RENORMALIZATION, FREEZING PHASE TRANSITIONS AND FIBONACCI QUASICRYSTALS.

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Abstract. We examine the renormalization operator determined by the Fibonacci substitution within the full shift on two symbols $\Sigma := \{0,1\}^{\mathbb{N}}$. We exhibit a fixed point and determine its stable leaf (under iteration of the operator acting on potentials $V : \Sigma \to \mathbb{R}$), which is completely determined by the germ near the attractor of the substitution. Then we study the thermodynamic formalism for potentials in this stable leaf, and prove they have a freezing phase transition at finite temperature, with ground state supported on the attracting quasi-crystal associated to the Fibonacci substitution.

Resumé. Nous étudions les relations entre renormalisation, substitutions et transitions de phase: nous montrons que la substitution de Fibonacci dans le shift plein à deux symboles $\Sigma := \{0,1\}^{\mathbb{N}}$ génère un opérateur de renormalisation sur les potentiels $V : \Sigma \to \mathbb{R}$. Nous montrons que cet opérateur possède un point fixe, uniquement déterminé par son germe proche de l'attracteur associé à la substitution de Fibonacci. Nous déterminons aussi la feuille stable de ce point fixe. Dans un second temps, nous montrons que tous les potentiels dans cette feuille stable présentent une transition de phase congelante. En particulier, cela donne un nouvel exemple d'obtention d'un état fondamental porté par un quasi-cristal avant le zéro absolu.

1. Introduction

1.1. **Background.** The present paper studies phase transitions from an ergodic theory and dynamical systems point of view. It investigates renormalization, substitutions and phase transition initiated in [2] and continued in [5].

Phase transitions are an important topic in statistical mechanics and also in probability theory (see *e.g.* [10, 11, 22, 26]). The viewpoint presented here is different for several reasons. One of them is that, here, the geometry of the lattice is not relevant[1], whereas in statistical mechanics, the geometry of the lattice is the most important part.

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[1] and we only consider a one-dimensional lattice.

During the 1970's, motivated by problems in statistical mechanics, Bowen, Ruelle and Sinai (see [3, 23, 25]) introduced thermodynamic formalism into ergodic theory. Given a dynamical system (X,T) and a potential $\varphi : X \to \mathbb{R}$, the pressure function is given by

$$\mathcal{P}(\beta) := \sup \left\{ h_\mu(T) + \beta \int \varphi \, d\mu \right\},$$

where the supremum is taken over the invariant probability measures μ, $h_\mu(T)$ is the Kolmogorov entropy and β is a real parameter. Any measure realizing the supremum is then called *an equilibrium state* for $\beta\varphi$.

For a uniformly hyperbolic dynamical system (X,T) and a Hölder continuous potential φ, the pressure function $\beta \mapsto \mathcal{P}(\beta)$ is analytic (see *e.g.* [3, 23, 14]) and there is a unique equilibrium state $\mu_{\beta\varphi}$ (for every β). This equilibrium also satisfies a *Gibbs property*; in the dynamical systems language this condition expresses how the measure of n-cylinders scale: There is $K > 0$ such that

$$K^{-1} \leq \frac{\mu_{\beta\varphi}(Z_n)}{\exp(\beta \sum_{i=0}^{n-1} \varphi \circ T^i(x) - n\mathcal{P}(\beta))} \leq K \tag{1}$$

for all $n \geq 1$, all n-cylinders Z_n and $x \in Z_n$.

Since the late 1970s, people in dynamical systems focused on extending the notions and results of thermodynamics to non-uniformly hyperbolic dynamical systems. This started with the work of Hofbauer [12, 21] proving non-analyticity of pressure for a non-Hölder potential φ on the shift-space $(\{0,1\}^{\mathbb{N}}, \sigma)$. This example is closely related to the Manneville-Pomeau map, and an associated renormalization procedure, presented in [2], was the starting point of the project this paper is part of. Ledrappier [16] showed that any finite number of equilibrium can co-exist in similar examples (cf. also [4, 9] for the co-existence of multiple equilibirum states in other settings). Weakening the Gibbs property may be necessary as well. For instance, Yuri [29], in the setting of maps with neutral fixed points, used a version of *weak Gibbs* in which the K in (1) is replaced by K_n with $\lim_n \frac{1}{n} \log K_n = 0$. See [6, Section 3.1] for similar results to smooth interval maps with critical points.

More recently, the original motivation came back into focus, and the question of phase transitions is now a very active theme in ergodic theory. Nevertheless, due to equivalences or interdependences in the "classical" settings between unique existence of Gibbs measures, unique existence of equilibrium states and regularity of the pressure function, and also due to historical or inspiration models (*e.g.* the Ising model in probability or Erhenfest vs. Gibbs classification in statistical mechanics), the notion of phase transition may vary in the literature. In this paper, we adopt a largely accepted definition now in dynamical systems: a *phase transition* is characterized by a lack of analyticity of the pressure function

Although analyticity is usually considered as a very rigid property and thus quite rare, it turns out that proving non-analyticity for the pressure function is not so easy. Currently, this has become an important challenge in smooth ergodic theory to produce and study phase transitions, see *e.g.* [18, 8, 5] and also [13, Sec. 6] for the possible shapes of the pressure function. We also refer to [24] for results on the regularity of the pressure in the non-compact setting and [29, 20] for uniqueness of the equilibrium state, again in the non-compact case.

To observe phase transitions, one has to weaken hyperbolicity of the system or of regularity of the potential; it is the latter one that we continue to investigate here. Our dynamical system is the full shift, which is uniformly hyperbolic. The first main question we want to investigate is thus which potentials φ will produce phase transitions. More precisely, we are looking for a machinery to produce potentials with phase transitions.

The main purpose of [2] was to investigate possible relation between *renormalization* and phase transitions. In the shift space $(\{0,1\}^{\mathbb{N}}, \sigma)$, a renormalization is a function H for which there is an integer $k \geq 2$ such that

$$\sigma^k \circ H = H \circ \sigma. \tag{2}$$

The link with potentials was made in [2] by introducing a renormalization operator $\mathcal{R}$ acting on potentials and related to a solution H for (2). It is easy to check that constant length k substitutions are solutions to (2). In [5], we studied the Thue-Morse substitution, which has constant length 2. Here we investigate the Fibonacci substitution, which is not of constant length. Several reasons led us to study the Fibonacci case:

• Together with the Thue-Morse substitution, the Fibonacci substitution is the most "famous" substitution and it has been well-studied. In particular, the dynamical properties of their respective attracting sets are well-known and this will be used extensively in this paper. As a result, we were able to describe the relevant fixed point of renormalization exactly. Information of the left and right-special words in these attractors is a key ingredient to prove existence of a phase transition; it is a crucial issue in the relations between substitutions and phase transitions.

• The type of phase transition we establish is a *freezing phase transition.* This means that beyond the phase transition (*i.e.,* for large β), the pressure function is affine and equal to its asymptote, and the equilibrium state (*i.e.,* ground state) is the unique shift-invariant measure supported on an aperiodic subshift space, sometimes called *quasi-crystal.* One open question in statistical mechanics (see [27]) is whether freezing phase transitions can happen and whether *quasi-crystal ground state* can be reach at *positive temperature*. An affirmative answer was given for the Thue-Morse quasi-crystal in [5]; here we show that this also holds for the Fibonacci quasi-crystal.

• We think that Fibonacci shift opens the door to study more general cases. One natural question is whether any quasi-crystal can be reached as a ground state at positive temperature. In this context we emphasize that the Fibonacci substitution space is also Sturmian shift, that is, it encodes the irrational rotation (with angle the golden mean $\gamma := \frac{1+\sqrt{5}}{2}$). We expect that the machinery developed here can be extended to the Sturmian shift associated to general irrational rotation numbers (although those with bounded entries in the continued fraction expansion will be the easiest), possibly to rotations on higher-dimensional tori, and also to more general substitutions.

1.2. **Results.** Let $\Sigma = \{0,1\}^{\mathbb{N}}$ be the full shift space; points in Σ are sequences $x := (x_n)_{n\geq 0}$ or equivalently infinite *words* $x_0x_1\dots$. Throughout, let $\overline{x}_j = 1 - x_j$ denote the opposite symbol. The dynamics is the left-shift

$$\sigma : x = x_0x_1x_2\dots \mapsto x_1x_2\dots.$$

Given a word $w = w_0\dots w_{n-1}$ of *length* $|w| = n$, the corresponding *cylinder* (or *n-cylinder*) is the set of infinite words starting as $w_0\dots w_{n-1}$. We use the notation $C_n(x) = [x_0\dots x_{n-1}]$ for the n-cylinder containing $x = x_0x_1\dots$ If $w = w_0\dots w_{n-1}$ is a word with length n and $w' = w'_0\dots$ a word of any length, the *concatenation* ww' is the word $w_0\dots w_{n-1}w'_0\dots$

The Fibonacci substitution on Σ is defined by:

$$H : \begin{cases} 0 \to 01 \\ 1 \to 0 \end{cases}$$

and extended to words by the concatenation rule $H(ww') = H(w)H(w')$. It is convenient for us to count the Fibonacci numbers starting with index -2:

$$F_{-2} = 1,\ F_{-1} = 0,\ F_0 = 1,\ F_1 = 1,\ F_2 = 2,\ F_{n+2} = F_{n+1} + F_n. \tag{3}$$

We have

$$F_n^a := |H^n(a)| = \begin{cases} F_{n+1} & \text{if } a = 0, \\ F_n & \text{if } a = 1. \end{cases} \tag{4}$$

The Fibonacci substitution has a unique fixed point

$$\rho = 0\ 1\ 0\ 01\ 010\ 01001\ 01001010\ 0100101001001\dots$$

We define the orbit closure $\mathbb{K} = \overline{\cup_n \sigma^n(\rho)}$; it forms a subshift of (Σ, σ) associated to ρ, supporting a unique shift-invariant probability measure $\mu_{\mathbb{K}}$. More properties on $\mathbb{K}$ are given in Section 2.

We define the renormalization operator acting on potentials $V : \Sigma \to \mathbb{R}$ by

$$(\mathcal{R}V)(x) = \begin{cases} V\circ\sigma\circ H(x) + V\circ H(x) & \text{if } x \in [0], \\ V\circ H(x) & \text{if } x \in [1]. \end{cases}$$

We are interested in finding fixed points for $\mathcal{R}$ and, where possible, studying their stable leaves, *i.e.,* potentials converging to the fixed point under iterations of $\mathcal{R}$.

Contrary to the Thue-Morse substitution, the Fibonacci substitution is not of constant length. This is the source of several complications, in particular for the correct expression for $\mathcal{R}^n$.

