PLANAR EMBEDDINGS OF CHAINABLE CONTINUA

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ABSTRACT. We prove that for a chainable continuum $X = \varprojlim([0,1], f_i)$ where every $x \in X$ has only finitely many coordinate projections contained in a zigzag there exists a planar embedding $\varphi : X \to \varphi(X) \subset \mathbb{R}^2$ such that $\varphi(x)$ is accessible. This partially answers a question of Nadler and Quinn from 1972. Two embeddings $\varphi, \psi : X \to \mathbb{R}^2$ are called strongly equivalent if $\varphi \circ \psi^{-1} : \psi(X) \to \varphi(X)$ can be extended to a homeomorphism of $\mathbb{R}^2$. We prove that every nondegenerate indecomposable chainable continuum can be embedded in the plane in uncountably many ways that are not strongly equivalent.

1. Introduction

It is well-known that every chainable continuum can be embedded in the plane, see [6]. In this paper we develop methods to study nonequivalent planar embeddings, similar to methods used by Lewis in [14] and Smith in [26] for the study of planar embeddings of the pseudo-arc. Following Bing's approach from [6] (see Lemma 3.1), we construct nested intersections of discs in the plane which are small tubular neighborhoods of polygonal lines obtained from the bonding maps. Later we show that this approach produces all possible planar embeddings of chainable continua which can be covered with planar chains with *connected* links, see Theorem 8.5. From that we can produce uncountably many nonequivalent planar embeddings of the same chainable continuum.

Definition 1.1. *Let X be a chainable continuum. Two embeddings $\varphi, \psi : X \to \mathbb{R}^2$ are called* equivalent *if there is a homeomorphism h of $\mathbb{R}^2$ such that $h(\varphi(X)) = \psi(X)$. They are* strongly equivalent *if $\psi \circ \varphi^{-1} : \varphi(X) \to \psi(X)$ can be extended to a homeomorphism of $\mathbb{R}^2$.*

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That is, equivalence requires some homeomorphism between $\varphi(X)$ and $\psi(X)$ to be extended to $\mathbb{R}^2$ whereas strong equivalence requires the homeomorphism $\psi\circ\varphi^{-1}$ between $\varphi(X)$ and $\psi(X)$ to be extended to $\mathbb{R}^2$.

Clearly, strong equivalence implies equivalence, but in general not the other way around, see for instance Remark 9.19. We say a nondegenerate continuum is *indecomposable*, if it is not the union of two proper subcontinua.

Question 1. *Are there uncountably many nonequivalent planar embeddings of every chainable indecomposable continuum?*

This question is listed as Problem 141 in a collection of continuum theory problems from 1983 by Lewis [15] and was also posed by Mayer in his dissertation in 1983 [16] (see also [17]) using the standard definition of equivalent embeddings.

We give a positive answer to the adaptation of the above question using strong equivalence, see Theorem 9.14. If the continuum is the inverse limit space of a unimodal map and not hereditarily decomposable, then the result holds for both definitions of equivalent, see Remark 9.20.

In terms of equivalence, this generalizes the result in [2], where we prove that every unimodal inverse limit space with bonding map of positive topological entropy can be embedded in the plane in uncountably many nonequivalent ways. The special construction in [2] uses symbolic techniques which enable direct computation of accessible sets and prime ends (see [3]). Here we utilize a more direct geometric approach.

One of the main motivations for the study of planar embeddings of tree-like continua is the question of whether the *plane fixed point property* holds. The problem is considered to be one of the most important open problems in continuum theory. Is it true that every continuum $X \subset \mathbb{R}^2$ not separating the plane has the fixed point property, *i.e.,* every continuous $f : X \to X$ has a fixed point? There are examples of tree-like continua without the fixed point property, see *e.g.* Bellamy's example in [5]. It is not known whether Bellamy's example can be embedded in the plane. Although chainable continua are known to have the fixed point property (see [11]), insight in their planar embeddings may be of use to the general setting of tree-like continua.

Another motivation for this study is the following long-standing open problem. For this we use the following definition.

Definition 1.2. *Let $X \subset \mathbb{R}^2$. We say that $x \in X$ is* accessible *(from the complement of X) if there exists an arc $A \subset \mathbb{R}^2$ such that $A \cap X = \{x\}$.*

Question 2 (Nadler and Quinn 1972, pg. 229 in [25])**.** *Let X be a chainable continuum and $x \in X$. Can X be embedded in the plane such that x is accessible?*

We will introduce the notion of a *zigzag* related to the admissible permutations of graphs of bonding maps and answer Nadler and Quinn's question in the affirmative for the class of *non-zigzag* chainable continua (see Corollary 7.4). From the other direction, a promising possible counterexample to Question 2 is the one suggested by Minc (see Figure 15 and the description in [20]). However, the currently available techniques are insufficient to determine whether the point $p \in X_M$ can be made accessible or not, even with the use of thin embeddings, see Definition 8.2.

Section 2 gives basic notation, and we review the construction of natural chains in Section 3. Section 4 describes the main technique of permuting branches of graphs of linear interval maps. In Section 5 we connect the techniques developed in Section 4 to chains. Section 6 applies the techniques developed so far to accessibility of points of chainable planar continua; this is the content of Theorem 6.1 which is used as a technical tool afterwards. Section 7 introduces the concept of zigzags of a graph of an interval map. Moreover, it gives a partial answer to Question 2 and provides some interesting examples by applying the results from this section. Section 8 gives a proof that the permutation technique yields all possible thin planar embeddings of chainable continua. Furthermore, we pose some related open problems at the end of this section. Finally, Section 9 completes the construction of uncountably many planar embeddings that are not equivalent in the strong sense, of every chainable continuum which contains a nondegenerate indecomposable subcontinuum and thus answers Question 1 for strong equivalence. We conclude the paper with some remarks and open questions emerging from the study in the final section.

2. Notation

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ be the positive and nonnegative integers. Let $f_i : I = [0, 1] \to I$ be continuous surjections for $i \in \mathbb{N}$ and let *inverse limit space*

$$X_\infty = \varprojlim\{I, f_i\} = \{(x_0, x_1, x_2, \dots) : f_i(x_i) = x_{i-1}, i \in \mathbb{N}\}$$

be equipped with the subspace topology endowed from the product topology of I^∞. Let $\pi_i : X_\infty \to I$ be the *coordinate projections* for $i \in \mathbb{N}_0$.

Definition 2.1. *Let X be a metric space. A* chain *in X is a set $\mathcal{C} = \{\ell_1 \dots, \ell_n\}$ of open subsets of X called* links*, such that $\ell_i \cap \ell_j \neq \emptyset$ if and only if $|i - j| \le 1$. If also $\cup_{i=1}^n \ell_i = X$, then we speak of a* chain cover *of X. We say that a chain $\mathcal{C}$ is* nice *if additionally all links are open discs (in X).*

The mesh *of a chain $\mathcal{C}$ is $mesh(\mathcal{C}) = \max\{\operatorname{diam} \ell_i : i = 1, \dots, n\}$. A continuum X is* chainable *if there exist chain covers of X of arbitrarily small mesh.*

We say that $\mathcal{C}' = \{\ell'_1, \dots, \ell'_m\}$ *refines* $\mathcal{C}$ and write $\mathcal{C}' \preceq \mathcal{C}$ if for every $j \in \{1, \dots, m\}$ there exists $i \in \{1, \dots, n\}$ such that $\ell'_j \subset \ell_i$. We say that $\mathcal{C}'$ *properly refines* $\mathcal{C}$ and write $\mathcal{C}' \prec \mathcal{C}$ if additionally $\ell'_j \subset \ell_i$ implies that the closure $\overline{\ell'_j} \subset \ell_i$.

Let $\mathcal{C}' \preceq \mathcal{C}$ be as above. The *pattern of $\mathcal{C}'$ in $\mathcal{C}$*, denoted by $Pat(\mathcal{C}', \mathcal{C})$, is the ordered m-tuple $(a_1, \dots, a_m)$ such that $\ell'_j \subset \ell_{a(j)}$ for every $j \in \{1, \dots, m\}$ where $a(j) \in \{1, \dots, n\}$. If $\ell'_j \subset \ell_i \cap \ell_{i+1}$, we take $a(j) = i$, but that choice is just by convention.

For chain $\mathcal{C} = \{\ell_1, \dots, \ell_n\}$, write $\mathcal{C}^* = \cup_{i=1}^n \ell_i$.

3. Construction of natural chains, patterns and nested intersections

First we construct *natural chains* $\mathcal{C}_n$ for every $n \in \mathbb{N}$ (the terminology originates from [9]). Take some nice chain cover $C_0 = \{l_1^0, \dots, l_{k(0)}^0\}$ of I and define $\mathcal{C}_0 := \pi_0^{-1}(C_0) = \{\ell_1^0, \dots, \ell_{k(0)}^0\}$, where $\ell_i^0 = \pi_0^{-1}(l_i^0)$. Note that $\mathcal{C}_0$ is a chain cover of X_∞ (but the links are not necessarily connected sets in X_∞).

Now take a nice chain cover $C_1 = \{l_1^1, \dots, l_{k(1)}^1\}$ of I such that for every $j \in \{1, \dots, k(1)\}$ there exists $j' \in \{1, \dots, k(0)\}$ such that $f_1(\overline{l_j^1}) \subset l_{j'}^0$ and define $\mathcal{C}_1 := \pi_1^{-1}(C_1)$. Note that $\mathcal{C}_1$ is a chain cover of X_∞. Also note that $\mathcal{C}_1 \prec \mathcal{C}_0$ and $Pat(\mathcal{C}_1, \mathcal{C}_0) = \{a_1^1, \dots, a_{k(1)}^1\}$ where $f_1(\pi_1(\ell_j^1)) \subset \pi_0(\ell_{a_j^1}^0)$ for each $j \in \{1, \dots, k(1)\}$. So the pattern $Pat(\mathcal{C}_1, \mathcal{C}_0)$ can easily be calculated by just following the graph of f_1.

Inductively we construct $\mathcal{C}_{n+1} = \{\ell_1^{n+1}, \dots, \ell_{k(n+1)}^{n+1}\} := \pi_{n+1}^{-1}(C_{n+1})$, where $C_{n+1} = \{l_1^{n+1}, \dots, l_{k(n+1)}^{n+1}\}$ is some nice chain cover of I such that for every $j \in \{1, \dots, k(n+1)\}$ there exists $j' \in \{1, \dots, k(n)\}$ such that $f_{n+1}(\overline{l_j^{n+1}}) \subset l_{j'}^n$. Note that $\mathcal{C}_{n+1} \prec \mathcal{C}_n$ and $Pat(\mathcal{C}_{n+1}, \mathcal{C}_n) = (a_1^{n+1}, \dots, a_{k(n+1)}^{n+1})$, where $f_{n+1}(\pi_{n+1}(\ell_j^{n+1})) \subset \pi_n(\ell_{a_j^{n+1}}^n)$ for each $j \in \{1, \dots, k(n+1)\}$.

Throughout the paper we use the straight letter C for chain covers of the interval I and the script letter $\mathcal{C}$ for chain covers of the inverse limits space. Note that the links of C_n can be chosen small enough to ensure that $\text{mesh}(\mathcal{C}_n) \to 0$ as $n \to \infty$ and note that $X_\infty = \cap_{n\in\mathbb{N}_0} \mathcal{C}_n^*$.

