Classification of one dimensional dynamical systems by countable structures

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Abstract

We study the complexity of the classification problem of conjugacy on dynamical systems on some compact metrizable spaces. Especially we prove that the conjugacy equivalence relation of interval dynamical systems is Borel bireducible to isomorphism equivalence relation of countable graphs. This solves a special case of the Hjorth's conjecture which states that every orbit equivalence relation induced by a continuous action of the group of all homeomorphisms of the closed unit interval is classifiable by countable structures. We also prove that conjugacy equivalence relation of Hilbert cube homeomorphisms is Borel bireducible to the universal orbit equivalence relation.

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1 Introduction

Measuring the complexity of relations on structures is a very general task. In this paper we use the notion of *Borel reducibility* (see Definition 1) and the results of Invariant descriptive set theory to compare the complexities of classification problems. For more

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details on Invariant descriptive set theory we refer to the book by Gao [Gao09]. For a short and nice introduction to the theory of Borel reductions we refer to a paper by Foreman [For18].

Several equivalence relations became milestones in this theory. Let us mention four of those, which describe an increasing chain of complexities:

- the equality on an uncountable Polish space,
- the equality of countable sets,
- the S_∞-universal orbit equivalence relation (S_∞ is the group of permutations on $\mathbb{N}$),
- the universal orbit equivalence relation,

Let us give several *examples* to make the reader more familiar with the above relations. A classical example is a result of Gromov (see e.g. [Gao09, Theorem 14.2.1]) who proved that the isometry equivalence relation of compact metric spaces is a *smooth* equivalence relation, which means that it is Borel reducible to the equality of real numbers (or equivalently of an uncountable Polish space). The isomorphism relation of countable graphs or the isomorphism relation of countable linear orders are Borel bireducible to the S_∞-universal orbit equivalence. The homeomorphism equivalence relation of compact metrizable spaces and the isometry relation of separable complete metric spaces were proved by Zielinski in [Zie16] and by Melleray in [Mel07], respectively, to be Borel bireducible to the universal orbit equivalence relation (see the survey paper by Motto Ros [MR17]).

In order to capture all the structures in one space we need some sort of *coding.* This can be done by considering some universal space (e.g. the Hilbert cube or the Urysohn space) and all its subspaces with some natural Polish topology or Borel structure (e.g the hyperspace topology or the Effros Borel structure). Sometimes there are other natural ways to encode a given structure. For example the class of separable complete metric spaces can be coded by the set of all metrics on $\mathbb{N}$ where two metrics are defined to be equivalent if the completions of the respective spaces are isometric. Fortunately in this case, by [Gao09, Theorem 14.1.3] it does not matter which coding we choose. It is generally believed that this independence on a natural coding is common to other structures and thus the statements are usually formulated for all structures without mentioning the current coding. Nevertheless, for the formal treatment some coding is always necessary.

The aim of this paper is to determine the complexity of some classification problems of dynamical systems up to conjugacy. Dynamical systems of a fixed compact metrizable space X can be naturally coded as a space of continuous functions mapping X into itself, with the uniform topology. This one as well as the subspace of all self-homeomorphisms is well known to be a Polish space.

Let us mention several results which are dealing with the complexity of conjugacy equivalence relation. It was proved by Hjorth that conjugacy equivalence relation of homeomorphisms of [0,1] is classifiable by countable structures [Hjo00, Section 4.2] (in fact Borel bireducible to the universal S_∞-orbit equivalence relation) but conjugacy of homeomorphisms of $[0,1]^2$ is not [Hjo00, Section 4.3]. By a result of Camerlo and Gao, conjugacy equivalence relation of both selfmaps and homeomorphisms of the Cantor set are Borel bireducible to the S_∞-universal orbit equivalence relation [CG01, Theorem 5]. Kaya proved that conjugacy of pointed minimal Cantor dynamical systems is Borel bireducible to the equality of countable subsets of reals [Kay17b]. Conjugacy of odometers is smooth due to Buescu and Stewart [BS95]. The complexity of conjugacy of Toepliz subshifts was treated several times – by Thomas, Sabok and Tsankov, and by Kaya [Tho13, ST17, Kay17a]. Conjugacy of two-sided subshifts is Borel bireducible to the universal countable Borel equivalence relation due to Clemens [Cle09]. There is an extensive exposition of results on the complexity of conjugacy equivalence relation on subshifts of 2^G for a countable group G in the book by Gao, Jackson and Seward [GJS16, Chapter 9]. Recently, during the 8th Visegrad Conference on Dynamical Systems in 2019 it was announced by Dominik Kwietniak that conjugacy of shifts with specification is Borel bireducible to the universal countable Borel equivalence relation.

In this paper, we deal with some of the missing parts. By mainly elementary and standard tools, we prove that conjugacy of interval maps is Borel bireducible to the S_∞-universal orbit equivalence relation. Conjugacy of homeomorphisms as well as of selfmaps of the Hilbert cube is proved to be Borel bireducible to the universal orbit equivalence relation. To this end we use some tools of infinite dimensional topology and a result of Zielinski on the complexity of homeomorphism equivalence relation of metrizable compacta [Zie16] combining with some ideas of P. Krupski and the second author [KV20]. Finally we make a small overview on the complexity of conjugacy equivalence relation of dynamical systems on the Cantor set, on the arc, on the circle and on the Hilbert cube.