For $\alpha > 0$, let $\mathcal{X}_\alpha$ be the set of functions $V : \Sigma \to \mathbb{R}$ such that $V(x) \sim n^{-\alpha}$ if $d(x, \mathbb{K}) = 2^{-n}$. More precisely, $\mathcal{X}_\alpha$ is the set of functions V such that:

(1) V is continuous and non-negative.
(2) There exist two continuous functions $g, h : \Sigma \to \mathbb{R}$, satisfying $h_{|\mathbb{K}} \equiv 0$ and $g > 0$, such that
$$V(x) = \frac{g(x)}{n^\alpha} + \frac{h(x)}{n^\alpha} \quad \text{when} \quad d(x, \mathbb{K}) = 2^{-n}.$$

We call g the *α-density*, or just the *density* of $V \in \mathcal{X}_\alpha$. Continuity and the assumption $h_{|\mathbb{K}} \equiv 0$ imply that $h(x)/n^\alpha = o(n^{-\alpha})$.

Our first theorem achieves the existence of a fixed point for $\mathcal{R}$ and shows that the germ of V close to $\mathbb{K}$, *i.e.,* its α-density, allows us to determine the stable leaf of that fixed point.

Given a finite word w, let $\kappa_a(w)$ denote the number of symbols $a \in \{0, 1\}$ in w. If $x \in \Sigma \setminus \mathbb{K}$, we denote by $\widetilde{\kappa}_a(x)$ the number of symbols a in the finite word $x_0 \dots x_{n-1}$ where $d(x, \mathbb{K}) = 2^{-n}$.

Theorem 1. *If $V \in \mathcal{X}_\alpha$, with α-density function g, then*
$$\lim_{k\to\infty} \mathcal{R}^k V(x) = \begin{cases} \infty & \text{for all } x \in \Sigma \setminus \mathbb{K} \text{ if } \alpha < 1; \\ 0 & \text{for all } x \in \Sigma \text{ if } \alpha > 1; \\ \int g \, d\mu_{\mathbb{K}} \cdot \widetilde{V}(x) & \text{for all } x \in \Sigma \text{ if } \alpha = 1, \end{cases}$$
where $\widetilde{V} \in \mathcal{X}_1$ is a fixed point for $\mathcal{R}$, given by
$$\widetilde{V}(x) = \begin{cases} \log\left(\frac{\widetilde{\kappa}_0(x) + \frac{1}{\gamma}\widetilde{\kappa}_1(x) + \gamma}{\widetilde{\kappa}_0(x) + \frac{1}{\gamma}\widetilde{\kappa}_1(x) + \gamma - 1}\right) & \text{if } x \in [0]; \\ \log\left(\frac{\gamma\widetilde{\kappa}_0(x) + \widetilde{\kappa}_1(x) + \gamma^2}{\gamma\widetilde{\kappa}_0(x) + \widetilde{\kappa}_1(x) + \gamma^2 - 1}\right) & \text{if } x \in [1]. \end{cases} \tag{5}$$

This precise expression of $\widetilde{V}$ corresponds to a α-density $\tilde{g}(x) = \gamma^2/(2\gamma - 1)$ if $x \in [0] \cap \mathbb{K}$ and $\tilde{g}(x) = \gamma/(2\gamma - 1)$ if $x \in [1] \cap \mathbb{K}$, and $\int \widetilde{V}(x) d\mu_{\mathbb{K}} = 1$.

A *freezing phase transition* is characterized by the fact that the pressure is of the form
$$\mathcal{P}(\beta) = a\beta + b \quad \text{for } \beta \geq \beta_c$$
and that the equilibrium state is fixed for $\beta \geq \beta_c$. The word "freezing" comes from the fact that in statistical mechanics β is the inverse of the temperature (so the temperature goes to 0 as $\beta \to +\infty$) and that a *ground-state* is reached at positive

temperature $1/\beta_c$, see [26, Chap. 2]. In mathematics, one usually talks about maximizing measure (see *e.g.* [7, Prop. 29])

Theorem 2. *Any potential $\varphi := -V$ with $V \in \mathcal{X}_1$ admits a freezing phase transition at finite β: there exists $\beta_c > 0$ such that*

(1) *for $0 \leq \beta < \beta_c$ the map $\mathcal{P}(\beta)$ is analytic, there exists a unique equilibrium state for $\beta\varphi$ and this measure has full support;*
(2) *for $\beta \geq \beta_c$, the pressure $\mathcal{P}(\beta) = 0$ and $\mu_{\mathbb{K}}$ is an equilibrium state for $\beta\varphi$; it is unique for $\beta > \beta_c$.*

The equilibrium states $\mu_{\beta\varphi}$ for $\beta < \beta_c$ satisfy a local Gibbs property. Namely, (1) remains true but only for cylinders of the form $Z_{r^n(x)}(x) \subset J$, where J is a cylinder with empty intersection with $\mathbb{K}$ and r^n is a n^{th}-return time into J. In that case, the constant K also depends on the cylinder J. This holds because these equilibrium states have conditional measures with respect to J satisfying a Gibbs property for the induced map (see Section 4 and [17]).

The Gibbs property is less practical for the equilibrium state $\mu_{\mathbb{K}}$ for $\beta \geq \beta_c$ because the pressure $\mathcal{P}(\beta) = 0$ and the potential φ is identically zero on the support of $\mu_{\mathbb{K}}$. Therefore a weak Gibbs property holds trivially with $K_n = 2n$ for every n-cylinder intersecting $\mathbb{K} = \text{supp}(\mu_{\mathbb{K}})$. The proof of the existence of a phase transition depends on the estimate of a complicated series; its convergence for large β with limit value strictly less than 1 ensures that a phase transition has taken place at some β_c. Regularity of the pressure function (or number of equilibrium states) for $\beta = \beta_c$ depends on the convergence of another complicated series (related to the first one). Unfortunately, it requires a far more precise understanding of these series to obtain a reliable estimate on the value of β_c and whether the second series converges or diverges at $\beta = \beta_c$.

The phenomenon that the same class of potentials are both fixed by the renormalization operator $\mathcal{R}$ and exhibit phase transitions remains intriguing. We are not aware of a proof that the one implies the other. Rather, we believe that they both follow from a deeper structure, basically that the power α in the definition of V_α expresses the dimension of the lattice. This is one more incentive to study the structure of renormalization and phase transitions for substitutive systems in higher dimensional lattices.

1.3. **Outline of the paper.** In Section 2 we recall and prove various properties of the Fibonacci subshift and its special words. We establish the form of H^n and $\mathcal{R}^n V$ for arbitrary n. We define a notion of *accident* and link it to special words in the Fibonacci shift. In Section 3, after clarifying the role of accidents on the computation of $\mathcal{R}^n V$, we prove Theorem 1. Section 4 deals with the thermodynamic formalism. Following the strategy of [17] we specify and estimate the required (quite involved) quantities that are the core of the proof of Theorem 2.

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2. Properties of H, $\mathbb{K}$ and $\mathcal{R}$

2.1. **The set $\mathbb{K}$ as Sturmian subshift.** In addition to being a substitution subshift, $(\mathbb{K}, \sigma)$ is the Sturmian subshift associated to the golden mean rotation, $T_\gamma : x \mapsto x + \gamma \pmod 1$. The golden mean is $\gamma = \frac{1+\sqrt{5}}{2}$ and it satisfies $\gamma^2 = \gamma + 1$.

Fixing an orientation on the circle $\mathbb{S}^1$, let $\widehat{ab}$ denote the arc of points between a and b in the circle in that orientation. Define the itinerary $e(x) = e_0 e_1 \dots$ of a point $x \in \mathbb{S}^1$ as

$$e_i = \begin{cases} 0 & \text{if } T^i_\gamma(x) \in \widehat{0\gamma} \\ 1 & \text{if } T^i_\gamma(x) \in \widehat{\gamma 0}\ . \end{cases}$$

Then it turns out that $e(2\gamma) = \rho$, the fixed point of the substitution.

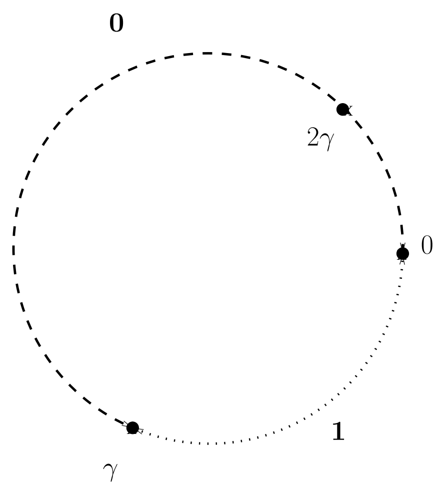

Figure 1. Coding for Fibonacci Sturmian subshift.

There is an almost (*i.e.,* up to a countable set) one-to-one correspondence between points in $\mathbb{K}$ and codes of orbits of $(\mathbb{S}^1, T_\gamma)$, expressed by the commutative diagram

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{T_\gamma} & \mathbb{S}^1 \\ \pi \downarrow & \circlearrowleft & \downarrow \pi \\ \mathbb{K} & \xrightarrow{\sigma} & \mathbb{K} \end{array}$$

and π is a bijection, except at points $T_\gamma^{-n}(\gamma) \in \mathbb{S}^1$, $n \geq 0$. Since Lebesgue measure is the unique T_γ-invariant probability measure, $\mu_{\mathbb{K}} := \mathrm{Leb} \circ \pi^{-1}$ is the unique invariant probability measure of $(\mathbb{K}, \sigma)$.

We will use the same terminology for both $\mathbb{K}$ and $\mathbb{S}^1$. For instance, a cylinder $C_n(x)$ for $x \in \mathbb{S}^1$ is an interval, with the convention that $C_n(x) = \pi^{-1}(C_n(\pi(x)))$, and we may confuse a point $x \in \mathbb{S}^1$ and its image $\pi(x) \in \mathbb{K}$.

Definition 2.1. *Let $\mathcal{A}_{\mathbb{K}}$ denote the set of finite words that appear in ρ. A word $\omega := \omega_0 \dots \omega_{n-1} \in \mathcal{A}_{\mathbb{K}}$ is said to be* left-special *if $0w$ and $1w$ both appear in $\mathcal{A}_{\mathbb{K}}$. It is* right-special *if $w0$ and $w1$ both appear in $\mathcal{A}_{\mathbb{K}}$. A left and right-special word is called* bi-special. *A* special *word is either left-special or right-special.*

Since ρ has $n+1$ subwords of length n (a characterization of Sturmian words), there is exactly one left-special and one right-special word of length n. They are of the form $\rho_0 \dots \rho_{n-1}$ and $\rho_{n-1} \dots \rho_0$ respectively, which can be seen from the forward itinerary $e(x)$ for $x \in \mathbb{S}^1$ close to γ and backward itinerary $e(x)$ for $x \in \mathbb{S}^1$ close to 0. Sometimes the left and right-special word merge into a single bi-special word ω, but only one of the two words $0\omega 0$, and $1\omega 1$ appears in $\mathcal{A}_{\mathbb{K}}$, see [1, Section 1].

