# The existence of absolutely continuous invariant measures is not a topological invariant for unimodal maps

Henk Bruin*
Mathematisches Institut, Universität Erlangen-Nürnberg
Bismarckstraße $1\frac{1}{2}$, 91054 Erlangen, Germany
bruin@mi.uni-erlangen.de

March 28, 2001

**Abstract**

Within the class of S-unimodal maps with fixed critical order it is shown that the existence of an absolutely continuous invariant probability measure is not a topological invariant.

## 1 Introduction

Rigidity is one of the main themes of interest in (one-dimensional) dynamics. We speak of *rigidity* if a certain topological property, within some smoothness class of mappings, guarantees a much stronger metrical or measure-theoretical structure; or if a certain metric or measure-theoretical property is preserved under topological conjugacies. A good example is the theory of circle diffeomorphisms. Here, the mere fact that the rotation number of a circle diffeomorphism satisfies a Diophantine condition guarantees smoothness of the conjugacy with a circle rotation, see e.g. [6, 17]. In a non-invertible setting, a much less orderly picture is to be expected. Let us restrict ourselves to the class $\mathcal{F}$ of $C^3$ S-unimodal maps of the interval with a fixed critical order. Contrary to the case of circle diffeomorphisms, conjugacies between smooth unimodal maps and tent-maps, i.e. piecewise linear unimodal maps, are in general not absolutely continuous, let alone $C^1$. There can be weaker forms of rigidity though.

Let $f$ be a unimodal map on the interval $I = [0, 1]$ with critical point $c$. Then $f$ satisfies the *Collet-Eckmann condition* if $\liminf \frac{1}{n} \log |Df^n(f(c))| > 0$. The question whether the Collet-Eckmann condition is a topological invariant, is one of the major open questions in interval dynamics. Some partial answers were given in [12, 16]. In [15] it was shown that the Collet-Eckmann condition is invariant under quasisymmetric conjugacies. A conjugacy $h : I \to I$ is *quasisymmetric* if there exists $K \geq 1$ such that $\frac{1}{K} \leq \frac{|h(x+\varepsilon)-h(x)|}{|h(x)-h(x-\varepsilon)|} \leq K$ for all $x \in I$ and $\varepsilon > 0$. This brings us to a second open question: Under what conditions two conjugate maps in $\mathcal{F}$ are also quasisymmetrically conjugate? Many results with respect to this question were obtained in e.g. [5, 8, 9].

It is known that a Collet-Eckmann map admits an absolutely continuous invariant probability measure (henceforth called *acip*). But an acip can exist under weaker conditions too, cf. [14]. In this paper we prove

*Supported by the Deutsche Forschungsgemeinschaft (DFG)

**Theorem 1** *The existence of an absolutely continuous invariant probability measure is not a topological invariant in $\mathcal{F}$.*

In other words, there exist two conjugate maps $f, g \in \mathcal{F}$ such that $f$ has an acip, but $g$ has not. In fact Theorem 1 applies to very general families of maps. The only things we will use in the proof are negative Schwarzian derivative, the abundance of long-branched maps (see Definition 1) in families of unimodal maps, and the fact that fixed points of conjugate but different maps usually have different multipliers. As an example we use the standard quadratic and a sine family. Note also that the order of the critical point is the same for both maps. We will show that the result holds for $f = a \sin \pi x$ and $g = bx(1-x)$ for appropriate values of $a$ and $b$. If we allow different critical orders, there is a counter-example in the so-called Fibonacci map: A quadratic Fibonacci map has an acip [10], while a Fibonacci map with sufficiently large critical order has not [4]. In fact, it can even have a *wild* attractor. It would be interesting to know if two conjugate long-branched maps are always quasisymmetrically conjugate. An affirmative answer would prove the conjecture: Within $\mathcal{F}$, the existence of an acip is not preserved under quasisymmetric conjugacies.

Theorem 1 was contained in our PhD.-thesis [2], but in this paper we try to keep the proof as concise as possible. We want to thank the referee for his suggestions.

## 2 Preliminaries

A map $f : I \to I$ is *unimodal* if there exists a unique critical point, $c$, such that $f$ is increasing on the left and decreasing on the right of $c$. Assuming that $f$ is $C^1$, $Df$ must vanish at $c$. Let $c_i := f^i(c)$, and the *critical orbit* $orb(c) := \{c, c_1, c_2, \ldots\}$. We say that $c$ has *critical order* $\ell$ if there exist $0 < O_1 \leq O_2$ such that

$$\ell O_1 \leq \frac{|Df(x)|}{|x-c|^{\ell-1}} \leq \ell O_2 \text{ and } O_1 \leq \frac{|f(x)-f(c)|}{|x-c|^{\ell}} \leq O_2. \quad (1)$$