Lemma 3.1. *Let X and Y be compact metric spaces and let $\{\mathcal{C}_n\}_{n\in\mathbb{N}_0}$ and $\{\mathcal{D}_n\}_{n\in\mathbb{N}_0}$ be sequences of chains in X and Y respectively such that $\mathcal{C}_{n+1} \prec \mathcal{C}_n$, $\mathcal{D}_{n+1} \prec \mathcal{D}_n$ and $Pat(\mathcal{C}_{n+1}, \mathcal{C}_n) = Pat(\mathcal{D}_{n+1}, \mathcal{D}_n)$ for each $n \in \mathbb{N}_0$. Assume also that mesh$(\mathcal{C}_n) \to 0$ and mesh$(\mathcal{D}_n) \to 0$ as $n \to \infty$. Then $X' = \cap_{n\in\mathbb{N}_0} \mathcal{C}_n^*$ and $Y' = \cap_{n\in\mathbb{N}_0} \mathcal{D}_n^*$ are nonempty and homeomorphic.*

Proof. To see that X' and Y' are nonempty note that they are nested intersections of nonempty closed sets. Define $\mathcal{C}_k = \{\ell_1^k, \dots, \ell_{n(k)}^k\}$ and $\mathcal{D}_k = \{L_1^k, \dots, L_{n(k)}^k\}$ for each $k \in \mathbb{N}_0$. Let $x \in X'$. Then $x = \cap_{k\in\mathbb{N}_0} \ell_{i(k)}^k$ for some $\ell_{i(k)}^k \in \mathcal{C}_k$ such that $\overline{\ell_{i(k)}^k} \subset \ell_{i(k-1)}^{k-1}$ for each $k \in \mathbb{N}$. Define $h : X' \to Y'$ as $h(x) := \cap_{k\in\mathbb{N}_0} L_{i(k)}^k$. Since the patterns agree and diameters tend to zero, h is a well-defined bijection. We show that it is continuous. First note that $h(\ell_{i(m)}^m \cap X') = L_{i(m)}^m \cap Y'$ for every $m \in \mathbb{N}_0$ and every $i(m) \in \{1, \dots, n(m)\}$, since if $x = \cap_{k\in\mathbb{N}_0} \ell_{i(k)}^k \subset \ell_{i(m)}^m$, then there is $k' \in \mathbb{N}_0$ such that $\ell_{i(k)}^k \subset \ell_{i(m)}^m$ for each $k \geq k'$. But then $L_{i(k)}^k \subset L_{i(m)}^m$ for each $k \geq k'$, thus $h(x) = \cap_{k\in\mathbb{N}_0} L_{i(k)}^k \subset L_{i(m)}^m$. The other direction follows analogously. Now let $U \subset Y'$ be an open set and $x \in h^{-1}(U)$. Since diameters tend to zero, there is $m \in \mathbb{N}_0$ and $i(m) \in \{1, \dots, n(m)\}$ such that

$h(x) \in L^m_{i(m)} \cap Y' \subset U$ and thus $x \in \ell^m_{i(m)} \cap X' \subset h^{-1}(U)$. So $h^{-1}(U) \subset X'$ is open and that concludes the proof. □

In the following sections we will construct nested intersections of nice planar chains such that their patterns are the same as the patterns of refinements $\mathcal{C}_n \prec \mathcal{C}_{n-1}$ of natural chains of X_∞ (as constructed at the beginning of this section) and such that the diameters of links tend to zero. By the previous lemma, this gives embeddings of X_∞ in the plane. We note that the previous lemma holds in a more general setting (with an appropriately generalized definition of patterns), *i.e.,* for graph-like continua and graph chains, see *e.g.* [19].

4. Permuting the graph

Let $C = \{l_1, \dots, l_n\}$ be a chain cover of I and let $f : I \to I$ be a continuous surjection which is piecewise linear with finitely many critical points $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$ (so we include the endpoints of $I = [0,1]$ in the set of critical points). In the rest of the paper we work with continuous surjections which are piecewise linear (so with finitely many critical points); we call them *piecewise linear surjections.* Without loss of generality we assume that for every $i \in \{0, \dots, m\}$ and $l \in C$, $f([t_i, t_{i+1}]) \not\subset l$.

Define $H_j = f([t_j, t_{j+1}]) \times \{j\}$ for each $j \in \{0, \dots, m\}$ and $V_j = \{f(t_j)\} \times [j-1, j]$ for each $j \in \{1, \dots, m\}$. Note that H_{j-1} and H_j are joined at their left endpoints by V_j if there is a local minimum of f in t_j and they are joined at their right endpoints if there is a local maximum of f in t_j (see Figure 1). The line $H_0 \cup V_1 \cup H_1 \cup \dots \cup V_m \cup H_m =: G_f$ is called the *flattened graph of f in $\mathbb{R}^2$.*

Definition 4.1. *A permutation $p : \{0, 1, \dots, m\} \to \{0, 1, \dots, m\}$ is called a C-*admissible permutation *of G_f if for every $i \in \{0, \dots, m-1\}$ and $k \in \{0, \dots, m\}$ such that $p(i) < p(k) < p(i+1)$ or $p(i+1) < p(k) < p(i)$ it holds that:*

1. *$f(t_{i+1}) \notin f([t_k, t_{k+1}])$, or*
2. *$f(t_{i+1}) \in f([t_k, t_{k+1}])$ but $f(t_k)$ or $f(t_{k+1})$ is contained in the same link of C as $f(t_{i+1})$.*

Denote a C-admissible permutation of G_f by

$$p^C(G_f) = p(H_0) \cup p(V_1) \cup \dots \cup p(V_m) \cup p(H_m),$$

for $p(H_j) = f([\tilde{t}_j, \tilde{t}_{j+1}]) \times \{p(j)\}$ and $p(V_j) = \{f(\tilde{t}_j)\} \times [p(j-1), p(j)]$, where $\tilde{t}_j$ is chosen such that $f(t_j)$ and $f(\tilde{t}_j)$ are contained in the same link of C, and such that $p^C(G_f)$ has no self intersections for every $j \in \mathbb{N}$. A line $p^C(G_f)$ will be called a *permuted graph of f with respect to C*. Let $E(p^C(G_f))$ be the endpoint of $p(H_0)$ corresponding to $(f(\tilde{t}_0), p(0))$.

Note that $p(V_j)$ from Definition 4.1 is a vertical line in the plane which joins the endpoints of $p(H_{j-1})$ and $p(H_j)$ at $f(\tilde{t}_j)$, see Figure 1.

Definition 4.2. *If $p(j) = m$, we say that H_j is* at the top of *$p^C(G_f)$.*

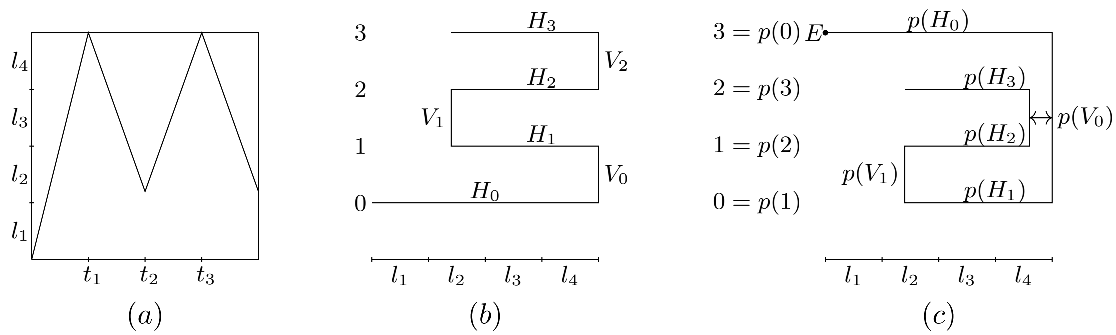

FIGURE 1. Flattened graph and its permutation. Note that H_0 is at the top of $p^C(G_f)$.

Note that a flattened graph G_f is just a graph of f for which its critical points have been extended to vertical intervals. These vertical intervals were introduced for the definition of a permuted graph. After permuting the flattened graph, we can quotient out the vertical intervals in the following way.

For every $i = 1, \dots, m$, pick a point $q_i \in p(V_i)$. There exists a homotopy $F : I \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(1, y) = q_i$ for every $y \in p(V_i)$ and every $i = 1, \dots, m$, and for every $t \in I$, $F(t, \cdot) : p^C(G_f)) \to \mathbb{R}^2$ is injective, and such that $\pi_x(F(t, (x, y))) = x$ for every $(x, y) \in \mathbb{R}^2$ and $t \in I$. Here $\pi_x(x, y)$ denotes a projection on the first coordinate. From now on $p^C(G_f)$ will always stand for the quotient $F(1, p^C(G_f))$, but for clarity in the figures of Sections 5 and 6 we will continue to draw it with long vertical intervals.

5. Chain refinements, their composition and stretching

Definition 5.1. *Let $f : I \to I$ be a piecewise linear surjection, p an admissible C-permutation of G_f and $\varepsilon > 0$. We call a nice planar chain $\mathcal{C} = \{\ell_1, \dots, \ell_n\}$ a* tubular ε-chain with nerve *$p^C(G_f)$ if*

- *$\mathcal{C}^*$ is an ε-neighborhood of $p^C(G_f)$, and*
- *there exists $n \in \mathbb{N}$ and arcs $A_1 \cup \ldots \cup A_n = p^C(G_f)$ such that ℓ_i is the ε-neighborhood of A_i for every $i \in \mathbb{N}$.*

Denote a nerve $p^C(G_f)$ of $\mathcal{C}$ by $\mathcal{N}_\mathcal{C}$. When there is no need to specify ε and $\mathcal{N}_\mathcal{C}$ we just say that $\mathcal{C}$ is tubular.

Definition 5.2. *A planar chain $\mathcal{C} = \{\ell_1, \dots, \ell_n\}$ will be called* horizontal *if there are $\delta > 0$ and a chain of open intervals $\{l_1, \dots, l_n\}$ in $\mathbb{R}$ such that $\ell_i = l_i \times (-\delta, \delta)$ for every $i \in \{1, \dots, n\}$.*

Remark 5.3. *Let $\mathcal{C}$ be a tubular chain. There exists a homeomorphism $\widetilde{H} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\widetilde{H}(\mathcal{C})$ is a horizontal chain and $\widetilde{H}^{-1}(\mathcal{C}')$ is tubular for every tubular $\mathcal{C}' \prec \widetilde{H}(\mathcal{C})$.*

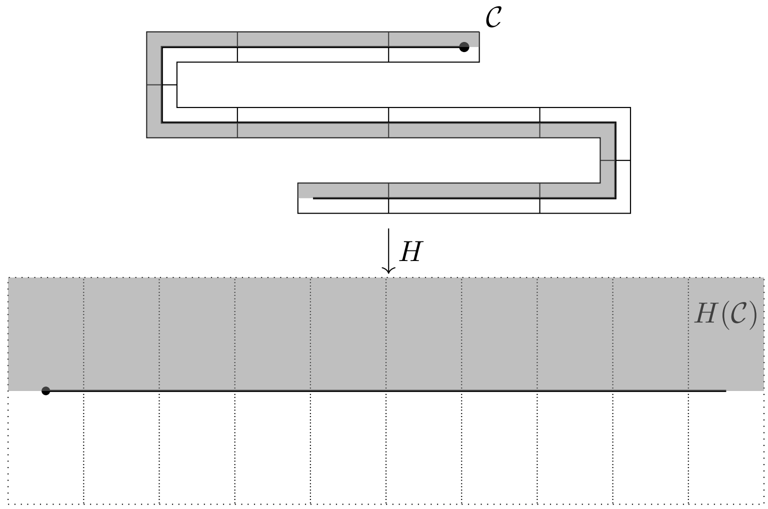

FIGURE 2. Stretching the chain $\mathcal{C}$. Recall that the vertical intervals are actually identified with points and thus $\tilde{H}^{-1}(\mathcal{C}')$ is tubular for every tubular $\mathcal{C}' \prec \tilde{H}(\mathcal{C})$.

Moreover, for $\mathcal{C} = \{\ell_1, \ldots, \ell_n\}$ denote by $\mathcal{N}_{\widetilde{H}(\mathcal{C})} = I \times \{0\}$. Note that $\mathcal{C} \setminus (\ell_1 \cup \ell_n \cup \mathcal{N}_{\mathcal{C}})$ has two components and thus it makes sense to call the components upper and lower. Let S be the upper component of $\mathcal{C} \setminus (\ell_1 \cup \ell_n \cup \mathcal{N}_{\mathcal{C}})$.

There exists a homeomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$ which has all the properties of a homeomorphism $\widetilde{H}$ above and in addition satisfies:

- *the endpoint $H(E(p^C(G_f))) = (0,0)$ (recall E from Definition 4.1) and*
- *$H(S)$ is the upper component of $H(\mathcal{C}^*) \setminus (H(\ell_1) \cup H(\ell_n) \cup H(A))$.*

Applying H to a chain $\mathcal{C}$ is called the stretching of $\mathcal{C}$ *(see Figure 2).*

Remark 5.4. *Let $X_\infty, \{C_i\}_{i \in \mathbb{N}_0}, \{\mathcal{C}_i\}_{i \in \mathbb{N}_0}$ be as defined in Section 3. For $i \in \mathbb{N}_0$, let $\mathcal{D}_i$ be a horizontal chain with the same number of links as $\mathcal{C}_i$ and such that $p^{C_i}(G_{f_{i+1}}) \subset \mathcal{D}_i^*$ for some C_i-admissible permutation p. Fix $\varepsilon > 0$ and note that, after possibly dividing links of $\mathcal{C}_{i+1}$ into smaller links (*i.e., *refining the chain C_{i+1} of I), there exists a tubular chain $\mathcal{D}_{i+1} \prec \mathcal{D}_i$ with nerve $p^{C_i}(G_{f_{i+1}})$ such that $Pat(\mathcal{D}_{i+1}, \mathcal{D}_i) = Pat(\mathcal{C}_{i+1}, \mathcal{C}_i)$ and $mesh(\mathcal{D}_{i+1}) < \varepsilon$, see Figure 3.*

Definition 5.5. *Let $H : \mathbb{R}^2 \to \mathbb{R}^2$ be a stretching of some tubular chain $\mathcal{C}$. If $\mathcal{C}'$ is a nice chain in $\mathbb{R}^2$ refining $\mathcal{C}$ and there is an interval map $g : I \to I$ such that $p^C(G_g)$ is a nerve of $H(\mathcal{C}')$, then we say that $\mathcal{C}'$* follows $p^C(G_g)$ in $\mathcal{C}$.