2 Definitions and notations

A *Polish space* is a separable completely metrizable topological space. Recall that a *standard Borel space* is a measurable space $(X, \mathcal{S})$ such that there is a Polish topology τ on X for which the family (X, τ) of Borel subsets is equal to $\mathcal{S}$. In order to compare the complexities of equivalence relations we use the notion of Borel reducibility.

Definition 1. Suppose that X and Y are sets and let E, F are equivalence relations on X and Y respectively. We say that E is *reducible* to F, and we denote this by $E \leq F$, if there exists a mapping $f\colon X \to Y$ such that

$$xEx' \iff f(x)Ff(x'),$$

for every $x, x' \in X$. The function f is called a *reduction* of E into F. If the sets X and Y are endowed with Polish topologies (or standard Borel structures), we say that E is *Borel* *reducible* to F, and we write $E \leq_B F$, if there is a reduction $f\colon X \to Y$ of E into F which is Borel measurable. We say that E is *Borel bireducible* to F, and we write $E \sim_B F$, if E is Borel reducible to F and F is Borel reducible to E.

In a similar fashion we define being *continuously reducible* if in addition X and Y are Polish spaces and f is continuous.

In the whole paper we set $I = [0, 1]$ and denote the closure operator by Cl. For a separable metric space X we denote by $\mathcal{K}(X)$ the hyperspace of all compacta in X with the Hausdorff distance d_H and the corresponding Vietoris topology. The space $C(X)$ is the space of all continuous functions of X into itself.

The equality equivalence relation of real numbers is denoted as $E_=$. We denote by $E_{=^+}$ the equivalence relation on $\mathbb{R}^{\mathbb{N}}$ defined by $(a_n)E_{=^+}(b_n)$ if and only if $\{a_n\colon n \in \mathbb{N}\} = \{b_n\colon n \in \mathbb{N}\}$. The last equivalence relations is called the *equality of countable sets*.

We say, that an equivalence relation E defined on a standard Borel space X is *classifiable by countable structures* if there is a countable relation language $\mathcal{L}$ such that E is Borel reducible to the isomorphism relation of $\mathcal{L}$-structures whose underlying set is $\mathbb{N}$. An equivalence relation E on a standard Borel space X is said to be an *orbit equivalence relation* if there is a Borel action of a Polish group G on X such that xEx' if and only if there is some $g \in G$ for which $gx = x'$.

Let $\mathcal{C}$ be a class of equivalence relations on standard Borel spaces. An element $E \in \mathcal{C}$ is called *universal* for $\mathcal{C}$ if $F \leq_B E$ for every $F \in \mathcal{C}$. It is known that for every Polish group G there is an equivalence relation (denoted by E_G) on a standard Borel space that is universal for all orbit equivalence relations given by continuous G-actions. We are particularly interested in the *universal S_∞-equivalence relation* E_{S_∞}, where S_∞ is the group of all permutations of $\mathbb{N}$. It is known that an equivalence relation is classifiable by countable structures if and only if it is Borel reducible to E_{S_∞}. Moreover E_{S_∞} is known to be Borel bireducible to isomorphism equivalence relation of countable graphs. Also there exists a *universal orbit equivalence relation* which is denoted by E_{G_∞}. We should also note that all the mentioned equivalence relations are analytic sets, i.e. images of standard Borel spaces with respect to a Borel measurable map. We have a chain of complexities

$$E_= \leq_B E_{=^+} \leq_B E_{S_\infty} \leq_B E_{G_\infty}$$

and it is known that none of these Borel reductions can be reversed.

3 Interval dynamical systems

In this section we prove that conjugacy of interval dynamical systems is classifiable by countable structures. The strategy of our proof is as follows. In the first part we describe a natural reduction of interval dynamical systems to some kind of countable structures.

We assign to every $f \in C(I)$ a countable invariant set $C_f \subseteq I$ of some dynamically exceptional points for f. Since the set C_f does not need to be dense in I we do not have enough information to capture the dynamics of f by restricting to C_f. On the other hand the dynamics on the maximal open intervals of $I \setminus C_f$ is quite simple. Hence it will be enough to define an invariant countable dense subset D_f in $I \setminus \mathrm{Cl}(C_f)$ arbitrarily. Consequently, we get that for f conjugate to g there exists a conjugacy of f to g which sends the set $C_f \cup D_f$ onto $C_g \cup D_g$. Finally it is enough to assign to every $f \in C(I)$ a countable structure $\Psi(f)$ whose underlying set is $C_f \cup D_f$ and which is equipped with one binary relation $\leq\restriction_{C_f \cup D_f}$ and one function $f\restriction_{C_f \cup D_f}$ (which can be as usual considered as a binary relation). We will prove then that if two such structures $\Psi(f)$ and $\Psi(g)$ are isomorphic then f and g are conjugate.

In the second part we prove that this reduction can be modified using some sort of coding so that the assigned countable structures share the same support and so that the new reduction is Borel. To this end we use Lusin-Novikov selection theorem [Kec95, Theorem 18.10] several times.

For $g \in C(I)$ we denote by $\mathrm{Fix}(g)$ the set of fixed points of g, i.e. those points for which $g(x) = x$. We omit the proof of the following "folklore" lemma. The key idea of the proof is the back and forth argument.