Moreover, $\frac{O_2}{O_1}$ can be taken arbitrarily close to 1, provided $x$ is sufficiently close to $c$. Let $x \mapsto \hat{x}$ be defined as $\hat{c} = c$, and for $x \neq c$, $\hat{x} \neq x$ is the unique point such that $f(\hat{x}) = f(x)$. If $f$ has critical order $\ell$ for some $\ell > 0$, then $|x - c| \mapsto |\hat{x} - c|$ is Lipschitz. For simplicity we will assume that $f$ is *symmetric*, i.e. $|x - c| = |\hat{x} - c|$. A $C^3$ map $f$ has *negative Schwarzian derivative* if $\frac{D^3 f(x)}{Df(x)} - \frac{3}{2}(\frac{D^2 f(x)}{Df(x)})^2 < 0$ whenever $Df(x) \neq 0$. A unimodal map with negative Schwarzian derivative is called *S-unimodal*. An interval $J$, $c \in J \subset I$, is called *restrictive* if $f^n(J) \subset J$ for some $n > 1$. If a restrictive interval exists, $f$ is called *renormalizable*. It can be derived from the results in [1], that if $f$ is not renormalizable and has no periodic attractor, then the following property holds:

$$\text{If } |A| > 0, \text{ then } \bigcup_i f^{-i}(A) \text{ has full Lebesgue measure.} \quad (2)$$

Here $|\ |$ denotes Lebesgue measure. If $f$ is finitely renormalizable, then (2) remains valid if $A$ is taken in the smallest restrictive interval of $f$. We will fix $1 < \ell < \infty$ throughout the paper, and let $\mathcal{F}$ be the class of symmetric $C^3$ S-unimodal maps with critical order $\ell$, having no periodic attractor.

An interval $T$ is called a *monotonicity interval* of $f^n$ if it is a maximal interval on which $f^n$ is monotone. We will need the notions of *cutting times* $\{\mathcal{S}_k\}_{k \geq 0}$ and *closest precritical points* $\{z_k\}_{k \geq 0}$. Define $\mathcal{S}_0 := 1$, $z_0 := f^{-1}(c) \cap (0, c)$ and inductively

$$\begin{aligned} \mathcal{S}_{k+1} &:= \min\{n > \mathcal{S}_k \mid f^n((z_k, c)) \ni c\}, \\ z_{k+1} &:= f^{-\mathcal{S}_{k+1}}(c) \cap (z_k, c). \end{aligned}$$

In particular, $f^n|_{(z_k,c)}$ and $f^n|_{(c,\hat{z}_k)}$ are monotone for every $n \leq \mathcal{S}_{k+1}$. We claim that $f^{\mathcal{S}_{k-1}}(z_k)$ is a closest precritical point. Clearly $y := f^{\mathcal{S}_{k-1}}(z_k) \in f^{\mathcal{S}_{k-1}-\mathcal{S}_k}(c)$, so it is a precritical point. If $y$ is not a closest precritical point, then there exists $z_n \in (y,c)$ (or $\hat{z}_n \in (c,y)$) such that $\mathcal{S}_n < \mathcal{S}_k - \mathcal{S}_{k-1}$. Pulling back $z_n$ (or $\hat{z}_n$) via the branch $f^{\mathcal{S}_{k-1}}|_{(z_{k-1},z_k)}$ yields a point in $f^{-\mathcal{S}_n-\mathcal{S}_{k-1}}(c) \cap (z_{k-1}, z_k)$. This contradicts that $z_k$ is a closest precritical point, proving the claim. As $y \in f^{\mathcal{S}_{k-1}-\mathcal{S}_k}(c)$ is a closest precritical point, it follows that $\mathcal{S}_k - \mathcal{S}_{k-1}$ is again a cutting time. We can define the *kneading map* $Q : \mathbf{N} \to \mathbf{N}$ by

$$\mathcal{S}_k - \mathcal{S}_{k-1} = \mathcal{S}_{Q(k)}.$$

In particular, we have

$$f^{\mathcal{S}_{k-1}}(c) \in (z_{Q(k)-1}, z_{Q(k)}) \cup (\hat{z}_{Q(k)}, \hat{z}_{Q(k)-1}). \tag{3}$$

In many cases we need to estimate Koebe space (see the Koebe Principle below) for intervals containing the critical value $c_1$. Let $H_{n-1}$ be the monotonicity interval of $f^{n-1}$ containing $c_1$. By construction, $f^{n-1}(H_{n-1} \cap [c_2, c_1]) \ni c$ whenever $n$ is a cutting time. If however $f^{n-1}(H_{n-1} \setminus [c_2, c_1]) \ni c$, then we say that $n$ is a *co-cutting time.* We denote them by

$$\{\mathcal{T}_k\}_{k \geq 0}, \text{ where } \mathcal{T}_0 := \min\{n > 1 \mid c_n \in (c, c_1]\}.$$

Using the above arguments, one can show that the difference $\mathcal{T}_k - \mathcal{T}_{k-1}$ is also a cutting time for each $k \geq 1$.

# 3 Long-branched Maps

**Definition 1** *A map $f$ is called* long-branched *if there exists $\beta > 0$ such that for every $n \geq 0$ and every monotonicity interval $T$ of $f^n$, $|f^n(T)| \geq \beta$.*

**Lemma 1** *A unimodal map $f$ is long-branched if and only if there exists $B$ such that $Q(k) \leq B$ for all $k \in \mathbf{N}$.*