Now we discuss compositions of chain refinements. Let $f, g : I \to I$ be piecewise linear surjections. Let $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = 1$ be the critical points of f and let $0 = s_0 < s_1 < \ldots < s_n < s_{n+1} = 1$ be the critical points of g. Let C_1 and C_2 be nice

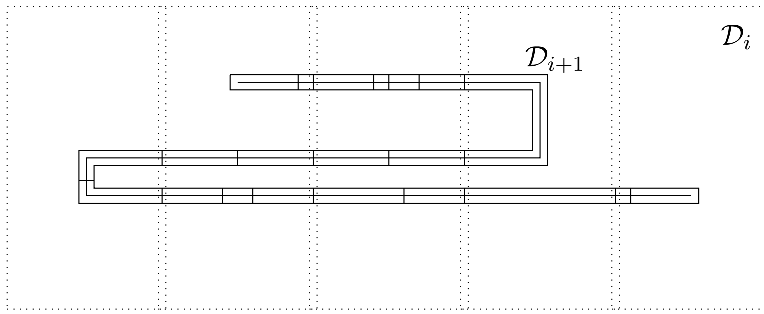

FIGURE 3. Constructing a tubular chain with nerve $p^C(G_f)$. Recall that vertical intervals represent points.

chain covers of I, let $p_1 : \{0,1,\dots,m\} \to \{0,1,\dots,m\}$ be an admissible C_1-permutation of G_f and let $p_2 : \{0,1,\dots,n\} \to \{0,1,\dots,n\}$ be an admissible C_2-permutation of G_g.

Assume $\mathcal{C}'' \prec \mathcal{C}' \prec \mathcal{C}$ are nice chains in $\mathbb{R}^2$ such that $\mathcal{C}$ is horizontal and $p_1^{C_1}(G_f) \subset \mathcal{C}^*$ (recall that $\mathcal{C}^*$ denotes the union of the links of $\mathcal{C}$), $\mathcal{C}'$ is a tubular chain with $\mathcal{N}_{\mathcal{C}'} = p_1^{C_1}(G_f)$, and $\mathcal{C}''$ follows $p_2^{C_2}(G_g)$ in $\mathcal{C}'$. Then $\mathcal{C}''$ follows $f \circ g$ in $\mathcal{C}$ with respect to a C_1-admissible permutation of $G_{f\circ g}$ which we will denote by $p_1 * p_2$ (see Figures 4 and 5).

Define
$$A_{ij} = \{x \in I : x \in [s_i, s_{i+1}], g(x) \in [t_j, t_{j+1}]\},$$
for $i \in \{0,1,\dots,n\}, j \in \{0,1,\dots,m\}$, *i.e.*, A_{ij} are maximal intervals on which $f \circ g$ is injective and possibly $A_{ij} = \emptyset$. Let H_{ij} be the horizontal branch of $G_{f\circ g}$ corresponding to the interval A_{ij}.

We want to see which branch H_{ij} corresponds to the top of $(p_1 * p_2)^{C_1}(G_{f\circ g})$. Denote the top of $p_1^{C_1}(G_f)$ by $p_1(H_{T_1})$, *i.e.*, $p_1(T_1) = m$. Denote the top of $p_2^{C_2}(G_g)$ by $p_2(H_{T_2})$, *i.e.*, $p_2(T_2) = n$. By the choice of orientation of H, the top of $(p_1 * p_2)^{C_1}(G_{f\circ g})$ is $H_{T_2 T_1}$ (see Figures 4 and 5).

6. Construction of the embeddings

Let $X_\infty = \varprojlim\{I, f_i\}$ where for every $i \in \mathbb{N}$ the map f_i is a continuous piecewise linear surjection with critical points $0 = t_0^i < t_1^i < \dots < t_{m(i)}^i < t_{m(i)+1}^i = 1$. Let $I_k^i = [t_k^i, t_{k+1}^i]$ for every $i \in \mathbb{N}$ and every $k \in \{0,\dots,m(i)\}$. We construct chains $(C_n)_{n\in\mathbb{N}_0}$ and $(\mathcal{C}_n)_{n\in\mathbb{N}_0}$ as before, such that for each $i \in \mathbb{N}_0$, $k \in \{0,\dots,m(i+1)\}$ and $l \in C_i$, $f_{i+1}(I_k^{i+1}) \not\subset l$. The flattened graph of f_i will be denoted by $G_{f_i} = H_0^i \cup V_1^i \cup \dots \cup V_{m(i)}^i \cup H_{m(i)}^i$ for each $i \in \mathbb{N}_0$.

Theorem 6.1. *Let $x = (x_0, x_1, x_2, \dots) \in X_\infty$ be such that for each $i \in \mathbb{N}_0$, $x_i \in I_{k(i)}^i$ and there exists an admissible permutation (with respect to C_{i-1}) $p_i : \{0,\dots,m(i)\} \to \{0,\dots,m(i)\}$ of G_{f_i} such that $p_i(k(i)) = m(i)$. Then there exists a planar embedding of X_∞ such that x is accessible.*

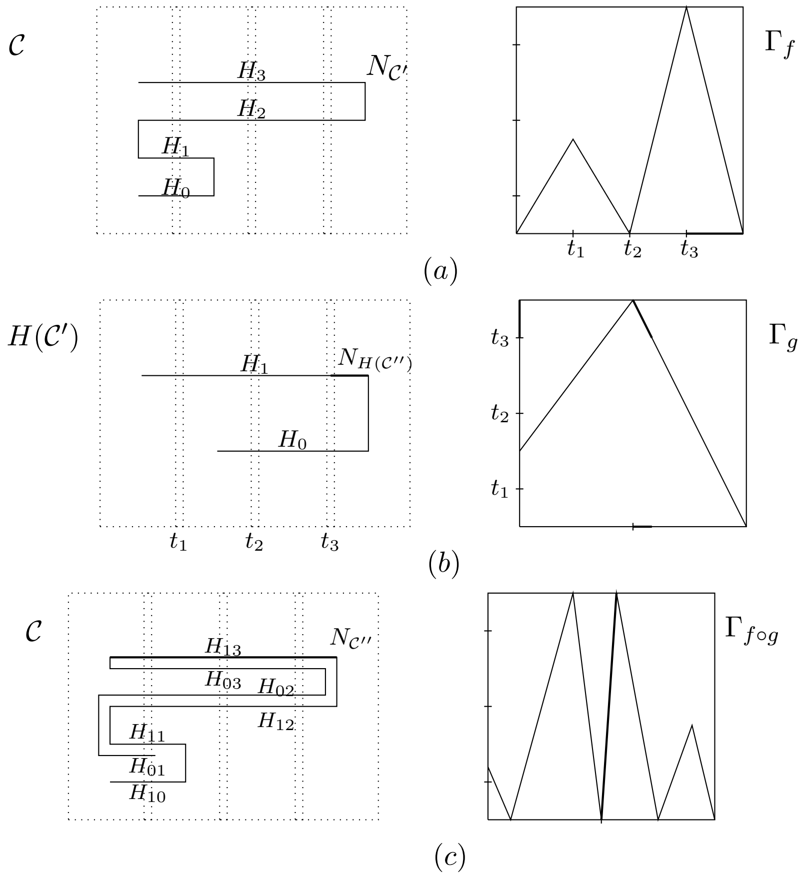

FIGURE 4. Composing refinements. In (a) the horizontal chain $\mathcal{C}$ and a nerve of $\mathcal{C}'$ are drawn. Nerve $\mathcal{N}_{\mathcal{C}'}$ equals $G_f^{C_1}$, a flattened version of the graph Γ_f. In (b) we draw $\mathcal{C}'$ as a horizontal chain by applying H. Also, nerve $\mathcal{N}_{H(\mathcal{C}'')}$ is given as $G_g^{C_2}$, a flattened version of the graph Γ_g. In (c) we draw $N_{\mathcal{C}''}$ in $\mathcal{C}$. In bold we trace the arc which is the top of $(id * id)^{C_1}(G_{f\circ g}) = N_{\mathcal{C}''}$.

Proof. Fix a strictly decreasing sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ such that $\varepsilon_i \to 0$ as $i \to \infty$. Let $\mathcal{D}_0$ be a nice horizontal chain in $\mathbb{R}^2$ with the same number of links as $\mathcal{C}_0$. By Remark 5.4 we can find a tubular chain $\mathcal{D}_1 \prec \mathcal{D}_0$ with nerve $p_1^{C_0}(G_{f_1})$, such that $Pat(\mathcal{D}_1, \mathcal{D}_0) = Pat(\mathcal{C}_1, \mathcal{C}_0)$ and $\mathrm{mesh}(\mathcal{D}_1) < \varepsilon_1$. Note that $p_1(k(1)) = m(1)$.

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a stretching of $\mathcal{D}_1$ (see Remark 5.3). Again using Remark 5.4 we can define $F(\mathcal{D}_2) \prec F(\mathcal{D}_1)$ such that $\mathrm{mesh}(\mathcal{D}_2) < \varepsilon_2$ (F is uniformly continuous), $Pat(F(\mathcal{D}_2), F(\mathcal{D}_1)) = Pat(\mathcal{C}_2, \mathcal{C}_1)$ and nerve of $F(\mathcal{D}_2)$ is $p_2^{C_1}(G_{f_2})$. Thus $H^2_{k(2)}$ is the top of $N_{F(\mathcal{D}_2)}$. By the arguments in the previous section, the top of $N_{\mathcal{D}_2}$ is $H_{k(2)k(1)}$.

As in the previous section, denote the maximal intervals of monotonicity of $f_1 \circ \ldots \circ f_i$ by

$$A_{n(i)\ldots n(1)} := \{x \in I : x \in I^i_{n(i)}, f_i(x) \in I^{i-1}_{n(i-1)}, \ldots, f_1 \circ \ldots \circ f_{i-1}(x) \in I^1_{n(1)}\},$$

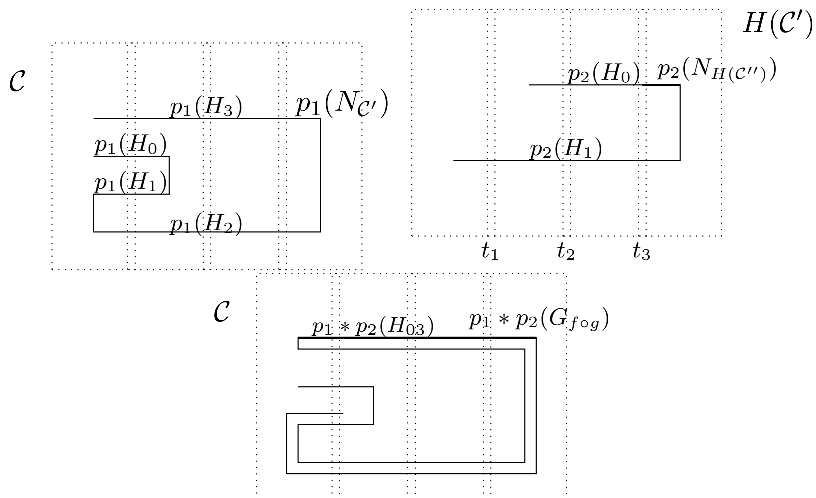

FIGURE 5. Composing permuted refinements. Here $p_1 = (0\ 2)$ and $p_2 = (0\ 1)$ are admissible. The top of $p_1(N_{\mathcal{C}'})$ is $p_1(H_3)$, so $T_1 = 3$. The top of $p_2(N_{H(\mathcal{C}'')})$ is $p_2(H_0)$, so $T_2 = 0$. Thus, the top of $(p_1 * p_2)^{C_1}(G_{f\circ g})$ is $H_{T_2T_1} = H_{03}$ (in bold).

and denote the corresponding horizontal intervals of $G_{f_1\circ\ldots\circ f_i}$ by $H_{n(i)\ldots n(1)}$.