Proposition 2.2. *Bi-special words in $\mathcal{A}_{\mathbb{K}}$ are of the form $\rho_0 \dots \rho_{F_m - 3}$ and for each $m \geq 3$, $\rho_0 \dots \rho_{F_m-3}$ is bi-special.*

We prove this proposition at the end of Section 2.2

2.2. **Results for H^n.** We recall that $\kappa_a(w)$ is the number of symbol a in the finite word w.

Lemma 2.3. *For any finite word w, the following recursive relations hold:*

$$\begin{aligned} \kappa_0(H^n(w)) &= F_n \kappa_0(w) + F_{n-1}\kappa_1(w); \\ \kappa_1(H^n(w)) &= F_{n-1}\kappa_0(w) + F_{n-2}\kappa_1(w); \\ |H^n(w)| &= F_{n+1}\kappa_0(w) + F_n \kappa_1(w) = |H^{n-1}(w)| + |H^{n-2}(w)|, \end{aligned}$$

where $|H^0(w)| = |w|$, $|H^1(w)| = |H(w)|$.

Since we have defined $F_{-2} = 1$ and $F_{-1} = 0$, see (3), these formulas hold for $n = 0$ and $n = 1$ as well.

Proof. Since $H^n(0)$ contains F_{n+1} zeroes and F_{n-1} ones, while $H^n(0)$ contains F_{n-1} zeroes and F_{n-2} ones, the first two lines follow from concatenation. The third line is the sum of the first two, and naturally the recursive relation follows from the same recursive relation for Fibonacci numbers. $\square$

Since $(\mathbb{K}, \sigma, \mu_{\mathbb{K}})$ is uniquely ergodic, and isomorphic to $(\mathbb{S}^1, T_\gamma, \mathrm{Leb})$, we immediately get that

$$\lim_{n\to+\infty} \frac{\kappa_a(H^n(w))}{|H^n(w)|} = \begin{cases} |\ \widehat{0\gamma}\ | = \frac{1}{\gamma} & \text{if } a = 0, \\ |\ \widehat{\gamma 0}\ | = 1 - \frac{1}{\gamma} & \text{if } a = 1. \end{cases} \tag{6}$$

Lemma 2.4. *Assume that x and y have a maximal common prefix w. Then $H^n(x)$ and $H^n(y)$ coincide for $T_n(w) + F_{n+2} - 2$ digits, where $T_n(w)$ is defined by*

$$T_0(w) = |w|, \; T_1(w) = |H(w)|, \; T_{n+2}(w) = T_{n+1}(w) + T_n(w). \tag{7}$$

Proof. For $x = w0\dots$ and $y = w1\dots$, we find

$$\begin{array}{c} w \\ w \end{array} \boxed{\begin{array}{c} 0 \\ 1 \end{array}} \xrightarrow{H} \begin{array}{cc} H(w) & 0 \\ H(w) & 0 \end{array} \boxed{\begin{array}{c} 1 \\ 0 \end{array}} \xrightarrow{H} \begin{array}{cccc} H^2(w) & 0 & 1 & 0 \\ H^2(w) & 0 & 1 & 0 \end{array} \boxed{\begin{array}{c} 0 \\ 1 \end{array}}$$

$$\xrightarrow{H} \begin{array}{ccccccc} H^3(w) & 0 & 1 & 0 & 0 & 1 & 0 \\ H^3(w) & 0 & 1 & 0 & 0 & 1 & 0 \end{array} \boxed{\begin{array}{c} 1 \\ 0 \end{array}} \xrightarrow{H} \dots$$

where we used that $H(a)$ starts with 0 for both $a = 0$ and $a = 1$. For $T_n(w) := |H^n(w)|$, the recursive formula (7) follows as in Lemma 2.3.

Iterating the words 01 and 10 by H, we find:

$$\boxed{\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}} \xrightarrow{H} \begin{array}{c} 0 \\ 0 \end{array} \boxed{\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}} \xrightarrow{H} \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 1 & 0 \end{array} \boxed{\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}} \xrightarrow{H} \begin{array}{cccccc} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{array} \boxed{\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}}. \tag{8}$$

Thus $|H^n(10)| = |H^n(01| = F_{n+2}$ and the common prefix of $H^n(10)$ and $H^n(01)$ has length $F_{n+2} - 2$. Therefore, if $x = w01$ and $y = w10$, then $H^n(x)$ and $H^n(y)$ coincide for $T_n(w) + F_{n+2} - 2$ digits. A similar argument works for $x = w00$ and $y = w10$. $\square$

Corollary 2.5. *For $x \in \mathbb{K}$ and $n \in \mathbb{N}$, $H^n(x)$ and ρ coincide for at least $F_{n+3} - 2$ digits if $x \in [0]$ and for at least $F_{n+2} - 2$ digits if $x \in [1]$.*

Proof. If $x \in [0]$, then, by Lemma 2.4, $H^n(x)$ coincides with $H^n(\rho) = \rho$ for at least $T_n(0) + F_{n+2} - 2$ digits. But $T_n(0) = |H^n(0)| = F_{n+1}$, so $T_n(0) + F_{n+2} - 2 = F_{n+3} - 2$. If $x \in [1]$, then $H(x) \in [0]$ and the previous argument gives that $H^n(x)$ coincides with $H^n(\rho) = \rho$ for at least $F_{n+2} - 2$ digits. $\square$

Proof of Proposition 2.2. We iterate the blocks $0 \cdot 01$, $0 \cdot 10$ and $1 \cdot 01$ under H:

$$\boxed{\begin{array}{cccc} 0 & \cdot & 0 & 1 \\ 0 & \cdot & 1 & 0 \\ 1 & \cdot & 0 & 1 \end{array}} \xrightarrow{H} \boxed{\begin{array}{cc} 0 & 1 \\ 0 & 1 \\ & 0 \end{array}} \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \boxed{\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{array}} \xrightarrow{H} \boxed{\begin{array}{cc} \dots 1 & 0 \\ \dots 1 & 0 \\ 0 & 1 \end{array}} \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array} \boxed{\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{array}} \xrightarrow{H} \cdots,$$

so the common central block here is bi-special, and it is the same as the common block v of $H^n(01)$ and $H^n(10)$ of length $F_{n+2} - 2$ in the proof of Lemma 2.4. Thus we have found the bi-special word of length $F_{n+2} - 2$, and every prefix and suffix of v is left and right-special respectively. The fact that these are the only bi-special words can be derived from the Rauzy graph for this Sturmian shift, see *e.g.* [1, Sec. 1]. In their notation, there is a bi-special word of length k if the two special nodes in the Rauzy graph coincide: $D_k = G_k$. The lengths of the two "buckles" of non-special nodes between $D_k = G_k$ are two consecutive Fibonacci numbers minus one, as follows from the continued fraction expansion

$$\gamma = 1 + \cfrac{1}{1 + \cfrac{1}{1+\ddots}}.$$

Therefore, the complexity satisfies

$$k + 1 = p(k) = \#\{\text{nodes of Rauzy graph of order } k\} = F_n - 1 + F_{n-1} - 1 + 1,$$

so indeed only the numbers $k = F_{n+1} - 2$ can be the lengths of bi-special words. $\square$

2.3. **Iterations of the renormalization operator.** The renormalization operator for potentials can be rewritten as (recall the definition of F_n^a, from (4))

$$\mathcal{R}V|_{[a]} = \sum_{j=0}^{F_1^a - 1} V \circ \sigma^j \circ H|_{[a]}. \tag{9}$$

This leads to an expression for $\mathcal{R}^n V$. The main result here is Lemma 2.7, which shows that

$$(\mathcal{R}^n V)(x) = \sum_{j=0}^{F_{n^*} - 1} V \circ \sigma^j \circ H^n(x), \tag{10}$$

where

$$n^* = \begin{cases} n+1 & \text{if } x \in [0], \\ n & \text{if } x \in [1]. \end{cases} \tag{11}$$

The substitution H solves a renormalization equation of the form (2). If $x = 0x_1 \dots$, then $H(x) = 01H(x_1)\dots$ and $\sigma^2 \circ H(x) = H \circ \sigma(x)$. If $x = 1x_1 \dots$ then $\sigma \circ H(x) = H \circ \sigma(x)$. The renormalization equation is thus more complicated than for the constant length case. We need an expression for iterates of H and σ.

Lemma 2.6. *Given $k \geq 0$ and $a = 0, 1$, let $w = w_1 w_2 \dots w_{F_k^a} = H^k(a)$. Then*

$$H \circ \sigma^i \circ H^k|_{[a]} = \sigma^{|H(w_1 \dots w_i)|} \circ H^{k+1}|_{[a]}$$

for every $0 \leq i < F_k^a$.

Proof. For $k = 0$ this is true by default and for $k = 1$, this is precisely what is done in the paragraph before the lemma. Let us continue by induction, assuming that the statement is true for k. Then σ^i removes the first i symbols of $w = H^k(a)$, which otherwise, under H, would be extended to a word of length $|H(w_1 \dots w_i)|$. We need this number of shifts to remove $H(w_1 \dots w_i)$ from $H([w]) = H^{k+1}([a])$. $\square$

Lemma 2.7. *For every $k \geq 0$ and $a = 0, 1$, we have*

$$\mathcal{R}^k V|_{[a]} = S_{F_k^a} V \circ H^k|_{[a]},$$

where $S_n V = \sum_{i=0}^{n-1} V \circ \sigma^i$ denotes the n-th ergodic sum.

Proof. For $k = 0$ this is true by default. For $k = 1$, this follows by the definition of the renormalization operator $\mathcal{R}$. Let us continue by induction, assuming that the statement is true for k. Write $w = H^k(a)$ and $t_i = |H(w_i)| = F_{w_i}$. Then

$$\begin{aligned}
\mathcal{R}^{k+1} V|_{[a]} &= (\mathcal{R}V) \circ S_{F_k^a} V \circ H^k|_{[a]} && \text{(induction assumption)} \\
&= \sum_{i=0}^{F_k^a - 1} \left(\sum_{j=0}^{t_i - 1} V \circ \sigma^j \circ H \right) \sigma^i \circ H^k|_{[a]} && \text{(by formula (9))} \\
&= \sum_{i=0}^{F_k^a - 1} \left(\sum_{j=0}^{t_i - 1} V \circ \sigma^{j + |H(w_1 \ldots w_i)|} \circ H \right) \circ H^k|_{[a]} && \text{(by Lemma 2.6)} \\
&= \sum_{l=0}^{F_{k+1}^a - 1} V \circ \sigma^l \circ H^{k+1}|_{[a]},
\end{aligned}$$

as required. □

2.4. **Special words are sources of accidents.** Overlaps of ρ with itself are strongly related to bi-special words. They are of prime importance to determine the fixed points of $\mathcal{R}$ and their stable leaves, see *e.g.* formula (12) below. Dynamically, they correspond to what we call *accident* in the time-evolution of the distance between the orbit and $\mathbb{K}$. For most x close to $\mathbb{K}$, $d(\sigma(x), \mathbb{K}) = 2d(x, \mathbb{K})$, but the variation of $d(\sigma^j(x), \mathbb{K})$ is not always monotone with respect to j. When it decreases, it generates an accident:

Definition 2.8. *Let $x \in \Sigma$ and $d(x, \mathbb{K}) = 2^{-n}$. If $d(\sigma(x), \mathbb{K}) \leq 2^{-n}$, we say that we have an* accident *at $\sigma(x)$. If there is an accident at $\sigma^j(x)$, then we shall simply say we have an accident at j.*

The next lemma allows us to detect accidents.