*Proof:* Taking $B$ such that $|f(z_B) - f(c)| \leq \beta$ shows the only if part. Conversely, suppose that $T$ is a monotonicity interval of $f^n$. Then there exist $a < b < n$ such that $c \in f^a(\partial T), f^b(\partial T)$. More precisely, $f^a(T) = (z_k, c)$ or $(c, \hat{z}_k)$ for some $k$, and $f^b(T) = (c, c_{\mathcal{S}_k})$. By (3), $c_{\mathcal{S}_k} \notin (z_{Q(k+1)}, \hat{z}_{Q(k+1)}) \supset (z_B, \hat{z}_B)$. So $f^b(T) \ni z_B$ or $\hat{z}_B$, and $|f^b(T)| \geq |c - z_B| > 0$. Furthermore, $n - \mathcal{S}_k \leq \mathcal{S}_B$, and because $f$ is non-singular, also $|f^n(T)|$ is uniformly bounded away from 0. □

**Lemma 2** *Let $f \in \mathcal{F}$ be long-branched. There exists $\gamma > 0$ such that if $T$ is a monotonicity interval such that $f^n(T) \ni z_k$ (resp. $\hat{z}_k$), then $f^n(T) \supset (z_k - \gamma, z_k)$ (resp. $(\hat{z}_k \hat{z}_k + \gamma)$).*

*Proof:* Take $\beta > 0$ such that $|f^n(T)| \geq \beta$ for every branch, and let $K$ be such that $|f(c) - f(z_K)| \leq \beta$. Then it is clear that we only have to check the point $z_k$ with $k \leq K$. Let $L = \max_x |Df(x)| < \infty$, and $\gamma = L^{-\mathcal{S}_K - 2}\beta$. If $f^n(T) \ni z_k$, but has an endpoint in $(z_k - \gamma, z_k)$, then $T$ contains a interval of monotonicity $T'$ of $f^{n+\mathcal{S}_k+1}$ such that $|f^{n+\mathcal{S}_k+1}(T')| \leq L^{\mathcal{S}_k+1}\gamma < \beta$, contradicting long-branchedness. □

The main reason to consider long-branched maps is that they have relatively nice distortion properties.

**Definition 2** *Let $g : J \to g(J)$ be a monotone $C^1$ map. The quantity*

$$dis(g, J) := \sup_{x,y \in J} \frac{|Dg(x)|}{|Dg(y)|}$$

*is called the* distortion *of $g$ on $J$.*

We will use a standard tool (see e.g. [13]) to obtain bounds on the distortion:

**Lemma 3 (Koebe Principle)** *Suppose $g : T \to g(T)$ is monotone and has negative Schwarzian derivative. Assume that the interval $J \subset T$ is such that both components of $g(T \setminus J)$ have size $\geq \delta|g(J)|$. Then there exists $K = K(\delta)$ such that $dis(g, J) \leq K$.*

The components of $g(T \setminus J)$ are called the *Koebe space*. The Koebe Principle states that the existence of (relative) Koebe space gives a bound on the distortion of the middle part of the branch.

## 4 Construction of the Measure

Define annuli $A_k := (z_{k-1}, z_k) \cup (\hat{z}_k, \hat{z}_{k-1})$. Define the *induced map* $F$ as

$$F : \bigcup_{k \geq 1} A_k \to (z_0, \hat{z}_0), \quad F|_{A_k} = f^{\mathcal{S}_{k-1}}.$$

Clearly $F(A_k) = (z_{Q(k)}, c)$ or $(c, \hat{z}_{Q(k)})$. Hence $F$ preserves the partition $\bigcup_k A_k$ of $(z_0, \hat{z}_0)$: $F$ is a *Markov map.*

**Lemma 4** *If $f$ is long-branched, then also the induced map is long-branched, and there exists $K > 0$ such that for every $n \geq 0$, the distortion of every branch of $F^n$ is bounded by $K$.*

*Proof:* The long-branchedness of $F$ is obvious. To be precise, every branch-image $F^n(J)$ has the form $(z_k, c)$ or $(c, \hat{z}_k)$ for some $k \leq B$. Now for the distortion, let $J$ be any branch-domain of any iterate $F^n$, say $F^n(J) = f^m(J) = (z_k, c)$. Let $T$ be the monotonicity interval of $f^m$ containing $J$. Then by Lemma 2 and long-branchedness, $f^m(T)$ contains also $(z_k - \gamma, z_k]$ and $[c, \hat{z}_B)$. This gives Koebe space on both sides. So the lemma follows from the Koebe Principle. □

As a result, we can conclude that the induced map has an acip. This follows from a 'Folklore Theorem':

**Theorem 2 (Folklore Theorem)** *Let $F$ be a Markov map on the interval. If there exist $\beta > 0$ and $K \geq 1$ such that $dis(F^n, J) \leq K$ and $|F^n(J)| \geq \beta$ for all $n$ and all branches $F^n|_J$, then $F$ admits an acip, $m$, and the Radon-Nikodým derivative $\frac{dm}{dx}$ is bounded.*

The proof can be found in e.g. [13, Theorem V 2.1].