Assume that we have constructed a sequence of chains $\mathcal{D}_i \prec \mathcal{D}_{i-1} \prec \ldots \prec \mathcal{D}_1 \prec \mathcal{D}_0$. Take a stretching $F : \mathbb{R}^2 \to \mathbb{R}^2$ of $\mathcal{D}_i$ and define $F(\mathcal{D}_{i+1}) \prec F(\mathcal{D}_i)$ such that $\text{mesh}(\mathcal{D}_{i+1}) < \varepsilon_{i+1}$, $Pat(F(\mathcal{D}_{i+1}), F(\mathcal{D}_i)) = Pat(\mathcal{C}_{i+1}, \mathcal{C}_i)$ and such that a nerve of $F(\mathcal{D}_{i+1})$ is $p_{i+1}^{C_i}(G_{f_{i+1}})$, which is possible by Remark 5.4. Note that the top of $\mathcal{N}_{\mathcal{D}_{i+1}}$ is $H_{k(i+1)\ldots k(1)}$.

Since $Pat(F(\mathcal{D}_{i+1}), F(\mathcal{D}_i)) = Pat(\mathcal{D}_{i+1}, \mathcal{D}_i)$ for every $i \in \mathbb{N}_0$ and by the choice of the sequence (ε_i), Lemma 3.1 yields that $\cap_{n\in\mathbb{N}_0}\mathcal{D}_n^*$ is homeomorphic to X_∞. Let $\varphi(X_\infty) = \cap_{n\in\mathbb{N}_0}\mathcal{D}_n^*$.

To see that x is accessible, note that $H = \lim_{i\to\infty} H_{k(i)\ldots k(1)}$ is a well-defined horizontal arc in $\varphi(X_\infty)$ (possibly degenerate). Let $H = [a, b] \times \{h\}$ for some $h \in \mathbb{R}$. Note that for every $y = (y_1, y_2) \in \varphi(X_\infty)$ it holds that $y_2 \le h$. Thus every point $p = (p_1, h) \in H$ is accessible by the vertical planar arc $\{p_1\} \times [h, h+1]$. Since $x \in H$, the construction is complete. □

7. ZIGZAGS

Definition 7.1. *Let $f : I \to I$ be a continuous piecewise monotone surjection with critical points $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = 1$. Let $I_k = [t_k, t_{k+1}]$ for every $k \in \{0, \ldots, m\}$. We say that I_k is* inside a zigzag *of f if there exist critical points a and e of f such that $a < t_k < t_{k+1} < e \in I$ and either*

(1) *$f(t_k) > f(t_{k+1})$ and $f|_{[a,e]}$ assumes its global maximum at a and its global minimum at e, or*
(2) *$f(t_k) < f(t_{k+1})$ and $f|_{[a,e]}$ assumes its global minimum at a and its global maximum at e.*

Then we say that $x \in \mathring{I}_k = I_k \setminus \{t_k, t_{k+1}\}$ is inside a zigzag of f *(see Figure 6). We also say that f* contains a zigzag *if there is a point inside a zigzag of f.*

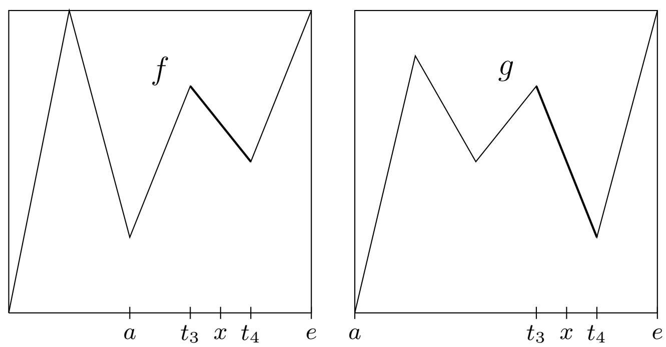

FIGURE 6. The interval $[t_3, t_4]$ is inside a zigzag of f and g.

Lemma 7.2. *Let $f : I \to I$ be a continuous piecewise linear surjection with critical points $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = 1$. If there is $k \in \{0, \ldots, m\}$ such that $I_k = [t_k, t_{k+1}]$ is not inside a zigzag of f, then there exists an admissible permutation p of G_f (with respect to any nice chain C) such that $p(k) = m$.*

Proof. Assume that I_k is not inside a zigzag of f. Without loss of generality assume that $f(t_k) > f(t_{k+1})$. If $f(a) \geq f(t_{k+1})$ for each $a \in [0, t_k]$ (or if $f(e) \leq f(t_k)$ for each $e \in [t_{k+1}, 1]$) we are done (see Figure 7) by simply reflecting all H_i, $i < k$ over H_k (or reflecting all H_i, $i > k$ over H_k in the second case).

Therefore, assume that there exists $a \in [0, t_k]$ such that $f(a) < f(t_{k+1})$ and there exists $e \in [t_{k+1}, 1]$ such that $f(e) > f(t_k)$. Denote the largest such a by a_1 and the smallest such e by e_1. Since I_k is not inside a zigzag, there exists $e' \in [t_{k+1}, e_1]$ such that $f(e') \leq f(a_1)$ or there exists $a' \in [a_1, t_k]$ such that $f(a') \geq f(e_1)$. Assume the first case and take e' for which $f|_{[t_{k+1}, e_1]}$ attains its global minimum (in the second case we take a' for which $f|_{[a_1, t_k]}$ attains its global maximum). Reflect $f|_{[a_1, t_k]}$ over $f|_{[t_k, e']}$ (in the second case we reflect $f|_{[t_{k+1}, e_1]}$ over $f|_{[a', t_{k+1}]}$). Then, H_k becomes the top of $G_{f|_{[a_1, e_1]}}$ (see Figure 8).

If $f(a) \geq f(e')$ for each $a \in [0, a_1]$ (or if $f(e) \leq f(a')$ for all $e \in [e_1, 1]$ in the second case), we are done. So assume that there is $a_2 \in [0, a_1]$ such that $f(a_2) < f(e')$ and take the largest such a_2. Then there exists $a'' \in [a_2, a_1]$ such that $f(a'') \geq f(e_1)$, and take a'' for which $f|_{[a_2, a_1]}$ attains its global maximum. If $f(e) \leq f(a'')$ for each $e \in [e_1, 1]$, we reflect $f|_{[a_2, a'']}$ over $f|_{[e_1, 1]}$ and are done. If there is (minimal) $e_2 > e_1$ such that $f(e_2) > f(a'')$, then there exists $e'' \in [e_1, e_2]$ such that $f(e'') \leq f(a_2)$ and for

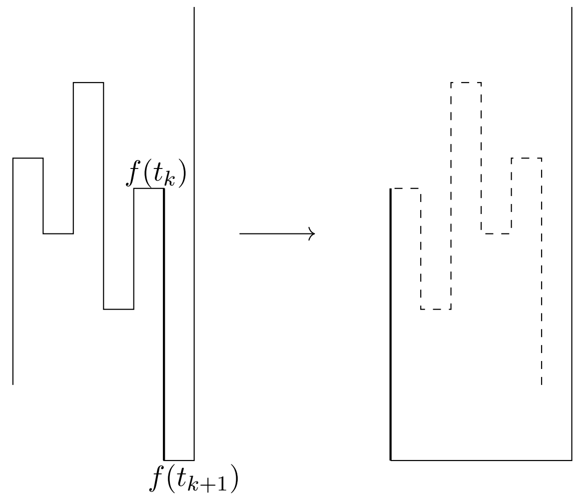

FIGURE 7. Reflections in the first part of the proof of Lemma 7.2.

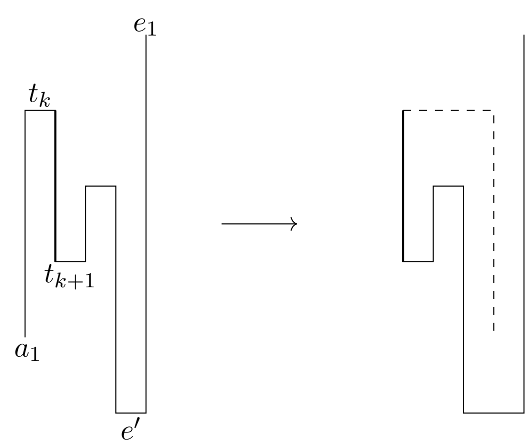

FIGURE 8. Reflections in the second part of the proof of Lemma 7.2.

which $f|_{[e_1,e_2]}$ attains a global minimum. In that case we reflect $f|_{[a'',t_k]}$ over $f|_{[t_k,e']}$ and $f|_{[a_2,a'']}$ over $f|_{[t_k,e'']}$ (see Figure 9).

Thus we have constructed a permutation such that H_k becomes the top of $G_{f|_{[a_2,e_2]}}$. We proceed by induction. $\square$

Theorem 7.3. *Let $X_\infty = \varprojlim\{I, f_i\}$ where each $f_i : I \to I$ is a continuous piecewise linear surjection. If $x = (x_0, x_1, x_2, \dots) \in X_\infty$ is such that for each $i \in \mathbb{N}$, x_i is not inside a zigzag of f_i, then there exists an embedding of X_∞ in the plane such that x is accessible.*

Proof. The proof follows by Lemma 7.2 and Theorem 6.1. $\square$

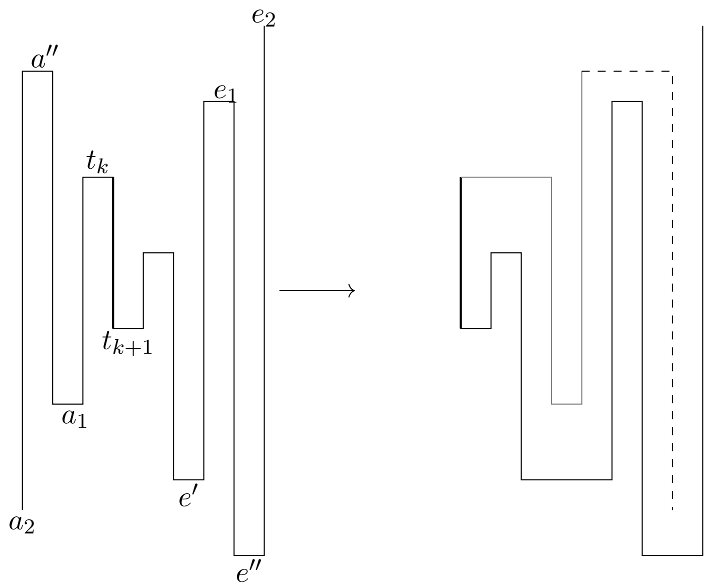

FIGURE 9. Reflections in the third part of the proof of Lemma 7.2.

Corollary 7.4. *Let $X_\infty = \varprojlim\{I, f_i\}$ where each $f_i : I \to I$ is a continuous piecewise linear surjection which does not have zigzags. Then, for every $x \in X_\infty$ there exists an embedding of X_∞ in the plane such that x is accessible.*

Remark 7.5. *Note that if $T : I \to I$ is a unimodal map and $x \in \varprojlim(I, T)$, then $\varprojlim(I, T)$ can be embedded in the plane such that x is accessible by the previous corollary. This is Theorem 1 of* [2]*. This easily generalizes to an inverse limit of open interval maps (*e.g. *generalized Knaster continua).*

The following lemma shows that given arbitrary chains $\{C_i\}$, the zigzag condition from Lemma 7.2 cannot be improved.

