THE TOPOLOGICAL ENTROPY OF BANACH SPACES

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Abstract. We investigate some properties of (universal) Banach spaces of real functions in the context of topological entropy. Among other things, we show that any subspace of $C([0,1])$ which is isometrically isomorphic to ℓ_1 contains a functions with infinite topological entropy. Also, for any $t \in [0,\infty]$, we construct a (one-dimensional) Banach space in which any nonzero function has topological entropy equal to t.

1. Introduction

Let $C([0,1])$ denote the set of all continuous functions $f : [0,1] \to \mathbb{R}$ equipped with the supremum norm. A theorem of Banach and Mazur [2] states that the Banach space $C([0,1])$ is universal, *i.e.,* every real, separable Banach space X is isometrically isomorphic to a closed subspace of $C([0,1])$. It is known that one can require more properties of the functions of $C([0,1])$ in the image of X: a universal space containing only the zero function and nowhere differentiable functions [7], resp. consisting of the zero function and nowhere approximatively differentiable and nowhere Hölder functions [3] has been proved. On the other hand, no universal space can consist of functions of bounded variation [4] and every isometrically isomorphic copy of ℓ_1 (*i.e.,* the space of sequences with 1-norm) in $C([0,1])$ contains a function which is non-differentiable at every point of a perfect subset of $[0,1]$, see [6].

In this paper we are going to investigate some properties of (universal) Banach spaces of real functions of a real variable in the context of topological entropy. We show how to construct a universal Banach space using the zero function and functions with infinite topological entropy - Theorem 3, and as a supplement of the result from [6] we show that any subspace of $C([0,1])$ which is isometrically isomorphic to ℓ_1 contains a functions with infinite topological entropy - Theorem 2. Finally, for any $t \in [0,\infty]$ we construct a (one-dimensional) Banach space in which any nonzero function has its topological entropy equal to t - Theorem 4.

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2. Preliminaries and auxiliary results

Let $C_b(X)$ denote the set of all *bounded* continuous functions $f : X \to \mathbb{R}$ equipped with the supremum norm. Clearly, $C_b(\mathbb{R})$ is a non-separable Banach space. Let $[a, b]$ be a closed finite subinterval of $\mathbb{R}$. We identify $f\colon [a, b] \to \mathbb{R}$ with its extension

$$(\mathrm{Ex}\, f)(x) = \begin{cases} f(x) & \text{if } x \in [a, b]; \\ f(b) & \text{if } x \geq b; \\ f(a) & \text{if } x \leq a. \end{cases} \tag{1}$$

Under this identification, $C([a, b]) \subset C_b(\mathbb{R})$. We will deal with the topological entropy of maps from $C_b(\mathbb{R})$ defined as $h_{top}(f) := h_{top}(f|_{\overline{f(\mathbb{R})}})$ - see [1, Chapter 4].

The well known Banach - Mazur Theorem states that the Banach space $C([0, 1])$ is universal, *i.e.,* every real, separable Banach space $\mathcal{D}$ is isometrically isomorphic to a closed subspace of $C([0, 1])$. Since by our convention, $C([0, 1])$ is a closed subspace of $C_b(\mathbb{R})$, the non-separable space $C_b(\mathbb{R})$ is also universal. In our paper we will restrict ourselves to separable universal Banach spaces only.

Following [1] we recall the notion of horseshoe.

Definition 1. *A function $f \in C_b(\mathbb{R})$ is said to have a d-*horseshoe *if there exist d subintervals $I_1, I_2, \dots, I_d$ of $\mathbb{R}$ with disjoint interiors such that $f(I_i) \supset I_j$ for all $1 \leq i, j \leq d$.*

Proposition 1. [5] *If $f \in C_b(\mathbb{R})$ has a d-*horseshoe *then $h_{top}(f) \geq \log d$.*

In the next lemma we denote by $F(X)$ a linear space of functions $f\colon X \to \mathbb{R}$.

