MATCHING IN A FAMILY OF PIECEWISE AFFINE INTERVAL MAPS

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Abstract. We consider a class of simple one parameter families of interval maps, and we study how metric (resp. topological) entropy changes as the parameter varies. We show that in many cases the entropy displays a semi-regular behaviour, i.e., it is smooth on an open and dense set. This feature is due to a combinatorial property called *matching*, which was first observed in the parametric family of α-continued fractions introduced by Nakada (see [22]).

1. Introduction

In the setting of the dynamics of piecewise smooth one-dimensional maps, the phenomenon of "matching" refers to the property that upper and lower orbits of the discontinuity points merge with coinciding one-sided derivatives, and it becomes particularly interesting when it happens over non-trivial intervals in parameter space. This is observed in the family of shifted β-transformations $x \mapsto \beta x + \alpha \pmod 1$ for the orbits of 0 and 1, which may be thought of as the unique jump discontinuity of a map defined on the circle. Similarly, matching for the orbits of α and $\alpha-1$, is well-studied in the family of Nakada's α-continued fractions (which is probably the setting where this phenomenon was noted for the first time)

$$T_\alpha : [\alpha-1, \alpha] \to [\alpha-1, \alpha], \qquad x \mapsto \frac{1}{|x|} - \lfloor \frac{1}{|x|} + 1 - \alpha \rfloor,$$

see [21, 22, 15, 10]. Other families of α-continued fractions displaying matching were studied recently[1] [16, 17] (see also [7]), and [6], which considers a case where the underlying group is not the modular group.

The structure of the matching set in parameter space is often connected to various number-theoretic properties and bifurcation properties of the (complex) logistic family $z \mapsto z^2 + c$, see [2, 8, 9, 10, 23]. (See also Dajani & Kalle [12].)

Matching is the cause that these maps have piecewise smooth (or even piecewise constant) invariant densities, very much like the situation when a Markov partition would have existed. Also, entropy depends monotonically on the parameter in the interior of each component of the matching set.

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[1] depending on authors, matching is sometimes referred as "cycle property" or "synchronization".

Both phenomena were observed by Botella-Soler et al. [3, 4] in the family of piecewise affine maps $(G_\beta)_{\beta\in\mathbb{R}}$ defined by

$$G_\beta(x) = \begin{cases} G_\beta^-(x) = x+2 & \text{if } x \le 0, \\ G_\beta^+(x) = \beta - sx & \text{if } x > 0. \end{cases} \quad (s>1) \tag{1}$$

when the slope s of the expanding branch attains some specific values ($s=2$, $s=\frac{\sqrt{5}+1}{2}$, etc.).

For us, it is more convenient to change coordinates: for fixed $s>1$ we set

$$Q_\gamma(x) = \begin{cases} x+1, & x \le \gamma, \\ 1+s(1-x), & x > \gamma, \end{cases} \tag{2}$$

see Figure 1.

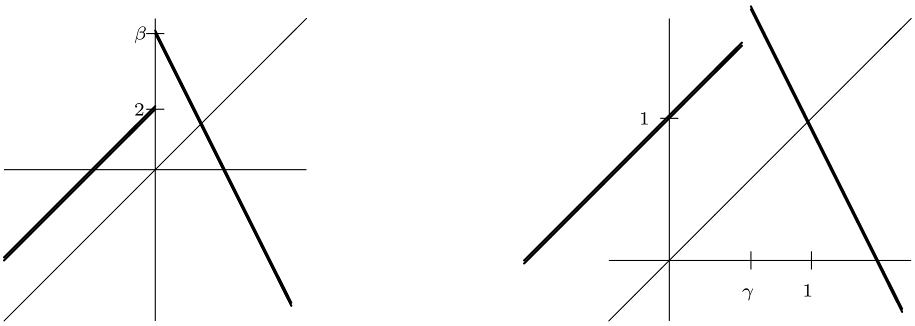

FIGURE 1. The conjugate families G_β and Q_γ for slope $s=2$.

These families are conjugate via a linear change of coordinates

$$H \circ Q_\gamma = G_\beta \circ H \quad \text{for } H(x) := 2(x-\gamma),\ \beta := 2(1+s)(1-\gamma).$$

Note that this change of coordinates $\beta = \beta(\gamma)$ reverses the orientation in parameter space; therefore, when passing from the family (Q_γ) to the family (G_β), the results about the monotonicity of entropy (see Theorem 1.9) will reverse accordingly.

The matching in the family (Q_γ) is between the upper and lower orbit of the discontinuity point γ.

As one can guess from Figure 2, there are specific values of the slope s for which matching occurs for most parameters. Another interesting feature, evident from Figure 2, are the intervals on which entropy is constant, called *plateaux*. We will see that this is due to a property we call *neutral matching*, and plateaux appear to be the closure of countably many components of the matching set (matching intervals), and they consist of much more than single (or even single "cascades" of) matching intervals.

All these features are best described in the case that the slope $s \ge 2$ is an integer. Indeed, in all these cases we can actually prove that the matching occurs for Lebesgue-a.e. value γ, and the set $\mathcal{E}$ where matching fails (called the *bifurcation set*) has zero measure (and yet the Hausdorff dimension of $\mathcal{E}$ equals 1). As it is often the case in dynamical problems, the bifurcation set has a fractal structure, and understanding the structure (and self-similarities) of this set will be important in order to characterize the plateaux of the

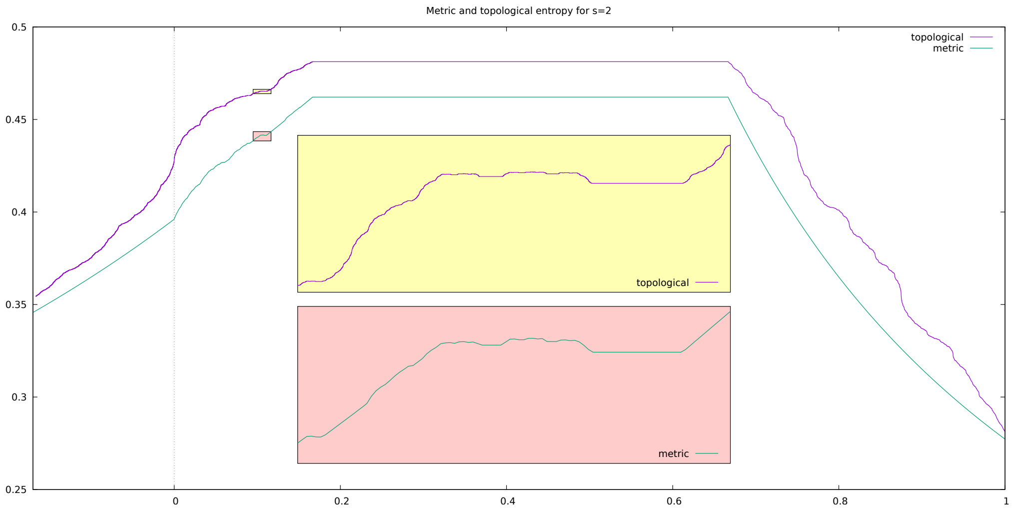

FIGURE 2. Topological and metric entropies of Q_γ for $s = 2$ as functions of γ.

entropy function. To accomplish this goal we will use methods from symbolic dynamics, including substitutions.

As we mentioned before, matching occurs in various one dimensional dynamical systems. The purpose of this paper is to investigate extensively all the features connected to the matching phenomenon in the most simple setting, namely piecewise affine maps.

We start with a result, which applies quite in general, showing that matching for a piecewise affine expanding map T implies that the density of any invariant measure is a simple function (i.e., it is piecewise constant). Then we will consider the phenomenon of matching in connection with the family Q_γ; in this setting we shall analyze its effects on the shape of the entropy, but we will also investigate for which slopes s matching occurs, and how frequent it is. The most complete picture will be given in the case the slope $s \geq 2$ is an integer, however numerical experiments show that also the cases of algebraic non-integer slopes such as the golden mean or other Pisot numbers are particularly intriguing and deserve further investigation.

1.1. Piecewise constant densities.

Let $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If T is a piecewise continuous map the *upper and lower orbits* of a point c are respectively

$$T^k(c^+) := \lim_{x \to c^+} T^k(x), \quad T^k(c^-) := \lim_{x \to c^-} T^k(x), \quad (k \in \mathbb{N}).$$

Definition 1.1. *Let $\mathbb{X}$ denote either the circle $\mathbb{S}^1$ or a compact interval. Let $T : \mathbb{X} \to \mathbb{X}$ be a piecewise smooth map, and let the* discontinuity set *$\mathcal{D}$ be the set of points where T or T' is discontinuous. We say that T satisfies the* matching condition *at a point $c \in \mathcal{D}$ if there exist positive integers $\kappa^\pm$ such that the following holds:*

$$T^{\kappa^-}(c^-) = T^{\kappa^+}(c^+) \quad \textit{and} \quad (T^{\kappa^-})'(c^-) = (T^{\kappa^+})'(c^+). \tag{3}$$

The minimal integers $\kappa_c^\pm \geq 1$ for which (3) *holds are called* matching exponents *of c.*

We say that T satisfies the matching condition *if for every $c \in \mathcal{D} \cap \overline{T(\mathbb{X})}$, the map T satisfies the matching condition at c. In this case we define the* prematching set *as*

$$\mathrm{PM} := \bigcup_{c \in \mathcal{D} \cap \overline{T(\mathbb{X})}} \left(\bigcup_{j=1}^{\kappa_c^- - 1} T^j(c^-) \cup \bigcup_{j=1}^{\kappa_c^+ - 1} T^j(c^+) \right). \tag{4}$$

The collection of complementary intervals of PM *is called the* prematching partition*.*

Let $T : \mathbb{X} \to \mathbb{X}$ be piecewise C^2 and eventually expanding T (i.e., there is $n \geq 1$ such that $\inf_{x \in \mathbb{X}} |(T^n)'(x)| > 1$). It is well-known that T has an absolutely continuous invariant probability measure (acip). We will denote this measure by μ and assume it is ergodic (otherwise we take an ergodic component). We can write $d\mu(x) = h(x)dx$ where h is a density of bounded variation. If the map is also piecewise affine and satisfies the matching condition we can say more:

Theorem 1.2. *Let T be a piecewise affine, eventually expanding map of the circle or interval, such that the matching condition holds. Then its acip μ has a density $h = \frac{d\mu}{dx}$ which is constant on elements of the prematching partition.*

The proof of Theorem 1.2 will be given in Section 2.

Remark 1.3. *By piecewise affine we mean that there is a* **finite** *partition such that T is affine on each partition element. For* **countably** *piecewise affine expanding maps, the existence of an acip is not guaranteed (cf.* [18]*), yet in many cases (*e.g. *with a finite image partition) the existence of an acip can still be proven. If T is not piecewise affine, but just piecewise smooth (such as is the case in the α-continued fractions, see* [21]*), then the same argument implies that the density h is smooth on each element of the matching partition. The α-continued fractions have a dense set of (pre)periodic points (a property used in our proof), as a consequence of density of full cylinders.*

Remark 1.4. *For some families of intervals maps (such as shifted β-transformations* [5] *or α-continued fractions* [22]*), one studies matching between the two boundary points of the interval. These points are not discontinuity points, but they belong to the prematching set (they are in fact $T(c^{\pm})$ for some $c \in \mathcal{D}$). However, one can also identify the endpoints of the interval to obtain a circle map, and use Theorem 1.2 with $\mathbb{X} = \mathbb{S}^1$: in this case the gluing point becomes the only jump discontinuity of a map of the circle, and the orbit of the left/right endpoint of the interval can be seen as the upper/lower orbit of this discontinuity.*

1.2. **Matching and monotonicity of entropy.** Let us now consider the family Q_γ for some fixed $s > 1$. The interval

$$\mathbb{X} := [1 + s(1-\theta), \theta] \quad \text{for} \quad \theta := \max\{\gamma + 1, 1 + s(1-\gamma)\} = \sup_x Q_\gamma(x) \tag{5}$$

is invariant and some iterate of Q_γ is uniformly expanding on $\mathbb{X}$. Therefore, as remarked above, Q_γ preserves an acip μ_γ; in fact μ_γ is unique, see Lemma 3.3. Therefore it is natural to study the *metric entropy function* $\gamma \mapsto h(\gamma) := h_{\mu_\gamma}(Q_\gamma)$, and it turns out that the matching property leads to monotonicity of the entropy function $\gamma \mapsto h(\gamma)$.

In this framework the matching condition is just $Q_\gamma^{\kappa^-}(\gamma^-) = Q_\gamma^{\kappa^+}(\gamma^+)$ with coinciding one-sided derivatives (for suitable positive integers $\kappa^\pm$), and the prematching set is

$$\mathrm{PM}_\gamma := \bigcup_{j=1}^{\kappa^- -1} Q_\gamma^j(\gamma^-) \cup \bigcup_{j=1}^{\kappa^+ -1} Q_\gamma^j(\gamma^+).$$

The set of γ for which Q_γ satisfies the matching condition (3) at $c = \gamma$ is possibly empty: indeed a necessary condition for (3) to hold is that s must be an algebraic integer (see Theorem 1.10). However, as soon as the slope s is compatible with the phenomenon of matching, a mild additional condition guarantees that the matching is stable:

Proposition 1.5. *Let $\gamma_0 \in \mathbb{R}$ be such that*

(1) Q_{γ_0} *satisfies the matching condition* (3) *(with $c = \gamma_0$),*
(2) $\gamma_0 \notin \mathrm{PM}_{\gamma_0}$.

Then for all γ in an open neighborhood of γ_0, the map Q_γ satisfies the same matching condition as Q_{γ_0}.

Proof. The branches $Q_{\gamma_0}^{\kappa^-}$ and $Q_{\gamma_0}^{\kappa^+}$ on either side of the discontinuity have exactly the same affine form, and since $\gamma_0 \notin \mathrm{PM}_{\gamma_0}$, this form will not change as γ moves in a small neighborhood of γ_0. □

Definition 1.6. *We call* matching set *the set of parameter values which satisfy the hypotheses (1) and (2) of Proposition 1.5. The matching set is open, its complement is denoted by $\mathcal{E}$ and it is called the* bifurcation set*. The connected components of the matching set will be called* matching intervals*.*

All points belonging to the same matching interval J satisfy the same matching condition; in particular, the matching exponents $\kappa^\pm$ are the same for all $\gamma \in J$. The difference $\Delta := \kappa^+ - \kappa^-$ is called the matching index *of the matching interval J. If $\Delta = 0$ on some matching interval J, then J is called a* neutral matching interval*.*

Remark 1.7. *Note that when γ approaches a boundary point γ_* of a matching interval, then the condition that $\gamma \notin \mathrm{PM}_\gamma$ fails for Q_{γ_*}. However, the prematching partition turns into a Markov partition when $\gamma = \gamma_*$.*

Example 1.8. *The following examples show that, for $s \geq 2$ an integer, matching occurs on some intervals:*
(1) $\gamma < 0$.

$$\begin{cases} \gamma \overset{-}{\mapsto} 1+\gamma \overset{+}{\mapsto} 1-s\gamma \overset{+}{\mapsto} 1+s^2\gamma, \\ \gamma \overset{+}{\mapsto} 1+s-s\gamma \overset{+}{\mapsto} 1-s^2+s^2\gamma \underbrace{\overset{-}{\mapsto} \dots \overset{-}{\mapsto}}_{s^2 \text{ times}} 1+s^2\gamma, \end{cases}$$

with metric entropy $h_\mu(Q_\gamma) = \frac{4\log s}{s^2+3-2\gamma(s^2-1)}$, *see Proposition 3.7.*
(2) $\gamma > \frac{s}{s+1}$.

$$\begin{cases} \gamma \overset{-}{\mapsto} 1+\gamma \overset{+}{\mapsto} 1-s\gamma \underbrace{\overset{-}{\mapsto} \dots \overset{-}{\mapsto}}_{s \text{ times}} 1+s-s\gamma, \\ \gamma \overset{+}{\mapsto} 1+s-s\gamma, \end{cases}$$

with $h_\mu(Q_\gamma) = \frac{2\log s}{2\gamma(s+1)-s+1}$, *see Proposition 3.7.*
(3) $\gamma \in (\frac{s-1}{s+1}, \frac{s}{s+1})$. *(Neutral matching)*

$$\begin{cases} \gamma \mapsto 1+\gamma \overset{+}{\mapsto} 1-s\gamma \underbrace{\overset{-}{\mapsto} \dots \overset{-}{\mapsto}}_{s-1 \text{ times}} s-s\gamma \overset{+}{\mapsto} 1+s-s^2+s^2\gamma, \\ \gamma \overset{+}{\mapsto} 1+s-s\gamma \overset{+}{\mapsto} 1-s^2+s^2\gamma \underbrace{\overset{-}{\mapsto} \dots \overset{-}{\mapsto}}_{s \text{ times}} 1+s-s^2+s^2\gamma. \end{cases}$$

with $h_{top}(Q_\gamma) = \log(\lambda_*)$ *where* λ_* *is the leading root of* $p(x) = x^s - (x^{s-1} + x^{s-2} + \dots + x + 1)$ *(so* $h_{top}(Q_\gamma) = \log \frac{1+\sqrt{5}}{2}$ *if* $s = 2$*), see Example 4.32 below. The metric entropy* $h_\mu(Q_\gamma) = \frac{2}{s+1}\log s$, *see Proposition 3.7 combined with the fact that* $\frac{s}{s+1}$ *is the right endpoint of the top plateau* $M_s = [\frac{s}{s+1} - \frac{1}{s}, \frac{s}{s+1}]$, *see Theorem 1.15.*
(4) $\gamma \in (\frac{s-2s^2+s^3}{1+s^3}, \frac{s-1}{s+1}) = (\frac{s^2-s}{s^2-s+1}\frac{s-1}{s+1}, \frac{s-1}{s+1})$. *(Neutral matching)*

$$\begin{cases} \gamma \overset{-}{\mapsto} 1+\gamma \overset{+}{\mapsto} 1-s\gamma \underbrace{\overset{-}{\mapsto} \dots \overset{-}{\mapsto}}_{s-2 \text{ times}} s-1-s\gamma \overset{+}{\mapsto} 1+2s-s^2+s^2\gamma \overset{+}{\mapsto} \\ \qquad \overset{+}{\mapsto} 1-2s^2+s^3-s^3\gamma \underbrace{\overset{-}{\mapsto} \dots \overset{-}{\mapsto}}_{s \text{ times}} 1+s-2s^2+s^3-s^3\gamma, \\ \gamma \overset{+}{\mapsto} 1+s-s\gamma \overset{+}{\mapsto} 1-s^2+s^2\gamma \underbrace{\overset{-}{\mapsto} \dots \overset{-}{\mapsto}}_{2s-1 \text{ times}} 2s-s^2+s^2\gamma \overset{+}{\mapsto} 1+s-2s^2+s^3-s^3\gamma. \end{cases}$$

The topological and metric entropies are the same as in the previous case, because $(\frac{s-2s^2+s^3}{1+s^3}, \frac{s-1}{s+1}) \subset M_s$.