Lemma 7.6. *Let $f : I \to I$ be a continuous piecewise linear surjection with critical points $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = 1$. If $I_k = [t_k, t_{k+1}]$ is inside a zigzag for some $k \in \{0, \ldots, m\}$, then there exists a nice chain C covering I such that $p(k) \neq m$ for every admissible permutation p of G_f with respect to C.*

Proof. Take a nice chain cover C of I such that mesh $(C) < \min\{|f(t_i) - f(t_j)| : i, j \in \{0, \ldots, m+1\}, f(t_i) \neq f(t_j)\}$. Assume without loss of generality that $f(t_k) > f(t_{k+1})$ and let $t_i < t_k < t_{k+1} < t_j$ be such that minimum and maximum of $f|_{[t_i,t_j]}$ are attained at t_i and t_j respectively. Assume t_i is the largest and t_j is the smallest index with such properties. Let p be some permutation. If $p(i) < p(j) < p(k)$, then by the choice of C, $p(H_j)$ intersects $p(V_{i'})$ for some $i' \in \{i, \ldots, k\}$. We proceed similarly if $p(j) < p(i) < p(k)$. □

Remark 7.7. *Let $X_\infty = \varprojlim\{I, f_i\}$ and $x = (x_0, x_1, x_2, \ldots) \in X_\infty$. If there exist piecewise linear continuous surjections $g_i : I \to I$ and a homeomorphism $h : X_\infty \to$*

$\varprojlim\{I, g_i\}$ *such that every projection of* $h(x)$ *is not in a zigzag of* g_i*, then* X_∞ *can be embedded in the plane such that* x *is accessible. We have the following two corollaries. See also Examples 7.10-7.12.*

Corollary 7.8. *Let* $X_\infty = \varprojlim\{I, f_i\}$ *where each* $f_i : I \to I$ *is a continuous piecewise linear surjection. If* $x = (x_0, x_1, x_2, \dots) \in X_\infty$ *is such that* x_i *is inside a zigzag of* f_i *for at most finitely many* $i \in \mathbb{N}$*, then there exists an embedding of* X_∞ *in the plane such that* x *is accessible.*

Proof. Since $\varprojlim\{I, f_i\}$ and $\varprojlim\{I, f_{i+n}\}$ are homeomorphic for every $n \in \mathbb{N}$, the proof follows using Theorem 7.3. □

Corollary 7.9. *Let* f *be a continuous piecewise linear surjection with finitely many critical points and* $x = (x_0, x_1, x_2, \dots) \in X_\infty = \varprojlim\{I, f\}$*. If there exists* $k \in \mathbb{N}$ *such that* x_i *is not inside a zigzag of* f^k *for all but finitely many* i *, then there exists a planar embedding of* X_∞ *such that* x *is accessible.*

Proof. Note that $\varprojlim\{I, f^k\}$ and X_∞ are homeomorphic. □

We give applications of Corollary 7.9 in the following examples.

Example 7.10. *Let* f *be a piecewise linear map such that* $f(0) = 0$*,* $f(1) = 1$ *and with critical points* $\frac{1}{4}, \frac{3}{4}$*, where* $f(\frac{1}{4}) = \frac{3}{4}$ *and* $f(\frac{3}{4}) = \frac{1}{4}$ *(see Figure 10).*

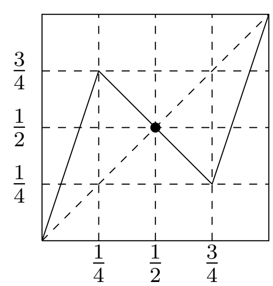

FIGURE 10. Graph of f from Example 7.10.

Note that $X = \varprojlim\{I, f\}$ *consists of two rays compactifying on an arc and therefore, for every* $x \in X$*, there exists a planar embedding making* x *accessible. However, the point* $\frac{1}{2}$ *is inside a zigzag of* f*. Figure 11 shows the graph of* f^2*. Note that the point* $\frac{1}{2}$ *is not inside a zigzag of* f^2 *and that gives an embedding of* X *such that* $(\frac{1}{2}, \frac{1}{2}, \dots)$ *is accessible.*

Let $x = (x_0, x_1, x_2, \dots) \in X$ *be such that* $x_i \in [1/4, 3/4]$ *for all but finitely many* $i \in \mathbb{N}_0$*. Then, the embedding in Figure 11 will make* x *accessible. For other points* $x = (x_0, x_1, x_2, \dots) \in X$ *there exists* $N \in \mathbb{N}$ *such that* $x_i \in [0, 1/4]$ *for each* $i > N$ *or* $x_i \in [3/4, 1]$ *for each* $i > N$ *so the standard embedding makes them accessible. In fact, the embedding in Figure 11 will make every* $x \in X$ *accessible.*

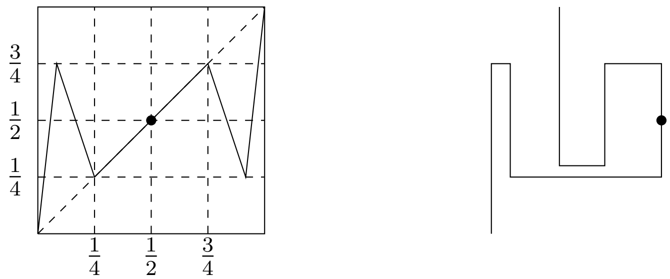

FIGURE 11. Graph of f^2 from Example 7.10.

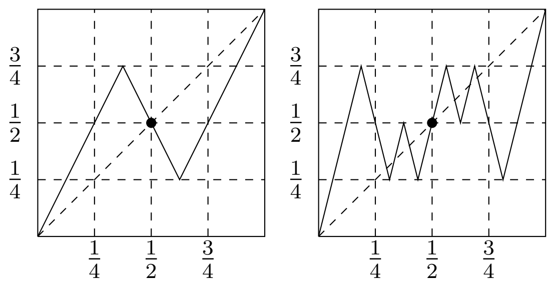

FIGURE 12. Graph of f and f^2 in Example 7.11.

Example 7.11. *Assume that f is a piecewise linear map with $f(0)=0$, $f(1)=1$ and critical points $f(\frac{3}{8})=\frac{3}{4}$ and $f(\frac{5}{8})=\frac{1}{4}$ (see Figure 12).*

Note that $X=\varprojlim\{I,f\}$ consists of two Knaster continua joined at their endpoints together with two rays both converging to these two Knaster continua. Note that $(\frac{1}{2},\frac{1}{2},\ldots)$ can be embedded accessibly with the use of f^2, see Figure 12. However, as opposed to the previous example, X cannot be embedded such that every point is accessible (this follows already by a result of Mazurkiewicz [18] *which says that there exist nonaccessible points in any planar embedding of a nondegenerate indecomposable continuum). It is proven by Minc and Transue in* [21] *that such an embedding of a chainable continuum exists if and only if it is* Suslinean, i.e., *contains at most countably many mutually disjoint nondegenerate subcontinua.*

Example 7.12 (Nadler)**.** *Let $f:I\to I$ be as in Figure 13. This is Nadler's candidate from* [25] *for a negative answer to Question 2. However, in what follows we show that every point can be made accessible via some planar embedding of $\varprojlim(I,f)$.*

Let $n\in\mathbb{N}$. If $J\subset I$ is a maximal interval such that $f^n|_J$ is increasing, then J is not inside a zigzag of f^n, see e.g. *Figure 13.*

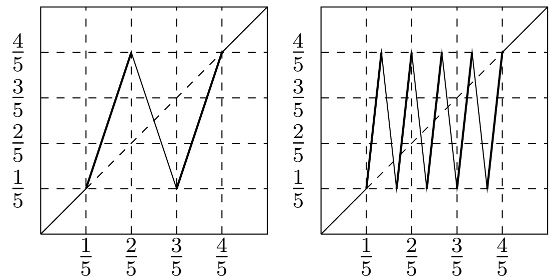

FIGURE 13. Map f and its second iterate. Bold lines are increasing branches of the restriction to $[1/5, 4/5]$. Note that they are not inside a zigzag of f or f^2 respectively.

We will code the orbit of points in the invariant interval $[1/5, 4/5]$ *in the following way. For* $y \in [1/5, 4/5]$ *let* $i(y) = (y_n)_{n\in\mathbb{N}_0} \subset \{0, 1, 2\}^\infty$*, where*

$$y_n = \begin{cases} 0, & f^n(y) \in [1/5, 2/5], \\ 1, & f^n(y) \in [2/5, 3/5], \\ 2, & f^n(y) \in [3/5, 4/5]. \end{cases}$$

The definition is somewhat ambiguous with a problem occurring at points $2/5$ *and* $3/5$*. Note, however, that* $f^n(2/5) = 4/5$ *and* $f^n(3/5) = 1/5$ *for all* $n \in \mathbb{N}$*. So every point in* $[1/5, 4/5]$ *will have a unique itinerary, except the preimages of* $2/5$ *(to which we can assign two itineraries* $a_1 \dots a_n \frac{0}{1} 2222 \dots$*) and preimages of* $3/5$*, (to which we can assign two itineraries* $a_1 \dots a_n \frac{1}{2} 0000 \dots$*), where* $\frac{0}{1}$ *means "0 or 1" and* $a_1, \dots, a_n \in \{0, 1, 2\}$*.*

Note that if $i(y) = 1y_2 \dots y_n 1$*, where* $y_i \in \{0, 2\}$ *for every* $i \in \{2, \dots, n\}$*, then* y *is contained in an increasing branch of* f^{n+1}*. This holds also if* $n = 1$, i.e., $y_2 \dots y_n = \emptyset$*. Also, if* $i(y) = 0 \dots$ *or* $i(y) = 2 \dots$*, then* y *is contained in an increasing branch of* f*. See Figure 14.*

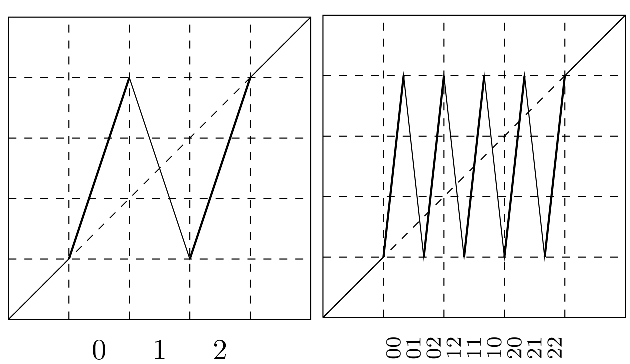

FIGURE 14. Map f and its iterate with symbolic coding of points. Note that points with itinerary $0\dots$ or $2\dots$ are contained in an increasing branch of f and points with itineraries $11\dots$ are contained in an increasing branch of f^2.

We extend the symbolic coding to X. For $x=(x_0,x_1,x_2,\dots)\in X$ with itinerary $i(x)$ let $(y_k)_{k\in\mathbb{Z}}$ be defined by

$$y_k=\begin{cases} i(x)_k, & k\geq 0, \text{ and for } k<0,\\ 0, & x_{-k}\in[1/5,2/5],\\ 1, & x_{-k}\in[2/5,3/5],\\ 2, & x_{-k}\in[3/5,4/5].\end{cases}$$

Again, the assignment is injective everywhere except at preimages of critical points $2/5$ or $3/5$.

Now fix $x=(x_0,x_1,x_2,\dots)\in X$ with its backward itinerary $\overleftarrow{x}=\dots y_{-2}y_{-1}y_0$ (assume the itinerary is unique, otherwise choose one of the two possible backward itineraries). Assume first that $y_k\in\{0,2\}$ for every $k\leq 0$. Then, for every $k\in\mathbb{N}_0$ it holds that $i(x_k)=0\dots$ or $i(x_k)=2\dots$ so x_k is in an increasing branch of f and thus not inside a zigzag of f. By Theorem 7.3 it follows that there is an embedding making x accessible. Similarly, if there exists $n\in\mathbb{N}$ such that $y_k\neq 1$ for $k<-n$, we use that X is homeomorphic to $\varprojlim\{I,f_j\}$ where $f_1=f^n$, $f_j=f$ for $j\geq 2$.

Assume that $\overleftarrow{x}=\dots 1(\frac{0}{2})^{n_3}1(\frac{0}{2})^{n_2}1(\frac{0}{2})^{n_1}$ where $\frac{0}{2}$ means "0 or 2" and $n_i\geq 0$ for $i\in\mathbb{N}$. We will assume that $n_1>0$; the general case follows similarly. Note that $i(x_{n_1-1})=(\frac{0}{2})^{n_1}\dots$ and so it is contained in an increasing branch of f^{n_1-1}. Note further that $i(x_{n_1+1+n_2})=1(\frac{0}{2})^{n_2}1(\frac{0}{2})^{n_1}\dots$ and so it is contained in an increasing branch of f^{n_2+2}. Also $f^{n_2+2}(x_{n_1+1+n_2})=x_{n_1-1}$. Further we note that $i(x_{n_1+1+n_2+1+n_3-1})=(\frac{0}{2})^{n_3}1(\frac{0}{2})^{n_2}1(\frac{0}{2})^{n_1}$ and so it is contained in an increasing branch of f^{n_3}. Furthermore, $f^{n_3}(x_{n_1+1+n_2+1+n_3-1})=x_{n_1+1+n_2}$.

In this way, we see that for every even $k\geq 4$ it holds that

$$i(x_{n_1+1+n_2+1+\dots+1+n_k})=1\left(\frac{0}{2}\right)^{n_k}1\dots 1\left(\frac{0}{2}\right)^{n_1}\dots$$

and so it is contained in an increasing branch of f^{n_k+2}. Also, $f^{n_k+2}(x_{n_1+1+n_2+1+\dots+1+n_k})=x_{n_1+1+n_2+1+\dots+1+n_{k-1}-1}$. Similarly,

$$i(x_{n_1+1+n_2+1+\dots+1+n_k+1+n_{k+1}-1})=\left(\frac{0}{2}\right)^{n_{k+1}}1\dots 1\left(\frac{0}{2}\right)^{n_1}\dots$$

so $x_{n_1+1+n_2+1+\dots+1+n_k+1+n_{k+1}-1}$ is in an increasing branch of $f^{n_{k+1}}$. Note also that $f^{n_{k+1}}(x_{n_1+1+n_2+1+\dots+1+n_k+1+n_{k+1}-1})=x_{n_1+1+n_2+1+\dots+1+n_k}$.