Lemma 1. *Given n linearly independent functions in $F(X)$, there exist n points $x_1, \dots, x_n \in X$ such that the vectors*

$$\begin{pmatrix} f_1(x_1) \\ f_1(x_2) \\ \vdots \\ f_1(x_n) \end{pmatrix}, \begin{pmatrix} f_2(x_1) \\ f_2(x_2) \\ \vdots \\ f_2(x_n) \end{pmatrix}, \dots, \begin{pmatrix} f_n(x_1) \\ f_n(x_2) \\ \vdots \\ f_n(x_n) \end{pmatrix}$$

are linearly independent in $\mathbb{R}^n$.

Proof. This is clear if $n = 1$. Assume now by induction that the lemma holds for $k < n$ and points $x_1, \dots, x_k$. For the unique linear combination such that $f_{k+1}(x_i) = a_1 f_1(x_i) + \dots + a_k f_k(x_i)$ for all $1 \leq i \leq k$. Now if $f_{k+1}(x) = a_1 f_1(x) + \dots + a_k f_k(x)$ for all $x \in X$, then $f_1, \dots, f_{k+1}$ are linearly dependent, contrary to our assumption. So there must be some other point x_{k+1} for which $f_{k+1}(x_{k+1}) \neq a_1 f_1(x_{k+1}) + \dots + a_k f_k(x_{k+1})$, which concludes the induction step. $\square$

3. The Main Theorems

Definition 2. *For a given set $\mathcal{B} \subset C_b(\mathbb{R})$, let*

$$\begin{aligned} h^+_{top}(\mathcal{B}) &= \sup\{h_{top}(f) : f \in \mathcal{B}\}, \\ h^-_{top}(\mathcal{B}) &= \inf\{h_{top}(f) : f \in \mathcal{B},\ f \text{ is non-zero}\}. \end{aligned}$$

Theorem 1. *If a linear space $\mathcal{B} \subset C_b(\mathbb{R})$ has dimension n, then*

$$h^+_{top}(\mathcal{B}) \geq \log(n-1).$$

In particular, $h^+_{top}(\mathcal{B}) = \infty$ if $\dim(\mathcal{B}) = \infty$.

Proof. Take $f_1, \ldots, f_n \in \mathcal{B}$ linearly independent and find points $x_1, \ldots, x_n$ as in Lemma 1. We can assume that $x_1 < x_2 < \cdots < x_n$. Form a linear combination $f = a_1 f_1 + \cdots + a_n f_n$ such that $f(x_i) = x_1$ if i is odd, and $f(x_i) = x_n$ if i is even. Then f has an $(n-1)$-horseshoe, so from Proposition 1 we get $h_{top}(f) \geq \log(n-1)$ as required. □

Example 1. *(i) Let $[a, b]$ be a closed subinterval of $\mathbb{R}$. Given a continuous function $f : \mathbb{R} \to \mathbb{R}$, let*

$$(Cr_{[a,b]} f)(x) = \begin{cases} f(x) & \text{if } x \in [a, b]; \\ f(b) & \text{if } x \geq b; \\ f(a) & \text{if } x \leq a, \end{cases}$$

be the cropped *version of f. Clearly $Cr_{[a,b]} f \in C([a, b]) \subset C_b(\mathbb{R})$. Let*

$$P^{n-1} = \{Cr_{[a,b]} p : p \in C(\mathbb{R}) \text{ is a polynomial of degree } \leq n-1\}.$$

Then P^{n-1} has dimension n, each $f \in P^{n-1}$ is at most $n-2$-modal, so our definition of the entropy $h_{top}(f)$ and [1, Theorem 4.2.4] *imply that $h_{top}(f) \leq \log(n-1)$. This shows that the bound in Theorem 1 is sharp.*

(ii) Let

$$P = \{Cr_{[a,b]} p : p \in C(\mathbb{R}) \text{ is a polynomial }\}.$$

Then P is a normed linear subspace of $C([a, b])$ and $\dim(P) = \infty$, hence by Theorem 1, $h^+_{top}(P) = \infty$. By the same argument as above, the entropy of any $p \in P$ satisfies $h_{top}(p) \leqslant \log \deg(p)$, so it is finite.

Theorem 1 does not answer the question whether every infinite dimensional Banach space $\mathcal{A} \subset C_b(\mathbb{R})$ contains a function with infinite entropy. Our next example shows that in general it is not the case.

