appendix is to prove the following result.

Theorem 3. *For $f \in \mathcal{A}$, the following properties hold.*

1. *For every $r > 1$ there is a constant $K > 0$ that depends only on $\ell_{\max}$, such that if $f \in \bigcup_N \mathcal{A}(N, \ell_{\max}, K)$, then f satisfies the backward contracting property with constant r.*
2. *If f satisfies the summability condition with exponent $\alpha > 0$, then there exists $\delta_0 > 0$, and a function $r : (0, \delta_0) \to (1, \infty)$ such that for every $\theta \in (0, 1)$ we have*
$$\sum_{n \gg 1} r(\theta^n)^{-\alpha} < \infty,$$
and such that f is backward contracting with growth function r.
3. *If f satisfies the Collet-Eckmann condition, then there are constants $\alpha \in (0, 1]$ and $C > 0$, such that f is backward contracting with growth function $r(\delta) = C\delta^{-\alpha}$.*

Parts 2 and 3 were proved for rational maps in [**R**, Theorem A].

For the proof of the theorem, let $f \in \mathcal{A}$. Given $v \in CV$ and $c \in \text{Crit}$ let

$$0 \leq k_1(v, c) < k_2(v, c) < \cdots$$

be all integers $k \geq 0$ satisfying the following property: If $r > 0$ is the smallest number so that the closure of $\hat{B}_r(c)$ contains $f^k(v)$, then the pull-back of the closure of $\hat{B}_r(c)$ by f^k to v is diffeomorphic. Observe that for certain v and c this sequence might be finite or non-existent. We denote by $\xi_i(v, c)$ the corresponding preimage of c by $f^{k_i(c,v)}$. Notice that when $\xi_i(v, c)$ is close to v, the integer $k_i(v, c)$ is large.

Let $\eta_0 = \eta_0(f) > 0$ be sufficiently small. Then for any $\eta \in (0, \eta_0)$, we have

$$|\hat{B}_\eta(c)| \approx 2\left(\frac{\eta}{A_c}\right)^{1/\ell_c}$$

and for $x \in \hat{B}_{\eta_0}(c)$,

$$|Df(x)| \approx A_c \ell_c |x - c|^{\ell_c - 1},$$

where $A_c = \lim_{y \to c} |f(y) - f(c)|/|x - c|^{\ell_c}$. Here by writing $C_1 \approx C_2$ we mean that $C_2/2 \leq C_1 \leq 2C_2$.

Lemma 10. *There is a constant $C > 0$ only depending on $\ell_{\max}$ such that the following property holds. For $\delta > 0$ small put*

$$\rho(\delta) = \min\left\{C \cdot A(\delta), \frac{\eta_0}{\delta}\right\},$$

where $A(\delta)$ is equal to

$$\inf_{\substack{dist(\xi_i(v,c),v) \geq \delta \\ f^{k_i(v,c)+1}(v) \in B_{\eta_0}(CV)}} \left(\frac{dist(\xi_i(v, c), v)}{\delta} |Df^{k_i(v,c)+1}(v)|\right).$$

Then for every $c \in Crit$, every $n \geq 0$ and every $z \in f^{-n}(c)$ such that for every $i = 0, \ldots, n-1$ we have $f^i(z) \notin B_\delta(CV)$, the pull-back of $\hat{B}_{\delta\rho(\delta)}(c)$ to z by f^n is diffeomorphic and disjoint from CV.