So we have the following sequence

$$\dots\xrightarrow{f^{n_5}}x_{n_1+1+\dots+1+n_4}\xrightarrow{f^{n_4+2}}x_{n_1+1+n_2+1+n_3-1}\xrightarrow{f^{n_3}}x_{n_1+1+n_2}\xrightarrow{f^{n_2+2}}x_{n_1-1}\xrightarrow{f^{n_1-1}}x_0,$$

where the chosen points in the sequence are not contained in zigzags of the corresponding bonding maps. Let

$$f_i = \begin{cases} f^{n_1-1}, & i = 1, \\ f^{n_i+2}, & i \text{ even}, \\ f^{n_i}, & i > 1 \text{ odd}. \end{cases}$$

Then, $\varprojlim\{I, f_i\}$ is homeomorphic to X and by Theorem 7.3 it can be embedded in the plane such that every $x \in \varprojlim\{I, f_i\}$ is accessible.

8. Thin embeddings

We have proven that if a chainable continuum X has an inverse limit representation such that $x \in X$ is not contained in zigzags of bonding maps, then there is a planar embedding of X making x accessible. Note that the converse is not true. The pseudo-arc is a counter-example, because it is homogeneous, so each of its points can be embedded accessibly. However, the crookedness of the bonding maps producing the pseudo-arc implies the occurrence of zigzags in every representation. Since the pseudo-arc is hereditarily indecomposable, no point is contained in an arc. To the contrary, in Minc's continuum X_M (see Figure 15), every point is contained in an arc of length at least $\frac{1}{3}$.

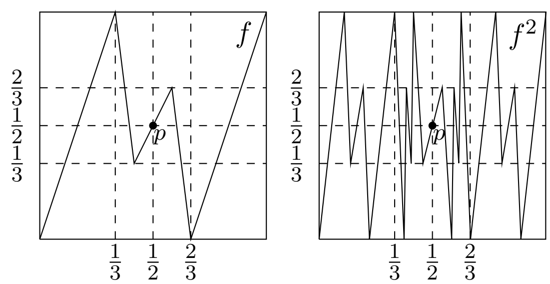

FIGURE 15. Minc's map and its second iteration (the example was given at the Spring Topology and Dynamical Systems Conference 2001 in a talk by Minc entitled "On embeddings of chainable continua into the plane").

In the next two definitions we introduce the notion of thin embedding, used under this name in *e.g.* [10]. In [1] the notion of thin embedding was referred to as *C-embedding.*

Definition 8.1. *Let $Y \subset \mathbb{R}^2$ be a continuum. We say that Y is* thin chainable *if there exists a sequence $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of chains in $\mathbb{R}^2$ such that $Y = \cap_{n\in\mathbb{N}}\mathcal{C}_n^*$, where $\mathcal{C}_{n+1} \prec \mathcal{C}_n$ for every $n \in \mathbb{N}$, mesh$(\mathcal{C}_n) \to 0$ as $n \to \infty$, and the links of $\mathcal{C}_n$ are connected sets in $\mathbb{R}^2$ (note that links are open in the topology of $\mathbb{R}^2$).*

Definition 8.2. *Let X be a chainable continuum. We say that an embedding φ : $X \to \mathbb{R}^2$ is a* thin embedding *if $\varphi(X)$ is thin chainable. Otherwise φ is called a* thick embedding.

Note that in [6] Bing shows that every chainable continuum has a thin embedding in the plane.

Question 3 (Minc, 2001)**.** *Is there a planar embedding of Minc's chainable continuum X_M which makes p accessible? Or as a special case, is there a* thin embedding *of X_M which makes p accessible?*

Example 8.3 (Bing, [6])**.** *An* Elsa continuum *(see* [24]*) is a continuum consisting of a ray compactifying on an arc (in* [8] *this was called an arc+ray continuum). An example of a thick embedding of an Elsa continuum was constructed by Bing (see Figure 16).*

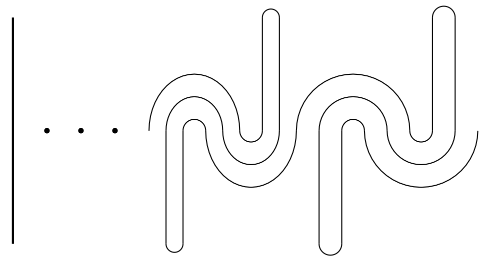

FIGURE 16. Bing's example from [6].

An example of a thick embedding of the 3-Knaster continuum was given by Dębski and Tymchatyn in [10]. An arc has a unique planar embedding (up to equivalence), so all of its planar embeddings are thin. Therefore, it is natural to ask the following question.

Question 4 (Question 1 in [1])**.** *Which chainable continua have a thick embedding in the plane?*

Definition 8.4. *Given a chainable continuum X, let $\mathcal{E}_C(X)$ denote the set of all planar embeddings of X obtained by performing admissible permutations of G_{f_i} for every representation X as $\varprojlim\{I, f_i\}$.*

The next theorem says that the class of all planar embeddings of chainable continuum X obtained by performing admissible permutations of graphs G_{f_i} is the class of all thin planar embeddings of X up to the equivalence relation between embeddings.

Theorem 8.5. *Let X be a chainable continuum and $\varphi : X \to \mathbb{R}^2$ a thin embedding of X. Then there exists an embedding $\psi \in \mathcal{E}_C(X)$ which is equivalent to φ.*

Proof. Recall that $\mathcal{C}_n^* = \bigcup_{\ell\in\mathcal{C}_n} \ell$. Let $\varphi(X) = \cap_{n\in\mathbb{N}_0}\mathcal{C}_n^*$, where the links of $\mathcal{C}_n$ are open, connected sets in $\mathbb{R}^2$. Furthermore, $\mathcal{C}_{n+1} \prec \mathcal{C}_n$ for every $n \in \mathbb{N}_0$. Note that we assume that links of $\mathcal{C}_n$ have a polygonal curve for a boundary, using a brick decomposition of the plane (see *e.g.* [23], pg. 34).

We argue that we can also assume that every $\mathcal{C}_n^*$ is simply connected. This goes in a few steps. In every step we first state the claim we can obtain and then argue in the rest of the step how to obtain it.

(1) Without loss of generality we can assume that the separate links of $\mathcal{C}_n$ are simply connected, by filling in the holes. That is, if a link $\ell \in \mathcal{C}_n$ is such that $\mathbb{R}^2 \setminus \ell$

separates the plane, instead of ℓ, we take $\ell \cup \bigcup_i V_i$, where the V_i are the bounded components of $\mathbb{R}^2 \setminus \ell$. Filling in the holes thus merges all the links contained in $\ell \cup \bigcup_i V_i$ into a single link. This does not change the mesh nor the pattern of the chain.

(2) We can assume that for every hole H between the links ℓ_i and ℓ_{i+1} (*i.e.,* a connected bounded component of $\mathbb{R}^2 \setminus (\ell_i \cup \ell_{i+1})$) it holds that if $H \cap \ell_j \neq \emptyset$ for some j, then $\ell_j \subset \ell_i \cup \ell_{i+1} \cup H$. Denote by U_i the union of bounded component of $\mathbb{R}^2 \setminus (\ell_i \cup \ell_{i+1})$ and note that $\{\ell_1 \cup U_1 \cup \ell_2, \ell_3 \cup U_3 \cup \ell_4, \dots, \ell_{k(n)-1} \cup U_{k(n)-1} \cup \ell_{k(n)}\}$ is again a chain. (It can happen that the first or last few links are merged into one link. Also, we can assume that n is even by merging the last two links if necessary.) Denote for simplicity $\tilde{\ell}_i = \ell_{2i-1} \cup U_{2i-1} \cup \ell_{2i}$ for every $i \in \{1, 2, \dots, n/2\}$. We claim that if $\tilde{\ell}_j \cap H \neq \emptyset$ for some hole between $\tilde{\ell}_i$ and $\tilde{\ell}_{i+1}$, then $\tilde{\ell}_j \subset \tilde{\ell}_i \cup \tilde{\ell}_{i+1} \cup H$. Assume the contrary, and take without loss of generality that $j > i+1$. Then necessarily $j = i+2$ and $\tilde{\ell}_{i+2}$ separates $\tilde{\ell}_{i+1}$ so that at least two components of $\tilde{\ell}_{i+1} \setminus \tilde{\ell}_{i+2}$ intersect $\tilde{\ell}_i$. That is, $\tilde{\ell}_{i+2}$ separates ℓ_{2i+1}. But this is a contradiction since $\tilde{\ell}_{i+2} = \ell_{2i+3} \cup U_{2i+3} \cup \ell_{2i+4}$ can only intersect ℓ_{2i+1} if $\ell_{2i+1} \subset U_{2i+3}$ in which case $\tilde{\ell}_{i+2}$ does not separate ℓ_{2i+1}.

(3) If there is a hole between links $\tilde{\ell}_i$ and $\tilde{\ell}_{i+1}$, then we can fill it in a similar way as in Step (1). That is, letting $\tilde{U}_i$ be the union of bounded components of $\mathbb{R}^2 \setminus (\tilde{\ell}_i \cup \tilde{\ell}_{i+1})$, the links of the modified chain are $\{\tilde{\ell}_1, \dots, \tilde{\ell}_{i-1}, \tilde{\ell}_i \cup \tilde{U}_i \cup \tilde{\ell}_{i+1}, \tilde{\ell}_{i+2}, \dots, \tilde{\ell}_{k(n)/2}\}$. (It can happen that $\tilde{\ell}_j \subset \tilde{U}_i$ for all $j > N \geq i+1$ or $j < N \leq i$, but then we merge all these links.) We do this for each $i \in \{1, \dots, k(n)/2\}$ where there is a hole between $\tilde{\ell}_i$ and $\tilde{\ell}_{i+1}$, so not just the odd values of i as in Step (2). Due to the claim in Step (2), the result is again a chain. These modified chains can have a larger mesh (up to four times the original mesh), but still satisfy $\mathcal{C}_{n+1} \prec \mathcal{C}_n$ for every $n \in \mathbb{N}_0$ and $\mathrm{mesh}(\mathcal{C}_n) \to 0$ as $n \to \infty$.

In the rest of the proof we construct homeomorphisms F_j, $0 \leq j \leq n \in \mathbb{N}_0$ and $G_n := F_n \circ \ldots \circ F_1 \circ F_0$, which straighten the chains $\mathcal{C}_n$ to horizontal chains. The existence of such homeomorphisms follows from the generalization of the piecewise linear Schoenflies' theorem given in *e.g.* [23, Section 3]. Take a homeomorphism $F_0 : \mathbb{R}^2 \to \mathbb{R}^2$ which maps $\mathcal{C}_0$ to a horizontal chain. Then $F_0(\mathcal{C}_1) \prec F_0(\mathcal{C}_0)$ and there is a homeomorphism $F_1 : \mathbb{R}^2 \to \mathbb{R}^2$ which is the identity on $\mathbb{R}^2 \setminus F_0(\mathcal{C}_0)^*$ (recall that $\mathcal{C}_n^*$ denotes the union of links of $\mathcal{C}_n$), and which maps $F_0(\mathcal{C}_1)^*$ to a tubular neighborhood of some permuted flattened graph with $\mathrm{mesh}(F_1(F_0(\mathcal{C}_1))) < \mathrm{mesh}(\mathcal{C}_1)$.
Note that $G_n(\mathcal{C}_{n+1}) \prec G_n(\mathcal{C}_n)$ and there is a homeomorphism $F_{n+1} : \mathbb{R}^2 \to \mathbb{R}^2$ which is the identity on $\mathbb{R}^2 \setminus G_n(\mathcal{C}_n)^*$ and which maps $G_n(\mathcal{C}_{n+1})^*$ to a tubular neighborhood of some flattened permuted graph with $\mathrm{mesh}(F_{n+1}(G_n(\mathcal{C}_{n+1}))) < \mathrm{mesh}(\mathcal{C}_{n+1})$.
Note that the sequence $(G_n)_{n \in \mathbb{N}_0}$ is uniformly Cauchy and $G := \lim_{n\to\infty} G_n$ is well-defined. By construction, $G : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism and $G \circ \varphi \in \mathcal{E}_C(X)$. $\square$

Question 5 (Question 2 in [1]). *Is there a chainable continuum X and a thick embedding ψ of X such that the set of accessible points of $\psi(X)$ is different from the set of accessible points of $\varphi(X)$ for any thin embedding φ of X?*

9. Uncountably many nonequivalent embeddings

In this section we construct, for every chainable continuum containing a nondegenerate indecomposable subcontinuum, uncountably many embeddings which are pairwise not strongly equivalent. Recall that $\varphi, \psi \colon X \to \mathbb{R}^2$ are strongly equivalent if $\varphi \circ \psi^{-1}$ can be extended to a homeomorphism of $\mathbb{R}^2$.