Example 2. *For $n \geqslant 1$ and $a \in \mathbb{R}$, let $f_{n,a} : \mathbb{R} \to \mathbb{R}$ be given by*

$$f_{n,a}(x) = \begin{cases} a \cdot (x - 2 + \frac{1}{n}) \cdot (2 - \frac{1}{n+1} - x) & \text{if } x \in J_n := [2 - \frac{1}{n}, 2 - \frac{1}{n+1}]; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $f_{n,a}$ is unimodal, so its entropy $h_{top}(f_{n,a}) \leq \log 2$. Consider the smallest Banach space Q (subspace of $C_b(\mathbb{R})$ with supremum norm) containing all finite sums

$f_{1,a_1} + f_{2,a_2} + \cdots + f_{n,a_n}$. Then $dim(Q) = \infty$, $\lim_{x \to 2_-} f(x) = 0$ for each $f \in Q$ and if

$$\max\{x \in \mathbb{R} \colon f(x) = 1\} \in J_n,$$

then since the modality of $f|_{[1,1-\frac{1}{n+1}]}$ is at most $2n$ and $f^2(x) = 0$ for $x \notin [1, 1-\frac{1}{n+1}]$, we conclude that $h_{top}(f) \leqslant \log(2n+1)$.

As a counterpart of the previous example we will prove the following theorem.

Theorem 2. *Let $\mathcal{A} \subset C([0,1])$ be isometrically isomorphic to ℓ_1. Then $\mathcal{A}$ contains a function with infinite topological entropy.*

Proof. Let Φ be an isometrical isomorphism ensured by the statement, so $\Phi(\ell_1) = \mathcal{A}$. For $i \in \mathbb{N}$ let $e_i = (e_{ij})_{j=1}^{\infty} \in \ell_1$ be defined by

$$e_{ij} = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Then for every $n \in \mathbb{N}$ and every choice of distinct positive integers $i(1), \dots, i(n)$

$$\| \pm e_{i(1)} \pm \cdots \pm e_{i(n)} \|_{\ell_1} = n.$$

Denote $f_i = \Phi(e_i) \in \mathcal{A} \subset C([0,1])$, $i \in \mathbb{N}$. Clearly $\|f_i\| = 1$; in particular for every $x \in [0,1]$,

$$|f_i(x)| \leqslant 1. \tag{2}$$

Claim 1. *For every $s = (s_i)_i \in \{1, -1\}^{\mathbb{N}}$ there exists a point $x \in [0,1]$ such that the sequence $(f_i(x))_{i \in \mathbb{N}}$ is equal to either s or $-s$.*

Proof. Assume that for some n,

$$\forall\ x \in [0,1] \colon (f_i(x))_{i=1}^n \neq (s_i)_{i=1}^n \text{ and } (f_i(x))_{i=1}^n \neq (-s_i)_{i=1}^n;$$

then (2) implies $|\sum_{i=1}^n s_i f_i(x)| < n$ for every $x \in [0,1]$. This contradicts the equalities

$$\| \sum_{i=1}^n s_i e_i \|_{\ell_1} = \| \sum_{i=1}^n s_i f_i \| = n. \tag{3}$$

Thus, for each $n \in \mathbb{N}$ one can find a point $x_n \in [0,1]$ for which either $(f_i(x_n))_{i=1}^n = (s_i)_{i=1}^n$ or $(f_i(x_n))_{i=1}^n = (-s_i)_{i=1}^n$. Taking a limit point x of the sequence $(x_n)_n$, from the continuity of the functions f_i we get either $(f_i(x)) = s$ or $(f_i(x)) = -s$. $\square$

For $n > 1$ define the matrix $A_n = (a^n_{ij})^n_{i,j=1}$ by $a_{ij} = (-1)^i$ for $1 \leqslant j < i$ and $a_{ij} = (-1)^{i+1}$ when $i \leqslant j \leqslant n$. For instance, the particular matrix A_8 is

$$\begin{pmatrix} +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ +1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 \\ +1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 \\ +1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & +1 & +1 \\ +1 & +1 & +1 & +1 & +1 & +1 & +1 & -1 \end{pmatrix}$$

One can easily verify the following fact.