**Proposition 1** *Let $f \in \mathcal{F}$ be long-branched. Then $f$ admits an acip if and only if the summability condition*

$$\sum_k \mathcal{S}_{k-1}|A_k| < \infty$$

*is satisfied. Every long-branched map has at least a $\sigma$-finite absolutely continuous invariant measure.*

*Proof:* Let $m$ be the acip of the induced map. We construct an invariant measure $\mu$ for the original map $f$ using Kac's formula: For any measurable set $A$, let

$$\mu(A) = \sum_{k\geq 0} \sum_{i=0}^{\mathcal{S}_{k-1}-1} m(f^{-i}(A) \cap A_k).$$

It is easy to verify that $\mu$ is absolutely continuous and invariant. For the finiteness of $\mu$ we check

$$\mu(I) = \sum_{k\geq 0} \sum_{i=0}^{\mathcal{S}_{k-1}-1} m(f^{-i}(I) \cap A_k) = \sum_k \mathcal{S}_{k-1} m(A_k) \leq \sum_k \mathcal{S}_{k-1} M|A_k|,$$

where $M$ is an upper bound of $\frac{dm}{dx}$. Because every branch of $F$ covers either $(z_B, c)$ or $(c, \hat{z}_B)$, there exists also $M' > 0$ such that $m(A) \geq M'|A|$, whenever $A = \hat{A} \subset (z_B, \hat{z}_B)$ is measurable. Therefore $\mu(I) \geq \sum_{k\geq B} \mathcal{S}_{k-1} M'|A_k|$. So $\mu$ is infinite if and only if $\sum_k \mathcal{S}_k |A_k| = \infty$.

The measure $\mu$ is $\sigma$-finite though. For a long-branched map, the critical orbit is nowhere dense, as was proven in [3, Lemma 6]. Every S-unimodal finitely renormalizable map with a nowhere dense critical orbit have a $\sigma$-finite absolutely continuous invariant measure, [7, 11]. We will give a proof for our case. We neglect the case that $f$ has a periodic attractor. In that case a $\sigma$-finite invariant measure, be it dissipative, is easily constructed. Let $U \subset (z_B, \hat{z}_B)$ be an interval such that $orb(c) \cap U = \emptyset$. Let $V$ be the middle third of $U$, and let $N$ be such that both components of $U \setminus V$ contain a point in $\bigcup_{n\leq N} f^{-n}(c)$. Assume that $f^n(A_k) \cap V \neq \emptyset$ for some $n \leq \mathcal{S}_{k-1}$. Because $orb(c) \cap U = \emptyset$, $f^n((z_{k-1}, c))$ contains a component of $U \setminus V$, and therefore a point of $\bigcup_{n\leq N} f^{-n}(c)$. Because $f^{\mathcal{S}_k}|_{(c, z_{k-1})}$ is monotone, it follows that $\mathcal{S}_{k-1} - n < N$. Hence $\mu(V) \leq \sum_k Nm(A_k) \leq N$. By (2), $\bigcup_n f^{-n}(V)$ has full measure. So we can write $I = \bigcup_{n\geq 0} V_n \cup W$, where $V_n := f^{-n}(V) \setminus \bigcup_{0\leq i<n} f^{-i}(V)$ has measure $\mu(V_n) \leq \mu(V)$, and $W$ is a nullset. So $\mu$ is indeed $\sigma$-finite.

As Lebesgue measure is ergodic [1], there can be only one (up to a multiplicative constant) absolutely continuous invariant measure, finite or $\sigma$-finite. This proves the proposition. □

# 5 Saddle Node Returns

A close return $c_n$ of $c$ to itself is called a *saddle node* return if the central branch is almost tangent to the diagonal, see figure 1. The central branch of $f^n$ doesn't cover $c$ in this case, so $n$ is not a cutting time. Let $\mathcal{S}_k$ be the smallest cutting time larger than $n$. Then $f^n$ is monotone on $(c, \hat{z}_{k-1})$, and $f^n(\hat{z}_k) = \hat{z}_r$ for some $r < k$. If $f$ is long-branched, then $r \leq B$. Call $\hat{z}_k = \hat{z}_{begin}$. Pulling back $\hat{z}_{begin}$ along the central branch of $f^n$, we get new closest precritical points. If $f_n$ is very close to the diagonal, these precritical points cluster together near the *funnel*. Past the funnel, the preimages get more separated again. Let $\hat{z}_{end}$ be the last of those preimages for which $|\hat{z}_{end} - c| \geq |c_n - c|$. The number of closest precritical points between $\hat{z}_{begin}$ and $\hat{z}_{end}$ can be arbitrarily large, provided we take the central branch of $f^n$ close enough to the diagonal.

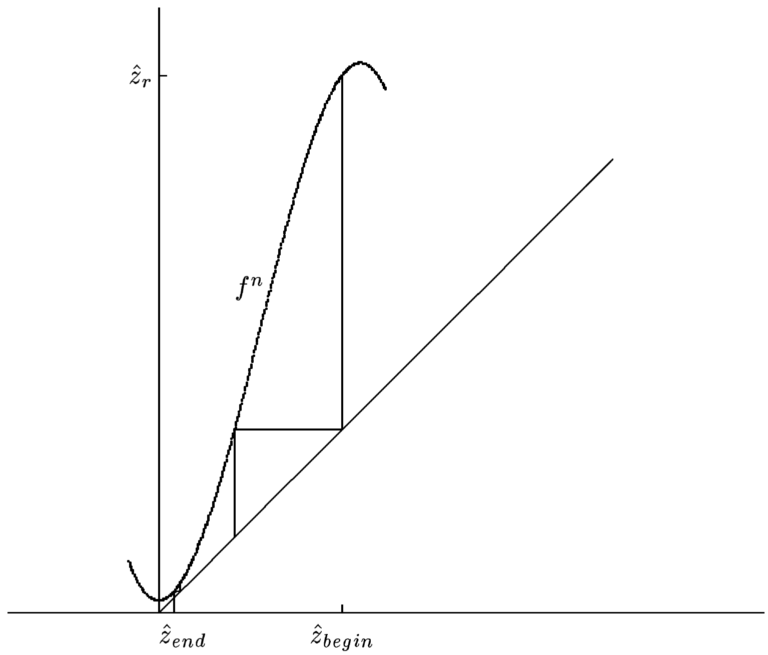


*Figure 1*

Let for $t \in \mathbf{R}$, $x_t \in (c, \hat{z}_{begin})$ be the point, closest to $c$, such that $Df^n(x_t) = t$.