Note that the metric as well as the topological entropy of Q_γ are monotone on regions where matching takes place (and a constant in the case of neutral matching), see Figure 2. This is indeed the general case, as the next theorem shows (see Section 3 for its proof).

Theorem 1.9. *If γ is in a matching interval*[2] *with matching index $\Delta = \kappa^+ - \kappa^-$, then the entropies*

$$h_\mu(Q_\gamma) \text{ and } h_{top}(Q_\gamma) \text{ are } \begin{cases} \text{increasing} & \text{if } \Delta > 0; \\ \text{constant} & \text{if } \Delta = 0; \\ \text{decreasing} & \text{if } \Delta < 0, \end{cases}$$

as function of γ.

As shown in Remark 2.6, the metric entropy is smooth on matching intervals, and when it is increasing or decreasing then it is also strictly increasing or strictly decreasing. For the topological entropy we also obtain this strict monotonicity, but our proof does not give smoothness. Cosper & Misiurewicz used a different method for piecewise affine maps similar to Q_γ to prove that entropy is locally constant at many parameter values [11].

[2] and here we really need $\gamma \notin \mathrm{PM}(\gamma)$

An interesting fact, to which we come back later, is that the entropy appears to stay constant on the parameter intervals which are covered (up to a zero measure set) by a countable union of neutral matching intervals. For instance, if the slope s is an integer, the entropy is constant on the parameter interval $M_s := [\frac{s}{s+1} - \frac{1}{s}, \frac{s}{s+1}]$ (called the top plateau, see Theorem 1.15), even though the intersection of M_s with the bifurcation set $\mathcal{E}$ has positive Hausdorff dimension.

1.3. **Occurrence of matching in the family $(Q_\gamma)_\gamma$.** We have seen that matching has interesting consequences on the dynamics of a system. It is not hard to see (numerically) that various algebraic values of the slope s frequently lead to matching. This suggests the following natural (but intriguing) question:

> For which values of the slope $s > 1$ does matching occur in the family (2), and what portion of parameter space is covered by matching intervals?

We can give the following necessary condition:

Theorem 1.10. *Let $s > 1$ be a fixed slope, and assume that Q_γ satisfies the matching condition. Then s is an algebraic integer (i.e., there is a* **monic** *polynomial $p \in \mathbb{Z}[x]$ such that $p(s) = 0$). In particular, matching cannot hold for any non-integer rational value s.*

Proof. Let R denote the first return map to $[\gamma, \infty)$ and let $d_j := R^j(\gamma+1) - R^j(\gamma^+)$; it is easy to check that

$$d_0 = 1, \quad d_{j+1} = -sd_j + k_j \quad \text{with } k_j \in \mathbb{Z}.$$

By induction $d_j = p_j(s)$ where $p_j \in \mathbb{Z}[x]$ is some monic polynomial. If matching holds $R^m(\gamma^+) = R^m(\gamma+1)$ for some $m \in \mathbb{N}$ where the matching of derivatives ensures that m is the same in the left and right hand side of this equality. Hence $p_m(s) = 0$, which means s is an algebraic integer. Moreover if s is rational then (by the Eisenstein criterion) it must be an integer. □

Even if we are not able to give a complete characterization of the slopes for which matching holds, we can prove that for integer values of s matching occurs, and is prevalent (see Section 1.4).

We can provide numerical evidence that matching holds and is prevalent also for other algebraic values such as the golden number $\phi := (\sqrt{5}+1)/2$ or the plastic constant (the real root of $x^3 = x+1$). We will also provide some examples of quadratic surd for which matching occurs but is not prevalent (see the final section).

1.4. **Prevalence of matching.** In order to understand the structure of the bifurcation set $\mathcal{E}$ let us note that:

(1) For $\gamma \leq 0$, the first return map (relative to Q_γ) to the interval $[\gamma, \gamma+1]$ is conjugate to $y \mapsto s^2 y + \gamma(s^2-1) \pmod 1$.
(2) For $\gamma \geq s/(s+1)$ the first return map (relative to Q_γ) to the interval $[\gamma, \gamma+1]$ is conjugate to $y \mapsto -sy + \gamma(s+1) \pmod 1$.
(3) If $2 \leq s \in \mathbb{N}$, then matching holds for all $\gamma \in (-\infty, 0) \cup (s/(s+1), \infty)$ (see also Example 1.8 (1) and (2) above).

Remark 1.11. *This means that for general values of s the structure of the bifurcation set outside the closed interval $[0, s/(s+1)]$ displays a periodic structure, and its structure inside each period can be studied analyzing the problem of matching for a family of generalized β-transformation with negative or positive slope. For the case of positive slope, such a study has been carried out in [5], and from the results contained there it follows that the intersection $\mathcal{E} \cap (-\infty, 0)$ has measure zero for various algebraic values of s, including $\phi := (\sqrt{5}+1)/2$, and also all other quadratic irrationals of Pisot type.*

For integer values of s, the bifurcation set $\mathcal{E}$ is contained in the interval $[0, s/(s+1)]$. Moreover in this case we can prove that matching is typical.

Theorem 1.12. *For $2 \leq s \in \mathbb{N}$, the bifurcation set $\mathcal{E} \subset [0, s/(s+1)]$ has zero Lebesgue measure and Hausdorff dimension $HD(\mathcal{E}) = 1$, although $HD(\mathcal{E} \setminus [0, \delta)) < 1$ for all $\delta > 0$.*

The proof of Theorem 1.12 will be given in Section 4.1.

The same kind of result seems to be true for other values of the slope s (such as $s = (\sqrt{5}+1)/2$ and other Pisot numbers) but we don't have a rigorous proof of prevalence in these cases.

1.5. **Pseudocenters.** In what follows we restrict to the integer case, namely $2 \leq s \in \mathbb{N}$; indeed in this setting one can provide a detailed explicit description of the fractal structure of the bifurcation set.

Recall that the components of $[0, \frac{s}{s+1}] \setminus \mathcal{E}$ are called *matching intervals*. We will see that every matching interval contains a unique s-adic rational of lowest denominator. We call this point the *pseudocenter*. Knowing the pseudocenter ξ and its **even-length** s-adic expansion $\xi = .w$, we can reconstruct the matching interval:

Theorem 1.13. *Let ξ be the pseudocenter of a matching interval V, with shortest* **even-length** *s-adic expansion $\xi = .w$. Then the boundary points $\partial V = \{\xi_L, \xi_R\}$ can be obtained from ξ:*

$$\begin{cases} \xi_L := .\overline{\check{v}v}, \\ \xi_R := .\overline{w}, \end{cases}$$

where v is the shortest **odd-length** *s-adic expansion of $1-\xi$, and $\check{v}$ is the bit-wise negation of v (i.e., $\check{a} = s-1-a$ for $a \in \{0, \dots, s-1\}$).*

The proof of Theorem 1.13 is obtained combining Lemma 4.8 and Proposition 4.9 of Section 4.2. According to Theorem 1.9, the matching index $\Delta = \Delta_\xi$ determines how the entropy depends on the parameter $\gamma \in V$. On the other hand, the following theorem (proved in Section 4.4) shows that the matching index Δ_ξ can be obtained from the s-expansion of ξ using a simple recipe. For a finite word w in the alphabet $\{0, 1, \dots, s-1\}$, define

$$\|w\| = \sum_{j=0}^{s-1} (s-1-2j)|w|_j \quad \text{for } |w|_j = \#\{i \in \{1, \dots, |w|\} : w_i = j\}. \tag{6}$$

Theorem 1.14. *Every s-adic pseudocenter ξ with even-length expansion $\xi = .w$, has matching index*

$$\Delta_\xi = \frac{s+1}{2} \|w\|,$$

which is always multiple of $s+1$*.*

As a consequence of this description, we will prove that on the left of each matching interval V there exists a period-doubling cascade of adjacent neutral matching intervals (see Proposition 4.27).

1.6. **Plateaux and self-similarities.** As mentioned before, there are intervals in parameter spaces, called *plateaux*, where the entropy is constant. In the previous section we saw that, for $2 \leq s \in \mathbb{N}$, there is neutral matching in every period-doubling cascade, but plateaux always represent more than a single period-doubling cascade. In fact we can give an explicit description of the highest plateau in the integer slope case:

Theorem 1.15. *For fixed integer slope* $s \geq 2$*, the family* Q_γ *has* top plateau

$$M_s := \left[\frac{s}{s+1} - \frac{1}{s}\ ,\ \frac{s}{s+1}\right] \tag{7}$$

where both the metric and the topological entropy are constant. This plateau consists of more than a single period-doubling cascade: in fact, the intersection $M_s \cap \mathcal{E}$ *has positive Hausdorff dimension.*

For instance for the particular family corresponding to the slope $s = 2$, both metric and topological entropy of Q_γ are constant for $\gamma \in [1/6, 2/3]$. This implies that the entropy of the maps G_β is constant for $\beta \in [2, 5]$, thus proving a conjecture which was stated in [4].

As a matter of fact Theorem 1.15 is just a particular case of Theorem 4.33, which shows that the entropy is constant on several other intervals; in Section 4.5 we shall give a detailed description of the plateaux. In order to do this, we are led to study some self-similar features of the graph of the entropy function which have also other interesting consequences.

2. Piecewise constant density

In this section, we prove Theorem 1.2, and then we give some general discussion of how to find the invariant density, based on the fact that the prematching partition plays the role of a Markov partition.

We start with a simple lemma.

Lemma 2.1. *Let* $T : \mathbb{X} \to \mathbb{X}$ *be a piecewise linear, eventually expanding map with a finite discontinuity set* $\mathcal{D}$*. Then the preperiodic points of* T *are dense.*

Proof. Take $p \in \mathbb{N}$ such that $S := T^p$ satisfies $|S'(x)| > 3$ almost everywhere. We shall prove that preperiodic points of S are dense, and this will imply our claim. Since every open interval eventually maps onto a neighborhood of some $c \in \mathcal{D}$, it is enough to prove that we can find (pre)periodic points arbitrarily close to any discontinuity. Indeed, fix $\epsilon > 0$ such that $F := \{(c-\epsilon, c), (c, c+\epsilon)\ :\ c \in C\}$ consists of disjoint intervals. We iterate any one of these intervals; its size will grow geometrically until it eventually covers some $c \in C$, and it covers a whole element of F. We repeat the argument iterating this new element, and since F has finite cardinality, we get that the original element will map onto another element which eventually maps onto itself. This leads to a preperiodic point. □

An interval J is called *nice* if $\mathrm{orb}(\partial J) \cap J^\circ = \emptyset$.

Corollary 2.2. *Under the assumptions of Lemma 2.1, every point $x \in \mathbb{X}$ has arbitrarily small nice neighborhoods.*

Proof. Let U be a neighborhood of x. By Lemma 2.1, either component of $U \setminus \{x\}$ contains a (pre)periodic point. Let us call them a and b respectively, and we can assume without loss of generality that $x \notin \mathrm{orb}(a) \cup \mathrm{orb}(b)$. Then the component of $U \setminus (\mathrm{orb}(a) \cup \mathrm{orb}(b))$ containing x is the required nice neighborhood. □

Lemma 2.3. *Let $T : \mathbb{X} \to \mathbb{X}$ be a piecewise affine, eventually expanding map satisfying the matching condition at each $c \in \mathcal{D} \cap \overline{T(\mathbb{X})}$. Let μ be an acip for T, let $J \subset supp(\mu)$ be a nice interval and $R : J \to J$ the first return map to J. If J is disjoint from the prematching set, then all* branches *(i.e., all maximal pieces of the graph of R where R is continuous and monotone) are affine and onto. Moreover R preserves Lebesgue measure, which is ergodic.*

Proof. Since $J \subset \mathrm{supp}(\mu)$ the first return map is defined μ-a.e. on J. Let $R : K \to J$ be a branch of R and assume by contradiction that $R|_K$ is not onto. Then there is $x \in \partial K$ such that $T^\tau(x) \in J^\circ$, where $\tau = \tau(K)$ is the return time of K to J.

If $\mathbb{X} = \mathbb{S}^1$ this can only happen if there exists $0 \leq j < \tau$ for which one of the following alternatives occurs:

- $T^j(x) \in \partial J$;
- $T^j(x) = c$ is a discontinuity (of T or of T').

However the first case cannot happen: since J is nice, $T^{\tau - j}(\partial J) \notin J^\circ$, which is incompatible with $T^\tau(x) \in J^\circ$.

On the other hand, if the latter case takes place, then $c \in \mathcal{D} \cap \overline{T(\mathbb{X})}$ and T satisfies the matching condition at c. Note that in this case the upper/lower orbit of x might have different return times to J, and yet, since J is disjoint from the prematching set, $\tau(x^\pm) - j \geq \kappa_c^\pm$. Furthermore, there is an interval K' adjacent to K such that $T^j(K')$ and $T^j(K)$ have c as common boundary point (say they are to the left and right of c, respectively). Due to the matching condition, $\overline{T^{j+\kappa_c^-}(K') \cup T^{j+\kappa_c^+}(K)}$ contains a two-sided neighborhood of $T^{\kappa_c^-}(c_-) = T^{\kappa_c^+}(c_+)$ and by (3), the derivative of R is constant on $K \cup K'$, but this contradicts the maximality of the branch over K.

It follows that R has only affine, surjective branches, as claimed. Since all the branches of R are affine and onto, Lebesgue measure is preserved and ergodic.

In the case when $\mathbb{X}$ is an interval, in principle one has to discuss one more case, namely that $T^j(x) = b \in \partial\mathbb{X}$ and $T^{\tau-j}(b) \in J^\circ$. However if j is the minimal iterate such that this happens, then $T^{j-1}(x) \in \mathcal{D}$ is a discontinuity for T and we actually are in the second case discussed above. The proof then is the same as in the case $\mathbb{X} = \mathbb{S}^1$. □

Remark 2.4. *The above argument actually shows that T-periodic points are dense in $supp(\mu)$*

Proof of Theorem 1.2. Since T is eventually expanding, T preserves an acip μ which is supported on a finite union of intervals. If $x \notin \operatorname{supp}(\mu)$ then the density $h = d\mu/dx$ vanishes at x, so we can restrict our attention to $x \in \operatorname{supp}(\mu)$; without loss of generality we may in fact assume that x belongs to the interior of $\operatorname{supp}(\mu)$.

Let $\kappa_c^\pm$ be the matching indices of the discontinuity points $c \in \mathcal{D} \cap T(\mathbb{X})$, so the prematching set is

$$\mathrm{PM} := \bigcup_{c \in \mathcal{D} \cap \overline{T(\mathbb{X})}} \left(\bigcup_{j=1}^{\kappa_c^- -1} T^j(c_-) \cup \bigcup_{j=1}^{\kappa_c^+ -1} T^j(c_+) \right).$$

Given any $x \in \operatorname{supp}(\mu) \setminus \mathrm{PM}$ we can use Corollary 2.2 to find a nice neighborhood $J \subset \operatorname{supp}(\mu)$ of x disjoint from PM. By Lemma 2.3 the first return map R to J has only affine, surjective branches, and it preserves Lebesgue measure m. But R is a first return map, so $\mu|_J$ is also R-invariant and coincides with Lebesgue measure, up to a constant. This constant is in fact the density $h|_J$. Since x was arbitrary, it follows that h is constant away from the prematching set. □

Let us consider any finite partition $\{P_i\}_{i=1}^N$ of $\mathbb{X}$ which is finer than the partition generated by the discontinuity set of T. Then T is affine on each P_i, $|DT(x)| = t_i$ for all $x \in P_i$ and setting $\Pi_{i,j} = |T(P_i) \cap P_j|/|P_j|$ we get,

$$|P_i| = \frac{1}{t_i}|TP_i| = \sum_{j=1}^N \frac{\Pi_{ij}}{t_i}|P_j| \qquad (1 \le i \le N)$$

and thus $(|P_1|, ..., |P_N|)^T$ is a right eigenvector (corresponding to eigenvalue 1) of the matrix $A = (A_{i,j}) = (\frac{1}{t_i}\Pi_{i,j})$.