Claim 2. *For any $\beta \in \mathbb{R}^n$, the linear equation $A_n\alpha = \beta$ has a unique solution α given by the formulas*

$$\alpha_i = \frac{\beta_i + \beta_{i+1}}{(-1)^{i+1}2},\ i = 1, \dots, n-1, \quad \alpha_n = \frac{\beta_1 + (-1)^{n+1}\beta_n}{2}. \tag{4}$$

In particular, $\max|\alpha_i| \leqslant \max|\beta_i|$.

Let us denote the i-th row of the matrix A_n by $a^n_i = (a^n_{i1}, a^n_{i2}, \cdots, a^n_{in})$. By Claim 1, for each $n > 1$ there are distinct points $x^n_1, \dots, x^n_n \in [0, 1]$ such that either

$$f_1(x^n_i) = \cdots = f_{2^n}(x^n_i) = 1,\ (f_{2^n+1}(x^n_i), f_{2^n+2}(x^n_i), \cdots, f_{2^n+n}(x^n_i)) = a^n_i, \tag{5}$$

or

$$f_1(x^n_i) = \cdots = f_{2^n}(x^n_i) = -1,\ (f_{2^n+1}(x^n_i), f_{2^n+2}(x^n_i), \cdots, f_{2^n+n}(x^n_i)) = -a^n_i. \tag{6}$$

Put $X_n = \{x^n_1, \dots, x^n_n\}$. Since $n = \mathrm{card}(X_n)$ is growing to infinity, one can consider subsets $X'_n \subset X_n$ satisfying

$$\lim_{n\to\infty} \mathrm{card}(X'_n) = \infty,\ \lim_{n\to\infty} \mathrm{diam}(X'_n) = 0. \tag{7}$$

Passing to a subsequence if necessary, we can assume that $X'_n \to x_0 \in [0, 1]$, *i.e.*,

$$\forall\, \varepsilon > 0\ \exists\, n_0\ \forall\, n > n_0\colon X'_n \subset (x_0 - \varepsilon, x_0 + \varepsilon). \tag{8}$$

Now, using (5) and (6), we obtain that either $(f_i(x_0))_i = (1)_i$ or $(f_i(x_0))_i = (-1)_i$. Without loss of generality assume the first possibility. Notice that then

$$\forall\, n > 1\colon (f_{2^n+1}(x_0), f_{2^n+2}(x_0), \cdots, f_{2^n+n}(x_0)) = a^n_1. \tag{9}$$

We can formally put

$$e = x_0 e_1 + \sum_{n=2}^{\infty}\sum_{k=1}^{n} \alpha^n_k e_{2^n+k},\ \Phi(e) = f(x) = x_0 f_1(x) + \sum_{n=2}^{\infty}\sum_{k=1}^{n} \alpha^n_k f_{2^n+k}(x), \tag{10}$$

where coefficients $\alpha^n = (\alpha_1^n, \alpha_2^n, \cdots, \alpha_n^n)$ satisfy a linear equation $A_n \alpha^n = \beta^n$, $\beta^n = (\beta_1^n, \beta_2^n, \cdots, \beta_n^n) \in \mathbb{R}^n$. It can be easily seen that $f \in \mathcal{A}$ if and only if

$$\sum_{n=2}^{\infty} \sum_{k=1}^{n} |\alpha_k^n| < \infty.$$

Moreover, if $\beta_1^n = 0$ for each n and $f \in C([0,1])$ then $f(x_0) = x_0$ by the equation $x_0 f_1(x_0) = x_0$ and the property (9) implying

$$\sum_{k=1}^{n} \alpha_k^n f_{2^n+k}(x_0) = 0 \text{ for each } n.$$

Using Claim 2 we will show in the sequel that there exists a sequence $(\beta^n = (0, \beta_2^n, \cdots, \beta_n^n))_n$ such that the corresponding function f given by (10) satisfies $f \in \mathcal{A}$ and $h_{top}(f) = \infty$. In what follows we denote

$$g_1(x) = x_0 f_1(x),\ g_m(x) = g_1(x) + \sum_{n=2}^{m} \sum_{k=1}^{n} \alpha_k^n f_{2^n+k}(x),\ m \geqslant 2.$$