**Lemma 5** *Let $f \in \mathcal{F}$, and let $c_n$ be a closest return of saddle node type. Writing $\varepsilon = |\hat{z}_{begin} - c|$, there exist $K_1$, $K_2$ and $K_3$, independent of $n$, such that*

$$K_1 \varepsilon^{\frac{\ell}{\ell-1}} \leq |x_{2\ell} - c| \leq K_2 \varepsilon^{\frac{\ell}{\ell-1}},$$

*and if $r$ is the smallest iterate such that $f^{nr}(x_{2\ell}) \notin (c, \hat{z}_{begin})$, then $r \leq K_3 \log\log \frac{1}{\varepsilon}$.*

*Proof:* Write the central branch $f^n|_{(c,c+\varepsilon)}$ as $g \circ f|_{(c,c+\varepsilon)}$. Since $c_n$ is a closest return, the monotonicity interval $T \ni c_1$ of $f^{n-1}$ maps onto $(z_B, f^{n-\mathcal{S}_{k-1}}(c)) \supset (z_B, \hat{z}_B)$. Here $k$ is such that $\mathcal{S}_k > n > \mathcal{S}_{k-1}$. By Lemma 2 and the Koebe Principle, there exists a constant $K$, independent of $n$ and $\varepsilon$ such that the distortion $dis(g, (f(c+\varepsilon), f(c))) \leq K$. Let also $1 \geq \eta := |f^n(\hat{z}_{begin}) - c| \geq |\hat{z}_B - c|$. It follows by (1) that

$$|x_t - c|^{\ell-1} \frac{O_1}{O_2} \frac{\ell\eta}{K\varepsilon^\ell} \leq Df^n(x_t) = t \leq |x_t - c|^{\ell-1} \frac{O_2}{O_1} \frac{\ell K}{\varepsilon^\ell},$$

which proves the first part of the lemma. The above arguments show that $f^n|_{(c,c+\varepsilon)}$ has roughly the shape $\varphi : [0, \varepsilon] \to \mathbf{R}$, $\varphi(x) = \eta(\frac{x}{\varepsilon})^\ell + C\varepsilon^{\frac{\ell}{\ell-1}}$. For this function, $x_{2\ell} = (\frac{2}{\eta})^{\frac{1}{\ell-1}} \varepsilon^{\frac{\ell}{\ell-1}} := d_0 \varepsilon^{\frac{\ell}{\ell-1}}$. In general $\varphi(d\varepsilon^{\frac{\ell}{\ell-1}}) = (\eta d^\ell + C)\varepsilon^{\frac{\ell}{\ell-1}}$, so we get $\varphi^r(d_0 \varepsilon^{\frac{\ell}{\ell-1}}) = d_r \varepsilon^{\frac{\ell}{\ell-1}}$, where $d_r = \eta d_{r-1}^\ell + C$. An inductive proof shows that $d_r \geq 2^{(\ell^{r-1})} d_0$. It follows that

$$\begin{aligned}
\varphi^r(x_{2\ell}) \leq \eta \quad & \Rightarrow \quad 2^{(\ell^{r-1})} (\frac{2}{\eta})^{\frac{1}{\ell-1}} \varepsilon^{\frac{\ell}{\ell-1}} \leq \eta \\
& \Rightarrow \quad \ell^{r-1} \leq \frac{\ell}{\ell-1} \frac{1}{\log 2} \log \frac{\eta}{\varepsilon} \\
& \Rightarrow \quad r \leq \frac{1}{\log \ell} \{\log\log \frac{\eta}{\varepsilon} + \log \frac{\ell}{\ell-1} - \log\log 2\} + 1.
\end{aligned}$$

This proves the second statement. □

# 6 Proof of the Theorem 1

For the proof of the Theorem 1 we have to find, in view of Proposition 1, two conjugate maps, $f$ and $g$, such that $\sum_i \mathcal{S}_i|A_i(f)| < \infty$ and $\sum_i \mathcal{S}_i|A_i(g)| = \infty$. So there must be a significant difference between $f$ and $g$. The difference that we will exploit is the difference in slope at the fixed point. If the critical point approaches the fixed point very closely, this will have an effect on the rate at which certain closest precritical points converge to the critical point.