Lemma 2.9. *Let $x = x_0 x_1 \ldots$ coincide with some $y \in \mathbb{K}$ for d digits. Assume that the first accident occurs at b. Then $x_b \ldots x_{d-1}$ is a bi-special word in $\mathcal{A}_{\mathbb{K}}$. Moreover, the word $x_0 \ldots x_{d-1}$ is not right-special.*

Proof. By definition of accident, there exists y and y' in $\mathbb{K}$ such that $d(x, \mathbb{K}) = d(x, y)$ and $d(\sigma^b(x), \mathbb{K}) = d(\sigma^b(x), y')$.

Figure 2 shows that the word $x_b \ldots x_{d-1}$ is bi-special because its two extensions y and y' in $\mathbb{K}$ have different suffix and prefix for this word.

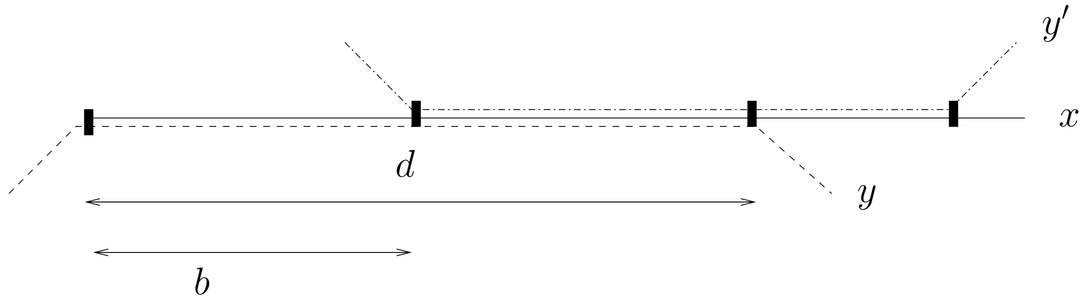

FIGURE 2. Accident and bi-special words

It remains to prove that $x_0 \dots x_{d-1}$ is not right-special. If it was, then $x_0 \dots x_{d-1}x_d = y_0 \dots y_{d-1}\overline{y_d}$ would a $\mathbb{K}$-admissible word, thus $d(x, \mathbb{K}) \leq 2^{-(d+1)} \neq 2^{-d}$. □

3. Proof of Theorem 1

3.1. **Control of the accidents under iterations of $\mathcal{R}$.** We compute $\mathcal{R}^n V$ and show that accidents do not crucially perturb the Birkhoff sum involved. This will follow from Corollaries 3.2 and 2.5.

Note that Lemma 2.4 shows that H is one-to-one. The next proposition explains the relation between the attractor $\mathbb{K}$ and its image by H.

Proposition 3.1. *The subshift $\mathbb{K}$ is contained in $H(\mathbb{K}) \cup \sigma \circ H(\mathbb{K})$. More precisely, if $[0] \cap \mathbb{K} \subset H(\mathbb{K})$ and $[1] \cap \mathbb{K} \subset \sigma \circ H(\mathbb{K})$.*

Proof. First note that Lemma 2.4 shows that H is one-to-one and recall that the word 11 is forbidden in $\mathbb{K}$. Hence, each digit 1 in $x = x_0x_1x_2 \dots \in \mathbb{K}$ is followed and preceded by a 0 (unless the 1 is in first position).

Assuming $x_0 = 0$, we can uniquely split x into blocks of the form 0 and 01. In this splitting, we replace each single 0 by 1 and each pair 01 by 0. This produces a new word, say y, and $H(y) = x$ by construction. Denote this operation by H^{-1}. It can be used on finite words too, provided that the last digit is 1. If $x_0 = 1$, we repeat the above construction with $0x$, and $x = \sigma \circ H(y)$.

It remains to prove that $y \in \mathbb{K}$. For every $x \in \mathbb{K}$, there is a sequence $k_n \to \infty$ such that $\sigma^{k_n}(\rho) \to x$. Assume again that $x_0 = 0$. Then we can find a sequence $l_n \sim k_n/\gamma$ such that $H \circ \sigma^{l_n}(\rho) = \sigma^{k_n}(\rho)$. Therefore $y := \lim_n \sigma^{l_n}(\rho) \in \mathbb{K}$ satisfies $H(y) = x$. Finally, for $x_0 = 1$, we repeat the argument with $0x$. □

Corollary 3.2. *If $d(x, \mathbb{K}) = d(x, y)$ with $y \in \mathbb{K}$, then $d(H^n(x), \mathbb{K}) = d(H^n(x), H^n(y))$ for $n \geq 0$.*

Proof. Write $x = wa$ and $y = w\overline{a}$ where a is an unknown digit and $\overline{a}$ its opposite. Note that $H^n(x)$ starts with 0 for any $n \geq 1$. Assume that there is some $z \in \mathbb{K}$ such that $d(H(x), z) < d(H(x), H(y))$.

Case 1: $x = w0\dots$ and $y = w1\dots$ Necessarily, $y = w10$. Therefore $H(x) = H(w)01\dots$ and $H(y) = H(w)001\dots$. By assumption, z coincides with $H(x)$ longer than $H(y)$, which shows that z starts as $z = H(w)01\dots$ Consequently, $H^{-1}(z) = w0\dots$ and this contradicts that $d(x, \mathbb{K}) = d(x, y)$.

Case 2: $x = w1b\dots$ and $y = w0\dots$. Then $H(x) = H(w)00\dots$ (since $H(b)$ starts with 0 regardless what b is) and $H(y) = H(w)01\dots$ Again z coincides with $H(x)$ longer than $H(y)$ and thus z starts as $H(w)00$. The 0 before last position is necessarily a single zero for the H^{-1}-procedure and thus $H^{-1}(z)$ coincide with x for longer than y. This is a contradiction.

Consequently, for both cases we have shown $d(H(x), \mathbb{K}) = d(H(x), H(y))$. The result follows by induction. $\square$

By (10), $\mathcal{R}^n V$ is given by a Birkhoff sum of F_{n^*} terms where $n^* = n + 1$ or n as in (11). To compute $(\mathcal{R}^n V)(x)$, we need an estimate for $d(\sigma^j(H^n(x)), \mathbb{K})$, for $0 \leq j \leq F_{n^*} - 1$. The key point is that no accident can occur for these j. This follows from the next lemma.

Lemma 3.3. *The sequence $H^n(x)$ has no accident in the first F_{n^*} entries.*

Proof. We give the proof for $x \in [1]$, so $n^* = n$. The proof for $x \in [0]$ is analogous. By Corollary 2.5, $H^n(x)$ coincides for at least $F_{n+2} - 2$ digits with ρ. If an accident happens in the first F_n digits, say at entry $0 \leq j < F_n$, then by Lemma 2.9, a bi-special word starts at j, which by Proposition 2.2 is a suffix of ρ of length $F_m - 2$ for some m. Since we have an accident, $j + F_m - 2 \geq F_{n+2} - 1$, so $m > n + 1$.

Hence $\rho_0 \dots \rho_{F_{n+2}-1}$ can be written as BBB' where B is the suffix of ρ of length j and B' is a suffix of ρ of length $\geq |B|/\gamma$. Clearly B starts with 0. We can split it uniquely into blocks 0 and 01, and B fits an integer number of such blocks, because if the final block would overlap with the second appearance of B, then B would start with 1, which it does not.

Therefore we can perform an inverse substitution H^{-1}, for each block B and also for B' because we can apply H^{-1} on $\rho_0 \dots \rho_{F_{n+2}-1}$. This gives $H^{-1}(BBB') = CCC'$ which has the same characteristics. Repeating this inverse iteration, we find that ρ starts with 0101, or with 00, a contradiction. $\square$

Let $N(x, n)$ be the integer such that $2^{-N(x,n)} = d(H^n(x), \mathbb{K})$. By the previous lemma $d(\sigma^j(H^n(x))\mathbb{K}) = 2^{-(N(x,n)-j)}$ for every $j < F_{n^*}$. For the largest value $j = F_{N^*}$, we have $d(\sigma^j(H^n(x)), \mathbb{K}) = 2^{-(T_n + F_{n+2} - 2 - F_{n^*})}$. Therefore, if g is the α-density function for V, then we obtain

$$(\mathcal{R}^n V)(x) = \sum_{j=0}^{F_{n^*}-1} \frac{g\circ\sigma^j\circ H^n(x)}{(N(x,n)-j)^\alpha} + o\left(\sum_{j=0}^{F_{n^*}-1} \frac{g\circ\sigma^j\circ H^n(x)}{(N(x,n)-j)^\alpha}\right). \tag{12}$$

3.2. **Proof of Theorem 1.**

3.2.1. *$\widetilde{V}$ is a fixed point.* Take $\widetilde{V}$ as in (5). We show that $\mathcal{R}$ fixes $\widetilde{V}$. Assume $x \notin \mathbb{K}$ is such that $\widetilde{\kappa}_0(x) = n$ and $\widetilde{\kappa}_1(x) = m$ (see the definition of $\widetilde{\kappa}_a$ above the statement of Theorem 1). Then, by Lemmas 2.3 and 2.4 and the fact that $H(x)$ starts with 0, we get

$$\begin{array}{ll} \widetilde{\kappa}_0(H(x)) = n+m+1 & \qquad \widetilde{\kappa}_0(\sigma\circ H(x)) = n+m, \\ \widetilde{\kappa}_1(H(x)) = n & \qquad \widetilde{\kappa}_1(\sigma\circ H(x)) = n. \end{array}$$