Proof. Consider a critical point $c \in \mathrm{Crit}$, an integer $n \geq 0$ and $z \in f^{-n}(c)$. Given $\delta > 0$ small and $0 < r \leq \eta_0/\delta$, put $U_0 = \hat{B}_{\delta r}(c)$ and consider the successive pull-backs $U_0, U_1, \ldots, U_n$, such that U_k contains $f^{n-k}(z)$. Let us suppose that for some $k = 0, \ldots, n$ the set U_k contains a critical value $v \in CV$, and let k be minimal with this property. Note that the restriction of f^k to U_k is a diffeomorphism onto its image. Moreover, $k = k_i(v, c)$ for some $i \geq 1$ and $\xi = \xi_i(v, c) \in U_k$ is equal to $f^{n-k}(z)$. Then for a constant $C_1 > 0$ only depending on f, we have by the One-sided Koebe Principle,

$$\frac{\mathrm{dist}(\xi, v)}{|\hat{B}_{\delta r}(c)|} \leq \frac{\mathrm{dist}(\xi, v)}{\mathrm{dist}(c, f^k(v))} \leq \left(\frac{\mathrm{dist}(\xi, v)}{\mathrm{dist}(c, f^k(v))}\right)^{\frac{\ell_c - 1}{\ell_c}} |Df^k(v)|^{-\frac{1}{\ell_c}} C(\tfrac{1}{2})^{\frac{1}{\ell_c}}.$$

There is thus a constant $C_1 > 0$ depending only on ℓ_c such that, if δ is such that δr is sufficiently small, then

$$\delta r > C_1 \mathrm{dist}(\xi, v) |Df^{k+1}(v)|.$$

By hypothesis $\mathrm{dist}(\xi, v) \geq \delta$, so if we take the constant C in the definition of ρ equal to C_1, then we have $\hat{B}_{\delta\rho(\delta)}(c) \subset U_0$ and for each $k = 0, \ldots, n$ the set U_k is disjoint from CV. □

Lemma 11. *There is a constant C_0 that depends only on $\ell_{\max}$ such that the following property holds. For $\delta > 0$ put*

$$r_0(\delta) = \\ = \min \left\{ 31^{-\ell_{\max}} \rho(\delta), C_0 \inf_{dist(\xi_i(v,c),v) < \delta} \left(\left(\frac{\delta}{dist(\xi_i(v,c), v)} \right)^{\ell_c - 1} |Df^{k_i(v,c)+1}(v)| \right) \right\}.$$

If for every $\delta > 0$ small we have $r_0(\delta) \geq 2$, then f is backward contracting with growth function r_0.

Proof. We will choose the constant $C_0 > 0$ below. Given $c \in \mathrm{Crit}$ and $\delta > 0$ small, consider successive pull-backs $U_0 = \hat{B}_{\delta r_0(\delta)}(c), U_1, \ldots U_k$, such that $U_k \cap B_\delta(CV) \neq \emptyset$. Since for every $\delta > 0$ small we have $r_0(\delta) \geq 2$, arguing by induction, it is enough to consider the case when for every $i = 0, \ldots, k-1$ we have $U_i \cap B_\delta(CV) = \emptyset$.

For each $i = 0, \ldots, k$ let U_i' and U_i'' be the corresponding pull-backs of $\hat{B}_{10^{\ell_{\max}} \delta r_0(\delta)}(c)$ and $\hat{B}_{31^{\ell_{\max}} \delta r_0(\delta)}(c)$ respectively, so that $U_i \subset U_i' \subset U_i''$. Since by definition $31^{\ell_{\max}} r_0(\delta) \leq \rho(\delta)$, the previous lemma implies that $f^k : U_k'' \to U_0''$ is a diffeomorphism. Observe that U_0'' contains a 1-scaled neighborhood of U_0' and that U_0' contains a 4-scaled neighborhood of U_0. In particular, the Koebe Principle implies that the distortion of f^k on U_k' is bounded by 5.

Case 1. U_k' is disjoint from CV. Since U_0' contains a 4-scaled neighborhood of U_0, it follows from the Macroscopic Koebe Principle (Proposition 1 (iv)) that U_k' contains a 1-scaled neighborhood of U_k. Since $U_k \cap B_\delta(CV) \neq \emptyset$ and $U_k' \cap CV = \emptyset$, it follows that $|U_k| < \delta$.