Lemma 2. *Let $f, g \in C(I)$ be increasing homeomorphisms such that $Fix(f) = Fix(g) = \{0, 1\}$ and let $A, B \subseteq (0, 1)$ be countable dense sets that are invariant in both directions for f and g respectively. Then there is a conjugacy h of f and g satisfying $h(A) = B$.*

Definition 3. For $f \in C(I)$ let us say that a point $z \in I$ is a *left sharp local maximum* of f if there is some $\delta > 0$ such that $f(x) < f(z)$ for $x \in (z - \delta, z)$ and $f(x) \leq f(z)$ for $x \in (z, z + \delta)$. In a similar fashion we define *left sharp local minimum*, *right sharp local minimum* and *right sharp local maximum*.

Notation 4. Let M_f be the union of $0, 1$ and the set of all left and right sharp local maxima and minima. It is easily shown that the set M_f is countable. For a closed set $F \subseteq I$ denote by $\mathrm{Acc}(F)$ the set of all accessible points of F in $\mathbb{R}$, i.e. those points $x \in I$ for which there exists an open interval $(a, b) \subseteq \mathbb{R} \setminus F$ for which $x = a$ or $x = b$.

For every $f \in C(I)$ let us denote by C_f the smallest set such that

a) $M_f \subseteq C_f$,

b) if $f^{-1}(y)$ contains an interval then $y \in C_f$,

c) if $n \in \mathbb{N}$ then $\mathrm{Acc}(\mathrm{Fix}(f^n)) \subseteq C_f$,

d) $f(C_f) \subseteq C_f$,

e) if $y \in C_f$ then $\mathrm{Acc}(f^{-1}(y)) \subseteq C_f$.

Lemma 5. *The set C_f is countable for every $f \in C(I)$.*

Proof. Let S_1 be the union of M_f, all the values of f at locally constant points and all the sets $\mathrm{Acc}(\mathrm{Fix}(f^n))$ for $n \in \mathbb{N}$. Clearly S_1 is countable. Let $S_{i+1} = S_i \cup f(S_i) \cup \bigcup\{\mathrm{Acc}(f^{-1}(y)) \colon y \in S_i\}$. Clearly $C_f = \bigcup\{S_i \colon i \in \mathbb{N}\}$ and thus it is countable. $\square$

Note that C_f depends only on the topological properties of I and the dynamics of f. That is if f and g are conjugate by some homeomorphism h, then $h(C_f) = C_g$. This is clear because h maps M_f onto M_g, locally constant intervals of f to locally constant intervals of g and periodic points of f to periodic points of g.

Let us denote by $\mathcal{J}_f$ be the collection of all maximal open subintervals of $I \setminus C_f$.

Lemma 6. *Let $J \in \mathcal{J}_f$. Then either $f \restriction_J$ is constant or $f \restriction_J$ is one to one and in this case $f(J) \in \mathcal{J}_f$. Also $f^{-1}(J)$ is the finite union (possibly the empty union) of elements of $\mathcal{J}_f$.*

Proof. Let us prove first that $f \restriction_J$ is either constant or one-to-one. Suppose that the contrary holds. Then there are points $x, y, z \in J$ such that $f(x) = f(y) \neq f(z)$ and $x \neq y$. Let us suppose without loss of generality that $x < y < z$ and $f(x) < f(z)$. Let $u = \min f \restriction_{[x,z]}$ and let $v = \max(f^{-1}(u) \cap [x, z])$. It follows that $v \in (x, z)$ is a right sharp local minimum. By Notation 4 a) it follows that $v \in C_f$ which is a contradiction since J is disjoint from C_f.

Suppose now that $f \restriction_J$ is one-to-one and let us prove that $f(J) \in \mathcal{J}_f$. Observe first that $f(J)$ is disjoint from C_f, otherwise there would be a point $y \in f(J) \cap C_f$ and since $f^{-1}(y)$ is a closed set not containing the whole set J there will be a point in $\mathrm{Acc} f^{-1}(y) \cap J$ which is a contradiction with Notation 4 e). We need to prove that $f(J)$ is a maximal interval disjoint from C_f. Suppose that $J = (a, b)$. Then there are $a_n, b_n \in C_f$ such that $a_n \to a, b_n \to b$. By continuity of f it follows that $f(a_n) \to f(a)$ and $f(b_n) \to f(b)$. Also $f(a_n), f(b_n) \in C_f$ by Notation 4 d). Thus the maximality follows.

Observe first that $f^{-1}(J)$ is a countable union of disjoint collection of open intervals and if we prove that each of the intervals is mapped by f onto J it will follow by continuity that such a collection is in fact finite. Denote $(a, b) = J$ and let (c, d) by a maximal interval in $f^{-1}(J)$. Clearly $(c, d) \cap C_f = \emptyset$ by Notation 4, so it is enough to prove that it is maximal with this property. Note that $f(c), f(d) \in \{a, b\}$ otherwise we get a contradiction with (c, d) being maximal interval in $f^{-1}(J)$. Also it can not happen that $f(c) = f(d)$ otherwise there will be a point of left local maximum or minimum in (c, d) which would produce a point in $M_f \cap J$. Hence $f(c, d) = J$. Moreover, by the first part of this proof we get that $f \restriction_{(c,d)}$ is one-to-one and thus it is either increasing or decreasing. Without loss of generality suppose the first case. Let us distinguish several possibilities If $f \geq f(c)$ on a left neighborhood of c then c is a point of right sharp local minimum and thus $c \in C_f$. Otherwise choose a sequence $a_n \in C_f$ such that $a_n \to a$. We define points