- If x starts with 0, then $H(x)$ starts with 01 and

$$\begin{aligned}
(\mathcal{R}\widetilde{V})(x) &= \widetilde{V}(H(x)) + \widetilde{V}\circ\sigma(H(x)) \\
&= \log\left(\frac{(n+m+1)+\frac{1}{\gamma}n+\gamma}{(n+m+1)+\frac{1}{\gamma}n+\gamma-1}\right) + \log\left(\frac{\gamma(n+m)+n+\gamma^2}{\gamma(n+m)+n+\gamma^2-1}\right) \\
&= \log\left(\frac{(n+m+1)+\frac{1}{\gamma}n+\gamma}{n+m+\frac{1}{\gamma}n+\gamma}\right) + \log\left(\frac{n+m+\frac{1}{\gamma}n+\gamma}{n+m+\frac{1}{\gamma}n+\gamma-\frac{1}{\gamma}}\right) \\
&= \log\left(\frac{n+m+1+\frac{1}{\gamma}n+\gamma}{n+m+\frac{1}{\gamma}n+\gamma-\frac{1}{\gamma}}\right) \\
&= \log\left(\frac{n(1+\frac{1}{\gamma})+m+\gamma+1}{n(1+\frac{1}{\gamma})+m+\gamma-\frac{1}{\gamma}}\right) \qquad \text{since } \gamma^2=\gamma+1 \\
&= \log\left(\frac{\gamma n+m+\gamma^2}{\gamma n+m+\gamma(\gamma-1)}\right) = \log\left(\frac{n+\frac{1}{\gamma}m+\gamma}{n+\frac{1}{\gamma}m+\gamma-1}\right) = \widetilde{V}(x).
\end{aligned}$$

- If x starts with 1, then $H(x)$ starts with 0 and

$$\begin{aligned}
(\mathcal{R}\widetilde{V})(x) &= \widetilde{V}(H(x)) = \log\left(\frac{(n+m+1)+\frac{1}{\gamma}n+\gamma}{(n+m+1)+\frac{1}{\gamma}n+\gamma-1}\right) \\
&= \log\left(\frac{\gamma(n+m+1)+n+\gamma^2}{\gamma(n+m+1)+n+\gamma^2-\gamma}\right) \\
&= \log\left(\frac{n(\gamma+1)+\gamma m+\gamma+\gamma^2}{n(\gamma+1)+\gamma m+\gamma^2}\right) \\
&= \log\left(\frac{\gamma^2 n+\gamma m+\gamma^3}{\gamma^2 n+\gamma m+\gamma^2}\right) \\
&= \log\left(\frac{\gamma n+m+\gamma^2}{\gamma n+m+\gamma}\right) = \log\left(\frac{\gamma n+m+\gamma^2}{\gamma n+m+\gamma^2-1}\right) = \widetilde{V}(x).
\end{aligned}$$

3.2.2. *Toeplitz summation.* Next, we compute $\mathcal{R}^n V$ for arbitrary $V \in \mathcal{X}_1$. (For $V \in \mathcal{X}_\alpha$ with $\alpha \neq 1$, we can perform the same computation, and obtain an extra factor $F_{n^*}^{1-\alpha}$ which will push the limit to 0 or ∞ according to whether $\alpha > 1$ or $\alpha < 1$.)

We will show that $\sum_{j=0}^{F_{n^*}-1} \frac{g\circ\sigma^j\circ H^n(x)}{N(x,n)-j}$ actually converges, which immediately yields that $o\left(\sum_{j=0}^{F_{n^*}-1} \frac{g\circ\sigma^j\circ H^n(x)}{N(x,n)-j}\right)$ converges to 0. We can thus ignore the little o terms.

Lemmas 2.3 and 2.4 and Corollary 3.2 give

$$N(x,n) := \log_2 d(H^n(x),\mathbb{K}) = T_n + F_{n+2} - 2 \quad \text{for } \ T_n := F_{n+1}\widetilde{\kappa}_0(x) + F_n\widetilde{\kappa}_1(x).$$

We thus have to compute the limit of

$$\sum_{j=0}^{F_{n^*}-1} \frac{g\circ\sigma^j\circ H^n(x)}{F_{n+1}\widetilde{\kappa}_0(x)+F_n\widetilde{\kappa}_1(x)+F_{n+2}-(j+2)}$$

as $n^* \to +\infty$, where $n^* = n+1$ if $x \in [0]$ and $n^* = n$ if $x \in [1]$. Moreover g is a non-negative continuous function, hence uniformly continuous. For $y \in \mathbb{K}$ closest to x, the point $\sigma^{F_{n^*}} \circ H^n(x)$ coincides with $\sigma^{F_{n^*}} \circ H^n(y)$ for at least $F_n - 2$ digits. There exists a sequence $\varepsilon_n \downarrow 0$ such that

$$\left| g\circ\sigma^j(H^n(x)) - g\circ\sigma^j(H^n(y)) \right| \leq \varepsilon_n,$$

for every $j \leq F_{n^*} - 1$.

Finally, Binet's formula $F_{n+1} - \gamma F_n = \sqrt{5}\gamma^{-(n-1)}$ shows that

$$\begin{aligned}
F_{n+1}\widetilde{\kappa}_0(x)+F_n\widetilde{\kappa}_1(x) &+ F_{n+2}-(j+2) \\
&= F_{n+1}(\widetilde{\kappa}_0(x)+\frac{1}{\gamma}\widetilde{\kappa}_1(x)+\gamma-\frac{j}{F_{n+1}})(1+\varepsilon'_n) \\
&= F_n(\gamma\widetilde{\kappa}_0(x)+\widetilde{\kappa}_1(x)+\gamma^2-\frac{j}{F_n})(1+\varepsilon''_n),
\end{aligned}$$

where ε'_n and ε''_n tend to 0 as $n \to +\infty$.

Combining ε, ε' and ε'' in a single $o(1)$, we can rewrite the above equalities to

$$\sum_{j=0}^{F_{n^*}-1} \frac{g \circ \sigma^j \circ H^n(x)}{F_{n+1}\widetilde{\kappa}_0(x) + F_n\widetilde{\kappa}_1(x) + F_{n+2} - (j+2)} = \frac{1+o(1)}{F_{n^*}} \sum_{j=0}^{F_{n^*}} \frac{g \circ \sigma^j \circ H^n(y)}{X_n - \frac{j}{F_{n^*}}}, \tag{13}$$

where $X_n = \widetilde{\kappa}_0(x) + \frac{1}{\gamma}\widetilde{\kappa}_1(x) + \gamma$ if $x \in [0]$ and $X_n = \gamma\widetilde{\kappa}_0(x) + \widetilde{\kappa}_1(x) + \gamma^2$ if $x \in [1]$.

3.2.3. *Convergence of the weighted sum in* (13). The reader can verify that we are here considering a Toeplitz summation method, with a regular matrix (see [15, Definition 7.5] and [28]), up to a renormalization factor, which is the limit of

$$\frac{1}{F_{n^*}} \sum_{j=0}^{F_{n^*}} \frac{1}{X_n - \frac{j}{F_{n^*}}}.$$

This expression is a Riemann sum, and converges to $\widetilde{V}(x)$ as $n^* \to \infty$.

Using notations from [28], coefficients of the Toeplitz matrix are

$$a_{n,k} := \frac{1}{\widetilde{V}(x)} \frac{1}{F_{n^*}X_n - k} \text{ for } k \leq F_{n^*} - 1 \text{ and } a_{n,k} = 0 \text{ otherwise,}$$

and they satisfy the Müller criterion (see [19] and [28, p.1 equality (1.4)])

$$\lim_{n\to+\infty} \sum_{k=1}^{+\infty} |a_{n,k} - a_{n,k+1}| \log k = 0.$$

Therefore

$$\lim_{n\to+\infty} \frac{1}{F_{n^*}} \sum_{j=0}^{F_{n^*}} \frac{g \circ \sigma^j(z)}{X_n - \frac{j}{F_{n^*}}} = \widetilde{V}(x) \int g \, d\mu_{\mathbb{K}} \qquad \mu_{\mathbb{K}}\text{-a.e.} \tag{14}$$

Nevertheless, our expression in (13) is different, because the point $z = H^n(y)$ depends on n. A priori, this may generate fluctuations in the convergence, but we prove here that this is not the case.

The main argument is that $(\mathbb{K}, \sigma)$ is uniquely ergodic. This implies that the convergence in (14) is uniform in z. Indeed, if it is not uniform, we can find $\varepsilon > 0$ and a sequence of z_n such that for every n, $|\frac{1}{F_{n^*}} \sum_{j=0}^{F_{n^*}} \frac{g\circ\sigma^j(z_n)}{X_n - \frac{j}{F_{n^*}}} - \widetilde{V}(x) \int g \, d\mu_{\mathbb{K}}| > \varepsilon$ for every n. Then any accumulation point μ_∞ of the family of measures

$$\mu_n := \frac{1}{F_{n^*}} \sum_{j=0}^{F_{n^*}} \frac{1}{X_n - \frac{j}{F_{n^*}}} \delta_{\sigma^j(z_n)}$$

is σ-invariant (because $F_{n^*} \to +\infty$), supported on $\mathbb{K}$, and $\int g \, d\mu_\infty \neq \int g \, d\mu_{\mathbb{K}}$. This contradicts the unique ergodicity for $(\mathbb{K}, \sigma)$.

Therefore, the convergence in (14) is uniform in z and this shows that

$$\frac{1}{F_{n^*}}\sum_{j=0}^{F_{n^*}}\frac{g\circ\sigma^j(H^n(y))}{X_n-\frac{j}{F_{n^*}}}\to\widetilde{V}(x)\cdot\int g\,d\mu_{\mathbb{K}}.$$

This finishes the proof of Theorem 1.

4. Proof of Theorem 2

4.1. **The case $-\log\frac{n+1}{n}$.** We first consider the potential $\varphi(x)=-\log\frac{n+1}{n}$ when $d(x,\mathbb{K})=2^{-n}$, leaving general potentials in $\mathcal{X}_1$ for later.

4.1.1. *Strategy, local equilibria.* Fix some cylinder J such that the associated word, say ω_J, does not appear in ρ (as *e.g.* 11). We follow the induction method presented in [17]. Let τ be the first return time into J (possibly $\tau(x)=+\infty$), and consider the family of transfer operators

$$\begin{array}{rcl}\mathcal{L}_{Z,\beta}:\psi & \mapsto & \mathcal{L}_{Z,\beta}(\psi)\\ x & \mapsto & \mathcal{L}_{Z,\beta}(\psi)(x):=\displaystyle\sum_{n=1}^{+\infty}\sum_{\substack{y\in J\ \ \tau(y)=n\\ \sigma^n(y)=x}}e^{\beta\cdot(S_n\varphi)(y)-nZ}\psi(y),\end{array}$$

which acts on the set of continuous functions $\psi:J\to\mathbb{R}$. Following [17, Proposition 1], for each β there exists $Z_c(\beta)$ such that $\mathcal{L}_{Z,\beta}$ is well defined for every $Z>Z_c(\beta)$. By [17, Theorem 1], $Z_c(\beta)\geq 0$ because the pressure of the dotted system (which in the terminology of [17] is the system restricted to the trajectories that avoid J) is larger or equal to the pressure of $\mathbb{K}$ which is zero.