Let $\omega(f, X) = \sup_{x,y \in X} |f(x) - f(y)|$ denote the oscillation of a function f on a set X. For a positive $\varepsilon(i)$ we use the notation $J(i) = [x_0 - \varepsilon(i), x_0 + \varepsilon(i)]$. The zero element in $\mathbb{R}^n$ is denoted by 0_n. Let $(\gamma_m)_{m \in \mathbb{N}}$ be a sequence of positive numbers satisfying for each m

$$\gamma_m > \sum_{i=m+1}^{\infty} 2(i+3)\gamma_i. \tag{11}$$

Step 0. $n(0) = 1$.

Step 1. We can find values $\varepsilon(1) > 0$ and $n(1) > n(0) + 1$ such that

$$\varepsilon(1) + \omega(g_{n(0)}, J(1)) < \gamma_1 - \sum_{i=2}^{\infty} 2(i+3)\gamma_i, \tag{12}$$

$$J(1) \cap X'_{n(1)} \supset \{x_{i(1)}^{n(1)} < x_{i(2)}^{n(1)} < x_{i(3)}^{n(1)} < x_{i(4)}^{n(1)}\}.$$

We put $\beta^n = 0_n$ for each $n(0) < n < n(1)$; the coefficients $\alpha_k^{n(1)}$, $k = 1, \ldots, n(1)$ are gained as the unique solution of the linear equation $A_{n(1)} \alpha^{n(1)} = \beta^{n(1)}$, where (as we already know) $\beta_1^{n(1)} = 0$, $\beta_{i(j)}^{n(1)} = (-1)^j \gamma_1$, $j = 1, 2, 3, 4$ and $\beta_i^{n(1)} = 0$ otherwise.

Step m. We can find values $\varepsilon(m) > 0$ and $n(m) > n(m-1) + 1$ such that

$$\varepsilon(m) + \omega(g_{n(m-1)}, J(m)) < \gamma_m - \sum_{i=m+1}^{\infty} 2(i+3)\gamma_i, \tag{13}$$

$$J(m) \cap X'_{n(m)} \supset \{x_{i(1)}^{n(m)} < x_{i(2)}^{n(m)} < \cdots < x_{i(m+2)}^{n(m)} < x_{i(m+3)}^{n(m)}\}.$$

We put $\beta^n = 0_n$ for each $n(m-1) < n < n(m+1)$; the coefficients $\alpha_k^{n(m)}$, $k = 1, \dots, n(m)$ are gained as the unique solution of the linear equation $A_{n(m)}\alpha^{n(m)} = \beta^{n(m)}$, where $\beta_1^{n(m)} = 0$, $\beta_{i(j)}^{n(m)} = (-1)^j\gamma_m$, $j = 1, \dots, m+3$ and $\beta_i^{n(m)} = 0$ otherwise.

Since by Claim 2, $\alpha_k^n = 0$ for $n \neq n(m)$, $|\alpha_k^{n(m)}| \leqslant \gamma_m$ and by (4) there are at most $2(m+3)$ nonzero coefficients $\alpha_k^{n(m)}$, one can see that by our choice of the β's

$$\sum_{n=2}^{\infty}\sum_{k=1}^{n}|\alpha_k^n| \leqslant \sum_{m=1}^{\infty}\sum_{k=1}^{n(m)}|\alpha_k^{n(m)}| \leqslant \sum_{m=1}^{\infty} 2(m+3)\gamma_m < \infty.$$

Thus, the function f given by the above coefficients α_k^n and the formula (10) belongs to the space $\mathcal{A}$. Using the equality $g_{n(m-1)}(x_0) = x_0$ and (11), (13) we get for $j \leqslant m+3$ odd

$$f(x_{i(j)}^{n(m)}) \leqslant x_0 + \omega(g_{n(m-1)}, J(m)) - \gamma_m + \sum_{i=m+1}^{\infty} 2(i+3)\gamma_i \leqslant x_0 - \varepsilon(m)$$

and analogously for $j \leqslant m+3$ even

$$f(x_{i(j)}^{n(m)}) \geqslant x_0 - \omega(g_{n(m-1)}, J(m)) + \gamma_m - \sum_{i=m+1}^{\infty} 2(i+3)\gamma_i \geqslant x_0 + \varepsilon(m).$$