Case 2. There is $v \in U_k' \cap CV$. Then there is $i \geq 1$ such that $k = k_i(v,c)$ and $\xi = \xi_i(v,c) \in U_k'$ is the unique k-th preimage of c in U_k'. Then we must have that $\xi \in B(v,\delta)$; otherwise Lemma 10 would imply that $U_k'' \cap CV = \emptyset$.

Suppose by contradiction that $|U_k'| \geq \delta$. Then, if $\delta > 0$ is such that $\delta r_0(\delta)$ is small, then we have

$$\begin{aligned} \frac{\delta}{|U_0|} &\leq 11\frac{|U_k'|}{|U_0'|} \leq 55\frac{\operatorname{dist}(\xi,v)}{\operatorname{dist}(c,f^k(v))} \\ &\leq 55 \cdot 5^{\frac{1}{\ell_c}} \left(\frac{\operatorname{dist}(\xi,v)}{\operatorname{dist}(c,f^k(v))}\right)^{\frac{\ell_c-1}{\ell_c}} |Df^k(v)|^{-\frac{1}{\ell_c}}. \end{aligned}$$

So, for a constant $C_2 > 0$ depending only on $\ell_{\max}$, we have

$$r_0(\delta) > C_2 \left(\frac{\delta}{\operatorname{dist}(\xi,v)}\right)^{\ell_c-1} |Df^{k+1}(v)|.$$

Since $\operatorname{dist}(\xi,v) < \delta$, letting $C_0 = C_2$ we obtain a contradiction. □

Proof of Theorem 3. Part 1 is a direct consequence of Lemmas 10 and 11.

2. Given $\theta \in (0,1)$ we have

$$\begin{aligned} &\sum_{n\gg 1} r(\theta^n)^{-\alpha} \leq \\ \leq\ & C' \sum_{i,v,c} \Bigg(\sum_{n\geq 0,\, \theta^n < \operatorname{dist}(\xi_i(v,c),v)} \left(\frac{\operatorname{dist}(\xi_i(v,c),v)}{\theta^n}\right)^{-\alpha} |Df^{k_i(v,c)+1}(v)|^{-\alpha} + \\ &+ \sum_{n\geq 0,\, \theta^n > \operatorname{dist}(\xi_i(v,c),v)} \left(\frac{\theta^n}{\operatorname{dist}(\xi_i(v,c),v)}\right)^{-\alpha(\ell_c-1)} |Df^{k_i(v,c)+1}(v)|^{-\alpha} \Bigg) \leq \\ \leq\ & C'' \sum_{i,v,c} |Df^{k_i(v,c)+1}(v)|^{-\alpha} < \infty. \end{aligned}$$

3. Let $C_0 > 0$ and $\lambda > 1$ be such that for every $v \in CV$ we have $|Df^k(v)| \geq C_0\lambda^k$. By [**NS1**, Theorem B] there are constants $C_1 > 0$ and $\theta \in (0,1)$ such that $\operatorname{dist}(\xi_i(v,c),v) \geq C_1\theta^{k_i(v,c)}$. Therefore there is $C_2 > 0$ and $\gamma \in (0,1)$ such that

$$|Df^{k_i(c,v)+1}(v)| \geq C_0\lambda^{k_i(v,c)} \geq C_2(\operatorname{dist}(\xi_i(v,c),v))^{-\gamma}.$$

Choose $\mu \in (0,1)$ and note that, if $\frac{\operatorname{dist}(\xi_i(v,c),v)}{\delta} \leq \delta^{-\mu}$, we have

$$|Df^{k_i(v,c)+1}(v)| \geq C_2 \operatorname{dist}(\xi_i(v,c),v)^{-\gamma} \geq C_2\delta^{-\gamma(1-\mu)}.$$

Thus there is a constant $C_4 > 0$ such that $r_0(\delta) \geq C_4\delta^{-\alpha}$, where $\alpha = \min(\mu, \gamma(1-\mu))$. □

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