The idea of the construction is to find uncountably many composants in some indecomposable planar continuum which can be embedded accessibly in more than a point. The conclusion then follows easily with the use of the following theorem.

Theorem 9.1 (Mazurkiewicz [18]). *Let $X \subset \mathbb{R}^2$ be an indecomposable planar continuum. There are at most countably many composants of X which are accessible in at least two points.*

Let $X = \varprojlim\{I, f_i\}$, where $f_i : I \to I$ are continuous piecewise linear surjections.

Definition 9.2. *Let $f : I \to I$ be a continuous surjection. An interval $I' \subset I$ is called a* surjective interval *if $f(I') = I$ and $f(J) \neq I$ for every $J \subsetneq I'$. Let $A_1, \dots, A_n$, $n \geq 1$, be the surjective intervals of f ordered from left to right. For every $i \in \{1, \dots, n\}$ define the* right accessible set *by $R(A_i) = \{x \in A_i : f(y) \neq f(x) \text{ for all } x < y \in A_i\}$ (see Figure 17).*

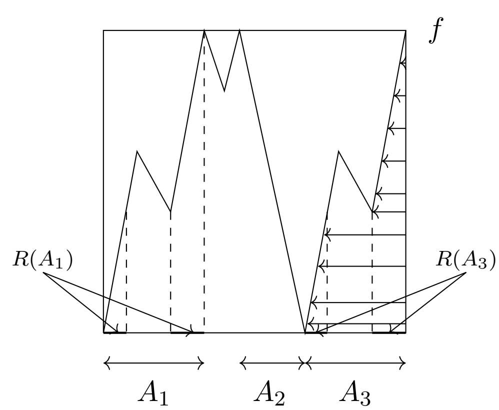

Figure 17. Map f has three surjective intervals. The right accessible sets in the surjective intervals A_1 and A_3 of f are denoted in the picture by $R(A_1)$ and $R(A_3)$ respectively. Note that $R(A_2) = A_2$.

We will first assume that the map f_i contains at least three surjective intervals for every $i \in \mathbb{N}$. We will later see that this assumption can be made without loss of generality.

Remark 9.3. *Assume that $f : I \to I$ has $n \geq 3$ surjective intervals. Then $A_1 \cap A_n = \emptyset$ and $f([l, r]) = I$ for every $l \in A_1$ and $r \in A_n$. Also $f([l, r]) = I$ for every $l \in A_i$ and $r \in A_j$ where $j - i \geq 2$.*

Lemma 9.4. *Let $J \subset I$ be a closed interval and $f : I \to I$ a map with surjective intervals $A_1, \dots A_n$, $n \geq 1$. For every $i \in \{1, \dots, n\}$ there exists an interval $J^i \subset A_i$ such that $f(J^i) = J$, $f(\partial J^i) = \partial J$ and $J^i \cap R(A_i) \neq \emptyset$.*

Proof. Consider the interval $J = [a, b]$ and fix $i \in \{1, \dots, n\}$. Let $a_i, b_i \in R(A_i)$ be such that $f(a_i) = a$ and $f(b_i) = b$. Assume first that $b_i < a_i$ (see Figure 18). Find the smallest $\tilde{a}_i > b_i$ such that $f(\tilde{a}_i) = a$. Then $J^i := [b_i, \tilde{a}_i]$ has the desired properties. If $a_i < b_i$, then take $J^i = [a_i, \tilde{b}_i]$, where $\tilde{b}_i > a_i$ is the smallest such that $f(\tilde{b}_i) = b$. $\square$

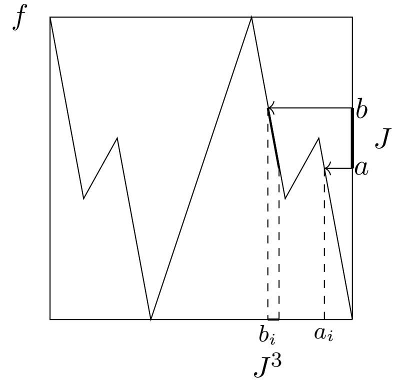

FIGURE 18. Construction of interval J^i from the proof of Lemma 9.4.

The following definition is a slight generalization of the notion of the "top" of a permutation $p(G_f)$ of the graph Γ_f.

Definition 9.5. *Let $f : I \to I$ be a piecewise linear surjection and for a chain C of I, let p be a admissible C-permutation of G_f. For $x \in I$ denote the point in $p(G_f)$ corresponding to $f(x)$ by $p(f(x))$. We say that x is* topmost in $p(G_f)$ *if there exists a vertical ray $\{f(x)\} \times [h, \infty)$, where $h \in \mathbb{R}$, which intersects $p(G_f)$ only in $p(f(x))$.*

Remark 9.6. *If $A_1, \dots, A_n$ are surjective intervals of $f : I \to I$, then every point in $R(A_n)$ is topmost. Also, for every $i = 1, \dots, n$ there exists a permutation of G_f such that every point in $R(A_i)$ is topmost.*

Lemma 9.7. *Let $f : I \to I$ be a map with surjective intervals $A_1, \dots A_n$, $n \geq 1$. For $[a, b] = J \subset I$ and $i \in \{1, \dots, n\}$ denote by J^i an interval from Lemma 9.4. There exists an admissible permutation p_i of G_f such that both endpoints of J^i are topmost in $p_i(G_f)$.*

Proof. Let $A_i = [l_i, r_i]$. Assume first that $f(l_i) = 0$ and $f(r_i) = 1$, thus $a_i < b_i$ (recall the notation $a_i, \tilde{a}_i$ and $b_i, \tilde{b}_i$ from the proof of Lemma 9.4). Find the smallest critical point m of f such that $m \geq \tilde{b}_i$ and note that $f(x) > f(a)$ for all $x \in A_i$, $x > m$. So we can reflect $f|_{[m,r_i]}$ over $f|_{[a_i,m]}$ and $f|_{[r_i,1]}$ over $f|_{[0,l_i]}$. This makes a_i and $\tilde{b}_i$ topmost, see Figure 19. In the case when $f(l_i)=1, f(r_i)=0$, thus $a_i > b_i$, we have that $f(x) < f(b)$ for all $x \in A_i$, $x > m$ so we can again reflect $f|_{[m,r_i]}$ over $f|_{[a_i,m]}$ making $\tilde{a}_i$ and b_i topmost. □

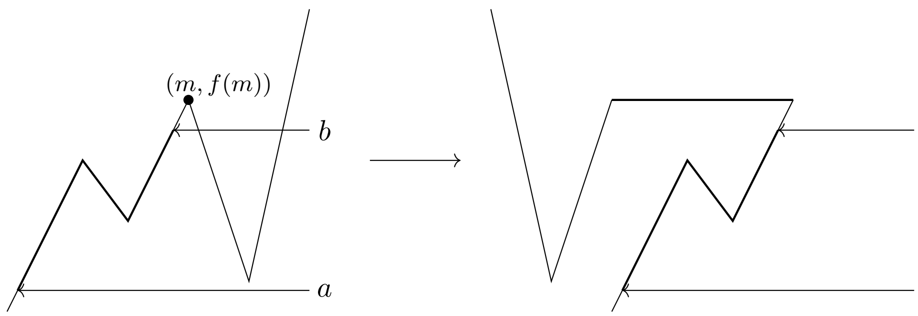

FIGURE 19. Making endpoints of J^i topmost.

Lemma 9.8. *Let $X = \varprojlim\{I, f_i\}$, where each $f_i : I \to I$ is a continuous piecewise linear surjection and assume that X is indecomposable. If f_i contains at least three surjective intervals for every $i \in \mathbb{N}$, then there exist uncountably many planar embeddings of X that are not strongly equivalent.*

Proof. For every $i \in \mathbb{N}$ let $k_i \geq 3$ be the number of surjective branches of f_i and fix $L_i, R_i \in \{1,\dots,k_i\}$ such that $|L_i - R_i| \geq 2$. Let $J \subset I$ and $(n_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}}\{L_i, R_i\}$. Then

$$J^{(n_i)} := J \xleftarrow{f_1} J^{n_1} \xleftarrow{f_2} J^{n_1 n_2} \xleftarrow{f_3} J^{n_1 n_2 n_3} \xleftarrow{f_4} \dots$$

is a well-defined subcontinuum of X. Here we used the notation $J^{nm} = (J^n)^m$. Moreover, Lemma 9.7 and Theorem 6.1 imply that X can be embedded in the plane such that both points in $\partial J \leftarrow \partial J^{n_1} \leftarrow \partial J^{n_1 n_2} \leftarrow \partial J^{n_1 n_2 n_3} \leftarrow \dots$ are accessible.

Remark 9.3 implies that for every $f : I \to I$ with surjective intervals $A_1, \dots, A_n$, every $|i-j| \geq 2$ and every $J \subset I$ it holds that $f([J^i, J^j]) = I$, where $[J^i, J^j]$ denotes the convex hull of J^i and J^j. So if $(n_i), (m_i) \in \prod_{i\in\mathbb{N}}\{L_i, R_i\}$ differ at infinitely many places, then there is no proper subcontinuum of X which contains both $J^{(n_i)}$ and $J^{(m_i)}$, *i.e.*, they are contained in different composants of X. Now Theorem 9.1 implies that there are uncountably many planar embeddings of X that are not strongly equivalent. □

Next we prove that the assumption of at least three surjective intervals can be made without loss of generality for every nondegenerate indecomposable chainable continuum. For $X = \varprojlim\{I, f_i\}$, where each $f_i : I \to I$ is a continuous piecewise linear surjection, we show that there is $X' = \varprojlim\{I, g_i\}$ homeomorphic to X such that g_i has at least three surjective intervals for every $i \in \mathbb{N}$. We will build on the following remark.

Remark 9.9. *Assume that $f, g : I \to I$ each have at least two surjective intervals. Note that then $f \circ g$ has at least three surjective intervals. So if f_i has two surjective* *intervals for every* $i \in \mathbb{N}$*, then* X *can be embedded in the plane in uncountably many nonequivalent ways.*

Definition 9.10. *Let* $\varepsilon > 0$ *and let* $f : I \to I$ *be a continuous surjection. We say that* f *is* P_ε *if for every two segments* $A, B \subset I$ *such that* $A \cup B = I$ *it holds that* $d_H(f(A), I) < \varepsilon$ *or* $d_H(f(B), I) < \varepsilon$*, where* d_H *denotes the Hausdorff distance.*

Remark 9.11. *Let* $f : I \to I$ *and* $\varepsilon > 0$*. Note that* f *is* P_ε *if and only if there exist* $0 \le x_1 < x_2 < x_3 \le 1$ *such that one of the following holds*

(a) $|f(x_1) - 0|, |f(x_3) - 0| < \varepsilon$, $|f(x_2) - 1| < \varepsilon$, *or*
(b) $|f(x_1) - 1|, |f(x_3) - 1| < \varepsilon$, $|f(x_2) - 0| < \varepsilon$.

For $n < m$ denote by $f_n^m = f_n \circ f_{n+1} \circ \ldots \circ f_{m-1}$.

Theorem 9.12 (Kuykendall [13])**.** *The inverse limit* $X = \varprojlim\{I, f_i\}$ *is indecomposable if and only if for every* $\varepsilon > 0$ *and every* $n \in \mathbb{N}$ *there exists* $m > n$ *such that* f_n^m *is* P_ε*.*

Furthermore, we will need the following strong theorem.