At the same time $[x_{i(j)}^{n(m)}, x_{i(j+1)}^{n(m)}] \subset J(m) = [x_0 - \varepsilon(m), x_0 + \varepsilon(m)]$, hence the function f has an $(m+2)$-horseshoe (created by the points $x_{i(1)}^{n(m)}, x_{i(2)}^{n(m)}, \dots, x_{i(m+3)}^{n(m)}$) on the interval $J(m)$. It means that $h_{top}(f) \geqslant \log(m+2)$ and m can be arbitrarily large. □

Theorem 3. *There is a universal Banach space $\mathcal{A} \subset C_b(\mathbb{R})$ such that $h_{top}(f) = \infty$ for every non-zero f from $\mathcal{A}$.*

Proof. Take $p_n = 2^{-n}$ for $n \geq 0$ and $\{q_n\}_{n\geq 0}$ a decreasing sequence such that $q_0 = 1$, $q_n \geq p_n$ for all n, $q_n/p_n \to \infty$, but $q_n \to 0$. Choose intervals $I_n = [\frac{3}{4}p_n, \frac{5}{4}p_n]$ and $J_n = (\frac{2}{3}p_n, \frac{4}{3}p_n) \supset I_n$, both 'centered' at p_n. Notice also that the J_n's are adjacent: $\frac{2}{3}p_n$ is the common boundary point of J_n and J_{n+1}. Now for a function $f \in C([0,1])$, construct $g := \Psi(f) \in C_b(\mathbb{R})$ as follows, see Figure 1:

$$g(y) = \begin{cases} 0 & \text{if } y = 0; \\ q_n \cdot f(\frac{2y}{p_n} - \frac{3}{2}) & \text{if } y \in I_n \text{ for some } n \geq 0; \\ 0 & \text{if } y \in \cup_n \partial J_n; \\ 0 & \text{if } y \geq \frac{4}{3}; \\ \text{by linear interpolation} & \text{if } y \in \cup_n (J_n \setminus I_n); \\ g(-y) & \text{if } y < 0; \end{cases}$$

Let $\mathcal{A} = \Psi(C([0,1])) \subset C([-\frac{4}{3}, \frac{4}{3}])$ equipped with the norm ($q_0 = 1$)

$$\sup_{y\in\mathbb{R}} |g(y)| = \|g\| = \sup_{y\in I_0} |g(y)| = \|f\|,$$

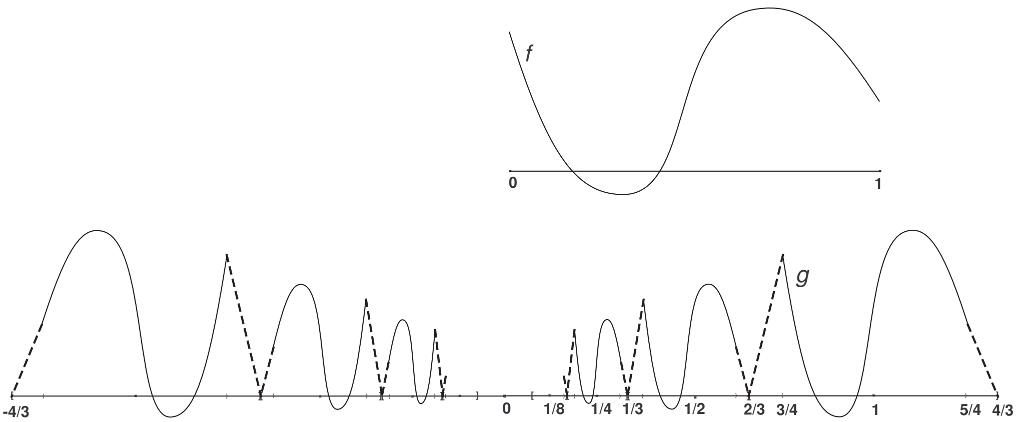

FIGURE 1. The maps $f \in C([0,1])$ and $\Psi(f) = g \in C([-\frac{4}{3}, \frac{4}{3}])$, $p_n = (\frac{1}{2})^n$, $q_n = (\frac{2}{3})^n$, $n \geqslant 0$.

so Ψ is an isometrical isomorphism and $\mathcal{A}$ is a separable Banach space.