**Lemma 6** *Let $f \in \mathcal{F}$, and let $p$ be its orientation reversing fixed point. For each $\varepsilon > 0$, there exists $N$ with the following property: Suppose $n$ is a cutting time, and $n+1$ a co-cutting time. Suppose also that $c_j \in (c,1)$ for every $n \le j \le m$. Then for each $k$ such that $n+N \le \mathcal{S}_k \le m-N$,*

$$|Df(p)|^{-\frac{2}{\ell}} - \varepsilon \le \frac{|z_{k+1}-c|}{|z_k-c|} \le |Df(p)|^{-\frac{2}{\ell}} + \varepsilon.$$

*Proof:* Since $n$ is a cutting and $n+1$ is a co-cutting time, $f^{n-1}(H_{n-1}) \supset (c,\hat{z}_0)$. Let

$$c = q_0 < q_2 < q_4 < \ldots < p < \ldots < q_5 < q_3 < q_1 = \hat{z}_0$$

be the precritical points closest to $p$. In particular $f(q_{i+1}) = q_i$ for every $i \ge 0$, and $\frac{|q_{i+2}-p|}{|q_i-p|} \to |Df(p)|^{-2}$ as $i \to \infty$. Let $r = m-n$. Because $c_j \in (c,1)$ for all $n \le j \le m$, $c_n$ must be contained in $(q_r, q_{r-1})$. Furthermore, the precritical points $z_k$, $n \le \mathcal{S}_k \le m$, must be mapped onto $q_0$, $q_2$, etc. Let $\varepsilon > 0$ be arbitrary. Then there exists $N$, independent of $n$ and $m$, such that we can carry out the following steps:

- For all $i \ge N$,
$$\left|\frac{|q_{i+2}-p|}{|q_i-p|} - \frac{1}{|Df(p)|^2}\right| \le \frac{\varepsilon}{10}$$
- For every $i \le r-N$, $\left|\frac{|q_i-c_n|}{|q_i-p|} - 1\right| \le \frac{\varepsilon}{10}$. By this condition also
$$\left|\frac{|q_{i+2}-c_n|}{|q_i-c_n|} - \frac{1}{|Df(p)|^2}\right| \le \frac{\varepsilon}{3}$$
for every $N \le i \le r-N$.
- The distortion of $f^{n-1}|_H$ is bounded by $1+\frac{\varepsilon}{10}$ where $c_1 \in H \subset H_{n-1}(c_1)$ is the interval that is mapped onto $(q_N, q_{N-1})$. Using the Koebe Principle, this interval can be found. It follows that the points $f(z_k)$, $f(z_{k+1}), \ldots$, for $n+N \le \mathcal{S}_k < \mathcal{S}_{k+1} \ldots \le m-N$, accumulate on $f(c)$ with exponential rate approximately $|Df(p)|^{-2}$.
- For all $x, y \in f^{-1}(H) \setminus \{c\}$, we have by (1) $\left|\frac{|f(x)-f(c)|}{|x-c|^\ell}\frac{|y-c|^\ell}{|f(y)-f(c)|} - 1\right| \le \frac{\varepsilon}{10}$.

The lemma follows. □

For the proof of Theorem 1, we have to construct maps that will alternate close saddle node returns and close approaches to the fixed point. In order to prove that such maps exist, at least in a combinatorial setting, we will construct a kneading invariant that exhibits this behaviour. In particular we will prove that there exist long-branched maps with a recurrent critical point. Let $\{v_k\}_{k\geq 1}$ and $\{w_k\}_{k\geq 1}$ be integer sequences, to be defined inductively. The *kneading invariant* $\nu(f)$ of a unimodal map $f$ is the symbolic itinerary of the critical point: $\nu(f) = \nu_1(f)\nu_2(f)\ldots \in \{0,1\}^{\mathbf{N}}$, where $\nu_i(f) = 0$ if $c_i < c$ and $\nu_i = 1$ if $c_i > c$. We will construct a kneading invariant $\nu$ as the limit of finite strings $\nu_1 \ldots \nu_{v_k}$ and $\nu_1 \ldots \nu_{w_k}$ of increasing length. These strings are constructed according to the following algorithm:

(i) Let $k = 1$, $v_k = v_1 = 3$ and $\nu_1\nu_2\nu_3 = 101$.

(ii) Consider the concatenation $\nu_1 \ldots \nu_{v_k}\nu_1 \ldots \nu_{v_k}\nu_1 \ldots \nu_{v_k} \ldots$ and choose $w_k$ arbitrary such that the $w_k$-th and the $w_{k+3}$-th entry of this concatenation are zeroes. Let $\nu_1 \ldots \nu_{w_k}$ be the string consisting of the first $w_k$ entries of the concatenation.

(iii) Increase $k$ by 1.

(iv) Let $v_k > w_{k-1} + 3$ be arbitrary, but such that $v_k - w_{k-1}$ is odd. Let $\nu_1 \ldots \nu_{v_k}$ be the concatenation of $\nu_1 \ldots \nu_{w_{k-1}}$ and $v_k - w_{k-1}$ ones.

(v) Go to step (ii).

Let $\nu$ be the limit sequence. We call $\nu$ *admissible* if there exists a unimodal map having $\nu$ as kneading invariant. If $f$ indeed has the above kneading invariant $\nu$, then $c_{v_k}$ will be very close to $c$ if $w_k - v_k$ is large, and $c_{w_k}$ will be very close to $p$ if $v_k - w_{k-1}$ is large.