Example 2.5. *In the case $s = 2$ and $\gamma \in [-\frac{1}{3}, 0]$, we use the partition $\{[-3+4\gamma, -2+4\gamma],\ [-2+4\gamma, -1+4\gamma],\ [-1+4\gamma, 4\gamma],\ [4\gamma, \gamma], [\gamma, 1+\gamma],\ [1+\gamma, 1-2\gamma],\ [1-2\gamma, 3-2\gamma]\}$, with transition matrix*

$$\Pi = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1-\theta & 1 & 0 \\ 1 & 1 & 1 & 1 & \theta & 0 & 0 \end{pmatrix} \quad \textit{and} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1-\theta}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{\theta}{2} & 0 & 0 \end{pmatrix},$$

where $\theta = |T(P_3) \cap P_5|/|P_5| = 1 + 3\gamma$.

If we make the ansatz that T has an invariant measure $d\mu(x) = \rho(x)dx$ with density $\rho(x) = \sum r_i 1_{P_i}(x)$ (for indicator functions 1_P), then the condition $\mu(P_j) = \mu(T^{-1}(P_j))$ implies that

$$r_j = \sum_i \frac{r_i}{t_i}\Pi_{i,j}, \qquad 1 \le j \le N. \tag{8}$$

This means that $(r_1, ..., r_N)$ must be a left eigenvector (corresponding to eigenvalue 1) of the matrix A defined above. Since $1 \in \sigma(A) = \sigma(A^T)$, equation (8) always admits nontrivial solutions, and it is well known that if $\{P_i\}_{i=1}^N$ is a Markov partition for T (i.e., if $\Pi_{ij} \in \{0, 1\}$ for all i, j) any such nontrivial (normalized) solution corresponds to an invariant (probability) measure for T.

Obviously the Markov property cannot hold in general. However, if T satisfies the hypotheses of Theorem 1.2 and the partition $\{P_i\}_{i=1}^N$ is determined by the union of the discontinuity set and prematching set then the existence of an invariant density is guaranteed by Theorem 1.2 and equation (8) provides an effective way of computing the invariant density.

Remark 2.6. *Theorem 1.2 proves that the maps Q_γ have an acip with density constant outside the prematching partition (and zero outside the invariant interval $\mathbb{X} := [1+s(1-\theta), \theta]$ from* (5)*). Thus, if γ varies in a matching interval and we consider the partition generated by the points $\{\gamma\} \cup \mathrm{PM}_\gamma$, then we see from* (8) *that the elements of the matrix A change smoothly as γ varies in the matching interval, and so does also the unique normalized solution of equation* (8)*. The entropy can be computed by the Rokhlin formula*

$$h(\gamma) = \int \log |DQ_\gamma(x)| d\mu_\gamma(x) = (\log s) \sum_{P_i \subset (\gamma,\infty)} r_i |P_i|$$

and thus it is smooth as well, see also Corollary 3.6.

Example 2.7. *In the case $s = 2$ and $\gamma \in [\frac{2}{3}, 1]$, we have the partition $\{[1-2\gamma, 2-2\gamma]\ ,\ [2-2\gamma, \gamma]\ ,\ [\gamma, 1+\gamma]\}$ and matrices*

$$\Pi = \begin{pmatrix} 0 & 1 & \theta \\ 0 & 0 & 1-\theta \\ 1 & 1 & \theta \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & \theta \\ 0 & 0 & 1-\theta \\ \frac{1}{2} & \frac{1}{2} & \frac{\theta}{2}, \end{pmatrix}$$

so in this case there are three intervals whose boundary maps to the matching point (here $\theta = 3(1-\gamma)$). The normalized left and right eigenvectors are, respectively

$$\frac{1}{5-2\theta} \begin{pmatrix} 1 & 2 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1-\theta \\ 1 \end{pmatrix}$$

Note that $\theta = 0$ as $\gamma = 1$, and $\gamma = \frac{2}{3}$ corresponds to $\theta = 1$. The metric entropy of μ is $h_\mu(Q_\gamma) = \frac{2\log 2}{6\gamma - 1}$ by the Rokhlin formula.

3. Monotonicity of entropy

From now on we shall focus on the family of maps $(Q_\gamma)_{\gamma\in\mathbb{R}}$ defined in the introduction. Note that using this notation we omit the dependence on the slope: it is implicit that different choices of the slope $s > 1$ will give rise to different families. We will show how the matching property affects some dynamical invariants of these families, such as topological or metric entropy.

Lemma 3.1. *Let us say that the upper (resp. lower) Q_γ-orbit of γ* closes in $p \geq 1$ steps *if $\gamma = Q_\gamma^p(\gamma^+)$ (resp. $\gamma = Q_\gamma^p(\gamma^-)$) and p is the minimal positive integer with such property. If γ belongs to the matching set then the upper Q_γ-orbit of γ closes in p_u steps if and only if the lower Q_γ-orbit of γ closes in p_ℓ steps, where*

$$p_u \geq \kappa^+, \quad p_\ell \geq \kappa^-, \quad p_u - p_\ell = \kappa^+ - \kappa^- = \Delta,$$

and $\kappa^\pm$ denote the matching exponents of γ. Moreover

$$(Q_\gamma^{p_u})'(\gamma^+) = (Q_\gamma^{p_\ell})'(\gamma^-).$$

Proof. Suppose that the upper Q_γ-orbit of γ closes after p_u steps. Since $\gamma \notin \mathrm{PM}_\gamma$ by the definition of matching set, $p_u \geq \kappa^+$, i.e., the orbit cannot close before matching takes place. Therefore we can change the upper orbit into the lower orbit just substituting the initial κ^+ terms of the first with the initial κ^- terms of the latter; this shows that also the lower orbit closes, and $p_\ell \le p_u - \kappa^+ + \kappa^-$. Repeating the same argument for the lower orbit we get that the closing exponents satisfy $p_u - p_\ell = \kappa^+ - \kappa^-$, yet the multiplier does not change, because of the matching of derivatives. □

Remark 3.2. *Note that $\gamma = Q_\gamma^{p_u}(\gamma^+)$ implies that the upper orbit of γ is periodic of period p_u (if $Q_\gamma^{p_u}$ is order preserving near γ) or period $2p$ (if $Q_\gamma^{p_u}$ is order reversing near γ). However if $\delta > 0$ is sufficiently small then for all $\gamma' \in (\gamma - \delta, \gamma)$, the orbit of γ under $Q_{\gamma'}$ is periodic, and coincides with $\{Q_\gamma^k(\gamma^+),\ \ 0 \le k < p_u\}$. Similarly, if $\delta > 0$ is sufficiently small then for all $\gamma'' \in (\gamma, \gamma + \delta)$, the orbit of γ under $Q_{\gamma''}$ is periodic, and coincides with $\{Q_\gamma^k(\gamma^-),\ \ 0 \le k < p_\ell\}$. So the matching index reflects how the period of a periodic orbit changes when the parameter overtakes it; however the multiplier of the orbit does not change.*

Ergodic properties of Q_γ can be obtained by standard arguments:

Lemma 3.3. *For every value of the slope $s > 1$ the maps Q_γ are Lebesgue ergodic; hence there is a unique acip μ_γ. Moreover, Q_γ is topologically transitive on supp(μ_γ).*

Proof. Let $\mathbb{X} := [1 + s(1 - \theta), \theta]$ be the invariant interval from (5), and recall that some iterate, say $S := Q_\gamma^p$, is expanding on $\mathbb{X}$. In the terminology of [25], S is an AFU map, and according to [25, Lemma 4] there are finitely many S-invariant sets with disjoint interiors, each of them supporting an ergodic absolutely continuous measure. However, since every point has a neighborhood that will be eventually mapped onto a neighborhood of the single discontinuity γ, there can be only one acip.

Take $U \subset \mathrm{supp}(\mu_\gamma)$ an open interval. Then $\cup_{j\ge 0} Q_\gamma^j(U)$ is forward invariant and of positive measure and by ergodicity of μ_γ, it has full measure and is therefore dense in $\mathrm{supp}(\mu_\gamma)$. This gives topological transitivity on $\mathrm{supp}(\mu_\gamma)$. □

In some cases (which will be relevant later on) we can even get a stronger result:

Lemma 3.4. *If $s > (1 + \sqrt{5})/2$ and $\gamma \in (0, s/(s + 1))$, then Q_γ is topologically mixing (hence also topologically transitive) on the interval $\mathbb{X} = [1 - s^2(1 - \gamma), 1 + s(1 - \gamma)]$.*

In connection with Lemma 3.3, this implies that $\mathbb{X} = \mathrm{supp}(\mu_\gamma)$.

Proof. Mutatis mutandis, this result is essentially contained in Appendix A of [4], but for the reader's convenience we reproduce the proof here. Since $\gamma \le s/(s + 1)$, $\theta = Q_\gamma(\gamma^+) = 1 + s(1 - \gamma)$ is the supremum of Q_γ. Therefore $\mathbb{X} := [Q_\gamma^2(\gamma^+), Q_\gamma(\gamma^+)] = [1 - s^2(1 - \gamma), 1 + s(1 - \gamma)]$ is invariant:

$$\begin{aligned} Q_\gamma &: [Q_\gamma^2(\gamma^+), \gamma] \to [Q_\gamma^2(\gamma^+) + 1, \gamma + 1] \subset [Q_\gamma^2(\gamma^+), Q_\gamma(\gamma^+)], \\ Q_\gamma &: (\gamma, Q_\gamma(\gamma^+)] \to [Q_\gamma^2(\gamma^+), Q_\gamma(\gamma^+)). \end{aligned}$$

We now claim that for any open interval $U \subset \mathbb{X}$ there is $k_1 \in \mathbb{N}$ such that

$$Q_\gamma^{k_1}(U) \text{ contains an open neighborhood of the fixed point } 1. \tag{9}$$

From this topological mixing follows, since $Q_\gamma^{k_2}(U) \supset \mathbb{X}$ for a suitable $k_2 \ge k_1 \in \mathbb{N}$.

Let us prove claim (9). Since $\mathbb{X}$ is bounded and Q_γ is eventually expanding there exists a minimal exponent $k_0 \geq 0$ such that $Q_\gamma^{k_0}(U)$ covers a neighborhood of the discontinuity point γ; we have that $|Q^{k_0}(U)| \geq |U|$ and $Q_\gamma^{k_0}(U)$ is an interval by the minimality of k_0.

Let us call V^+ and V^- the right and left components of $Q^{k_0}(U) \setminus \{\gamma\}$. We can assume that $V^+ \subset (\gamma, 1)$, since otherwise (9) holds. Moreover $|V^+| \geq \frac{1}{s+1}|U|$ or $|V^-| \geq \frac{s}{s+1}|U|$. In the first case $Q_\gamma^2(V^+)$ is an interval and $|Q_\gamma^2(V^+)| \geq \frac{s^2}{s+1}|U|$; in the second case $Q_\gamma(V^-)$ is an interval and $|Q_\gamma(V^-)| \geq \frac{s^2}{s+1}|U|$.

Set $\beta := \frac{s^2}{s+1}$, which is > 1 by the assumption on s. Calling $U_0 := U$ and U_1 the largest of $Q_\gamma^2(V^+)$ and $Q^\gamma(V^-)$, we have proved that

$$Q^{j_1}(U_0) \supset U_1 \quad \text{and} \quad |U_1| \geq \beta |U_0|,$$

and $U_1 \subset \mathbb{X}$. Repeating this argument, we get a sequence $(U_\ell)_{\ell \geq 0}$ of open intervals contained in $\mathbb{X}$ and a sequence $(j_\ell)_{\ell \geq 1}$ of iterates such that

$$Q^{j_\ell}(U_{\ell-1}) \supset U_\ell, \qquad |U_\ell| \geq \beta |U_{\ell-1}|.$$

This implies that for ℓ sufficiently large, $U_\ell \ni 1$ or $U_\ell \ni 0$ (so $Q_\gamma(U_\ell) \ni 1$). This proves the claim. □

Lemma 3.5. *Let $T : H \to H$ be a piecewise affine, transitive and eventually expanding map, where $H \subset \mathbb{X}$ is a finite union of intervals (or $H = \mathbb{S}^1$). Let $J \subset H$ be a non-degenerate closed interval and $b \in \partial J$. Define*

$$\tilde{T}(x) = \begin{cases} b & \textit{if } x \in J; \\ T(x) & \textit{otherwise.} \end{cases}$$

Then $h_{top}(\tilde{T}) < h_{top}(T)$.

Proof. Since $T : H \to H$ is transitive and some iterate is uniformly expanding, T supports a unique measure of maximal entropy μ and $\mu(J) > 0$, see [13]. Now $\tilde{T}$ is entropy-preservingly semiconjugate (say via ψ) to a map with slope $\pm e^{\tilde{h}_{top}(\tilde{T})}$. Let $\tilde{\nu}$ be the measure of maximal entropy of this map, and $\nu = \tilde{\nu} \circ \psi$. Then $0 = \nu(\tilde{T}(J)) \geq \nu(J)$, because ν is non-atomic. It follows that $\mathrm{supp}(\nu) \cap J^\circ = \emptyset$, and definitely $\nu \neq \mu$. However, ν is both $\tilde{T}$-invariant and T-invariant. Since μ is the unique measure of maximal entropy of T, it follows that $h_{top}(\tilde{T}) < h_{top}(T)$. □

We shall now consider the map $\gamma \mapsto h_\mu(\gamma) := h(Q_\gamma, \mu_\gamma)$ which associates to every parameter γ the metric entropy of Q_γ. Classical general results [14] ensure that this map is Hölder continuous (of any exponent $\eta < 1$); yet on intervals where matching holds the entropy function is much more regular, in fact it is analytic (see Remark 2.6 and Corollary 3.6).

But the most evident feature displayed by the metric entropy is its monotonicity on each matching interval (and whether it is increasing or decreasing is determined by the sign of the matching index). Different matching intervals are intertwined in a complicated way, so that in the end the global regularity of the entropy is no better than Hölder continuous.

Proof of Theorem 1.9. Let γ belong to a matching interval of exponents (κ^+, κ^-). By the matching condition we know that the function

$$\phi(x) := \begin{cases} Q_\gamma^{\kappa^-}(x) & \text{for } \ x \le \gamma, \\ Q_\gamma^{\kappa^+}(x) & \text{for } \ x > \gamma, \end{cases}$$

is affine on a neighborhood of γ, and $\delta := \mathrm{dist}(\mathrm{PM}_\gamma, \gamma) > 0$.

Any interval J containing γ but not intersecting PM_γ is contained in the same element of the prematching partition containing γ, and therefore is contained in the support of μ_γ. For any such interval we set $J_\gamma^+ := \{x \in J, x > \gamma\}$ and $J_\gamma^- := \{x \in J, x \le \gamma\}$.

By Corollary 2.2 we can choose J nice and so small that the Q_γ-orbit of $J_\gamma^\pm$ shadows $Q_\gamma^j(\gamma^\pm)$ for $1 \le j \le \kappa^\pm - 1$ so closely that

$$\mathrm{dist}(Q_\gamma^j(J^\pm), J) > \frac{\delta}{2} \quad \text{for all } \ 1 \le j \le \kappa^\pm - 1. \tag{10}$$

This implies that ϕ is continuous on J, since it is continuous in γ and no $x \in J_\gamma^\pm$ returns to J before iterate $\kappa^\pm$. The same argument also shows that no $x \in J$ returns to J before passing through $\tilde{J} := \phi(J)$, hence the first return map $R : J \to J$ decomposes as $R := \tilde{R} \circ \phi$, a composition of the affine map ϕ with the return map $\tilde{R} : \tilde{J} \to J$ defined as

$$\tilde{R}(x) := Q_\gamma^{\tilde{\tau}(x)}(x) \quad \text{with} \quad \tilde{\tau}(x) := \min\{k \ge 0 : \quad Q_\gamma^k(x) \in J\}.$$

Note that the only discontinuities of $\tilde{R}$ are in $\tilde{R}^{-1}(\partial J)$, and since J is nice, $\tilde{R}$ only has surjective affine branches.

If we move γ slightly within J, the map $\tilde{R}$ does not change, but the gluing point $\phi(\gamma)$ moves inside J, provoking a change in the return times. Indeed, $R(x) = Q_\gamma^{\tau(x)}(x)$ with

$$\tau(x) = \begin{cases} \kappa^- + \tilde{\tau}(\phi(x)) & \text{for } x \in J_\gamma^-, \\ \kappa^+ + \tilde{\tau}(\phi(x)) & \text{for } x \in J_\gamma^+. \end{cases} \tag{11}$$

Let us treat the metric and topological entropy separately.

Monotonicity of metric entropy. By Lemma 2.3, all branches of R are affine onto and R preserves Lebesgue measure m. Formula (11) shows that $\int_J \tau \, dm$ decreases (increases resp. remains the same) according to whether $\Delta > 0$ ($\Delta < 0$ resp. $\Delta = 0$).

Therefore, using Abramov's formula, we obtain that

$$h_\mu(Q_\gamma) = \frac{1}{\int_J \tau \, dm} h_m(R) \tag{12}$$

increases (decreases resp. remains the same) accordingly, proving the monotonicity of $h_\mu(Q_\gamma)$.