We shall prove

Proposition 4.1. *There exists β_0 such that $\mathcal{L}_{0,\beta}(\mathbb{1}_J)(x)<1$ for every $\beta>\beta_0$ and $x\in J$.*

We claim that if Proposition 4.1 holds, then [17, Theorem 4] proves that $\mathcal{P}(\beta)=0$ for every $\beta>\beta_0$, and $\mu_{\mathbb{K}}$ is the unique equilibrium state for $\beta\varphi$.

To summarize [17] (and adapt it to our context), the pressure function[2] satisfies $Z_c(\beta)\leq\mathcal{P}(\beta)$, $\mathcal{P}(\beta)\geq 0$ and $\mathcal{P}(\beta)=0$ if $\log(\mathcal{L}_{0,\beta}(\mathbb{1}_J))<0$ (see Figure 3). As long as $\mathcal{P}(\beta)>0$, there is a unique equilibrium state and it has full support. In particular this shows that the construction does not depend on the choice of J. If Proposition 4.1 holds, then $\mathcal{P}(\beta)=0$, for $\beta>\beta_c$ and no equilibrium state gives positive weight to J (and also to any cylinder which does not intersect $\mathbb{K}$). Therefore, $\mu_{\mathbb{K}}$ is the unique equilibrium state.

[2] We will see that $\mathcal{L}_{0,\beta}(\mathbb{1}_J)$ is a constant function on J.

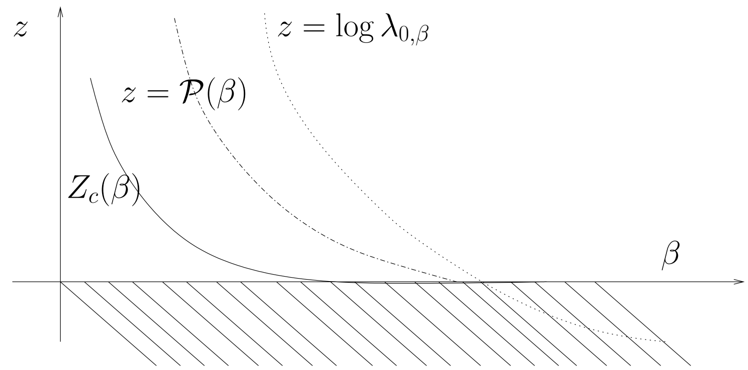

FIGURE 3. The pressure between $Z_c(\beta)$ and $\log \lambda_{0,\beta} := \mathcal{L}_{0,\beta}(\mathbb{1}_J)$

4.1.2. *Proof of Proposition 4.1-Step 1.* We reduce the problem to the computation of a series depending on β. Note that $\varphi(x)$ only depends on the distance from x to $\mathbb{K}$. This shows that if $x, x' \in J$ and $y, y' \in J$ are such that

$$y = \omega x \ , \quad y' = \omega x',$$

with $\omega \in \{0,1\}^n$, $\tau(y) = \tau(y') = n$, then

$$(S_n\varphi)(y) = (S_n\varphi)(y').$$

In other words, $\mathcal{L}_{Z,\beta}(\mathbb{1}_J)$ is a constant function, and then equal to the spectral radius $\lambda_{Z,\beta}$ of $\mathcal{L}_{Z,\beta}$.

Consequently, to compute $\lambda_{Z,\beta}$, it suffices to compute the sum of all $e^{\beta\cdot(S_n\varphi)(\omega)-nZ}$, where ω is a word of length $n + |\omega_J|$, starting and finishing with ω_J. Such a word ω can also be seen as a path of length n starting from J and returning (for the first time) to J at time n.

We split such a path in several sub-paths. We fix an integer N and say that the path is *free* at time k if $\omega_k \dots \omega_{n-1}\omega_J$ is at distance larger than 2^{-N} to $\mathbb{K}$. Otherwise, we say that we have an *excursion*. The path is thus split into intervals of free times and excursions. We assume that N is chosen so large that 0 is a free time. This also shows that for every $k \leq n$, $d(\sigma^k(\omega\omega_J), \mathbb{K})$ is determined by $\omega_k \dots \omega_{n-1}$.

If k is a free time, then $\varphi(\sigma^k(\omega\omega_J)) \leq A_N := -\log\left(1 + \frac{1}{N}\right)$. Denote by k_0 the maximal integer such that k is a free time for every $k \leq k_0$. Then $S_{k_0+1}\varphi \leq (k_0+1)A_N$ and there at most 2^{k_0+1} such prefixes of length $k_0 + 1$.

Now, assume that every j for $k_0 + 1 \leq j \leq k_0 + k_1$ is an excursion time, and assume that k_1 is the maximal integer with this property. To the contribution $(S_{k_0+1}\varphi)(\omega\omega_J)$ we must add the contribution $(S_{k_1}\varphi)(\sigma^{k_0+1}(\omega\omega_J))$ of the excursion. Then we have a new interval of free times, and so on. We can compute $\mathcal{L}_{0,\beta}(\mathbb{1}_J)$ by grouping together paths with the same number of free periods and excursions. If we denote by C_E the total contribution of all paths with exactly one excursion (and only starting at the beginning of the excursion), then we have

$$\lambda_{0,\beta} = \mathcal{L}_{0,\beta}(\mathbb{1}_J) \leq \sum_{k=1}^{+\infty} \left(\sum_{k_0=0}^{+\infty} e^{(k_0+1)(\beta A_N + \log 2)} \right)^{k+1} C_E^k. \tag{15}$$

The sum in k accounts for $k+1$ free intervals with k excursions between them. The sum in k_0 accounts for the possible length k_0+1 for an interval of free times. These events are potentially not independent but the sum in (15) includes all paths, possible or not, and therefore yields an upper bound.

The integer N is fixed, and we can take β so large that $\beta A_N < -\log 2$. This shows that the sum in k_0 in (15) converges and is as close to 0 as we want if β is taken sufficiently large.

To prove Proposition 4.1, it is thus sufficient to prove that C_E can be made as small as we want if β increases.

4.1.3. *Proof of Proposition 4.1-Step 2.* We split excursions according to their number of accidents, see Definition 2.8. Let x be a point at a beginning of an excursion.

Let $B_0 := 0 = b_0$, $B_1 := b_1 > b_0$, $B_2 := b_1 + b_2 > b_1$, $B_3 := b_1 + b_2 + b_3, \dots, B_M := b_1 + b_2 + \cdots + b_M$ be the times of accidents in the excursion. There is $y_0 \in \mathbb{K}$ such that x shadows y_0 at the beginning of the excursion, say for d_0 iterates. Let $y_i \in \mathbb{K}$, $i = 1, \dots, M$, be the points that x starts to shadow at the i-th accident, for d_i iterates.

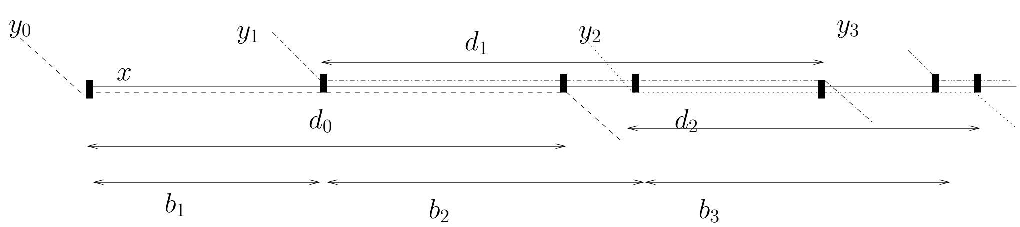

FIGURE 4. Accidents during an excursion.

By Lemma 2.9, $x_{b_{i+1}} \dots x_{d_i}$ is bi-special and by Proposition 2.2, $d_i - b_{i+1} = F_{n_{i+1}} - 2$ for some n_{i+1}.

Remark 1. We emphasize that the first d_i entries of y_i form neither a right-special word (due to Lemma 2.9) nor a left-special word, because otherwise there would be an accident earlier. ■

If there are $M+1$ accidents (counting the first as 0), the ergodic sums for φ are

$$
\begin{aligned}
(S_{b_{i+1}}\varphi)(\sigma^{B_i}(x)) &= \sum_{k=0}^{b_{i+1}-1} \varphi\circ\sigma^{B_i+k}(x) \\
&= \sum_{k=0}^{b_{i+1}-1} -\log\frac{d_i+1-k}{d_i-k} \\
&= -\log\frac{d_i+1}{d_i+1-b_{i+1}} = -\log(1+\frac{b_{i+1}}{d_i+1-b_{i+1}}),
\end{aligned}
$$

for $0\le i\le M-1$, while the ergodic sum of the tail of the excursion is

$$
(S_{d_M}\varphi)(\sigma^{B_M}(x)) = \sum_{k=0}^{d_M-1}\varphi\circ\sigma^{B_M+k}(x) = -\log\frac{d_M+1}{N+1}. \tag{16}
$$

We set $\mathbf{e}_i := e^{\beta\cdot(S_{b_i}\varphi)(\sigma^{B_{i-1}}(x))}$ for $i=1\dots M$ and $\mathbf{e}_{M+1} := e^{\beta\cdot(S_{d_M}\varphi)(\sigma^{B_M}(x))}$. Computing C_E, we can order excursions according to their number of accidents $(M+1)$ and then according to the contribution of each accident. Let E_i stand for the total contribution of all possible $\mathbf{e}_i$'s between accidents $i-1$ and i. Then

$$
C_E = \sum_{M=0}^{+\infty}\prod_{i=1}^{M+1} E_i. \tag{17}
$$

4.1.4. *Proof of Proposition 4.1-Step 3.* Let us now find an upper bound for E_i. By definition, E_i is the sum over the possible d_{i-1} and b_i of $\mathbf{e}_i$.

Recall that $d_{i-1}-b_i = F_{n_i}-2$, so b_i and F_{n_i} determine d_{i-1}. The key idea is that F_{n_i} and $F_{n_{i+1}}$ determine the possible values of b_i. This implies that E_i can be written as an expression over the F_{n_i} and $F_{n_{i+1}}$.

• For $2\le i\le M$ each $\mathbf{e}_i$ depends on F_{n_i} and b_i. Let us show that for $2\le i\le M$, b_i depends on n_i and n_{i-1}. Indeed, the sequence $y_i\in\mathbb{K}$ coincides with ρ for $F_{n_i}-2$ initial symbols, and from entry b_{i+1} has another $d_i-b_{i+1}=F_{n_{i+1}}-2$ symbols in common with the head of ρ, but differs from $x_{B_i+d_i}$ at entry d_i, see Figure 4. Thus we need to find all the values of $d_i > F_{n_i}-2$ such that $\rho_0\dots\rho_{d_i-1}$ ends the bi-special word $\rho_0\dots\rho_{F_{n_{i+1}-3}}$ but is itself not bi-special. The possible starting positions of this appearance of $\rho_0\dots\rho_{F_{n_{i+1}-3}}$ are the required numbers b_{i+1}.