**Lemma 7** *For every choice of $\{v_k\}$ and $\{w_k\}$ in the above algorithm, the resulting kneading invariant $\nu$ is admissible, and the corresponding map is long-branched.*

*Proof:* As shown in [3], the cutting times of a map can be retrieved from the kneading invariant as follows:

$$\mathcal{S}_0 = 1 \text{ and } \mathcal{S}_k = \min\{n > \mathcal{S}_{k-1} \mid \nu_n \neq \nu_{n-\mathcal{S}_{k-1}}\}.$$

It can be easily checked that $\mathcal{S}_k - \mathcal{S}_{k-1} \in \{1,2\}$ in our case. By Lemma (1), this yields long-branchedness. The co-cutting times can be found as

$$\mathcal{T}_0 = \min\{n > 1 \mid \nu_n = 1\} \text{ and } \mathcal{T}_k = \min\{n > \mathcal{T}_{k-1} \mid \nu_n \neq \nu_{n-\mathcal{T}_{k-1}}\}.$$

In our case the co-cutting times are

$$v_1, w_1 + 3, w_1 + 5, \ldots, v_2, w_2 + 3, w_2 + 5, \ldots v_3, w_3 + 3, \ldots$$

According to [3, Section 4], $\nu$ is admissible if the differences $\mathcal{S}_k - \mathcal{S}_{k-1}$ and $\mathcal{T}_k - \mathcal{T}_{k-1}$ are all cutting times. These differences are 1, 2 or $w_k + 3 - v_k$. Since every integer $n$ such that $\nu_n = 0$ is a cutting time, also $w_k + 3 - v_k$ is easily seen to be a cutting time. □

*Proof of Theorem 1:* Take $f, g \in \mathcal{F}$ conjugate, having a kneading invariant of the type described in Lemma 7. This means that we still have to determine the sequences $v_k$ and $w_k$. Assume also that $|Df(p_f)| > |Dg(p_g)|$, where $p_f$ and $p_g$ denote the orientation reversing fixed points of $f$ and $g$ respectively. This can be done as follows: Let $f_a = a\sin(\pi x)$ and $g_b = bx(1-x)$ be two unimodal families. It is well-known, see e.g. [13, Theorem II 4.1], that $f_a$ and $g_b$ are *full*, i.e. for every admissible kneading invariant $\nu$, there exist $a$ and $b$ such that $f_a$ and $g_b$ have kneading invariant $\nu$. Also, the kneading invariants of $f_a$ and $g_b$ depend continuously on $a$ and $b$. Now take e.g. $w_1 = 17$, so that $\nu$ begins with 10110110110110110111. It can be checked that $\nu(f_{0.92}) < \nu < \nu(f_{0.93})$ and $\nu(g_{3.82}) < \nu < \nu(g_{3.83})$, where $<$ denotes the usual order relation between kneading invariants.

Moreover $Df(p_f) \le -1.85 < -1.83 \le Dg(p_g)$ whenever $a \ge 0.92$ and $b \le 3.83$. Therefore there exist $\lambda_f$ and $\lambda_g$ such that, under the conditions of Lemma 6,

$$\frac{|z_{k+1}(f) - c|}{|z_k(f) - c|} \le \lambda_f < \lambda_g \le \frac{|z_{k+1}(g) - c|}{|z_k(g) - c|}. \tag{4}$$

Using these bounds we determine (below) the sequences $\{v_k\}$ and $\{w_k\}$, and therefore $\nu$. By the fullness of $f_a$ and $g_b$, there indeed exist parameters $a \in [0.92, 0.93]$ and $b \in [3.82, 3.83]$, for which $\nu(f_a) = \nu(g_b) = \nu$. (According to [5], $b$ is uniquely determined. For the sine family no uniqueness result is known.)

The maps $f$ and $g$ will exhibit an infinite sequence of saddle node returns, and close approaches to the fixed point in between. So let $\{a_k\}_{k\ge1}$ and $\{b_k\}_{k\ge1}$ be the integers corresponding to the subsequent $\hat{z}_{begin}$ and $\hat{z}_{end}$ from Section 5. For instance, $a_1 = 2$. The choices of $\{a_k\}$ and $\{b_k\}$ determine the choice of $\{v_k\}$ and $\{w_k\}$ and vice versa. By (4), we can choose $a_{k+1} > 2b_k + 4N$ so large that $|z_{a_{k+1}}(g) - c| \le \frac{1}{2}|z_{b_k}(g) - c|$ and

$$|z_{a_{k+1}}(f) - c| \le \frac{1}{(k+1)^2}|z_{a_{k+1}}(g) - c|.$$

Then we can choose $b_{k+1}$ so large that

$$1 \le b_{k+1}|z_{a_{k+1}}(g) - c|^{\frac{\ell}{\ell-1}} \le 2.$$

By Lemma 5,

$$\begin{aligned} \sum_i \mathcal{S}_i|A_i(g)| &\ge \sum_i i|A_i(g)| \ge \sum_k \frac{1}{2} b_k |z_{b_k}(g) - c| \\ &\ge \frac{1}{2}K_1 \sum_k b_k |z_{a_k}(g) - c|^{\frac{\ell}{\ell-1}} \ge \frac{1}{2}K_1 \sum_k 1 = \infty. \end{aligned}$$