Monotonicity of topological entropy. Define

$$h_{per}(Q_\gamma, J) := \limsup_{n\to\infty} \frac{1}{n} \log \#\mathrm{Per}_n(J),$$

where $\mathrm{Per}_n(J)$ is the set of n-periodic points whose orbit intersects J. The fact that $h_{top}(Q_\gamma) = \limsup_{n\to\infty} \frac{1}{n} \log \#\mathrm{Per}_n(\mathbb{X})$ justifies this definition (note that [20] prove, for more general interval maps, that $h_{top}(T) \le \limsup_{n\to\infty} \frac{1}{n} \log \#\mathrm{Per}_n$, but since in our case

Q_γ is eventually expanding, every branch of Q_γ^n can contain at most one n-periodic point and that gives equality.)

Let $R : \cup_i J_i \to J$ be the first return map to an *open* nice interval $J \ni \gamma$ as above, i.e., $\{J_i\}_{i\geq 0}$ denote the domains of the branches. Assume for the moment that $\gamma \in J \setminus \cup_i J_i$. Then the return times $\tau_i := \tau|_{J_i}$ are constant, and the dynamics of R can be seen in a transition graph way as one vertex J from which loops of length τ_i emerge. An n-periodic orbit intersecting J is thus a combination of such loops. The exponential growth rate $\lambda^* := \exp(h_{per}(Q_\gamma, J)) > 1$ is the (unique positive) solution of the equation

$$\sum_i \lambda^{-\tau_i} = 1. \tag{13}$$

This fact is difficult to find in the literature: the first appearance of the formula seems to be [24] without proof, whereas in [19, Exercise 4.3.7] it appears as an exercise. It can also be derived from [1, Theorem 1.7].

Suppose $J =: (a, b)$. As explained in the part for metric entropy, if γ increases from a to b, each τ_i decreases by Δ, while there are no periodic points on $J \setminus \cup_i J_i$. Therefore $h_{per}(Q_a, J) < h_{per}(Q_b, J)$ ($h_{per}(Q_a, J) > h_{per}(Q_b, J)$ resp. $h_{per}(Q_a, J) = h_{per}(Q_b, J)$) if $\Delta > 0$ ($\Delta < 0$ resp. $\Delta = 0$).

To deal with the set $\mathrm{Per}_n^c(J)$ of periodic points whose orbits don't intersect J, define $\bar{Q}_\gamma$ as $\tilde{Q}_\gamma(x) = \inf_J Q_\gamma$ for $x \in J$ and $\tilde{Q}_\gamma(x) = Q_\gamma(x)$ otherwise. Then we apply Lemma 3.5 to conclude that $h_{top}(\tilde{Q}_\gamma) = \limsup_{n\to\infty} \frac{1}{n} \log \#\mathrm{Per}_n^c(J) < h(Q_\gamma)$. But this means that $h_{top}(Q_\gamma) = h_{per}(Q_\gamma, J)$.

Finally, if $\gamma' < \gamma''$ are arbitrary parameters in the same matching interval as γ, then we can write the interval (γ', γ'') as a countable union of *adjacent* nice intervals. To each of these nice intervals, the above argument applies. Therefore $h_{top}(Q_{\gamma'}) < h_{top}(Q_{\gamma''})$ ($h_{top}(Q_{\gamma'}) > h_{top}(Q_{\gamma''})$ resp. $h_{top}(Q_{\gamma'}) = h_{per}(Q_{\gamma''}, J)$) if $\Delta > 0$ ($\Delta < 0$ resp. $\Delta = 0$) and the proof is complete. □

Closer inspection of equation (12) shows that $\frac{d}{d\gamma}\left(\frac{1}{h_\mu(Q_\gamma)}\right) = \frac{\kappa^+ - \kappa^-}{h_m(R)}$ does not depend on γ, and this has an interesting consequence:

Corollary 3.6. *The function $\gamma \mapsto 1/h_\mu(\gamma)$ is locally affine on every matching interval. Therefore, if (a, b) is a matching interval, the metric entropy is given by*

$$h(\gamma) = \frac{h(b)h(a)(b-a)}{(b-\gamma)h(b) + (\gamma - a)h(a)} \qquad \textit{for all } \gamma \in (a, b), \tag{14}$$

where, for sake of readability, we abbreviated $h(\gamma) := h_\mu(Q_\gamma)$.

Formula (14) is particularly interesting in practice: indeed, even if matching fails at the endpoints of a matching interval (a, b), when $\gamma \in \{a, b\}$ the map Q_γ admits a Markov partition. Hence both $h(a)$ and $h(b)$ can be computed in a standard way.

Theorem 1.9 implies that the metric entropy is constant on every neutral matching interval, but it turns out that -in many cases- the intervals where the entropy is constant are clusters of countably many neutral matching intervals. We shall describe this phenomenon later on, providing rigorous proofs in the case $2 \leq s \in \mathbb{N}$. A peculiar feature of the families with integer slope is the presence of two unbounded matching intervals, where the metric entropy can be explicitly computed:

Proposition 3.7. *Let $s \in \mathbb{N}$, $s \geq 2$. The metric entropy of Q_γ is*

$$h_\mu(Q_\gamma) = \begin{cases} \frac{4\log s}{s^2+3-2\gamma(s^2-1)} & \text{if } \gamma \leq 0, \\ \frac{2\log s}{2\gamma(s+1)-s+1} & \text{if } \gamma \geq \frac{s}{s+1}. \end{cases}$$

Proof. First assume that $\gamma \leq 0$. We use the first return map $R = Q_\gamma^\tau : [\gamma, 1+\gamma) \to [\gamma, 1+\gamma)$ with first return time $\tau(x) = \min\{n \geq 1 : Q_\gamma^n(x) \in [\gamma, 1+\gamma]\}$, i.e.,

$$R(x) = s^2 x + k(x) \qquad \text{where } k(x) = \lfloor \gamma - s^2 x \rfloor$$

This map has constant slope s^2 and preserves Lebesgue measure, so its entropy $h_m(R) = 2\log s$. Using Abramov's formula $h_\mu(Q_\gamma)\int \tau\, dm = h_m(R)$. A somewhat tedious computation gives the above answer.

For $\gamma \geq \frac{s}{s+1}$, the first return map R to $[\gamma, 1+\gamma)$ has slope $-s$ and metric entropy $h_m(R) = \log s$. Again, Abramov's formula gives the required answer. $\square$

4. Integer slopes

Throughout this section, $s \geq 2$ will be an integer, and $\mathbb{Q}_{\mathrm{s}} = \{p/s^m : p \in \mathbb{Z}, m \geq 0\}$ will denote the set of *s-adic rationals*.

When the slope $s \in \mathbb{N}$ we can give a quite complete account of the phenomenon of matching and related features. Many (but not all) of these features can also be observed for other values of the slope, but for these cases we still have no rigorous proofs (see Section 5 for a more detailed discussion of these issues).

Let us point out that in this integer slope case the bifurcation set is bounded[3]. Indeed, from Example 1.8, it follows that matching holds on the two half lines $(-\infty, 0)$ and $(\frac{s}{s+1}, \infty)$. Hence $\mathcal{E} \subset [0, \frac{s}{s+1}]$.

4.1. **Matching is typical.**

The main focus of this section is proving Theorem 1.12. A key ingredient to reach this goal is a neat characterization of the bifurcation set. Additionally, this characterization has some consequences that will be very useful in Section 4.6.

Let us observe that, since 1 is a fixed point for Q_γ for all $\gamma < 1$, if $\gamma \in \mathbb{Q}_{\mathrm{s}} \cap [0, s/(s+1)]$, then both the upper and lower orbit of γ end up in 1, and derivatives can be made to match. Furthermore, $\gamma \notin \mathrm{PM}_\gamma$, see (4). This implies that the matching persist under a small perturbation in γ, and hence matching is an open and dense condition.

Define

$$g : [0,1) \to [0,1), \quad x \mapsto s(1-x) \bmod 1. \tag{15}$$

Lemma 4.1. *Let $x \in (0,1)$ and let $R(x)$ denote the first return of $Q_\gamma^k(x)$ to $[0,1)$. Then*

$$R(x) := \begin{cases} g(x) & \text{if } x \in (0,\gamma], \\ g^2(x) & \text{if } x \in (\gamma, 1). \end{cases}$$

[3]This is not true for general slope.

Note that R is not defined for $x = 0$ because in this case $Q_\gamma^k(x) = 1$ for all $k \geq 1$ (it never returns to $[0,1)$).

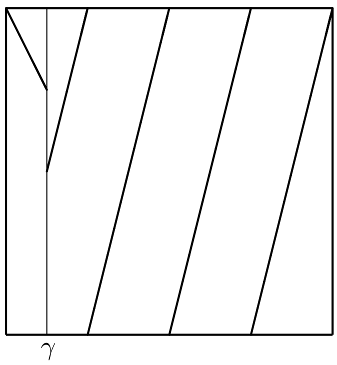

FIGURE 3. The map R for slope $s = 2$ and $\gamma = 1/8$.

Proposition 4.2. *Let $\gamma \in [0,1]$ be fixed. Then the following conditions are equivalent:*

(i) *$g^k(\gamma) < \gamma$ for some $k \in \mathbb{N}$;*
(ii) *γ belongs to the matching set.*

In other words, the bifurcation set is

$$\mathcal{E} = \{\gamma \in [0,1] \ : \ g^k(\gamma) \geq \gamma \ \forall k \in \mathbb{N}\}. \tag{16}$$

Proof. It is immediate to check that if $\gamma = p/s^m$ then both conditions (i) and (ii) hold, so let us assume that $\gamma \in [0,1]$ is not of this particular form, which is the same as assuming that $g^k(\gamma) > 0$ for all $k \in \mathbb{N}$.

(ii) $\Rightarrow$ **(i)**. Assume that $g^k(\gamma) \geq \gamma$ for all $k \in \mathbb{N}$. It follows by induction that for all integer value $j \geq 1$ there is $\delta_j > 0$ such that

$$\begin{array}{ll} R^j(x) = g^{2j}(x) > \gamma & \forall x \in (\gamma, \gamma + \delta_j), \\ R^j(x) = g^{2j-1}(x) > \gamma & \forall x \in (\gamma - \delta_j, \gamma). \end{array}$$

This implies that the upper orbit $\{R^j(\gamma^+), \ \ j \in \mathbb{N}\}$ coincides with even powers of g, while the lower orbit $\{R^j(\gamma^-), \ \ j \in \mathbb{N}\}$ coincides with odd powers of g. Since the corresponding slopes are even and odd powers of s respectively, matching cannot take place.

(i) $\Rightarrow$ **(ii)**. Let us assume that $g^k(\gamma) < \gamma$ for some integer k, and let us set

$$k_0 := \min\{k \in \mathbb{N} \ : \ g^k(\gamma) \in (0, \gamma)\}.$$

We split the discussion depending on whether k_0 is even or odd.

If $k_0 = 2h_0$ then

$$\begin{array}{ll} R^j(\gamma^+) = g^{2j}(\gamma) & 1 \leq j \leq h_0, \\ R^j(\gamma^-) = g^{2j-1}(\gamma) & 1 \leq j \leq h_0 + 1. \end{array}$$

Since $g^{2h_0}(\gamma) < \gamma$ we get that $R^{h_0+1}(\gamma^+) = g^{2h_0+1}(\gamma) = R^{h_0+1}(\gamma^-)$, hence also the derivatives match $(R^{h_0+1})'(\gamma^+) = (-s)^{2h_0+1} = (R^{h_0+1})'(\gamma^-)$ and $\gamma \notin \mathrm{PM}_\gamma$.

If $k_0 = 2h_0 + 1$ then

$$\begin{aligned} R^j(\gamma^+) &= g^{2j}(\gamma) \qquad & 1 \le j \le h_0 + 1, \\ R^j(\gamma^-) &= g^{2j-1}(\gamma) \qquad & 1 \le j \le h_0 + 1. \end{aligned}$$

Since $g^{2h_0+1}(\gamma) < \gamma$ we get that $R^{h_0+2}(\gamma^-) = g^{2h_0+2}(\gamma) = R^{h_0+1}(\gamma^+)$. Hence also the derivatives match $(R^{h_0+2})'(\gamma^-) = (-s)^{2h_0+2} = (R^{h_0+1})'(\gamma^+)$ and $\gamma \notin \mathrm{PM}_\gamma$. □

The following lemma will be used in the proof of Theorem 4.33.

Lemma 4.3. *Let $\gamma_0 \in (0,1)$ and let $p_0 > \gamma_0$ be a periodic point for Q_{γ_0}. If $p_0 \in \mathcal{E}$ then p_0 is the right endpoint of a matching interval.*

Proof. Let j_0 be the period of p_0, so $Q_{\gamma_0}^{j_0}(p_0) = p_0$; let us recall that all elements of the form $Q_{\gamma_0}^{j}(p_0)$ falling in $(0,1)$ are of the form $g^k(p_0)$ for some k. In particular, since $p_0 \in \mathcal{E}$ implies $g^k(p_0) \ge p_0$, there are no elements of the orbit of p_0 in the interval $[\gamma_0, p_0)$. Therefore p_0 is j_0-periodic for all $\gamma \in [\gamma_0, p_0)$, and its orbit does not change as γ ranges in this interval, i.e., $Q_\gamma^{j_0}(p_0) = p_0$ for all $\gamma \in [\gamma_0, p_0)$. Moreover, since γ cannot belong to the orbit of p_0 we also have that $Q_\gamma^{j_0} = g^m$ in an open neighborhood of p_0. Define $U_\gamma := (\gamma, p_0)$. We consider the cases where g^m preserves and reverses orientation separately.

***m* even:** Choose $\delta > 0$ so small that $\gamma \notin Q_\gamma^j(U_\gamma)$ for all $0 \le j < j_0$ and $\gamma \in (p_0 - \delta, p_0)$. Thus Q_γ^j is continuous on U_γ for all $0 \le j \le j_0$; in particular $Q_\gamma^{j_0}(t) = g^m(t)$ for all $t \in U_\gamma$ and $\gamma \in (p_0 - \delta, p_0)$. Since g^m is expanding, orientation-preserving and $g^m(p_0) = p_0$, we get that $g^m(\gamma) < \gamma$ for all $\gamma \in (p_0 - \delta, p_0)$. By Proposition 4.2 this proves the claim.

***m* odd:** Choose $\delta > 0$ so small that $\gamma \notin Q_\gamma^j(U_\gamma)$ for all $0 \le j < 2j_0$ and $\gamma \in (p_0 - \delta, p_0)$. Again, Q_γ^j is continuous on U_γ for all $0 \le j \le 2j_0$; in particular $Q_\gamma^{2j_0}(t) = g^{2m}(t)$ for all $t \in U_\gamma$ as soon as $\gamma \in (p_0 - \delta, p_0)$. Since g^{2m} is expanding, orientation-preserving and $g^{2m}(p_0) = p_0$ the claim follows as in the previous case. □

Remark 4.4. *Remark 3.2 and the result above are useful to describe how periodic orbits change as γ changes. Indeed, a periodic orbit can only change when γ crosses it, and this crossing may take place in a matching interval, in which case the periodic orbit persists but its period decreases by Δ as γ increases, or at the right endpoint of a matching interval. In the latter case, which is quite rare, the periodic point may even disappear.*

Proof of Theorem 1.12. Recall from (16) that $\mathcal{E} = \{t \in [0,1] : g^k(t) \ge t \; \forall k \ge 0\}$. Lebesgue measure is preserved by $g : [0,1) \to [0,1)$, so the Ergodic Theorem implies that $\inf\{g^k(\gamma) : k \ge 1\} = 0$ for Lebesgue-a.e. γ, and therefore $\mathcal{E}$ has zero Lebesgue measure.

To estimate the Hausdorff dimension, let us define for $m, r \in \mathbb{N}$, m odd:

$$K_{m,r} := \{x \in [0,1) : g^{kr}(x) \ge s^{-m} \text{ for all } k \ge 0\}.$$

Since the set $K_{m,m}$ is the attractor of the iterated function system determined by all-but-one inverses of the s^m branches of g^m, we have that $\mathrm{HD}(K_{m,m}) = \frac{\log(s^m-1)}{\log s^m}$. Moreover $K_{m,1} \subset K_{m,m} \subset K_{3m,1}$. The first inclusion is straightforward. For the second, note that if $x \notin K_{3m,1}$, so $g^k(x) \in [0, s^{-3m})$ for some k, then $g^{k+2j}(x) \in [0, s^{-m})$ for some $j \le m$ such that $k + 2j$ is a multiple of m, whence $x \notin K_{m,m}$.

Since $\mathcal{E} \setminus [0, s^{-m}) \subset K_{m,1}$ we have $\mathrm{HD}(\mathcal{E} \setminus [0, s^{-m})) \leq \frac{\log(s^m-1)}{\log s^m} < 1$. On the other hand, $g^{3m}(\mathcal{E} \cap (0, s^{-3m})) \supset K_{3m,1}$ and since $g^{3m} : (0, s^{-3m}) \to (0,1)$, as a linear map, doesn't change Hausdorff dimension, we have $\mathrm{HD}(\mathcal{E} \cap [0, s^{-3m})) \geq \frac{\log(s^m-1)}{\log s^m} \to 1$ as $m \to \infty$. □

4.2. **Pseudocenters.** In the previous section we noted that $\mathcal{E} \subset [0, s/(s+1)]$; now we shall show that there is a canonical set of labels for the components of $[0, s/(s+1)] \setminus \mathcal{E}$ which turns out also to be useful to keep track of the matching index.