$c_n = \max([0,c] \cap f^{-1}(a_n))$. These are eventually well defined, $c_n \to c$ and $c_n \in \mathrm{Acc} f^{-1}(a_n)$. Hence by d) $c_n \in C_f$. We can proceed in a similar way with the point d and thus the interval (c,d) is maximal subinterval of $I \setminus C_f$. $\square$

Example 7. For the tent map $f(x) = \min\{2x, 2(1-x)\}$, the set C_f contains all the dyadic numbers in I, thus C_f is a dense subset of I and hence $\mathcal{J}_f = \emptyset$. For the map $g = \frac{1}{4} f$ we have

$$\begin{aligned} C_g &= \{2^{-n}, 1 - 2^{-n} \colon n \in \mathbb{N}\} \cup \{0,1\}, \\ \mathcal{J}_f &= \{(2^{-n-1}, 2^{-n}), (1 - 2^{-n}, 1 - 2^{-n-1}) \colon n \in \mathbb{N}\}. \end{aligned}$$

Notation 8. Let G_f be a directed graph on $\mathcal{J}_f$ where (J,K) forms an oriented edge if and only if $f(J) = K$. Let $E_f = \mathbb{Q} \cap I \setminus \mathrm{Cl}(C_f)$ and let

$$D_f = \bigcup_{n \in \mathbb{Z}} f^n(E_f).$$

Note that the union is taken over all integers. In spite of that it follows by Lemma 6 that D_f is countable. Let us define

$$\Psi(f) = (C_f \cup D_f, \leq\restriction_{C_f \cup D_f}, f \restriction_{C_f \cup D_f}).$$

Theorem 9. *The mapping Ψ is a reduction of orientation preserving conjugacy of interval dynamical systems to the isomorphism relation of countable structures.*

Proof. Suppose first that f is conjugate to g via some increasing homeomorphism h, that is $f = h^{-1} g h$. We want to find an isomorphism $\varphi \colon \Psi(f) \to \Psi(g)$. Since h does not need to map D_f to D_g so we need to do some more work. In fact we find a conjugacy $\bar{h}$ of f and g such that $\bar{h}(C_f \cup D_f) = C_g \cup D_g$. Then it will be enough to define a mapping $\varphi \colon C_f \cup D_f \to C_g \cup D_g$ as the restriction of $\bar{h}$. We will define $\bar{h}$ by parts. First of all we define $\bar{h}$ on the set $\mathrm{Cl}(C_f)$ in the same way as h.

Clearly h induces an isomorphism of the graphs $(\mathcal{J}_f, G_f)$ and $(\mathcal{J}_g, G_g)$. We will consider the components of the symmetrized graphs G_f and G_g. Note that $J, K \in \mathcal{J}_f$ are in the same component of G_f if there are $m, n \geq 0$ such that $f^m(J) = f^n(K)$.

Let us distinguish two cases for the components of G_f. If a component of G_f contains an oriented cycle, choose an element J in there (note that the cycle is unique). Hence there is $n \in \mathbb{N}$ such that $f^n(J) = J$. By using Notation 4 c) it follows that either all the points of J are fixed points for f_n or there are no fixed points of f^n in J and the same has to be true for g^n. Hence by using Lemma 2 there is a conjugacy $\bar{h} \restriction_J$ of $f^n \restriction_{\mathrm{Cl}(J)}$ and $g^n \restriction_{\mathrm{Cl}(J)}$ sending $D_f \cap J$ onto $D_g \cap h(J)$. In components which do not contain an oriented cycle we choose J arbitrarily, and let $\bar{h} \restriction_J$ be an arbitrary increasing homeomorphism $J \to h(J)$ which maps $J \cap D_f$ onto $h(J) \cap D_g$.

For any K that is in the same component as J find $m, n \geq 0$ such that $f^m(J) = f^n(K) \in \mathcal{J}_f$ and define $\bar{h}$ on K using the definition of $\bar{h}$ on J as

$$(g^{-n} \restriction_K) g^m \bar{h} (f^{-m} \restriction_J) f^n.$$

On the other hand suppose that φ is an isomorphism of the countable structure $\Psi(f)$ to $\Psi(g)$. Hence $\varphi\colon C_f \cup D_f \to C_f \cup D_g$ is a bijection preserving the order. Thus it can be extended to an increasing homeomorphism $\tilde{\varphi}\colon I \to I$. We claim that $\tilde{\varphi}$ conjugates f and g. Consider any point $x \in C_f \cup D_f$ and compute

$$g(\tilde{\varphi}(x)) = g(\varphi(x)) = \varphi(f(x)) = \tilde{\varphi}(f(x)).$$

Since the set $C_f \cup D_f$ is dense it follows by continuity that $g(\tilde{\varphi}(x)) = \tilde{\varphi}(f(x))$ for every $x \in I$. Hence f and g are conjugate. □

3.1 Borel coding

We need to verify that the mapping Ψ that was proved in Theorem 9 to be a reduction can be coded in a Borel way. We use standard notation for the Borel hierarchy, especially Σ^0_1 is used for the collection of all open sets, Σ^0_2 is used for the collection of countable unions of closed sets etc. For a set $B \subseteq X \times Y$ and $x \in X$ let us denote by B_x the set $\{y \in Y\colon (x, y) \in B\}$ and call it *vertical section* of B.

The following seems to be folklore in descriptive set theory, but for the sake of completeness we include a proof.