Lemma 4.2. *Let us denote by $b_{i+1}(j)$, $j\ge 1$, the j-th value that b_{i+1} can assume. Then*

$$
b_{i+1}(j) \ge \max(F_{n_i}-F_{n_{i+1}}, F_{n_i-1}) + jF_{n_{i+1}-2}. \tag{18}
$$

This will allow us to find an upper bound for E_i for $1 \le i \le M-1$ later in this section.

Proof. We abbreviate the bi-special words $L_k = \rho_0 \dots \rho_{F_k - 3}$ for $k \ge 4$. For the smallest value $d_i \ge F_{n_i} - 2$ so that $\rho_0 \dots \rho_{d_i-1}$ ends in (but is not identical to) a block $L_{n_{i+1}}$, this block starts at entry:

$$b_{i+1}(0) = \begin{cases} F_{n_i} - F_{n_{i+1}} & \text{if } n_{i+1} < n_i \text{ and } n_i - n_{i+1} \text{ is even,} \\ F_{n_i} - F_{n_{i+1}-1} & \text{if } n_{i+1} < n_i \text{ and } n_i - n_{i+1} \text{ is odd,} \\ F_{n_i - 1} & \text{if } n_{i+1} \ge n_i. \end{cases}$$

However, if $n_{i+1} < n_i$ then $d_i = F_{n_i} - 2$, and if $n_{i+1} \ge n_i$ then $d_i = F_{n_{i+1}+1} - 2$ and thus $\rho_0 \dots \rho_{d_i - 1}$ is right-special, contradicting Lemma 2.9. Therefore we need to wait for the next appearance of $L_{n_{i+1}}$. For the Rauzy graph of the Fibonacci shift, the bi-special word L_k is the single node connecting loops of length F_{k-1} and F_{k-2}, see [1, Section 1]. Therefore the gap between two appearances of L_k is always F_{k-2} or F_{k-1}. This gives $b_{i+1}(j+1) \ge b_{i+1}(j) + F_{n_{i+1}-2}$ for all $j \ge 0$ and (18) follows. $\square$

• For $i = 1$, we introduce the quantity n_0, coinciding with the overlap of the end of the previous "fictitious" word, say y_{-1}. The point is that y_0 is the "beginning" of the excursion, thus the first accident. Then $F_{n_0} \le N$ and $F_{n_1} > N$ which yields $n_0 < n_1$. Formula (18) can now be applied. Therefore $b_1 = F_{n_0} - 2 + \frac{j}{\gamma}(F_{n_1} - 2)$ with $j \ge 0$.

• The estimate

$$E_{M+1} = \sum_{d \ge 1} e^{-\beta \log(\frac{F_{n_M} + d}{N+1})} = \sum_{d \ge 1} \left(\frac{F_{n_M} + d}{N+1} \right)^{-\beta} \le \frac{N+1}{\beta - 1} \left(\frac{F_{n_M}}{N+1} \right)^{1-\beta}$$

follows from by (16), with $d_M = F_{n_M} + d$ and $d \ge 1$.

Recall that within excursions, all $F_{n_j} \ge N+1$ for all j, where N can be chosen as large as we want. We also remind Binet's formula $F_n = \frac{1}{\sqrt{5}}(\gamma^{n+1} - (-1/\gamma)^{n+1})$, which allows to replace F_n by $\frac{\gamma^{n+1}}{\sqrt{5}}$, and we to treat the quantities -1 as negligible compared to γ^{n_i} for large n_i, as it is the case. Therefore, assuming that $\beta > 1$, there exists a constant C such that

$$\begin{aligned} E_i &= \sum_{j \ge 1} e^{-\beta \log\left(1 + \frac{\max(F_{n_i} - F_{n_{i+1}}, F_{n_i - 1}) + jF_{n_{i+1} - 2}}{F_{n_{i+1}} - 1}\right)} \\ &\le C \sum_{j \ge 1} \left(1 + \max(\gamma^{n_i - n_{i+1}} - 1, \gamma^{n_i - n_{i+1} - 1}) + j/\gamma^2\right)^{-\beta} \\ &\le \frac{C\gamma^2}{\beta - 1} \left(1 + \max(\gamma^{n_i - n_{i+1}} - 1, \gamma^{n_i - n_{i+1} - 1})\right)^{1-\beta}. \end{aligned}$$

for $2 \le i \le M$

Let $P = \lfloor \frac{\log \frac{N}{\sqrt{5}}}{\log \gamma} \rfloor = \max\{n \in \mathbb{N} : F_n \le N\}$. Then (17) yields

$$C_E \le \sum_{M=0}^{+\infty} \left(\frac{\gamma^2}{\beta-1}\right)^M \frac{(N+1)}{\beta-1} \cdot$$
$$\sum_{\substack{n_1,\dots n_M > P\\ n_0 \le P}} E_1 \left(1 + \max(\gamma^{n_i - n_{i+1}} - 1, \gamma^{n_i - n_{i+1} - 1})\right)^{1-\beta} \gamma^{(P-n_M)(\beta-1)}. \tag{19}$$

4.1.5. *Proof of Proposition 4.1-Step 4.* We show that $C_E \to 0$ as $\beta \to +\infty$.

Proposition 4.3. *There exists $A = A(\beta) \in (0,1)$ with $\lim_{\beta\to+\infty} A = 0$ such that*

$$C_E \le 2P \frac{N+1}{\beta-1} \sum_{n=1}^{+\infty} \gamma^{-n(\beta-1)} \sum_{M=0}^{+\infty} A^M \sum_{i=0}^{M} \frac{n^i}{i!}.$$

Before proving this proposition, we show that it finishes the proof of Proposition 4.1. The series has only positive terms. Clearly, $\sum_{M=0}^{+\infty} A^M \sum_{i=0}^{M} \frac{n^i}{i!} \le \frac{1}{1-A} e^n$, so the main sum converges if $\gamma^{\beta-1} > e$. Thus Proposition 4.3 implies that $C_E \to 0$ as $\beta \to +\infty$.

Therefore, inequality (15) shows that if $\beta \to +\infty$, then $\lambda_{0,\beta} \to 0$ too, and hence Proposition 4.1 is proved.

The rest of this subsection is devoted to the proof of Proposition 4.3. The following lemma follows easily by induction:

Lemma 4.4. *Let η and y be positive real numbers. Then for every n,*

$$\int_y^\infty x^n e^{-\eta(x-y)} dx = \sum_{j=0}^{n} \frac{n!}{j!} \frac{y^j}{\eta^{n+1-j}}.$$

For some positive integer n and real numbers $\xi, \zeta > 0$, let $D_n = (d_{n,i,j})_{i=1,j=1}^{n+1,n}$ be the matrix with $n+1$ rows and n columns defined by

$$d_{n,i,j} := \begin{cases} \frac{(j-1)!}{(i-1)!}\zeta^{j-i+1} & \text{if } i \le j,\\ \frac{\xi}{j} & \text{if } i = j+1,\\ 0 & \text{if } i > j+1,\end{cases}$$

or in other words:

$$D_n = \begin{pmatrix} 0!\zeta & 1!\zeta^2 & 2!\zeta^3 & \dots & (j-1)!\zeta^j & \dots & (n-1)!\zeta^n \\ \xi & \zeta & \dots & & & & (n-2)!\zeta^{n-1} \\ 0 & \frac{\xi}{2} & \zeta & & & & \vdots \\ 0 & 0 & \frac{\xi}{3} & \ddots & \frac{(j-1)!}{(i-1)!}\zeta^{j-i+1} & & \vdots \\ \vdots & & 0 & \ddots & \ddots & & \vdots \\ \vdots & & & 0 & \frac{\xi}{j} & \zeta & \zeta^2 \\ 0 & & & & 0 & \frac{\xi}{n-1} & \zeta \\ 0 & 0 & \dots & \dots & 0 & 0 & \frac{\xi}{n} \end{pmatrix}.$$

We call $\mathbf{w}$ non-negative (and write $\mathbf{w} \succeq 0$) if all a entries of $\mathbf{w}$ are non-negative. This defines a partial ordering on vectors by

$$\mathbf{w}' \succeq \mathbf{w} \iff \mathbf{w}' - \mathbf{w} \succeq 0.$$

Lemma 4.5. *Assume $0 < \zeta < 1$ and set $K := \frac{1}{1-\zeta}$. Then, for every n,*

$$D_n \cdot \begin{pmatrix} \frac{K^{n-1}}{0!} \\ \frac{K^{n-1}}{1!} \\ \frac{K^{n-1}}{2!} \\ \vdots \\ \frac{K^{n-1}}{(n-1)!} \end{pmatrix} \preceq \begin{pmatrix} \frac{K^n}{0!} \\ \frac{K^n}{1!} \\ \frac{K^n}{2!} \\ \vdots \\ \frac{K^n}{n!} \end{pmatrix}.$$

Proof. This is just a computation. For the first row we get

$$\sum_{j=1}^{n} (j-1)!\zeta^j . \frac{K^{n-1}}{(j-1)!} \leq K^{n-1} . \frac{\zeta}{1-\zeta} \leq K^n.$$

For row $i > 1$ we get

$$\frac{1}{(i-1)} \frac{K^{n-1}}{(i-2)!} + \sum_{j=i}^{n} \frac{(j-1)!}{(i-1)!}\zeta^{j-i+1} \frac{K^{n-1}}{(j-1)!} = \frac{K^{n-1}}{(i-1)!}\left(1 + \zeta + \zeta^2 \dots\right) \leq \frac{K^n}{(i-1)!}.$$

□

Proposition 4.6. *Set $\zeta := \frac{1}{(\beta-1)\log\gamma}$ and $K = \frac{1}{1-\zeta}$. For every $M \geq 2$, consider integers $n_1, \dots n_M$, with $n_M > P$. Then,*

$$\sum_{n_1,\dots,n_{M-1} > P} \prod_{i=1}^{M-1} \left(1 + \max(\gamma^{n_i - n_{i+1}} - 1, \gamma^{n_i - n_{i+1} - 1})\right)^{1-\beta} \leq K^{M-1} \sum_{i=0}^{M-1} \frac{(n_M - P)^i}{i!}$$

Proof. Note that

$$
\begin{aligned}
&\sum_{n_1,\dots,n_{M-1}>P} \prod_{i=1}^{M} \left(1+\max(\gamma^{n_i-n_{i+1}}-1, \gamma^{n_i-n_{i+1}-1})\right)^{1-\beta} \\
&\quad = \sum_{n_{M-1}=1}^{\infty} \left(\dots \left(\sum_{n_2=1}^{\infty} \left(\sum_{n_1=1}^{\infty} \left(1+\max(\gamma^{n_1-n_2}-1, \gamma^{n_1-n_2-1})\right)^{1-\beta} \right) \cdot \right.\right. \\
&\quad\quad \left. (1+\max(\gamma^{n_2-n_3}-1, \gamma^{n_2-n_3-1}))^{1-\beta} \right) \dots \cdot \\
&\quad\quad \left. \left(1+\max(\gamma^{n_{M-1}-n_M}-1, \gamma^{n_{M-1}-n_M-1}\right)^{1-\beta} \right).
\end{aligned}
$$

This means that we can proceed by induction. Now

$$
\begin{aligned}
&\sum_{n_1=P+1}^{\infty} (1+\max(\gamma^{n_1-n_2}-1, \gamma^{n_1-n_2-1}))^{1-\beta} \\
&\quad \le \int_P^{n_2} (1+\gamma^{x-n_2-1}))^{1-\beta} dx + \int_{n_2}^{\infty} (\gamma^{x-n_2})^{1-\beta} dx \\
&\quad \le n_2 - P + \int_{n_2}^{\infty} e^{-(\beta-1)\,(x-n_2)\,\log\gamma}\, dx \\
&\quad = n_2 - P + \int_{n_2}^{\infty} e^{-\frac{x-n_2}{\zeta}}\, dx,
\end{aligned}
$$

because $\zeta = \frac{1}{(\beta-1)\log\gamma}$. This shows that the result holds for $M = 2$.