If f is not constant zero, then $g = \Psi(f)$ is not constant zero either and

$$\sup_{y \in I_n} |g(y)| = q_n \|f\| > 0.$$

Fix $d \in \mathbb{N}$ arbitrary. Since $q_n/p_n = q_n/2^{-n} \to \infty$, there is an $n \in \mathbb{N}$ such that

$$q_n \|f\| > 2^{-n+d} = p_{n-d}.$$

Since $\{q_i\}_i$ is decreasing and $g(\pm\partial J_i) = 0$ for all i (where $-J_i = \{y : -y \in J_i\}$), it follows that $g(I_i) = g(-I_i) \supset [0, \max J_{n-d+1}]$ or $[-\max J_{n-d+1}, 0]$ for all $n-d+1 \leq i \leq n$. Hence, within the intervals $J_{n-d+1}, \dots, J_n$, or within $-J_{n-d+1}, \dots, -J_n$, we can choose d intervals that form a d-horseshoe. This implies that $h_{top}(g) \geq \log d$. As d was arbitrary, $h_{top}(g) = \infty$.

For a real, separable Banach space $\mathcal{B}$ we will find an isometrical isomorphism $\Phi : \mathcal{B} \to \mathcal{A}$. Since by the Banach–Mazur Theorem the space $C([0,1])$ is universal, there is an isometrical isomorphism $\tilde{\Phi} : \mathcal{B} \to C([0,1])$. Using the above constructed isometrical isomorphism $\Psi : C([0,1]) \to \mathcal{A}$, the required Φ is just $\Psi \circ \tilde{\Phi}$. □

Remark 1. *Recall that $f \in C^\alpha(\mathbb{R})$ (f is α-*Hölder on $\mathbb{R}$*) for some $\alpha \in (0,1)$ if*

$$\sup \left\{ \frac{|f(x) - f(y)|}{|x-y|^\alpha} : x, y \in \mathbb{R},\ 0 < |x-y| \leqslant 1 \right\} < \infty.$$

For some fixed $\alpha \in (0,1)$, if we choose $q_n = p_n^\alpha$ and $f \in C^\alpha([0,1])$, then $\Psi(f)$ is α-Hölder on $\mathbb{R}$. Therefore $\mathcal{A}^\alpha := \Psi(C^\alpha([0,1])) \subset C_b^\alpha(\mathbb{R})$ is a normed (infinite dimensional) linear space such that $h_{top}(f) = \infty$ for every non-zero f from $\mathcal{A}^\alpha$.

4. Entropy of one-dimensional Banach spaces

Even if $\dim(\mathcal{B}) = 1$, it is still possible that $h^+_{top}(\mathcal{B}) = \infty$. As the following example shows, the upper bound for the entropy need not be attained.

Example 3. *Let $\mathcal{B}$ be spanned by $f(x) = \sin x$, then λf admits a d-horseshoe whenever $|\lambda| \geq 2\pi d$. Therefore $h^+_{top}(\mathcal{B}) = \infty$.*

The above example also shows that there is no sensible upper bound for $h^+_{top}(\mathcal{B})$ in terms of $\dim(\mathcal{B})$ only. However, $h^-_{top}(\mathcal{B}) = 0$ - see Definition 2.

In this section we will be investigating the equality $h^-_{top}(\mathcal{B}) = h^+_{top}(\mathcal{B})$ for one-dimensional subspaces $\mathcal{B}$ of $C_b(\mathbb{R})$: so far we know that for some $\mathcal{B}$,

- $h^-_{top}(\mathcal{B}) = h^+_{top}(\mathcal{B}) = \infty$ (easy consequence of Theorem 3)
- $h^-_{top}(\mathcal{B}) = h^+_{top}(\mathcal{B}) = 0$ ($\mathcal{B}$ is spanned by a monotone map)

The following statement shows that the entropy can behave extremely rigidly on a one-dimensional subspace of $C_b(\mathbb{R})$.