Proposition 10. *Let X, Y be Polish spaces and $B \subseteq X \times Y$ be a Borel set with countable vertical sections. Then the set $\bigcup_{x \in X} \{x\} \times Cl(B_x)$ is Borel as well.*

Proof. Let $\mathcal{B}$ be a countable base for the topology of Y. By the Lusin-Novikov selection theorem, we can assume that $B = \bigcup f_n$ for some Borel maps f_n. It follows that

$$(X \times Y) \setminus \left(\bigcup_{x \in X} \{x\} \times \mathrm{Cl}(B_x) \right) = \bigcup_{U \in \mathcal{B}} \bigcap_{n \in \mathbb{N}} ((X \setminus f_n^{-1}(U)) \times U).$$

Hence the set under discussion is Borel. □

Let us denote $\Gamma = \{(K, a) \in \mathcal{K}(I) \times I\colon a \in \mathrm{Acc}(K)\}$.

Lemma 11. *The set Γ is a Σ^0_2-set.*

Proof. The sets

$$L_n = \{(K, a) \in \mathcal{K}(I) \times I\colon a \in K, K \cap (a - 2^{-n}, a) = \emptyset\},$$

$$R_n = \{(K, a) \in \mathcal{K}(I) \times I\colon a \in K, K \cap (a, a + 2^{-n}) = \emptyset\}$$

are closed for every $n \in \mathbb{N}$. Hence the set $\bigcup (L_n \cup R_n)$ is a Σ^0_2-set. □

Notation 12. For a set $B \subseteq C(I) \times I$ let us define

$$\begin{aligned} B^{\rightarrow} &= \{(f, f(x)) \colon (f, x) \in B\}, \\ B^{\leftarrow} &= \{(f, x) \colon (f, f(x)) \in B\}, \\ B^{\Leftarrow} &= \{(f, x) \colon x \in Acc(f^{-1}(y)), (f, y) \in B\}. \end{aligned}$$

Lemma 13. *Let $B \subseteq C(I) \times I$ be a Borel set with countable vertical sections. Then the sets $B^{\rightarrow}, B^{\leftarrow}$ and $B^{\Leftarrow}$ are Borel as well.*

Proof. The evaluation mapping $e \colon C(I) \times I \to I$, $e(f, x) = f(x)$ is continuous. Hence the mapping $\Phi \colon (f, x) \mapsto (f, e(f, x))$ is continuous as well. Especially, the restriction of Φ to B is Borel and also countable-to-1. Since by [Kec95, 18.14] countable-to-1 image of a Borel set is Borel we conclude that $\Phi(B) = B^{\rightarrow}$ is Borel.

By the Lusin-Novikov selection theorem we can write $B = \bigcup F_n$ for some Borel maps F_n. It follows then that

$$B^{\leftarrow} = \bigcup_{n \in \mathbb{N}} \{(f, x) \colon e(f, x) = F_n(f)\}$$

and thus it is a Borel set.

The mapping $p \colon C(I) \times I \to \mathcal{K}(I)$, $p(f, y) = f^{-1}(y)$ is upper semicontinuous and hence it is Borel by [Kec95, 25.14]. The set Γ is Borel by Lemma 11 and it has nonempty and countable vertical sections. Hence $\Gamma = \bigcup b_n$ for some Borel mappings $b_n \colon \mathcal{K}(I) \to I$, by the Lusin-Novikov selection theorem. The mapping $\Psi \colon (f, y) \mapsto (f, b_n(f^{-1}(y))) = (f, b_n(p(f, y)))$ is a Borel mapping and its restriction to B is countable-to-1. Hence by [Kec95, 18.14] the set $\Psi(B) = B^{\Leftarrow}$ is Borel. □

Lemma 14. *The set*

$$A = \{(f, x) \in C(I) \times I \colon x \in C_f \cup D_f\}$$

is a Borel subset of $C(I) \times I$.

Proof. Let us prove first that the set $B_a := \{(f, x) \colon x \in M_f\}$ is Borel. As the set $\{(f, x) \colon x \text{ is a left sharp local maximum}\}$ can be written in the form

$$\bigcup_{\varepsilon > 0} \bigcap_{\eta > 0} \bigcup_{\delta > 0} \{(f, x) \colon \forall z \in [x - \varepsilon, x - \eta] : f(z) \leq f(x) - \delta \ \& \ \forall z \in [x, x + \varepsilon] : f(z) \leq f(x)\}$$

it follows that it is a Σ_3^0 set. By symmetry it follows that B_a is the union of four Σ_3^0-sets and thus it is Borel. The set $B_b := \{(f, y) \colon f^{-1}(y) \text{ contains an interval}\}$ is a Σ_2^0-set. Let $B_c := \{(f, x) \in C(I) \times I \colon f(x) = x\}$. The function $F_n \colon C(I) \to K(I)$, $F_n(f) = \mathrm{Fix}(f^n)$ is upper semicontinuous and thus Borel. Since Γ is a Borel set by Lemma 11 we conclude that the composition $\Gamma \circ F_n$ is Borel because the composition of a Borel binary relation and a Borel function (in that order) is a Borel relation. Hence $B_c = \bigcup_{n\in\mathbb{N}} \Gamma \circ F_n$ is Borel. Hence the set $B = B_a \cup B_b \cup B_c$ is Borel. Define recursively $B_1 = B$, $B_{n+1} = B_n \cup B_n^{\rightarrow} \cup B_n^{\Leftarrow}$ for $n \in \mathbb{N}$. All these sets are Borel by Lemma 13. It follows that $A_1 = \bigcup B_n = \{(f,x)\colon x \in C_f\}$ is Borel.