Theorem 9.13 (Mioduszewski, [22])**.** *Two continua* $\varprojlim\{I, f_i\}$ *and* $\varprojlim\{I, g_i\}$ *are homeomorphic if and only if for every sequence of positive integers* $\varepsilon_i \to 0$ *there exists an infinite diagram as in Figure 20, where* (n_i) *and* (m_i) *are sequences of strictly increasing*

$$\begin{tikzcd}
I & & I \arrow[ll, "f_{n_1}^{n_2}"'] \arrow[dl] & & I \arrow[ll, "f_{n_2}^{n_3}"'] \arrow[dl] & & I \arrow[ll, "f_{n_3}^{n_4}"'] \arrow[dl] & & \cdots \arrow[ll, "f_{n_4}^{n_5}"'] \\
& I \arrow[ul] & & I \arrow[ll, "g_{m_1}^{m_2}"] \arrow[ul] & & I \arrow[ll, "g_{m_2}^{m_3}"] \arrow[ul] & & I \arrow[ll, "g_{m_3}^{m_4}"] \arrow[ul] & & \cdots \arrow[ll, "g_{m_4}^{m_5}"]
\end{tikzcd}$$

FIGURE 20. Infinite (ε_i)-commutative diagram from Mioduszewski's theorem.

integers, $f_{n_i}^{n_{i+1}} = f_{n_i+1} \circ \ldots \circ f_{n_{i+1}}$, $g_{m_i}^{m_{i+1}} = g_{m_i+1} \circ \ldots \circ g_{m_{i+1}}$ *for every* $i \in \mathbb{N}$ *and every subdiagram as in Figure 21 is* ε_i*-commutative.*

$$\begin{tikzcd}
I & & I \arrow[ll, "f_{n_i}^{n_{i+1}}"'] \arrow[dl] & \cdots \arrow[l, "f_{n_{i+1}}^{n_{i+2}}"'] & & I \arrow[ll, "f_{n_k}^{n_{k+1}}"'] \arrow[dl] & \\
& I \arrow[ul] & \cdots \arrow[l, "g_{m_i}^{m_{i+1}}"] & & I \arrow[ll, "g_{m_{k-1}}^{m_k}"] & & I \arrow[ll, "g_{m_k}^{m_{k+1}}"] \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
& I \arrow[dl] & & \cdots \arrow[ll, "f_{n_{i+1}}^{n_{i+2}}"'] & & I \arrow[ll, "f_{n_k}^{n_{k+1}}"'] & & I \arrow[ll, "f_{n_{k+1}}^{n_{k+2}}"'] \arrow[dl] \\
I & & I \arrow[ll, "g_{m_i}^{m_{i+1}}"] \arrow[ul] & & \cdots \arrow[ll, "g_{m_{i+1}}^{m_{i+2}}"] & & I \arrow[ll, "g_{m_k}^{m_{k+1}}"] \arrow[ul] &
\end{tikzcd}$$

FIGURE 21. Subdiagrams which are ε_i-commutative for every $i \in \mathbb{N}$.

Theorem 9.14. *Every nondegenerate indecomposable chainable continuum* X *can be embedded in the plane in uncountably many ways that are not strongly equivalent.*

Proof. Let $X = \varprojlim\{I, f_i\}$, where each $f_i : I \to I$ is a continuous piecewise linear surjection. If all but finitely many f_i have at least three surjective intervals, we are done by Lemma 9.8. If for all but finitely many i the map f_i has two surjective intervals, we are done by Remark 9.9.

Now fix a sequence (ε_i) such that $\varepsilon_i > 0$ for every $i \in \mathbb{N}$ and $\varepsilon_i \to 0$ as $i \to \infty$. Fix $n_1 = 1$ and find $n_2 > n_1$ such that $f_{n_1}^{n_2}$ is P_{ε_1}. Such n_2 exists by Theorem 9.12. For every $i \in \mathbb{N}$ find $n_{i+1} > n_i$ such that $f_{n_i}^{n_{i+1}}$ is P_{ε_i}. The continuum X is homeomorphic to $\varprojlim\{I, f_{n_i}^{n_{i+1}}\}$. Every $f_{n_i}^{n_{i+1}}$ is piecewise linear and there exist $x_1^i < x_2^i < x_3^i$ as in Remark 9.11. Take them to be critical points and assume without loss of generality that they satisfy condition (a) of Remark 9.11. Define a piecewise linear surjection $g_i : I \to I$ with the same set of critical points as $f_{n_i}^{n_{i+1}}$ such that $g_i(c) = f_{n_i}^{n_{i+1}}(c)$ for all critical points $c \notin \{x_1, x_2, x_3\}$ and $g_i(x_1) = g_i(x_3) = 0$, $g_i(x_2) = 1$. Then g_i is ε_i-close to $f_{n_i}^{n_{i+1}}$. By Theorem 9.13, $\varprojlim\{I, f_{n_i}^{n_{i+1}}\}$ is homeomorphic to $\varprojlim\{I, g_i\}$. Since every g_i has at least two surjective intervals, this finishes the proof by Remark 9.9. $\square$

Remark 9.15. *Specifically, Theorem 9.14 proves that the pseudo-arc has uncountably many embeddings that are not strongly equivalent. Lewis* [14] *has already proven this with respect to the standard version of equivalence, by carefully constructing embeddings with different prime end structures.*

In the next theorem we expand the techniques from this section to construct uncountably many strongly nonequivalent embeddings of every chainable continuum that contains a nondegenerate indecomposable subcontinuum. First we give a generalisation of Lemma 9.7.

Lemma 9.16. *Let $f : I \to I$ be a surjective map and let $K \subset I$ be a closed interval. Let $A_1, \dots, A_n$ be the surjective intervals of $f|_K : K \to f(K)$, and let J^i, $i \in \{1, \dots, n\}$, be intervals from Lemma 9.4 applied to the map $f|_K$.*
Assume $n \geq 4$. Then there exist $\alpha, \beta \in \{1, \dots, n\}$ such that $|\alpha - \beta| \geq 2$ and such that there exist admissible permutations p_α, p_β of G_f such that both endpoints of J^α are topmost in $p_\alpha(G_{f|_K})$ and such that both endpoints of J^β are topmost in $p_\beta(G_{f|_K})$.

Proof. Let $K = [k_l, k_r]$ and $f(K) = [K_l, K_r]$. Let $x > k_r$ be the smallest local extremum of f such that $f(x) > K_r$ or $f(x) < K_l$. A surjective interval $A_i = [l_i, r_i]$ will be called increasing (decreasing) if $f(l_i) = K_l$ ($f(r_i) = K_l$).

Case 1. Assume $f(x) > K_r$ (see Figure 22). If $A_i = [l_i, r_i]$ is increasing, since $f(x) > K_r$, there exists an admissible permutation which reflects $f|_{[m,x]}$ over $f|_{[a_i,m]}$ and leaves $f|_{[x,1]}$ fixed. Here m is chosen as in the proof of Lemma 9.7. Since there are at least four surjective intervals, at least two are increasing. This finishes the proof.

Case 2. If $f(x) < K_l$ we proceed as in the first case but for decreasing A_i. $\square$

Theorem 9.17. *Let X be a chainable continuum that contains a nondegenerate indecomposable subcontinuum Y. Then X can be embedded in the plane in uncountably many ways that are not strongly equivalent.*

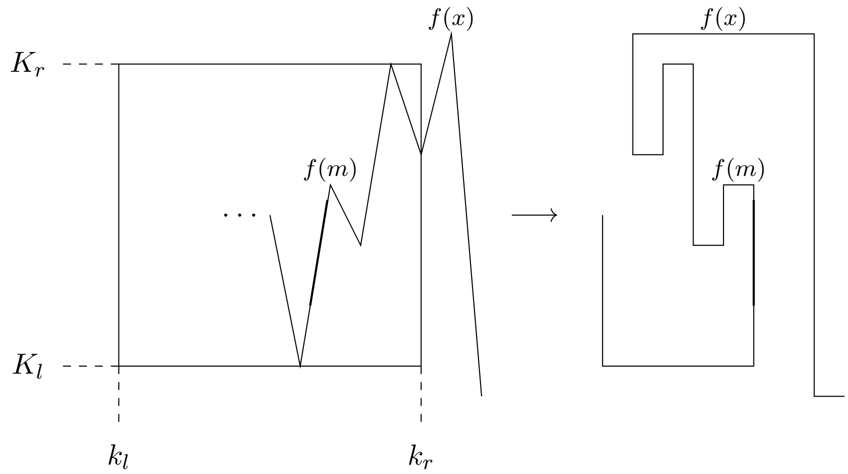

FIGURE 22. Permuting in the proof of Lemma 9.16.

Proof. Let
$$Y := Y_0 \xleftarrow{f_1} Y_1 \xleftarrow{f_2} Y_2 \xleftarrow{f_3} Y_3 \xleftarrow{f_4} \dots .$$
If $\varphi, \psi : X \to \mathbb{R}^2$ are strongly equivalent planar embeddings of X, then $\varphi|_Y, \psi|_Y$ are strongly equivalent planar embeddings of Y. We will construct uncountably many strongly nonequivalent planar embeddings of Y extending to planar embeddings of X, which will complete the proof.

According to Theorem 9.12 and Theorem 9.13 we can assume that $f_i|_{Y_i} : Y_i \to Y_{i-1}$ has at least four surjective intervals for every $i \in \mathbb{N}$. For a closed interval $J \subset Y_{j-1}$, let α_j, β_j be integers from Lemma 9.16 applied to $f_j : Y_j \to Y_{j-1}$, and denote the appropriate subintervals of Y_j by J^{α_j}, J^{β_j}. For every sequence $(n_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}}\{\alpha_i, \beta_i\}$ we obtain a subcontinuum of Y:
$$J^{(n_i)} := J \xleftarrow{f_1} J^{n_1} \xleftarrow{f_2} J^{n_1 n_2} \xleftarrow{f_3} J^{n_1 n_2 n_3} \xleftarrow{f_4} \dots$$
We use the notation of the proof of Lemma 9.8. Lemma 9.16 implies that for every sequence (n_i) there exists an embedding of Y such that both points of $\partial J \leftarrow \partial J^{n_1} \leftarrow \partial J^{n_1 n_2} \leftarrow \partial J^{n_1 n_2 n_3} \leftarrow \dots$ are accessible and which can be extended to an embedding of X. This completes the proof. □

We have proven that every chainable continuum containing a nondegenerate indecomposable subcontinuum has uncountably many embeddings that are not strongly equivalent. Thus we pose the following question.

Question 6. *Which hereditarily decomposable chainable continua have uncountably many planar embeddings that are not equivalent and/or strongly equivalent?*

Remark 9.18. *Mayer has constructed uncountably many nonequivalent planar embeddings (in both senses) in* [17] *of the* $\sin\frac{1}{x}$ *continuum by varying the rate of convergence of the ray. This approach readily generalizes to any Elsa continuum. We do not know whether the approach can be generalized to all chainable continua which contain a dense* *ray. Specifically, it would be interesting to see if* $\varprojlim\{I, f_{Feig}\}$ *(where* f_{Feig} *denotes the logistic interval map at the Feigenbaum parameter) can be embedded in uncountably many nonequivalent ways. However, this approach would not generalize to the remaining hereditarily decomposable chainable continua since there exist hereditarily decomposable chainable continua which do not contain a dense ray, see* e.g. [12].

Remark 9.19. *In Figure 23 we give examples of planar continua which have exactly* $n \in \mathbb{N}$ *or countably many nonequivalent planar embeddings. However, except for the arc, all the examples we know are not chainable.*

FIGURE 23. Left: Planar projection (Schlegel diagram) of the sides of the pyramid with $n \geq 4$ faces has exactly n embeddings that are not strongly equivalent, determined by the choice of the unbounded face. Actually any planar representation of a polyhedron with n faces would do in the previous example. We are indebted to Imre Péter Tóth for these examples. Continua with exactly $n = 2, 3$ nonequivalent planar embeddings in the strong sense are *e.g.* the letters H, X respectively. In the standard sense, there is only one planar embedding of each of these examples.
Right: the harmonic comb has countably many nonequivalent embeddings in both senses; any finite number of non-limit teeth can be flipped over to the left to produce a nonequivalent embedding.

Question 7. *Is there a non-arc chainable continuum for which there exist at most countably many nonequivalent planar embeddings?*

Remark 9.20. *For inverse limit spaces* X *with a single* **unimodal** *bonding map that are not hereditarily decomposable, Theorems 9.14 and 9.17 hold with the standard notion of equivalence as well, for details see* [2]. *This is because every self-homeomorphism of* X *is known to be pseudo-isotopic (two self-homeomorphisms* f, g *of* X *are called* pseudo-isotopic *if* $f(C) = g(C)$ *for every composant* C *of* X*) to a power of the shift homeomorphism (see* [4]*), and so every composant can only be mapped to one in a countable collection of composants. Hence, if uncountably many composants can be made accessible in at least two points, then there are uncountably many nonequivalent embeddings. In general there are no such rigidity results on the group of self-homeomorphisms of chainable continua. For example, there are uncountably many self-homeomorphisms of the pseudo-arc up to pseudo-isotopy, since it is homogeneous and all arc-components are degenerate. Thus we ask the following question.*

Question 8. *For which indecomposable chainable continua is the group of all self-homeomorphisms up to pseudo-isotopy at most countable?*

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