Assuming that the sum for $M = p$ is of the form $\sum_{j=0}^{p-1} a_j (n_p - P)^j$, we compute the sum for $M = p + 1$.

$$
\begin{aligned}
&\sum_{n_p=P+1}^{\infty} \sum_{j=0}^{p-1} a_j \frac{(n_p-P)^j}{(1+\max(\gamma^{n_p-n_{p+1}}-1, \gamma^{n_p-n_{p+1}-1}))^{\beta-1}} \\
&\quad \le \sum_j a_j \int_P^{n_{p+1}} \frac{(x-P)^j}{(1+\gamma^{x-n_{p+1}-1})^{\beta-1}}\, dx + \sum_j a_j \int_{n_{p+1}}^{\infty} \frac{(x-P)^j}{(\gamma^{x-n_{p+1}})^{\beta-1}}\, dx \\
&\quad \le \sum_j \frac{a_j (n_{p+1}-P)^{j+1}}{(j+1)} + \int_{n_{p+1}}^{\infty} (x-P)^j e^{-\frac{x-n_{p+1}}{\zeta}}\, dx.
\end{aligned}
$$

Set $\mathbf{w}\cdot\mathbf{w}' = \sum w_i w_i'$, for vectors $\mathbf{w} = (w_1,\dots,w_{p+1})$ and $\mathbf{w}' = (w_1',\dots,w_{p+1}')$. Lemma 4.4 yields

$$\begin{aligned}
\sum_{n_p=P+1}^{\infty}\sum_{j=0}^{p-1} a_j \frac{(n_p-P)^j}{(1+\max(\gamma^{n_p-n_{p+1}}-1,\gamma^{n_p-n_{p+1}-1}))^{\beta-1}}\\
\leq \sum_j \frac{a_j}{(j+1)}(n_{p+1}-P)^{j+1} + \sum_{i=0}^{j}\frac{j!}{i!}\zeta^{j-i+1}(n_{p+1}-P)^i\\
\leq D_p \begin{pmatrix} a_0\\ a_1\\ \vdots\\ a_{p-1}\end{pmatrix}\cdot\begin{pmatrix}1\\ n_{p+1}\\ \vdots\\ n_{p+1}^p\end{pmatrix}.
\end{aligned}$$

Lemma 4.5 concludes the proof of the induction. □

Proof of Proposition 4.3. We have just proved that

$$\begin{aligned}
\sum_{n_1,\dots n_M>P}\left(1+\max(\gamma^{n_i-n_{i+1}}-1,\gamma^{n_i-n_{i+1}-1})\right)^{1-\beta}\gamma^{(P-n_M)(\beta-1)}\\
\leq K^{M-1}\sum_{n_M=P+1}^{+\infty}\sum_{j=0}^{M-1}\frac{(n_M-P)^j}{j!}\gamma^{(n_M-P)(\beta-1)}.
\end{aligned}$$

It remains to sum over n_0. Note that in that case, there are only P terms of the form $\sum_{j=0}^{+\infty}\frac{1}{\left(1+\gamma^{n_0-n_1-2}+\frac{j}{\gamma}\right)^\beta}$ because $n_0\leq P<n_1$ for each possible n_0,

$$\begin{aligned}
\sum_{j=0}^{+\infty}\frac{1}{\left(1+\gamma^{n_0-n_1-1}+\frac{j}{\gamma^2}\right)^\beta} &= \frac{1}{\left(1+\gamma^{n_0-n_1-2}\right)^\beta}+\sum_{j=1}^{+\infty}\frac{1}{\left(1+\gamma^{n_0-n_1-2}+\frac{j}{\gamma^2}\right)^\beta}\\
&\leq 1+\frac{\gamma^2}{\beta-1}\frac{1}{\left(1+\gamma^{n_0-n_1-2}\right)^{\beta-1}}\\
&\leq 1+\frac{\beta-1}{2},
\end{aligned}$$

for $\beta\geq\sqrt{2}\gamma$. Finally, inequality (19) yields

$$C_E \leq 2P\,\frac{N+1}{\beta-1}\sum_{M=0}^{+\infty}A^M\sum_{n=1}^{+\infty}\gamma^{n(1-\beta)}\sum_{j=0}^{M-1}\frac{n^j}{j!},$$

with $A := \frac{\gamma}{\beta-1}K = \frac{\gamma}{\beta-1-\frac{1}{\log\gamma}}$. This tends to 0 as $\beta\to+\infty$. □

4.2. **End of the proof of Theorem 2.**

4.2.1. *End of the case* $-\log \frac{n+1}{n}$. Proposition 4.1 shows that there exists some minimal β_0 such that $\lambda_{0,\beta} < 1$ for every $\beta > \beta_0$. This also shows that $\mathcal{P}(\beta) = 0$ for $\beta > \beta_0$. Since $\mathcal{P}(\beta)$ is a continuous and convex function, it is constant for $\beta > \beta_0$. As $\mathcal{P}(0) = \log 2$, there exists a minimal $\beta_c > 0$ such that $\mathcal{P}(\beta) > 0$ for every $0 \leq \beta < \beta_c$. Clearly, $\beta_c \leq \beta_0$.

We claim that for $\beta < \beta_c$, there exists a unique equilibrium state and that it has full support. Indeed, there exists at least one equilibrium state, say μ_β, and at least one cylinder, say J, has positive μ_β-measure. Therefore, we can induce on this cylinder, and the form of potential (see [17, Theorem 4]) shows that there exists a unique local equilibrium state. It is a local Gibbs measure and therefore μ_β is uniquely determined on each cylinder, and unique and with full support (due to the mixing property).

We claim that the pressure function $\mathcal{P}(\beta)$ is analytic on $[0, \beta_c]$. Indeed, each cylinder J has positive μ_β-measure and the associated $Z_c(\beta)$ is the pressure of the dotted system (that is: restricted to the trajectories that avoid J). This set of trajectories has a pressure strictly smaller than $\mathcal{P}(\beta)$ because otherwise, several equilibrium states would coexist. Therefore $\mathcal{P}(\beta)$ is determined by the implicit equation $\lambda_{\mathcal{P}(\beta),\beta} = 1$ and $\mathcal{P}(\beta) > Z_c(\beta)$ for $\beta \in [0, \beta_c]$. The Implicit Function Theorem shows that $\mathcal{P}(\beta)$ is analytic.

For $\beta \geq \beta_c$, the pressure $\mathcal{P}(\beta) = 0$ and for cylinders J as above, we have $Z_c(\beta) \geq 0$. This shows that $Z_c(\beta) = 0$ for every $\beta \geq \beta_c$. Due to the form of the potential, $\lambda_{0,\beta}$ is continuous and decreasing in β.

Now, the next result finishes the proof of Theorem 2 in the case that $V(x) = -\log \frac{n+1}{n}$.

Lemma 4.7. *The parameters β_c and β_0 coincide.*

Proof. As mentioned above $\beta_c \leq \beta_0$. Assume by contradiction $\beta_c < \beta_0$. We claim that $\lambda_{0,\beta_c} > 1$. Indeed, $\beta \mapsto \lambda_{0,\beta}$ is strictly decreasing, and $\lambda_{0,\beta_c} \leq 1$ would yield that $\lambda_{0,\beta} < 1$ for every $\beta > \beta_c$. This would imply $\beta_c \geq \beta_0$ because β_0 is minimal with this property, and we have assumed $\beta_c < \beta_0$.

Now, for fixed β, $Z \mapsto \lambda_{Z,\beta}$ is continuous and strictly decreasing and goes to 0 as $Z \to +\infty$. Therefore, since $\lambda_{0,\beta_c} > 1$, there exists $Z > 0$ such that $\lambda_{Z,\beta_c} = 1$. The local equilibrium state for this Z generates some σ-invariant probability measure[3] with pressure for $\beta_c\varphi$ equal to Z, thus positive, and this contradicts $\mathcal{P}(\beta_c) = 0$. This proves that $\beta_c = \beta_0$ □

[3]Since $Z_c(\beta_c) = \mathcal{P}(\beta_c) = 0 < Z$, the expectation of the return time is comparable to $\left|\frac{\partial \mathcal{L}_{Z,\beta_c}(\mathbb{1}_J)}{\partial Z}\right|$, which converges.

4.3. **The general case $\boldsymbol{V \in \mathcal{X}_1}$.** For $V \in \mathcal{X}_1$, there exists $\kappa > 0$ such that

$$-V \leq \kappa\varphi.$$

This shows that the pressure function is constant equal to zero for $\beta \geq \beta_0/\kappa$. Again, the pressure is convex, thus non-increasing and continuous. We can define β'_c such that $\mathcal{P}(\beta) > 0$ for $0 \leq \beta \leq \beta'_c$ and $\mathcal{P}(\beta) = 0$ for $\beta \geq \beta'_c$.

The rest of the argument is relatively similar to the previous discussion. We deduce that for $\beta < \beta'_c$, there exists a unique equilibrium state, it has full support and $\mathcal{P}(\beta)$ is analytic on this interval. For $\beta \geq \beta'_c$, it is not clear that $\lambda_{0,\beta}$ decreases in β. However, we do not really need this argument, because if $\lambda_{0,\beta} > 1$, then the decrease of $Z \mapsto \lambda_{Z,\beta}$ (which follows from convexity argument and $\lim_{Z\to+\infty} \lambda_{Z,\beta} = 0$), is sufficient to produce a contradiction.

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