Since A_1 has countable vertical sections and it is Borel we conclude using Proposition 10 that $A_2 = \bigcup_{f\in C(I)}(\{f\}\times \mathrm{Cl}(A_{1,f}))$ is Borel as well. Consequently $A_3 = \{(f,x)\colon x \in E_f\} = (C(I)\times\mathbb{Q})\setminus A_2$ is Borel. By Lemma 13 we conclude that all the sets $A_{n+1} = A_n \cup A_n^{\rightarrow} \cup A_n^{\leftarrow}$, $n \geq 3$ are Borel. Finally $A = A_1 \cup \bigcup_{n\geq 3} A_n$ is a Borel set. □

Theorem 15. *The orientation preserving conjugacy of interval dynamical systems is Borel bireducible to the S_∞-universal orbit equivalence relation.*

Proof. By the result of [Hjo00, Section 4.2] conjugacy of interval homeomophisms is Borel bireducible to the S_∞-universal orbit equivalence relation. Hence especially the S_∞-universal orbit equivalence relation is Borel reducible to conjugacy of interval dynamical systems.

Let us argue for the converse. The set A from Lemma 14 is Borel and it has nonempty and countable vertical sections. Hence by the Lusin-Novikov selection theorem we can find Borel functions $F_n\colon C(I) \to I$ such that $\bigcup F_n = A$. Since all the vertical sections are infinite we can additionally suppose that for every pair $(f,x) \in A$ there is exactly one $n \in \mathbb{N}$ satisfying $F_n(f) = x$. Let

$$\Phi(f) = (\mathbb{N}, R, m),$$

where R is a binary relation and m is a unary function such that aRb iff $F_a(f) \leq F_b(f)$ and $m(a) = b$ iff $f(F_a(f)) = F_b(f)$ for $a, b \in \mathbb{N}$. There is a natural isomorphism $\Phi(f) \to \Psi(f)$, $a \mapsto F_a(f)$. Hence clearly Φ is a reduction. It is routine to verify that Φ is Borel by the fact that the functions F_n are Borel. □

Let us note that the same conclusion as in the previous theorem can be proved without assuming orientation preserving conjugacy but with just conjugacy. The reason is that in the proofs of Theorem 15 and Theorem 9 we can simply consider a ternary betweenness relation T instead of the binary relation of linear order $\leq$, i.e. $(x,y,z) \in T$ if and only if y is an element of the smallest interval containing x and z. This ternary relation is clearly forgetting the order of I. Also by [Hjo00, Exercise 4.14] E_{S_∞} is Borel reducible to conjugacy of interval homeomorphisms. Thus we get the following result.

Theorem 16. *The conjugacy of interval dynamical systems is Borel bireducible to the S_∞-universal orbit equivalence relation.*

We note that Theorem 16 is a special case of Hjorth's conjecture [Hjo00, Conjecture 10.6] stating that every equivalence relation induced by a continuous action of the group $\mathcal{H}(I)$ of all interval homeomorphisms on a Polish space is classifiable by countable structures. In this case the homeomorphism group acts on the space of continuous selfmaps by conjugacy. Similarly one can prove that the orbit equivalence relations induced by natural left or right composition actions of the homeomorphism group on the space of continuous selfmaps is Borel reducible to the S_∞-universal equivalence relation. Also it is known that the orbit equivalence induced by the homeomorphism group action $\mathcal{H}(I)$ on the hyperspace $\mathcal{K}(I)$ is Borel bireducible to the S_∞-universal orbit equivalence relation (see [Hjo00, Exercise 4.13] or [CG19] for a proof). All these are special cases of Hjorth's conjecture.

4 Hilbert cube dynamical systems

Since the homeomorphism equivalence relation of metrizable compacta is known to be Borel bireducible to the universal orbit equivalence relation, it is not surprising that conjugacy of dynamical systems on the Hilbert cube is of the same complexity, which is the main result of this section. In a dynamical system (X, f), a point x is called a *locally attracting fixed point* if $f(x) = x$ and there is a neighborhood U of x such that for every $z \in U$ the trajectory $(f^n(z))_{n\in\mathbb{N}}$ converges to x. The notion of a Z-set in the Hilbert cube Q plays an important role and it describes a kind of relative homotopical smallness. In fact, we do not need to recall the definition of a Z-set since only few properties on Z-sets are necessary. First, every homeomorphism of Z-sets can be extended to a homeomorphism of the Hilbert cube. Second, the Hilbert cube $Q \times I$ contains a topological copy of itself $Q \times \{0\}$ as a Z-set. Third, every closed subset of a Z-set in Q is a Z-set in Q.

The following proposition is a special case of [Zie16, Proposition 1] and it can be proved using the back and forth argument.

Proposition 17. *Let $K \subseteq A, L \subseteq B$ be four nonempty compact metrizable spaces such that $A \setminus K$ and $B \setminus L$ are dense sets of isolated points in A and B respectively. Then every homeomorphism of K onto L can be extended to a homeomorphism of A onto B.*

The following will be useful in the proof of Theorem 20.