Theorem 4. *For any $t \in [0, \infty]$, there exists a function $f \in C_b(\mathbb{R})$ such that for $\mathcal{B} = \{\lambda f\}_{\lambda \in \mathbb{R}}$ satisfies $h^-_{top}(\mathcal{B}) = h^+_{top}(\mathcal{B}) = t$.*

Proof. The case $t = 0$ and $t = \infty$ were covered previously, so let $t \in (0, \infty)$ arbitrary and take an odd integer $d > e^t$.

Let $\theta_a : [0, \infty) \to [0, \infty)$ be a one-parameter family (with $a \in [0, 1]$) of at most d-modal continuous maps such that for each $a \in [0, 1]$, $\theta_a([9, 10]) \subset [9, 10]$ and $\theta_a(x) = x$ whenever $x \notin (9, 10)$, θ_0 is the identity, and θ_1 has a full d-horseshoe on $[9, 10]$. In the C^1 topology for maps of fixed modality, topological entropy depends continuously on the map, see [1, Cor. 4.5.5], so there is no loss in generality in assuming that $h_{top}(\lambda \cdot \theta_a)$ is continuous in both $a \in [0, 1]$ and $\lambda \in [\frac{9}{10}, \frac{10}{9}]$. (Note that $h_{top}(\lambda \cdot \theta_a) \equiv 0$ for $\lambda \geqslant 0$ outside this interval.) Therefore $r_a = \sup_{\lambda \geq 0} h_{top}(\lambda \cdot \theta_a)$ is is continuous in a as well, and $r_0 = 0$, $r_1 = \log d > t$. Therefore there is a^* such that $r_{a^*} = t$. Fix $\Theta = \theta_{a^*}$.

Next let $\{\lambda_i\}_{i \geq 0}$ be a denumeration of the positive rationals such that $\lambda_1 = 1$ and

$$\lambda_{n+1} \leqslant 2\lambda_n \quad \text{for all } n \geq 0. \tag{14}$$

Let $x_n = 4^{-n}$ and $I_n = [0.9x_n, x_n]$ for $n \geq 0$. Now we set

$$f(x) = \begin{cases} \lambda_n \cdot \frac{x_n}{10} \cdot \Theta(\frac{10}{x_n} \cdot x) & \text{if } x \in I_n; \\ 0 & \text{if } x = 0; \\ 10 & \text{if } x \geq 10; \\ \text{by linear interpolation} & \text{if } x \in (0, 10) \setminus \cup_n I_n; \\ f(-x) & \text{if } x < 0. \end{cases}$$

Fix $\lambda > 0$. By assumption (14) we have that $\lambda f(x) \le \lambda f(y)$ for all $x \in I_{n+1}$, $y \in I_n$ and $n \ge 0$. It is not hard to see that every orbit with respect to λf can visit only finitely many intervals I_n, and at most one of them infinitely often. Therefore, if we choose some $x > 0$, then $\omega(x)$ can only belong to a single I_n, and only if the diagonal intersects the box $I_n \times \lambda f(I_n)$. By our choice of a^* (and hence Θ), $h_{top}(\lambda f|_{I_n}) \le t$. Since $x \ge 0$ is arbitrary, $h_{top}(\lambda f) \le t$.

For $\varepsilon > 0$ let λ^* satisfy $h_{top}(\lambda^* \Theta) \geqslant t - \varepsilon$. Since $\{\lambda_n\}_{n \ge 0}$ is dense in $[0, \infty)$ there is some interval I_m such that $\lambda_m \lambda$ is sufficiently close to λ^* hence $h_{top}(\lambda_m \lambda \Theta) \geqslant t - 2\varepsilon$ and also $h_{top}(\lambda f|_{I_m}) \geqslant t - 2\varepsilon$. This shows that $h_{top}(\lambda f) \ge t$, and so we have $h_{top}(\lambda f) = t$.

Finally, the dynamics of $-\lambda f$ on $(-\infty, 0]$ is conjugate to the dynamics of λf on $[0, \infty)$, so also $h_{top}(-\lambda f) = t$. □

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