Let u be a (finite or infinite) string composed with the alphabet $\mathcal{A}_s := \{0, 1, ..., s-1\}$, and let $\check{u}$ be the string obtained by flipping each digit by the involution $\epsilon \mapsto s - 1 - \epsilon$. For instance, in the case $s = 2$ if $u = 000101$, then $\check{u} = 111010$). Note that if w is an **infinite** string with digits in $\mathcal{A}_s$ and $x = .w$ is the corresponding expansion in base s, then $1 - x = .\check{w}$. The set of infinite strings on the alphabet $\mathcal{A}_s$ is totally ordered with respect to the usual lexicographic order; we will also define a partial order on **finite** strings: if u_1, u_2 are two finite strings on the alphabet $\mathcal{A}_s$, we say that $u_1 < u_2$ if every infinite string starting with u_1 is smaller than any infinite string starting with u_2.

Definition 4.5. *Let $\xi \in \mathbb{Q}_s \cap (0,1)$ and let w denote the shortest base s expansion of* **even** *length of ξ and v denote the shortest base s expansion of* **odd** *length of $1 - \xi$. We define the rational interval generated by ξ as the interval $I_\xi := (\xi_L, \xi_R)$ containing ξ where the endpoints are given by*

$$\xi_L := .\overline{\check{v}v}, \quad \xi_R := .\overline{w}.$$

Let us set $b := s - 1$ and $a := s - 2$; if $\xi = 1 - 1/s$ then $w = b0$, $v = 1$ and $\xi_L = .\overline{a1}$ while $\xi_R = .\overline{b0}$. This is (almost) the most degenerate example. In fact, for $\xi \in \mathbb{Q}_s \setminus \{1 - 1/s\}$, one can rephrase the definition of both ξ_L, ξ_R using the (even) expansion of ξ only. Indeed:

Lemma 4.6. *Let $\xi \in (0,1)$ be an s-adic rational with even s-adic expansion, so $\xi = 0.w = 0.\epsilon_1\epsilon_2 \ldots \epsilon_{2m-1}\epsilon_{2m}$. Define v to be the odd s-adic expansion of $1 - \xi$. Then*

$$v = \begin{cases} \check{\epsilon}_1 ... \check{\epsilon}_{2m-2}(\check{\epsilon}_{2m-1} + 1) & \text{if } \epsilon_{2m} = 0, \\ \check{\epsilon}_1 ... \check{\epsilon}_{2m-1}(\check{\epsilon}_{2m} + 1)0 & \text{if } \epsilon_{2m} \neq 0. \end{cases}$$

where $\check{\epsilon} = s - 1 - \epsilon$. Note that $\check{w} < v$ and $\check{v} < w$.

Proof. This is a straightforward computation. □

Example 4.7. *We give some examples for $s = 2$ in table-form:*

ξ	ξ_R	ξ_L
$\frac{1}{2} = .10$	$\frac{2}{3} = .\overline{10}$	$\frac{1}{3} = .\overline{01}$
$\frac{1}{4} = .01$	$\frac{1}{3} = .\overline{01}$	$\frac{2}{9} = .\overline{001110}$
$\frac{7}{32} = .001110$	$\frac{2}{9} = .\overline{001110}$	$\frac{7}{33} = .\overline{0011011001}$
$\frac{3}{16} = .0011$	$\frac{1}{5} = .\overline{0011}$	$\frac{2}{11} = .\overline{0010111010}$
$\frac{9}{64} = .001001$	$\frac{1}{7} = .\overline{001}$	$\frac{4334}{16383} = .\overline{00100011101110}$
$\frac{1}{8} = .0010$	$\frac{2}{15} = .\overline{0010}$	$\frac{1}{9} = .\overline{000111}$

The penultimate example in this table shows that right endpoint ξ_R can have a minimal period shorter than the length of ξ. In all of the above examples, the endpoints belong to the exceptional set, but this need not be the case, for instance if $\xi = 5/16 = .0101$ then $\xi_R = .\overline{0101} = 5/15 = 1/3$ but $\xi_L = .\overline{01000110110} \notin \mathcal{E}$. However, $I_{5/16} \subset I_{1/4}$; this follows from a general rule that we shall explain in Corollary 4.11 below.

Lemma 4.8. *If $\xi \in \mathbb{Q}_s \cap (0,1)$ then $I_\xi \cap \mathcal{E} = \emptyset$. Therefore*

$$\left[0, \frac{s}{s+1}\right] \setminus \mathcal{E} \supset \bigcup_{\xi \in \mathbb{Q}_s \cap (0,1)} I_\xi.$$

Proof. Let us consider the expanding map $f(x) = sx \pmod 1$ and the involution $\tau(x) := 1 - x$; it is easy to check that f and τ commute. Moreover, with g as in (15),

$$g^m = \begin{cases} f^m & \text{if } m \text{ is even,} \\ f^m \circ \tau & \text{if } m \text{ is odd.} \end{cases} \tag{17}$$

Let us first assume $\gamma \in [\xi, \xi_R)$. Let $\xi_R := .\overline{w}$ and let n denote the length of w. Since n is even, $g^n : \ [\xi, \xi_R] \to [0, \xi_R]$ is a continuous orientation-preserving expansive map, and $g^n(\xi_R) = \xi_R$. Therefore $g^n(\gamma) < \gamma$ for all $\gamma \in [\xi, \xi_R)$, and $\gamma \notin \mathcal{E}$ by Proposition 4.2. If $\gamma \in (\xi_L, \xi)$, let $m := |v|$. Since m is odd, it follows by equation (17) that $g^m : (\xi_L, \xi) \to (0, \xi_L)$ is an orientation-reversing homeomorphism, hence $g^m(\gamma) < g^m(\xi_L) = \xi_L < \gamma$ and $\gamma \notin \mathcal{E}$. ☐

Proposition 4.9. *Let $J = (c,d)$ be a connected component of $[0, s/(s+1)] \setminus \mathcal{E}$. Then there is a unique s-adic $\xi \in J \cap \mathbb{Q}_s$ of minimal denominator. Moreover*

- (i) *$c = \xi_L$, $d = \xi_R$ (i.e., not only $I_\xi \cap \mathcal{E} = \emptyset$ but I_ξ is maximal with respect to this property, since $I_\xi = J$);*
- (ii) *$g^k(\xi) \notin (0,d)$ for all $k \geq 1$;*
- (iii) *If $\xi' \in \mathbb{Q}_s \cap (c,d)$ is such that $g^k(\xi') \notin (0, \xi')$ for all $k \geq 1$, then $\xi' = \xi$.*

Definition 4.10. *If J is a connected component of $[0, s/(s+1)] \setminus \mathcal{E}$, the unique s-adic $\xi \in \mathbb{Q}_s \cap J$ of minimal denominator will be called the* **pseudocenter** *of J. We will denote by $\mathbb{Q}_s^{\max} \subset \mathbb{Q}_s \cap (0,1)$ the set of pseudocenters of components J of $[0, s/(s+1)] \setminus \mathcal{E}$.*

Pseudocenters provide a convenient way of labeling the connected components of $[0, s/(s+1)] \setminus \mathcal{E}$. Indeed, as a corollary of Lemma 4.8 and Proposition 4.9 we get

Corollary 4.11.

$$\left[0, \frac{s}{s+1}\right] \setminus \mathcal{E} = \bigcup_{\xi \in \mathbb{Q}_s^{\max}} I_\xi. \tag{18}$$

Proof of Proposition 4.9. Let $h_0 := \max\{k \in \mathbb{N} \ : \ g^k_{|_{[c,d]}} \text{ is continuous}\}$. We claim that if $0 \leq k \leq h_0$ then

$$g^k(x) \geq x \quad \text{ for all } x \in [c,d]. \tag{19}$$

Indeed, if k is odd then $g^k_{|_{[c,d]}}$ is a continuous orientation-reversing map and

$$x \leq d \leq g^k(d) \leq g^k(x) \quad \text{ for all } x \in [c,d].$$

Conversely, if k is even then $g^k_{|_{[c,d]}}$ is a continuous orientation-preserving expanding map, so $g^k(x) - g^k(c) > x - c$, whence $g^k(x) - x > g^k(c) - c \geq 0$ and $g^k(x) > x$.

Consider the set $Z := \{\frac{1}{s}, \frac{2}{s}, ..., \frac{s-1}{s}\}$ of discontinuity points of g, and note that $g^{h_0}([c,d]) \subset (0,1)$ but $Z \cap g^{h_0}([c,d]) \neq \emptyset$ (by maximality of h_0). Now, if in J there were more than one s-adic rational with minimal denominator, we can find a couple $\xi, \xi' \in \mathbb{Q}_{\rm s} \cap J$ such that $g^{h_0}(\xi), g^{h_0}(\xi') \in Z$ and $g^{h_0+1} : (\xi, \xi') \to (0,1)$ is onto. In particular, there exists $p \in J$ such that $g^{h_0+1}(p) = d$, whence $g^k(p) \geq p$ for all $k \geq 0$ which contradicts Proposition 4.2. Therefore there is a unique $\xi \in \mathbb{Q}_{\rm s}$ of minimal denominator in J. Moreover, ξ is the unique discontinuity of $g^{h_0+1}_{|_{[c,d]}}$ (by minimality of the pseudocenter). Thus $g^{h_0+1}_{|_{[c,\xi)}}$ and $g^{h_0+1}_{|_{(\xi,d]}}$ are continuous as well.

Now let us consider $\xi_L := .\overline{\check{v}v}$, $\quad \xi_R := .\overline{w}$ (where $w, v \in \{0, 1, ..., s-1\}^*$, $\xi = .w$, $1 - \xi = .v$, $|w|$ even, $|v|$ odd).

Since $[\xi_L, \xi_R] \subset [c,d]$ by Lemma 4.8, in order to prove **(i)** it is enough to check that ξ_L and ξ_R both belong to $\mathcal{E}$, i.e., they satisfy

$$g^k(\xi) \geq \xi \quad \text{ for all } k \in \mathbb{N}. \tag{20}$$

We split the discussion into two cases.

***h_0* is even:** then

$$|w| = h_0 + 2, \quad |v| = h_0 + 1.$$

By (19) $g^k(\xi_L) \geq \xi_L$ for $k \leq h_0$, and since $g^{h_0+1}(\xi_L) = \xi_L$, (20) holds for ξ_L.

Also ξ_R is periodic: $g^{h_0+2}(\xi_R) = \xi_R$. Thus we only have to check that $g^k(\xi_R) \geq \xi_R$ for $k \leq h_0 + 1$. The range $k \leq h_0$ is covered by (19); on the other hand since g^{h_0+1} is continuous and orientation-reversing on $(\xi, d]$ we get

$$g^{h_0+1}(\xi_R) \geq g^{h_0+1}(d) \geq d \geq \xi_R.$$

***h_0* is odd:** then

$$|w| = h_0 + 1, \quad |v| = h_0 + 2.$$

By (19) $g^k(\xi_R) \geq \xi_R$ for $k \leq h_0$, and since $g^{h_0+1}(\xi_R) = \xi_R$, (20) holds for ξ_R.

Also ξ_L is periodic: $g^{h_0+2}(\xi_L) = \xi_L$. Thus we only have to check that $g^k(\xi_L) \geq \xi_L$ for $k \leq h_0 + 1$. The range $k \leq h_0$ is covered by (19); on the other hand since g^{h_0+1} is a continuous and orientation-preserving map on $[a, \xi)$ we get $g^{h_0+1}(\xi_L) - g^{h_0+1}(c) \geq \xi_L - a$. Hence

$$g^{h_0+1}(\xi_L) - \xi_L \geq g^{h_0+1}(c) - c \geq 0,$$

i.e., $g^{h_0+1}(\xi_L) \geq \xi_L$ and we are done.

In order to prove **(ii)** we first point out that $g^k(\xi) \notin (c,d)$ for all $k \geq 1$. On the other hand $g^k(\xi) = 0$ for all $k \geq h_0 + 1$ while, by equation (19), $g^k(\xi) \geq \xi$ for $k \leq h_0$. Thus $g^k(\xi) \notin (0,d)$ for all $k \geq 1$.

Let us prove **(iii).** Note that if $\xi' \in (\xi_L, \xi)$ then $g^{|v|}(\xi') \in (0, \xi_L)$, so

$$0 < g^{|v|}(\xi') < \xi_L < \xi'.$$

On the other hand, if $\xi' \in (\xi, \xi_R)$ then $\xi_R - \xi' < g^{|w|}(\xi_R) - g^{|w|}(\xi')$ so

$$0 = g^{|w|}(\xi_R) - \xi_R > g^{|w|}(\xi') - \xi',$$

i.e., $g(\xi') < \xi'$, and we are done. □

Corollary 4.12. *Let $\xi \in \mathbb{Q}_s \cap (0, s/(s+1))$. Then $\xi \in \mathbb{Q}_s^{\max}$ if and only if $g^k(\xi) \notin (0,\xi)$ for all $k \geq 1$.*

Proof. The implication $\Rightarrow$ follows from Proposition 4.9-(ii). Conversely, if $\xi \in \mathbb{Q}_s \cap (0, s/(s+1))$ then ξ belongs to some component (c,d) of $[0, s/(s+1)] \setminus \mathcal{E}$. Thus $g^k(\xi) \notin (0,\xi)$ for all $k \geq 1$ implies, by Proposition 4.9-(iii), that ξ is the pseudocenter of (c,d). □

4.3. **Period doubling.** Another interesting consequence of the above characterization is the following:

Corollary 4.13. *Let $\xi \in \mathbb{Q}_s^{\max} \cap (0, s/(s+1))$ and let $I_\xi = (\xi_L, \xi_R)$ with $\xi_L = .\overline{\check{v}v}$. Then $\xi' := .\check{v}v \in \mathbb{Q}_s^{\max}$ as well.*

In other words, to the left of any matching interval there is an adjacent matching interval, hence there is a sequence of adjacent matching intervals. We shall refer to this phenomenon as *period-doubling bifurcation*, in analogy with period-doubling bifurcations in the quadratic family $z \mapsto z^2 + c$. Using Lemma 4.6 one can easily check that the first few elements of the period-doubling cascade are as follows:

$$\begin{aligned}
\xi_0 &= .w \\
\xi_1 &= .\check{v}v \\
\xi_2 &= .\check{v}\check{w}vw \\
\xi_3 &= .\check{v}\check{w}v\check{v}vw\check{v}v \\
\xi_4 &= .\check{v}\check{w}v\check{v}vw\check{v}\check{w}vw\check{v}v\check{v}\check{w}vw
\end{aligned} \tag{21}$$

This period-doubling phenomenon is just a particular case of tuning, we shall come back to it later on (see Proposition 4.27). A period-doubling cascade can also be described in terms of a substitution operator.

Remark 4.14. *Consider the substitution*

$$\chi : \begin{array}{ll} w \mapsto \check{v}v & \check{w} \mapsto v\check{v} \\ v \mapsto vw & \check{v} \mapsto \check{v}\check{w} \end{array}.$$

If ξ is a pseudocenter with even s-adic expansion w, then the s-adic code of the pseudocenter of the period-doubled matching interval adjacent to I_ξ is $\chi(w)$. Continuing this way, we find the s-adic codes of the pseudocenter of the matching interval in the cascade with seed ξ.

Remark 4.15. *This substitution factorizes over the Thue-Morse substitution* $\chi_{TM} : 0 \mapsto 01;\ 1 \mapsto 10$ *(via the change of symbols* $\pi(v) = \pi(\check{w}) = 0$, $\pi(\check{v}) = \pi(w) = 1$*), which in turn factorizes over the period-doubling substitution* $\chi_{PD} : 0 \mapsto 11;\ 1 \mapsto 10$.

Remark 4.16. *Denote the length of* $\chi^n(w)$ *by* l_n. *Since* $w \xrightarrow{\chi} \check{v}v \xrightarrow{\chi} \check{v}\check{w}vw$*, we find the recursive relation* $l_{n+2} = l_{n+1} + 2l_n$*, which is solved by* $l_n = 2^n \frac{|w|+2|v|}{3} + (-1)^n \frac{2|w|-2|v|}{3}$.

Proof of Corollary 4.13. By Corollary 4.12 it suffices to check that $g^k(\xi') \notin (0, \xi')$. Let us first point out that, for $m := |v|$, we have $g^m(\xi') = \xi$; hence (since $\xi \in \mathbb{Q}_s^{\max}$) $g^{m+k}(\xi') \notin (0, \xi)$. Thus we only have to check the orbit up to iterate m.

If $k < m$ then $g^k(\xi') = .\sigma$, where σ is a suffix of $\check{v}v$ of length $|\sigma| \geq m + 1$. On the other hand we know that $g^k(\xi_L) \geq \xi_L$. We claim that in fact $g^k(\xi_L) \geq \xi$. This is immediate if k is odd. For k even let us first remark that $g^k(\xi_L) \neq \xi_L$, because otherwise we would get $\xi_L = .\overline{p\sigma}$ (where p is the prefix of $\check{v}v$ of length k), so $.\overline{\sigma p} = g^k(\xi_L) = \xi_L = .\overline{p\sigma}$, would imply $g^k(\xi) = .\sigma < .\overline{\sigma p} = .\overline{p\sigma} = \xi_L$, contradicting that $\xi \in \mathbb{Q}_s^{\max}$. On the other hand $g^m : (\xi_L, \xi] \to [0, \xi_L)$ is an order reversing homeomorphism, so if $g^k(\xi_L) \in (\xi_L, \xi]$ we would also get $g^{k+m}(\xi_L) = g^m(g^k(\xi_L)) < g^m(\xi_L) = \xi_L$, another contradiction.