Now for $f$, let $x_{k,2\ell} \in (c, \hat{z}_{a_k})$ be the point, closest to $c$, such that $Df^n(x_{k,2\ell}) = 2\ell$. (Here $n$ is the iterate corresponding to the $k$-th saddle node return.) Then by Lemma 5, $|z_{b_k} - c| \le |x_{k,2\ell} - c| \le K_2|z_{a_k} - c|^{\frac{\ell}{\ell-1}}$. Furthermore, it takes at most $K_3 \log\log \frac{1}{|c-\hat{z}_{a_k}|}$ iterates of the $k$-th saddle node branch for $x_{k,2\ell}$ to leave $(c, \hat{z}_{a_k})$. As $|c - \hat{z}_{a_k}| \le \lambda_f^{(a_k - b_{k-1} - 2N)} \le \lambda_f^{a_k/2}$, the number of sets $A_i$ disjoint from $(\hat{x}_{k,2\ell}, x_{k,2\ell})$ is bounded by $K_4 a_k \log a_k$ for some $K_4$. Then, for some $C = C(K_1, K_2, K_4)$

$$\begin{aligned} \sum_{i\ge b_1} \mathcal{S}_i|A_i(f)| &\le \sum_{i\ge b_1} i|A_i(f)| \\ &\le \mathcal{S}_B \sum_{k\ge2} \left[ \sum_{i=a_k}^{a_k+K_4a_k\log a_k} i|A_i(f)| + \sum_{i=a_k+K_4a_k\log a_k}^{b_k} i|A_i(f)| + \sum_{i=b_{k-1}}^{a_k} i|A_i(f)| \right] \\ &\le \mathcal{S}_B \sum_{k\ge2} \left[ (a_k + K_4a_k\log a_k)|z_{a_k}(f) - c| + b_k|x_{k,2\ell} - c| + \sum_{i=1}^{a_k-b_{k-1}} i|A_{i+b_{k-1}}(f)| \right] \\ &\le \mathcal{S}_B \sum_{k\ge2} \Big[ (a_k + K_4a_k\log a_k)\lambda_f^{a_k-b_{k-1}-2N}|z_{b_{k-1}}(f) - c| \\ &\qquad + b_kK_2|z_{b_k}(f) - c| + \sum_{i=1}^{a_k-b_{k-1}} i\lambda_f^{i-2N}|z_{b_{k-1}}(f) - c| \Big] \end{aligned}$$

$$\begin{aligned} &\leq \mathcal{S}_B \sum_{k \geq 1} C b_k |z_{b_k}(f) - c| \leq \mathcal{S}_B \sum_{k} C K_2 b_k |z_{a_k}(f) - c|^{\frac{\ell}{\ell-1}} \\ &\leq \mathcal{S}_B \sum_{k \geq 1} C K_2 \frac{1}{k^2} b_k |z_{a_k}(g) - c|^{\frac{\ell}{\ell-1}} \leq \mathcal{S}_B \sum_{k \geq 1} 2 C K_2 \frac{1}{k^2} < \infty. \end{aligned}$$

This concludes the proof. □

## References

[1] A. Blokh, M. Lyubich *Measurable dynamics of S-unimodal maps of the interval,* Ann. Scient. Éc. Norm. Sup. **24** (1991) 545-573.

[2] H. Bruin, *Invariant measures for interval maps,* Thesis, Delft (1994).

[3] H. Bruin, *Combinatorics of the kneading map,* Int. Jour. of Bifur. and Chaos **5 no. 5** (1995) 1339-1349.

[4] H. Bruin, G. Keller, T. Nowicki, S. van Strien, *Wild Cantor attractors exist,* Ann. of Math. **143** (1996) 97-130.

[5] J. Graczyk, G. Świątek, *Hyperbolicity of real quadratic polynomial,* Preprint Penn State Univ. (1995).

[6] M. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations,* Publ. Math. IHES **49** (1979) 5-233.

[7] F. Hofbauer, G. Keller, *Some remarks on recent results about S-unimodal maps Ann. Inst. Henri Poincaré, Physique théorique,* **53** (1990) 413-425.

[8] M. Jakobson, *Quasisymmetric conjugacies for some one-dimensional maps inducing expansion,* Contemp. Math. **135** (1992) 203-211.

[9] M. Jakobson, G. Świątek, *Metric properties of non-renormalizable S-unimodal maps II: Quasisymmetric conjugacy classes,* Erg. Th. & Dyn. Sys. **15** (1995) 871-938.

[10] M. Lyubich, J. Milnor, *The Fibonacci unimodal map,* Jour. AMS. **6** (1993), 425-457.

[11] M. Martens, *The existence of σ-finite measures, applications to* 1*-dimensional dynamics,* Preprint IMPA. (1991).

[12] M. Misiurewicz, *Absolutely continuous measures for certain maps of an interval,* Publ. Math. I.H.E.S. **53** (1981) 17-51.

[13] W. de Melo, S. van Strien, *One-Dimensional Dynamics,* Springer, Berlin Heidelberg New York, (1993).

[14] T. Nowicki, S. van Strien, *Absolutely continuous measures under a summability condition,* Invent. Math. **93** (1988) 619-635.

[15] D. Sands, T. Nowicki, *Quasisymmetric invariance of the Collet-Eckmann Condition,* Preprint Paris (1995).

[16] D. Sands, *Topological conditions for positive Lyapunov exponent in unimodal maps,* Thesis (1994).

[17] J.-C. Yoccoz, *Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition Diophantienne,* Ann. Sci. Ec. Norm. Sup. **17** (1984) 333-361.