Proposition 18 ([GvM93, Theorem 2.6])**.** *If X is a nondegenerate Peano continuum then there exists a homotopy $H\colon \mathcal{K}(X) \times I \to \mathcal{K}(X)$ for which*

- *$H(A, 0) = A$ for every $A \in 2^X$,*
- *$H(A, t)$ is finite for every $t > 0$ and $A \in 2^X$.*

Recall that if $Y \subseteq X$ and $\varepsilon > 0$ we say that X is ε-deformable into Y if there exists a continuous mapping $\varphi\colon X \times [0, 1] \to X$ such that $\varphi(x, 0) = x$, $\varphi(x, 1) \in Y$ and the

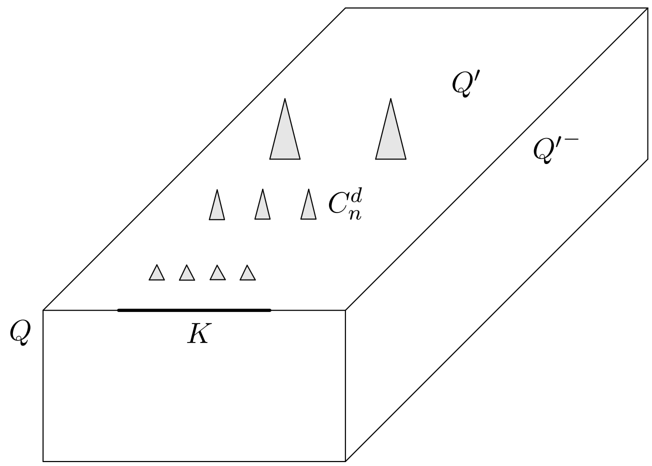

Figure 1: The compactum Q_K

diameter of $\varphi(\{x\} \times [0,1])$ is at most ε for every $x \in X$. The following proposition was proved in [Kra76, 1.1 and 1.3].

Proposition 19. *Let X be a compact space such that for every $\varepsilon > 0$ there exists an absolute neighborhood retract (absolute retract) $Y \subseteq X$ for which X is ε-deformable into Y. Then X is an absolute neighborhood retract (absolute retract, resp.).*

Some of the ideas for the proof of the following comes from the paper [KV20].

Theorem 20. *The conjugacy of Hilbert cube homeomorphisms (or selfmaps) is Borel bireducible to E_{G_∞}.*

Proof. For one direction it is enough to prove that the homeomorphism equivalence relation of metrizable compacta is Borel reducible to conjugacy of Hilbert cube homeomorphisms because the first relation is Borel bireducible to E_{G_∞} by the main result of [Zie16]. To this end let

$$\begin{aligned} Q &= \{x \in \ell_2 \colon 0 \leq x_n \leq 1/n\}, \\ Q' &= Q \times I, \\ Q'' &= Q \times I \times [-1,1], \\ Q'^{-} &= Q \times I \times [-1,0] \end{aligned}$$

and let $\|\cdot\|$ be the usual norm on ℓ_2.

Let us fix a homotopy $H\colon \mathcal{K}(Q)\times I\to\mathcal{K}(Q)$ given by Proposition 18 for the case $X=Q$. Let us fix $K\in\mathcal{K}(Q)$. We want to find a homeomorphism f_K of a Hilbert cube $Q_K\subseteq Q''$ such that the topological information about K is somehow encoded in the dynamics of f_K. Let $D_n^K=H(K,2^{-n})$, $n\in\mathbb{N}$. Let ε_n be the minimum of $1/n$ and the smallest distance of different points in D_n^K. For every $d\in D_n^K$ fix a set

$$B_n^d=\{(x,2^{-n},0)\in Q''\colon \|d-x\|\le\varepsilon_n/3\}.$$

It follows that B_n^d is always homeomorphic to the Hilbert cube since it is affinely homeomorphic to an infinite dimensional compact convex subset of a Hilbert space [Kel31]. Let C_n^d be the cone in Q'' with base B_n^d and with the vertex $(d,2^{-n},2^{-n})$, $d\in D_n^K$, $n\in\mathbb{N}$, i.e., the union of all segments with end points $(d,2^{-n},2^{-n})$ and p, $p\in B_n^d$. The cone over the Hilbert cube is homeomorphic to the Hilbert cube [vM01, Theorem 1.7.5], which applies to C_n^d. Let $Q_K^m=Q'^-\cup\bigcup\{C_n^d\colon n\in\mathbb{N},n\le m,d\in D_n^K\}$ and $Q_K=\bigcup\{Q_K^m\colon m\in\mathbb{N}\}$ (see Figure 1). By the result of [And67] the union of two Hilbert cubes, whose intersection is a Z-set homeomorphic to the Hilbert cube, is the Hilbert cube again. Hence we inductively obtain that all the spaces Q_K^m are homeomorphic to the Hilbert cube. Moreover, for every $\varepsilon>0$, Q_K is ε-deformable onto Q_K^m for some m and hence Q_K is an AR by Proposition 19. Moreover Q_K has the disjoint cell property (see [vM01, p. 294]) and thus Q_K is homeomorphic to the Hilbert cube by the famous Toruńczyk's theorem [vM01, Theorem 4.2.25].