So we can compare the s-adic expansion of $g^k(\xi_L) = .\sigma... \geq \xi = .w$ with that of $g^k(\xi') = .\sigma$. Since the length of w is $|w| \leq m + 1 \leq |\sigma|$ we immediately get $\xi = .w \leq .\sigma = g^k(\xi')$, and we are done. □

4.4. **Matching index.** Fix an integer $s \geq 2$, and recall $g(x) = s(1 - x) \bmod 1$ from (15). The first return map of Q_γ to the interval $[0, 1]$ has the form

$$R_\gamma(x) = \begin{cases} g^2(x) = Q_\gamma^{s^2-p+2}(x) & x \in (\gamma, 1) \cap [\frac{p-1}{s^2}, \frac{p}{s^2}),\ p = 1, \dots, s^2. \\ g(x) = Q_\gamma^{p+1}(x) & x \in (0, \gamma) \cap [\frac{p-1}{s}, \frac{p}{s}),\ p = 1, \dots, s. \end{cases} \tag{22}$$

Remark 4.17. *In particular, if we use the* s*-expansion of* γ *to code the domains of the branches of* g^2 *by blocks* ab, $a, b \in \{0, \dots, s-1\}$*, then substituting* $p = sa + b + 1$ *in the first formula and* $p = a + 1$ *in the second we get*

$$R_\gamma(x) = \begin{cases} Q_\gamma^{s^2-sa-b+1}(x) & \text{if } x \in (\gamma, 1) \cap [ab], \\ Q_\gamma^{a+2}(x) & \text{if } x \in (0, \gamma) \cap [ab]. \end{cases} \tag{23}$$

Proof of Theorem 1.14. Let $\xi = .w$ be the pseudocenter of a matching interval with s-adic expansion $w = \epsilon_1 ... \epsilon_{2m}$. As usual, we have to distinguish two cases (cf. Lemma 4.6):

Case 0: If $\epsilon_{2m} = 0$ then $v = \check{\epsilon}_1 ... \check{\epsilon}_{2m-2}(\check{\epsilon}_{2m-1} + 1)$. We will use (23) to compute $\kappa^\pm$, recalling that, since $|v| = |w| - 1$ by Lemma 4.6, matching occurs when ξ^+ reaches 1 under iteration of Q_γ and ξ^- reaches 1 for the second time:

$$\kappa^+ = \sum_{i=1}^{m-1}(s^2 - s\epsilon_{2i-1} - \epsilon_{2i} + 1) + (s^2 - s\epsilon_{2m-1} - \epsilon_{2m} + 1),$$
$$\kappa^- = \epsilon_1 + 2 + \sum_{i=1}^{m-1}(s^2 - s\check{\epsilon}_{2i} - \check{\epsilon}_{2i+1} + 1).$$

Now we compute the difference, keeping in mind that $\check\epsilon_k = s-1-\epsilon_k$:

$$\begin{aligned}\kappa^+ - \kappa^- &= \sum_{i=1}^{m-1}(s\check\epsilon_{2i} + \check\epsilon_{2i+1} - s\epsilon_{2i-1} - \epsilon_{2i}) - s\epsilon_{2m-1} - \epsilon_1 + s^2 - 1\\ &= (1+s)(s-1)(m-1) - s\sum_{i=1}^{m-1}(\epsilon_{2i}+\epsilon_{2i-1})\\ &\quad - \sum_{i=1}^{m-1}(\epsilon_{2i+1}+\epsilon_{2i}) - s\epsilon_{2m-1} - \epsilon_1 + (s+1)(s-1)\\ &= (s+1)\left[(s-1)m - \sum_{k=1}^{2m}\epsilon_k\right].\end{aligned}$$

Case 1: If $\epsilon_{2m}\neq 0$ then $v = \check\epsilon_1 ... \check\epsilon_{2m-1}(\check\epsilon_{2m}+1)0$. We will use equation (23) to compute $\kappa^\pm$, but this time $|v| = |w|+1$ and hence matching occurs when ξ^+ reaches 1 for the second time and ξ^- reaches 1.

$$\begin{aligned}\kappa^+ &= \textstyle\sum_{i=1}^{m-1}(s^2 - s\epsilon_{2i-1} - \epsilon_{2i} + 1) + (s^2 - s\epsilon_{2m-1} - \epsilon_{2m} + 1) + 1,\\ \kappa^- &= \epsilon_1 + 2 + \textstyle\sum_{i=1}^{m-1}(s^2 - s\check\epsilon_{2i} - \check\epsilon_{2i+1} + 1) + s^2 - s\check\epsilon_{2m} - s + 1.\end{aligned}$$

Since $\check\epsilon_k = s-1-\epsilon_k$ the difference gives:

$$\begin{aligned}\kappa^+ - \kappa^- &= \sum_{i=1}^{m-1}(s\check\epsilon_{2i} + \check\epsilon_{2i+1} - s\epsilon_{2i-1} - \epsilon_{2i}) - s\epsilon_{2m-1} - \epsilon_{2m} - \epsilon_1 + s\check\epsilon_{2m} + s - 1\\ &= (1+s)(s-1)(m-1) - s\sum_{i=1}^{m-1}(\epsilon_{2i}+\epsilon_{2i-1}) - \sum_{i=1}^{m-1}(\epsilon_{2i+1}+\epsilon_{2i})\\ &\quad - s\epsilon_{2m-1} - \epsilon_{2m} - \epsilon_1 + s(s-1) - s\epsilon_{2m} + s - 1\\ &= (s+1)\left[(s-1)m - \sum_{k=1}^{2m}\epsilon_k\right].\end{aligned}$$

In both cases we get the same expression. To conclude the proof it remains to check that it is equivalent to the formula given in Theorem 1.14 (which is quite immediate). □

Corollary 4.18. *To the left of every matching interval I_ξ, there is a neutral matching interval $I_{\xi'}$ obtained from period doubling, namely with $w' = \check v v$ being the even s-adic expansion of ξ'. In particular, there is a cascade of neutral matching interval to the left of each I_ξ.*

Proof. It follows immediately from Corollary 4.13 that $\xi_L = \xi'_R$ for ξ' as given in the statement. The shape of w' implies that $|w|_a = |w|_{\check a}$ and hence $\|w\| = 0$. By Theorem 1.14, the matching is neutral. □

4.5. **Tuning windows and plateaux.** Throughout this section ξ is some pseudocenter with even s-adic expansion $.w$. As usual we shall denote by $.v$ the odd s-adic expansion of $1-\xi$, and $\xi_R = .\overline{w}$.

Definition 4.19. *Let* $\xi_T := .\check{v}\overline{\check{w}}$*. The interval* $T_\xi := [\xi_T, \xi_R]$ *will be called* tuning window *generated by* $\xi \in \mathbb{Q}_s^{\max}$*.*

For instance the rightmost tuning window is $M_s = [\frac{s}{s+1} - \frac{1}{s}, \frac{s}{s+1}]$.

We will show that elements in $\mathcal{E} \cap [\xi_T, \xi_R]$ have s-adic expansion that can be easily described. Aiming at this, it is very useful first to consider the set $K(\xi_T) = \{x : g^k(x) \geq \xi_T \ \ \forall k \geq 0\}$; indeed it is easily seen that $\mathcal{E} \cap [\xi_T, \xi_R] \subset K(\xi_T) \cap [0, \xi_R]$.

Theorem 4.20. *Let* $\xi = .w \in \mathbb{Q}_s^{\max}$*, then the following conditions are equivalent:*

(i) $x \in K(\xi_T) \cap [0, \xi_R]$;
(ii) *x can be written as an infinite concatenation* $x = .\sigma_1\sigma_2\sigma_3...$ *where* $\sigma_1 \in \{w, \check{v}\}$, $\sigma_j \in \{w, v, \check{w}, \check{v}\}$ *for all* $j \geq 2$*, and adjacent blocks must avoid certain patterns, namely:*
$$\sigma_j\sigma_{j+1} \notin \{vv, v\check{w}, \check{v}\check{v}, \check{v}w, wv, w\check{w}, \check{w}\check{v}, \check{w}w\}.$$

Before going into the proof, let us remark that the s-adic expansion of a point x satisfying condition (ii) corresponds to an infinite path (starting with $\check{v}$ or w) in Figure 4. We shall refer to such expansion as *admissible expansion* or *admissible concatenation.*

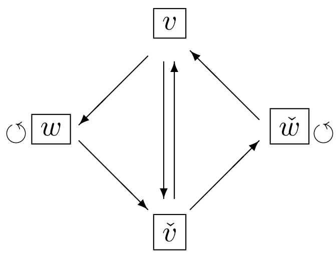

FIGURE 4. The graph for the codes of pseudocenters in the tuning window T_ξ for $\xi = .w$.

It is easy to check that any admissible expansion has the form
$$x = .w^{n_0}\check{v}\check{w}^{n_1}vw^{n_2}\check{v}\check{w}^{n_3}vw^{n_4}\check{v}\check{w}^{n_5}v...,$$
where w^n denotes the concatenation of n identical blocks w (possibly none, if $n = 0$). Here it must be understood that either $n_j \in \mathbb{N}_0$ for all $j \geq 0$, or $n_j \in \mathbb{N}_0$ for $0 \leq j < \ell$ and $n_\ell = \infty$. (in the case the expansion ends with an infinite tail of w or $\check{w}$). Note also that in an admissible expansion n_j is the exponent of w when j is even, of $\check{w}$ when j is odd. Moreover, if x has an admissible periodic expansion starting with $\check{v}$ then $x = .\overline{u}$ with $u = \check{v}\check{w}^{n_1}vw^{n_2}...\check{v}\check{w}^{n_{2\ell-1}}vw^{n_{2\ell}}$.

If $x = .\sigma_1, \sigma_2\sigma_3...$ is an admissible expansion then if $\sigma_j \in \{v, w\}$ then $\sigma_{j+1} \in \{w, \check{v}\}$, while if $\sigma_j \in \{\check{v}, \check{w}\}$ then $\sigma_{j+1} \in \{v, \check{w}\}$. Since by Lemma 4.6 $\check{w} < v$ and $\check{v} < w$, this means that the ordering between admissible expansions does not depend on the particular $\xi = .w$ which has been chosen. For instance, it is immediate that $\xi_T = .\check{v}\overline{\check{w}}$ corresponds to the smallest admissible expansion. More precisely, the following result holds:

Lemma 4.21. *Let us be given two admissible expansions*
$$\begin{aligned} x &= .w^{n_0}\check{v}\check{w}^{n_1}vw^{n_2}\check{v}\check{w}^{n_3}vw^{n_4}\check{v}\check{w}^{n_5}v... \\ x' &= .w^{n'_0}\check{v}\check{w}^{n'_1}vw^{n'_2}\check{v}\check{w}^{n'_3}vw^{n'_4}\check{v}\check{w}^{n'_5}v... \end{aligned}$$

Assume that there is $\bar{k}$ such that $n_{\bar{k}} < n'_{\bar{k}}$ but $n_j = n'_j$ for all $j < \bar{k}$. Then $x < x'$ if and only if $\bar{k}$ is even.

Proof. If $\bar{k}$ is odd, then

$$\begin{aligned} x = &\quad .p\breve{w}^{n_{\bar{k}}} v w^{n_{\bar{k}+1}} ..., \\ x' = &\quad .p\breve{w}^{n_{\bar{k}}} \breve{w}^{n'_{\bar{k}} - n_{\bar{k}}} v w^{n'_{\bar{k}+1}} ..., \end{aligned}$$

where p is a common prefix. Looking to the first block where the two expansions are different, we read a v for x and a $\breve{w}$ for x'. Since $\breve{w} < v$, we can conclude that $x' < x$. An analogous argument works when $\bar{k}$ is even. □

Remark 4.22. *Lemma 4.21 shows that, after identifying admissible expansions with the exponents $n_0 n_1 n_2 ...$, these elements are ordered according to the* alternate lexicographic order.

Definition 4.23. *Given a totally ordered alphabet $\mathcal{A}$, we define the* alternate lexicographic order $\preceq_{ALO}$ *on the space $\mathcal{A}^{\mathbb{N}}$ of infinite sequences as follows: if $\mathbf{a} = a_1 a_2 a_3 ...$ and $\mathbf{b} = b_1 b_2 b_3 ...$ we say that $\mathbf{a} \preceq_{ALO} \mathbf{b}$ if either $\mathbf{a} = \mathbf{b}$ or*

$$\exists k_0 : \quad a_k = b_k \quad \forall k < k_0 \qquad \textit{and} \qquad \begin{cases} a_{k_0} < b_{k_0} & \textit{if } k_0 \textit{ is even,} \\ a_{k_0} > b_{k_0} & \textit{if } k_0 \textit{ is odd.} \end{cases}$$

Take the alphabet $\mathcal{A} = \mathbb{N}$ the positive integers. We can identify an infinite sequence $a_1 a_2 a_3 ...$ with the continued fraction expansion $[0; a_1, a_2, a_3, ...]$; in this case the alternate lexicographic order corresponds to the usual order on the reals.

Lemma 4.24. *Let $\xi_T := .\check{v}\overline{\breve{w}}$ and $K(\xi_T) = \{x : g^k(x) \geq \xi_T \quad \forall k \geq 0\}$. If $x \in K(\xi_T) \cap [0, \xi_R]$, then x can be written as $x = .wu$ or $x = .\check{v}\check{u}$ for some $u \in \{0, 1, ..., s-1\}^{\mathbb{N}}$ such that $.u \in K(\xi_T) \cap [0, \xi_R]$.*

Proof. If $x \in [\xi, \xi_R] = [.w, .\overline{w}]$ then $x = .wu$. Moreover, since $g^{|w|} : [\xi, \xi_R] \to [0, \xi_R]$ is a homeomorphism and $K(\xi_T)$ is g-invariant, we see that $g^{|w|}(x) = .u \in K(\xi_T) \cap [0, \xi_R]$.

On the other hand, if $x \in [\xi_T, \xi]$ then $\tau x \in [1 - \xi, 1 - \xi_T] = [.v, .v\overline{w}]$, hence $\tau(x) = .vu$, i.e., $x = .\check{v}\check{u}$. Moreover, by (17) $g^{|v|} = f^{|v|} \circ \tau$, and since $\tau : [\xi_L, \xi] \to [.v, .v\overline{w}]$ and $f^{|v|} : [.v, .v\overline{w}] \to [0, \xi_R]$ are both homeomorphisms, we get that $g^{|v|} = f^{|v|}(\tau(x)) = .u \in K(\xi_T) \cap [0, \xi_R]$. □

Proof of Theorem 4.20. [(*i*) ⇒ (*ii*)] The fact that $\sigma_1 \in \{w, \check{v}\}$ is an immediate consequence of Lemma 4.24; the same is true for the fact that $\sigma_j \in \{w, v, \breve{w}, \check{v}\}$ for all j. Applying Lemma 4.24 twice, we see that the possible initial blocks in the expansion of x are $\check{v}\breve{w}, \check{v}v, ww, w\check{v}$.

Let us prove by induction that the arrows in Figure 4 represent all possible transitions. Indeed suppose $x = .\sigma_1 \sigma_2 ... \sigma_N ...$ is a concatenation of blocks which follows the arrows in the graph up to a certain N, then one of the following alternatives holds:

- $\sigma_1 ... \sigma_{N-1}$ has odd length p and $\sigma_N \in \{\breve{w}, v\}$.
- $\sigma_1 ... \sigma_{N-1}$ has even length p and $\sigma_N \in \{\check{v}, w\}$.

In the first case, $g^p(x) = .\check{\sigma}_N\check{\sigma}_{N+1}...$ and since $g^p(x) \in K(\xi_T) \cap [0, \xi_R]$ we get $\sigma_N\sigma_{N+1} \in \{vw, v\check{v}, \check{w}\check{w}, \check{w}v\}$. In the second case, $g^p(x) = .\sigma_N\sigma_{N+1}...$ and $\sigma_N\sigma_{N+1} \in \{\check{v}\check{w}, \check{v}v, ww, w\check{v}\}$. This proves the admissibility condition holds up to level $N + 1$.

$[(ii) \Rightarrow (i)]$ Let us recall that $\xi_T = .\check{v}\overline{\check{w}}$ is the smallest admissible expansion and let us prove that $\xi_T \in \mathcal{E}$. Assume by contradiction that ξ_T is inside a matching interval (ζ_L, ζ_R). Since $\zeta_R \in K(\xi_T) \cap [0, \xi_L]$ we get $\zeta_R = .\overline{\sigma_1...\sigma_\ell}$, where $\sigma_1..\sigma_\ell$ is the period of an admissible expansion starting with $\check{v}$ and ending with $\sigma_\ell \in \{v, w\}$ (because the transition $\sigma_\ell\sigma_1$ must be allowed as well). If $\sigma_\ell = v$ then $\zeta_L = .\sigma_1...\sigma_{\ell-1}\check{w}\check{\sigma}_1...\check{\sigma}_{\ell-1}w$ while if $\sigma_\ell = w$ then $\zeta_L = .\sigma_1...\sigma_{\ell-1}\check{v}\check{\sigma}_1...\check{\sigma}_{\ell-1}v$. In either case the s-adic expansion of ζ_L is an admissible concatenation of blocks starting with $\check{v}$. Therefore $\zeta_L > \xi_T$, which is a contradiction.

Now, if we consider an infinite admissible concatenation $x = .\sigma_1\sigma_2....$, we must check that $g^k(x) \geq \xi_T$. If $k = |\sigma_1...\sigma_\ell|$ then there is no problem, since $g^k(x)$ is again an admissible concatenation of blocks. Otherwise we can write $k = |\sigma_1...\sigma_{\ell-1}| + h$ with $1 \leq h \leq |\sigma_\ell| - 1$ and $g^k(x) = g^h(y)$, with $y = .\sigma_\ell\sigma_{\ell+1}...$ where $\sigma_\ell \in \{w, \check{v}\}$. If $\sigma_\ell = \check{v}$ then $g^h(y)$ belongs to the interval between $g^h(\xi_T)$ and $g^h(\xi)$ (which are both greater than ξ_T). Therefore $g^h(y) \geq \xi_T$. If $\sigma_\ell = w$, the conclusion follows by a similar argument. □

We recall that $\mathcal{E} \cap [\xi_T, \xi_R] \subset K(\xi_T) \cap [0, \xi_R]$, so the previous theorem gives a canonical representation for elements of $\mathcal{E}$ lying in the tuning window. It is then interesting to characterize the elements of $\mathcal{E} \cap [\xi_T, \xi_R]$, or also elements of $\mathbb{Q}_s^{\max} \cap [\xi_T, \xi_R]$, in terms of their period.

Theorem 4.25. *Let $x \in K(\xi_T) \cap [0, \xi_L]$ have admissible expansion*
$$x = .\check{v}\check{w}^{n_1}vw^{n_2}\check{v}\check{w}^{n_3}vw^{n_4}\check{v}\check{w}^{n_5}v...$$
Then the following conditions are equivalent

(i) $x \in \mathcal{E} \cap [\xi_T, \xi_R]$.
(ii) $n_1n_2n_3... \preceq_{ALO} n_kn_{k+1}n_{k+2}...$ *for all* $k \geq 0$.

Proof. $[(i) \Rightarrow (ii)]$ follows from Lemma 4.21 and Proposition 4.2.

$[(ii) \Rightarrow (i)]$ For any infinite admissible concatenation $x = \sigma_1\sigma_2....$ we must check that $g^k(x) \geq x$. If $k = |\sigma_1...\sigma_\ell|$ then there is no problem, since the expansion $g^k(x)$ is again an admissible concatenation of blocks which, by the hypothesis and Lemma 4.21, is no less than x. Otherwise we can write $k = |\sigma_1...\sigma_{\ell-1}|+h$ with $1 \leq h \leq |\sigma_\ell|-1$ and $g^k(x) = g^h(y)$, with $y = .\sigma_\ell\sigma_{\ell+1}... \geq x$ where $\sigma_\ell \in \{w, \check{v}\}$. If $y \in \mathcal{E}$ we are done; otherwise y lies in a matching interval $(\zeta_L, \zeta_R) \subset [\xi_T, \xi_R]$ and the same argument as in Proposition 4.9 leads to $g^h(y) \geq \zeta_R \geq x$. □

Corollary 4.26. *Let $\zeta = .z \in \mathbb{Q}_s^{\max} \cap [\xi_T, \xi_R]$. Then either $z = w$ or*
$$z = \check{v}\check{w}^{n_1}vw^{n_2}...\check{v}\check{w}^{n_{2\ell-1}}vw^{n_{2\ell}},$$
where n_j are non-negative integers such that $n_1n_2...n_{2\ell}$ is minimal among its cyclic permutations in the alternate lexicographic order.

Note that z ends either with the block w or with the block v (this latter suffix occurs if $n_{2\ell} = 0$).

For instance, given $\zeta \in \mathbb{Q}_s^{\max} \cap [\xi_T, \xi_R]$ we can well describe its period-doubling sequence of matching intervals in term of admissible expansions.

Proposition 4.27. *If $\zeta = .\sigma_1...\sigma_\ell \in \mathbb{Q}_s^{\max} \cap [\xi_T, \xi_R]$ then $\sigma_\ell \in \{v, w\}$,*

$$1 - \zeta = .\check{\sigma}_1...\check{\sigma}_{\ell-1}\sigma'_\ell, \quad \text{where } \sigma'_\ell = \begin{cases} v & \text{if } \sigma_\ell = w, \\ w & \text{if } \sigma_\ell = v. \end{cases}$$

Therefore the pseudocenter of the matching interval adjacent (on the left) to I_ζ is $\zeta_1 := .\sigma_1...\sigma_{\ell-1}\check{\sigma}'_\ell\check{\sigma}_1...\check{\sigma}_{\ell-1}\sigma'_\ell$.

Proof. By virtue of Corollary 4.13 it is enough to prove that $.\check{\sigma}_1...\check{\sigma}_{\ell-1}\sigma'_\ell$ is the expansion of $1 - \zeta$ of minimal odd length.

Indeed, since $\sigma_1...\sigma_\ell$ is the admissible factorization of the period of the right endpoint of I_ζ, it must have even length and it must end with $\sigma_\ell \in \{v, w\}$ (see Corollary 4.26), therefore $.\check{\sigma}_1...\check{\sigma}_{\ell-1}\sigma'_\ell$ has odd length. Moreover, since $.\sigma_\ell + .\sigma'_\ell = .w + .v = 1$, it is immediate to check that, setting $m := |\sigma_1...\sigma_{\ell-1}|$,

$$.\sigma_1...\sigma_{\ell-1}\sigma_\ell + .\check{\sigma}_1...\check{\sigma}_{\ell-1}\sigma'_\ell = .\sigma_1...\sigma_{\ell-1} + .\check{\sigma}_1...\check{\sigma}_{\ell-1} + s^{-m}(.\sigma_\ell + .\sigma'_\ell) = 1.$$

□

Using repeatedly this statement we can generate the formulas (21), which describe the first period-doubling cascade.

So far we have shown that, from a combinatorial point of view, all tuning windows look just the same. This has a bearing on the shape of the graph of the entropy, as we shall see soon. Before stating the next result let us introduce the following compact notation:

$$\mathbf{n} := n_1 n_2 ... n_{2\ell} \in \mathbb{N}_0^{2\ell}, \qquad [\![\mathbf{n}]\!] := \textstyle\sum_{j=1}^{\ell}(-1)^j n_j \qquad \mathbf{n}_w := \check{v}\check{w}^{n_1} v w^{n_2} ... \check{v}\check{w}^{n_{2\ell-1}} v w^{n_{2\ell}}.$$

Proposition 4.28. *Let $\zeta := .\overline{z} \in \mathcal{E} \cap [\xi_T, \xi_L]$, and write $z = \mathbf{n}_w$ (cf. Corollary 4.26). Then, with the notation $\|\ \|$ from (6), we have*

$$\|z\| = \|\mathbf{n}_w\| = [\![\mathbf{n}]\!]\ \|w\|.$$

Proof. Since clearly $\|u_1 u_2\| = \|u_1\| + \|u_2\|$ and $\|\check{u}\| = -\|u\|$, we have

$$\begin{aligned} \|z\| &= \|\check{v}\| + n_1\|\check{w}\| + \|v\| + n_2\|w\| + ... + \|\check{v}\| + n_{2\ell-1}\|\check{w}\| + \|v\| + n_{2\ell}\|w\| \\ &= \|w\| \sum_{j=1}^{\ell} (-1)^j n_j, \end{aligned}$$

as required. □

Remark 4.29. *This description allows us to see an unexpected link between the structure of $\mathcal{E}$ inside a tuning window and the set bifurcation set $\mathcal{E}_{CF}$ for the α-continued fractions of Nakada (see* [10]*). Indeed this latter bifurcation set can be characterized by means of the Gauss map G as*

$$\mathcal{E}_{CF} = \{x \in [0,1] : G^k(x) \geq x \quad \forall k \geq 0\}.$$

Considering the continued fraction expansion $x = [0; a_1, a_2, a_3, ...]$ *one can see that* $x \in \mathcal{E}_{CF}$ *if and only if the sequence* $a_1a_2a_3... \preceq_{ALO} a_{k+1}a_{k+2}a_{k+3}$ *for all* $k \geq 0$. *i.e., the string of partial quotients is minimal among its shifted copies with respect to the ALO order. The map* $\tau_w : \mathcal{E}_{CF} \to \mathcal{E} \cap [\xi_T(w), \xi_L(w)]$ *defined as*

$$\tau_w([0; a_1, a_2, ...]) = \check{v}\check{w}^{a_1-1}vw^{a_2-1}...$$

is an order preserving bijection.

Moreover, by virtue of Proposition 4.28, this correspondence has consequences for the shape of the entropy: matching intervals of positive, negative or zero index in the tuning window are intertwined exactly in the same way as the matching intervals for the α*-continued fractions.*

Following [10] let us define the set of untuned parameters $UT \subset \mathcal{E}$ as

$$UT := [0, \frac{s}{s+1}] \setminus \bigcup_{w \in \mathbb{Q}_s^{\max}} (\xi_T(w), \xi_R(w)).$$

Conjecture 4.30. *Every element* $\zeta \in UT \setminus \{\frac{s}{s+1}\}$ *is accumulated by non-neutral matching intervals.*

4.6. **Plateaux.**

Definition 4.31. *A* neutral window *for the family* $(Q_\gamma)_\gamma$ *is a maximal open interval* $J \subset (0, s/(s+1))$ *in parameter space such that* J *does not intersect any non-neutral matching interval.*

Example 4.32. *The maximal plateau* $M_s = [\frac{s}{s+1} - \frac{1}{s}, \frac{s}{s+1}]$ *from (7). For* $\gamma = \frac{s}{s+1}$, *the map* Q_γ *is continuous and has a Markov partition of* s *atoms* $[\frac{s}{s+1} - r, \ \frac{s}{s+1} - r + 1)$ *for* $r = 0, \dots, s-1$. *The transition matrix and characteristic polynomial are*

$$\begin{pmatrix} 0 & 1 & 0 & & & & \\ 0 & 0 & 1 & 0 & & & \\ \vdots & & 0 & 1 & 0 & & \\ & & & \ddots & \ddots & \ddots & \\ \vdots & & & & 0 & 1 & 0 \\ 0 & & & & & 0 & 1 \\ 1 & 1 & \dots & & \dots & 1 & 1 \end{pmatrix} \quad \text{and} \quad p(\lambda) = \lambda^s - \lambda^{s-1} - \lambda^{s-2} - \dots - 1.$$

Therefore $h_{top}(Q_\gamma)$ *is the logarithm of the leading root of* $p(\lambda)$. *We know already from Proposition 3.7 that the metric entropy is* $h_\mu(Q_\gamma) = \frac{2 \log s}{s+1}$. *Moreover* M_s *is a maximal plateau since it is accumulated by non-neutral matching intervals on the left and the adjacent non-neutral matching interval* $[\frac{s}{s+1}, \infty)$ *on the right.*

The question whether entropy is constant on the entire neutral windows (as the numerics suggest) or has some devil's staircase behavior is answered by the following:

Theorem 4.33. *If* J *is a neutral window for the family* $(Q_\gamma)_\gamma$ *then both the metric and the topological entropy are constant on the closure* $\bar{J}$.

Proof. By the results of [14] one can deduce that the dependence of the invariant density $d\mu_\gamma$ on γ is η-Hölder for any $\eta < 1$. Indeed, denoting P_t the transfer operator associated to the map Q_{γ_0+t}, our problem fits in the abstract setting of [14], which deals with a family of linear operators (P_t); in particular Corollary 1 of [14] guarantees that the eigenvector associated to the simple eigenvalue $\lambda = 1$ (i.e., the invariant density for Q_{γ_0+t}) is Hölder continuous in t (see also the remarks at page 143 of [14] to work out the details).

Consequently also the map $h(\gamma) := h_{\mu_\gamma}(Q_\gamma)$ is η-Hölder. Now let $J \subset [0, s/(s+1)]$ be a neutral window, since the origin is accumulated by non-neutral matching intervals it must happen that $\inf J \geq \delta > 0$, hence $\mathrm{HD}(J \cap \mathcal{E}) < 1$.

We have thus that by the Hölder property of h

$$\mathrm{HD}(h(J \cap \mathcal{E})) \leq \frac{1}{\eta} \mathrm{HD}(J \cap \mathcal{E}) < 1,$$

where the last inequality above is due to the fact that h is η-Hölder for any $\eta < 1$. On the other hand since J is a neutral window $h(J \cap \mathcal{E}) = h(J)$ is an interval, so the fact that $\mathrm{HD}(h(J \cap \mathcal{E})) < 1$ implies that $h(J)$ consists of a single point, i.e., h is constant on J.

Now for the topological entropy, let $\gamma_0 \in J$ be arbitrary, and $U \ni \gamma_0$ is a small neighborhood. The aim is to show that $\gamma \mapsto h_{top}(Q_\gamma)$ is constant on U, so that consequently $\gamma \mapsto h_{top}(Q_\gamma)$ is constant on the whole neutral window J. The idea is that as γ moves up through U, relatively few periodic orbits can change period, so that the exponential growth-rate of n-periodic points remains unchanged as γ varies in U. Although we need to adjust the size of U once in the proof below, it holds that $h_{top}(Q_\gamma)$ is locally constant at γ_0 and since γ_0 is arbitrary, $h_{top}(Q_\gamma)$ is constant on J.

Clearly, a periodic point p undergoes a bifurcation as $\gamma = p$, but we will show that:

(1) Q_γ displays matching if $\gamma < p$ is sufficiently close to p;
(2) if $\mathrm{orb}(p) \cap [0, p) \neq \emptyset$ for $\gamma < p$ close to p, then $\gamma \notin \mathrm{PM}_\gamma$;
(3) in this case, the period $n = \mathrm{per}(p)$ remains unchanged as γ "overtakes" p;
(4) for U sufficiently small, the exponential growth rate of n-periodic points $p \in U$ failing (2) is smaller than $\inf_{\gamma \in U} h_{top}(Q_\gamma)$.

For (1), we start with $\gamma < p$ so close to p that $(\gamma, p) \cap \mathrm{orb}(p) = \emptyset$. Throughout $n = \mathrm{per}(p)$. Let $V \ni p$ be the domain of the branch of Q_γ^n and $V' \subset V$ be an interval containing p so small that $Q_\gamma^{2n}(V') \subset V$. If we increase γ so that $\gamma < p$ and $\gamma \in V'$, then $0 < Q_\gamma^{2n}(\gamma) < \gamma < p$, so by Proposition 4.2, there is matching after at most $2n + 1$ iterates. Since J is a neutral tuning window, the matching index $\Delta = 0$.
To prove (2), if $0 < Q_\gamma^k(p) < p$ for some $k < n$, then also $0 < Q_\gamma^k(\gamma^+) < \gamma$ for $\gamma < p$ sufficiently close to p. So again by Proposition 4.2, there is matching with $\kappa^+ = \kappa^- \leq n$. Since $Q_\gamma^k(p) \neq p$ for $k < n$, also $Q_\gamma^k(\gamma^\pm) \neq \gamma$ for $k < n$, so $\gamma \notin \mathrm{PM}_\gamma$.
For (3), since all the matching in U is neutral by assumption, Lemma 3.1 shows that as γ overtakes p, neither the periodicity nor the period of p changes, see also Remark 4.4.
Finally, to show (4), let $V = [0, v]$ for some $0 < v < \inf U$, and define

$$\tilde{Q}_\gamma(x) = \begin{cases} 1 & \text{if } x \in V, \\ Q_\gamma(x) & \text{otherwise.} \end{cases}$$

Then p has a periodic orbit for $\tilde{Q}_\gamma$ if and only if p has a periodic orbit for Q_γ avoiding V. Since γ is in a neutral interval, $\gamma \in (0, \frac{s}{s+1})$, see Example 1.8. Therefore Lemma 3.4 implies that $Q_\gamma : \mathbb{X} \to \mathbb{X}$ is transitive for $\mathbb{X} = [1 - s^2(1 - \gamma), s(1 - \gamma) + 1]$. Also some iterate of Q_γ is expanding on $\mathbb{X}$. Therefore we can apply Lemma 3.5 to show that $h_{top}(\tilde{Q}_\gamma) < h_{top}(Q_\gamma)$. By taking U small we can assume by the continuity of $\gamma \mapsto h_{top}(Q_\gamma)$ that $h_{top}(\tilde{Q}_\gamma) < h_{top}(Q_{\gamma'})$ for all $\gamma, \gamma' \in U$. Now item (4) follows because $h_{top}(\tilde{Q}_\gamma) = \lim_n \frac{1}{n} \log \#\{n\text{-periodic points of } Q_\gamma \text{ avoiding } V\}$.

Observe also that if $\mathrm{orb}(p) \cap U = \emptyset$, then p undergoes no bifurcation as γ varies in U. This concludes the proof. □

The question whether every neutral window is indeed a tuning window will be discussed as Question (Q2) in the next section.

5. Numerical evidence and open problems

Before speaking about numerical evidence it is good to provide some background information on the objects we are interested in, and how we can explore them numerically.

5.1. **How do we compute?** Let us just recall that there are essentially three different ways of computing, namely (a) built-in hardware floating point arithmetic; (b) arbitrary precision arithmetic; (c) exact arithmetic (or symbolic) computations. The first method is the default, since it is fast and the precision, which is fixed, is largely adequate for most applications: the *double-precision* floating-point format available on most modern computers provides about 16 correct decimal digits. Method (b) can carry over computations using any (finite) number of correct digits, thus going beyond the built-in hardware precision. Finally, method (c) produces an exact result, let it be an algebraic number, a binary expansion or a kneading sequence. Method (a) relies on the built-in hardware representation of floating point numbers while methods (b) and (c) are computationally more expensive and only come with specific libraries or mathematical software such as Sage, Mathematica or Maple.

When computing with *finite precision* (i.e., employing methods (a) or (b)) we must be aware of the difference between the concepts of *precision* and *accuracy*: roughly speaking, the term *precision* indicates the number of digits used to represent floating point numbers, while the *accuracy* of a computation refers to the number of significant digits of its result. Often accuracy is just slightly smaller than precision, and yet there are cases where these two quantities differ strongly. If this happens we say we are facing an *ill-conditioned problem*. Overlooking this issue can even lead to computations which produce absurd results because they gain no significant digit at all.

5.2. **What do we compute?**

Invariant measure and metric entropy of Q_γ: In principle a numerical approximation of the invariant measure μ_γ can be obtained exploiting the fact that the frequency with which a typical orbit visits a small interval I is asymptotic to $\mu_\gamma(I)$. These computations also provide information about the entropy of Q_γ. Indeed, by the Rokhlin formula

$$h(Q_\gamma, \mu_\gamma) = \int_R \log |Q'_\gamma(x)| d\mu_\gamma(x) = \log(s)\mu_\gamma([\gamma, \infty)).$$

Unfortunately this general method is not very effective, and may even fail due to the fact that the computer might systematically choose non-typical points. This failure actually takes place if we use this strategy and compute with fixed precision the entropy of Q_γ when the slope $s = 2$: in this case the problem is caused by the correlation between the slope and the internal binary representation of floating point numbers.

However, when γ belongs to some matching interval, one can use an algorithm which is both more robust and much more effective in order to determine the invariant measure (and hence the entropy) of Q_γ. Indeed, we know *a priori* that the invariant density is constant on the complement of the prematching set, and computing the invariant density boils down to solving a linear system, an operation which can be easily done using exact arithmetic. Thus we used this method to compute numerically the metric entropy in cases when the matching condition is (or seems to be) dense (see Figure 2 and 6).

Formula (14) provides yet another approach to compute the metric entropy on matching intervals: indeed the entropy on a matching interval (a, b) only depends on $h(a)$ and $h(b)$, and these values can be computed in a standard way since, when γ equals one of the endpoints of a matching interval, then the map Q_γ admits a Markov partition.

Matching intervals: What we discussed just above shows that finding matching intervals for the parametric family $(Q_\gamma)_\gamma$ comes with some very precise information about the behavior entropy on such parameter values.

By Theorem 1.10, matching can occur in the family (Q_γ) if the slope s is an algebraic integer, and the quest for matching intervals is indeed an algebraic problem which can be dealt with using exact arithmetic in the algebraic number field $\mathbb{Q}[s]$.

In our numerical computations we adopt the following strategy: we fix a grid of points belonging to $\mathbb{Q}[s]$ and a safety threshold N, then for every $\bar{\gamma}$ belonging to the chosen grid we check if $Q_{\bar{\gamma}}$ satisfies the matching condition with matching exponents $\kappa^\pm \le N$; if this happens we then determine the endpoints of the matching interval containing $\bar{\gamma}$ by solving a system of linear equations in $\mathbb{Q}[s]$. We must use a threshold N in order to avoid that our algorithm gets stuck in an excessively long computation (or even infinite - in case $Q_{\bar{\gamma}}$ does not satisfy the matching property); and we will have to increase N as we go after smaller and smaller matching intervals.

In the particular case that the slope s is an integer, by the results of Section 1.4 we know that the endpoints of every matching intervals are rational and are easily deduced from the pseudocenter. This provides a much more efficient way of computing matching intervals: given an interval (c, d) with $c, d \in \mathbb{Q} \cap \mathcal{E}$ (for instance we might start setting $a := 0$ and $b := s/(s+1)$) we pick the unique $\xi \in \mathbb{Q}_s \cap (c, d)$ with lowest denominator, this ξ is the pseudocenter a matching interval $(\xi_L, \xi_R) \subset (c, d)$ (see Proposition 4.9); since both ξ_L and ξ_R are bifurcation values we can then repeat the same construction to find matching intervals inside (c, ξ_L) and (ξ_R, d) (if these are non-empty intervals). Going on with this bisection algorithm we can reach any fixed matching interval contained in (c, d) in a finite number of steps. Let us point out that all these computations are carried out in exact arithmetic (using expansions in base s), moreover the matching index relative to the matching intervals we find are computed by means of the closed formula of Theorem 1.14.

Let us mention that an analogous strategy works for searching tuning windows.

Kneading determinant and topological entropy: We compute the topological entropy through kneading invariants. For $s \in \mathbb{N}$ and $\gamma \in \mathbb{Q}$, this quantity can be computed with high accuracy: indeed in this case the map Q_γ admits a Markov partition and the kneading determinant is a rational function $R_\gamma(t) = p_\gamma(t)/q_\gamma(t)$ which we compute using exact arithmetic. On the other hand $h_{top}(Q_\gamma) = -\log(t^*)$ where t^* is the largest positive root of the polynomial p_γ. Therefore we compute the value of the topological entropy with the same accuracy we get for polynomial root finding.

5.3. **Questions about integer slopes.** The results of the previous sections provide a detailed description of the behavior of the entropy when $s \in \mathbb{N}$, yet some questions remain open. Indeed, even if the numerical evidence is quite clear we do not have yet a rigorous answer to the following questions:

(Q1) Do h_μ and h_{top} really attain their maximum values on the top plateau M_s?
(Q2) Does every neutral window coincide with the tuning window generated by some neutral interval?

Let us focus on the latter issue, which is more subtle and admits some partial result.

One can prove that if a neutral window J intersects a non-neutral tuning window (ξ_T, ξ_R) then $J \subset (\xi_T, \xi_R)$; thus, by virtue of the canonical homeomorphism described in Remark 4.29, we can use the results of [10] to conclude that J coincides with some neutral tuning window. In particular, if $w \in \mathbb{Q}_s^{\max}$ with $\|w\| \neq 0$, then the neutral tuning window of endpoints $\xi_L = .\overline{\check{v}v}$ and $\xi'_T = .\check{v}\check{w}\overline{v\check{v}}$ is a plateau for the entropy. Indeed, it only contains neutral intervals, it is adjacent to a non neutral interval on the right and is accumulated on the left by the non neutral intervals I_{ξ_n} with $\xi_n = .\check{v}\check{w}v(\check{v}v)^n$; for instance if $s = 2$ and $w = 0010$ we get that the entropy has a plateau on the interval $[125/1152, 1/9]$.

Question (Q2) admits a positive answer if and only if the following claim is true:

Claim: Every neutral tuning window which is *primitive* (i.e., it is not properly contained in another tuning window) is accumulated both on the right and on the left by non-neutral matching intervals.

This claim can indeed be checked in many particular cases, for instance if $s = 2$ and $w = 00001111$ we have that the tuning window $[\xi_T, \xi_R]$ has endpoints $\xi_T = .000011101\overline{11110000}$ and $\xi_R = .\overline{00001111}$ which are accumulated by the matching intervals with pseudocenters $.000011101(11110000)^n11110$ and $.0000111011(11100001)^n$, respectively.

5.4. **Irrational slopes.** As we mentioned in the introduction, the slope s does not need to belong to $\mathbb{N}$ for matching to occur. Note that matching may occur for a particular value of s without implying that matching is prevalent in the family $(Q_\gamma)_\gamma$. For instance, for $s = \sqrt{5}-1$ one can find a few matching intervals even if there is certainly no matching interval intersecting the half line $(-\infty, 0)$ (this last statement follows easily from Remark 1.11 together with the results of [5]).

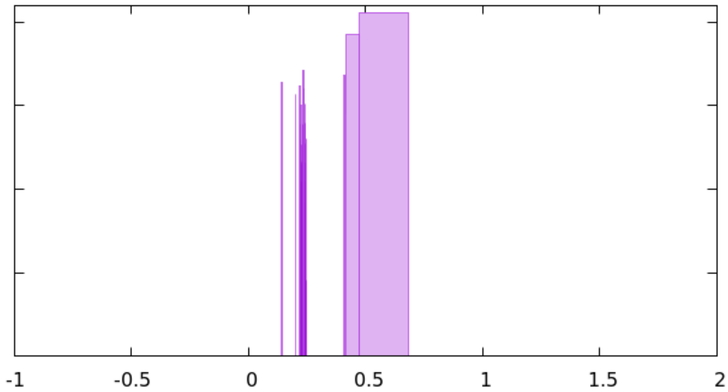

FIGURE 5. The boxes in this picture are built on the matching intervals of (Q_γ) for slope $s = \sqrt{5}+1$ and their heights are different, depending on the size of the interval. The largest matching interval, $(2\sqrt{5}-4, \frac{6}{5}\sqrt{5}-2)$, has matching exponents $(5,6)$ and has an adjacent neutral matching interval on the left, $(\frac{24+46\sqrt{5}}{31}, 2\sqrt{5}-4)$ with matching exponents $(9,9)$. Note that the matching set does not exhaust parameter space; in particular there are no matching intervals outside $[0,1]$.

On the other hand there are several choices for the slope s which seem to lead to prevalent matching in the family $(Q_\gamma)_\gamma$; in fact in such cases the entropy has the same self-similar features observed when the slope $s \geq 2$ is an integer value.

One first example of this can be observed when the slope s is a quadratic Pisot irrational ($s = (\sqrt{5}+1)/2$, for instance). Numerical evidence suggests that matching is prevalent, one can also observe the period-doubling phenomenon inside the window $[0, s/(s+1)]$, and it seems that the bifurcation set has complex fractal structure even inside every plateau of the entropy, but complete proofs of all these features are still missing.

The plateau which can be seen in Figure 6 contains many matching intervals; the largest being $(\frac{3-\sqrt{5}}{4}, \frac{3-\sqrt{5}}{2})$. Numerical evidence suggests the top plateau occurs for $\gamma \in [\frac{7-3\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}]$.

A peculiar feature which marks a difference with the integer slope cases is that when the slope is irrational the bifurcation set $\mathcal{E}$ is not bounded. Using Remark 1.11 and the results of [5] once again one can prove that $\mathcal{E}$ has in fact a periodic structure outside the bounded interval $[0, s/(s+1)]$.

With the same techniques can also obtain partial results about prevalence. For instance the result of [5] imply that for all values s which are quadratic Pisot, $\mathrm{HD}(\mathcal{E} \cap (-\infty, 0)) < 1$.

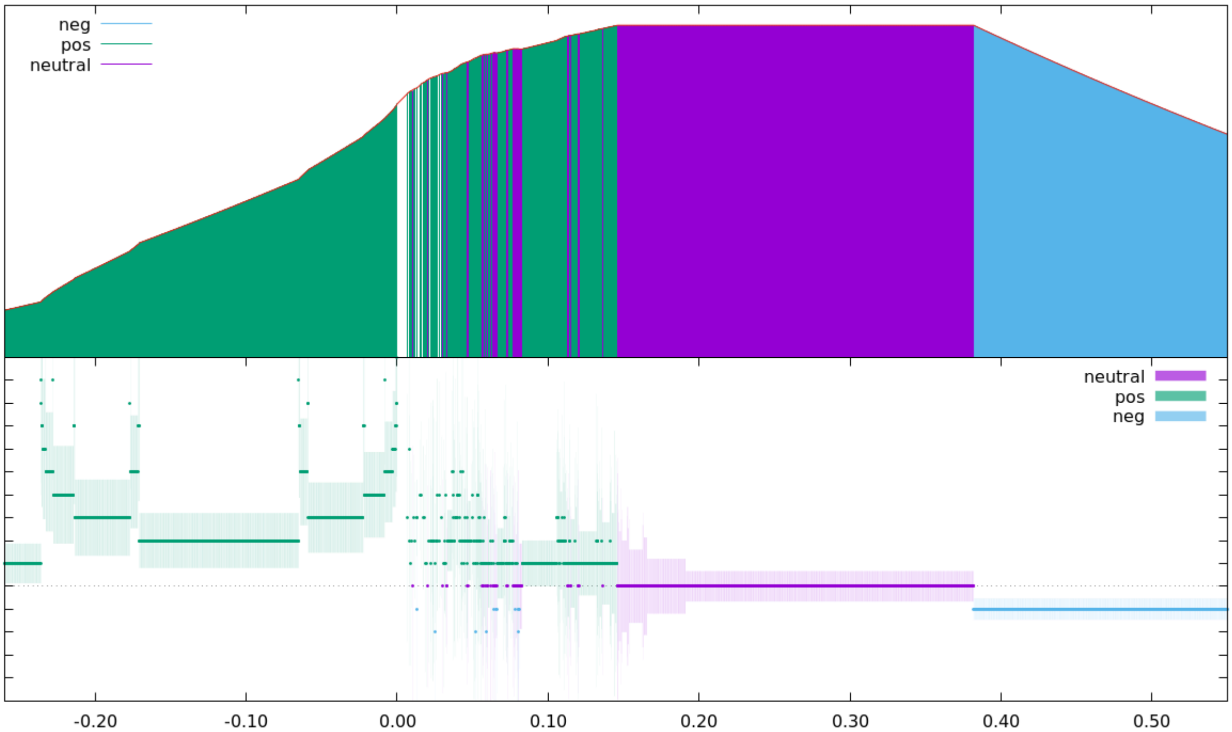

FIGURE 6. Q_γ with $s = (\sqrt{5}+1)/2$. Comparing entropy (above) and matching index (bottom); in the bottom picture a vertical whisker has been plotted around the value of the matching index, the size of the whisker is proportional to the sum of the matching exponents $\kappa^+ + \kappa^-$. Here we see that the entropy behaves very much like the integer slope case; one can detect the first few occurrences of period doubling, but one can also see that the top plateau contains more than a single period-doubling cascade.

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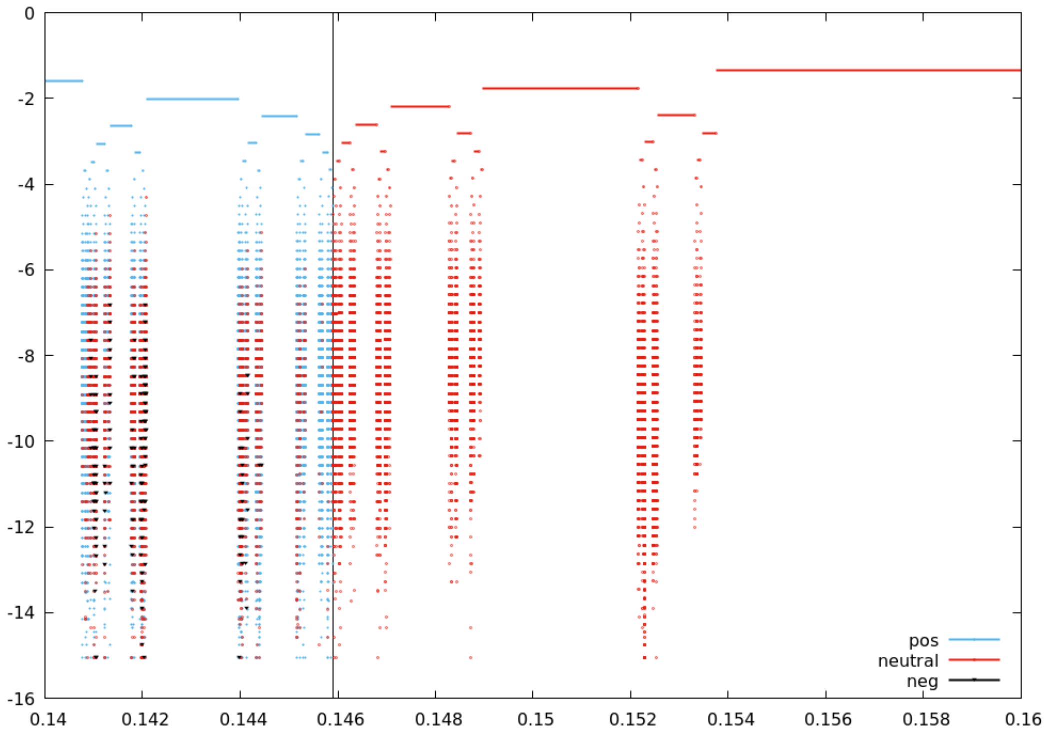

FIGURE 7. Q_γ with $s = (\sqrt{5}+1)/2$. Matching intervals plotted at different levels (accordingly to their sizes). It seems that the left endpoint of the first tuning window is $\frac{7-3\sqrt{5}}{2} = 0.145898033750315....$

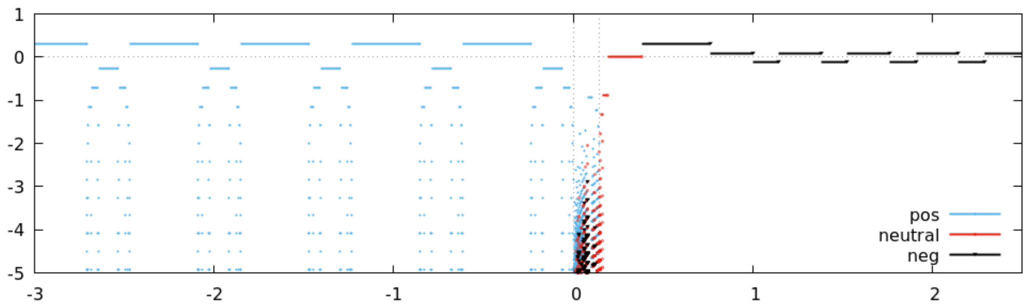

FIGURE 8. Q_γ with $s = (\sqrt{5}+1)/2$. Same picture as before, but zooming out it is now evident that the periodic structure extends both on the left and on the right.

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