Let $h(x)=\sqrt{x}, x\in I$ or any fixed homeomorphism of I with two fixed points $0,1$; and 1 being a locally attracting fixed point. We define

$$f_K(x)=\begin{cases}x, & x\in Q',\\ ((1-h(t))a+h(t)d,2^{-n},2^{-n}h(t)), & x=((1-t)a+td,2^{-n},2^{-n}t)\in C_n^d,\\ & \quad d\in D_n^K, t\in I, n\in\mathbb{N}.\end{cases}$$

All the points in Q' are fixed points for f_K and these are clearly not attracting. All the points in $\bigcup D_n^K$ are fixed points of f_K and these are attracting. There are no other fixed points of f_K. It follows that K is homeomorphic (or even equal) to the set of fixed points that are limits of attracting points but not attracting by itself (and thus defined only by dynamical notions). Hence if f_K and f_L are conjugate then K and L are homeomorphic, $K,L\in\mathcal{K}(Q)$.

On the other hand if K,L are homeomorphic compacta in Q then the sets $K\cup\bigcup_{n\in\mathbb{N}}D_n^K\times\{2^{-n}\}$ and $L\cup\bigcup_{n\in\mathbb{N}}D_n^L\times\{2^{-n}\}$ are homeomorphic by Lemma 17. This homeomorphism can be simply extended to a homeomorphism

$$\varphi\colon (K\times\{0\})\cup\bigcup\{B_n^d(K)\colon d\in D_n^K,n\in\mathbb{N}\}\to(L\times\{0\})\cup\bigcup\{B_n^d(L)\colon d\in D_n^L,n\in\mathbb{N}\}.$$

Both the sets in the domain and range of φ are Z-sets in Q'^{-} since these are closed subsets of the Z-set $Q' \times \{0\}$ [vM01, Lemma 5.1.2, Corollary 5.1.5]. Hence φ can be extended to a homeomorphism $\varphi' \colon Q' \to Q'$ [vM01, Theorem 5.3.7]. It remains to extend φ' linearly on the cones to obtain a homeomorphism φ''. It follows that φ'' conjugates f_K and f_L. To verify that the mapping $\mathcal{K}(Q) \to \mathcal{K}(Q'' \times Q'')$, $K \mapsto f_K$ is Borel is straightforward and it is left to the reader.

To conlude the proof it is enough to Borel reduce conjugacy of Hilbert cube maps to E_{G_∞}. Consider structures of the form (Q, R) where R is a closed binary relation on Q. Two such structures (Q, R) and (Q, S) are said to be isomorphic if there is a homeomorphisms $\psi \colon Q \to Q$ for which $(\psi \times \psi)(R) = S$. By a fairly more general result [RZ18] it follows that such isomorphism equivalence relation is Borel reducible to E_{G_∞}. There is a Borel (even continuous) reduction which takes a continuous map $f \colon Q \to Q$ and assigns $(Q, \mathrm{graph}(f))$ to it. Combining the two reductions we get the desired one. $\square$

5 Concluding remarks and questions

Let us summarize some of the results on the complexity of conjugacy equivalence relation in Table 1 in which we consider conjugacy equivalence relation of maps, homeomorphisms, and pointed transitive homeomorphisms of the arc, circle, Cantor set and Hilbert cube, respectively. Let us recall that a *pointed dynamical system* is a triple (X, f, x), where (X, f) is a dynamical system and $x \in X$. We say, that a pointed dynamical system (X, f, x) is *transitive* if the forward orbit of x in (X, f) is dense. Two pointed dynamical systems (X, f, x) and (Y, g, y) are called *conjugate* if there is a conjugacy of (X, f) and (Y, g) mapping x to y. We proceed by a series of simple notes as comments on Table 1.

	Homeomorphisms/maps	Pointed transitive homeomorphisms
Arc	E_{S_∞} [Hjo00], Theorem 16	$\emptyset$ Note 22
Circle	E_{S_∞} Note 23	$E_=$ Note 24
Cantor set	E_{S_∞} [CG01]	$E_{=^+}$ [Kay17b], Note 25
Hilbert cube	E_{G_∞} Theorem 20	? Question 26

Table 1: The complexity of conjugacy equivalence relation.

Note 21. Conjugacy of pointed transitive maps of the interval is smooth; indeed it is enough to assign to every pointed transitive dynamical system (I, f, x) the $\mathbb{N} \times \mathbb{N}$ matrix of true and false: $(f^m(x) < f^n(x))_{m,n \in \mathbb{N}}$ which determines f uniquely up to increasing conjugacy.

Note 22. There are no transitive homeomorphisms on the arc.

Note 23. The complexity result by Hjorth [Hjo00, Section 4.2] that conjugacy of interval homeomorphisms is Borel bireducible to E_{S_∞}, remains true for circle homeomorphisms simply by a modification of the original proof. A modification of the proof of Theorem 16 for circle maps is also possible and thus conjugacy of circle homeomorphisms is Borel bireducible to the S_∞-universal orbit equivalence relation.

Note 24. Transitive homeomorphisms of the circle are well known to be conjugate to irrational rotations. Hence the rotation number is a complete invariant and hence conjugacy of (pointed) transitive homeomorphisms of the circle is Borel bireducible to the equality on irrationals (or on an uncountable Polish space).

Note 25. By a result of Kaya [Kay17b], conjugacy of pointed minimal homeomorphisms of the Cantor set is Borel bireducible to the equality of countable sets $E_{=^+}$. Note that his proof works in the same vein for pointed transitive homeomorphisms of the Cantor set. Let us recall the main part of his construction in this case. Let X be the Cantor set and $\mathcal{B}$ the collection of all clopen sets in X. To a pointed transitive system (X, f, x) we assign the collection

$$\mathsf{Ret}(f, x) = \{\mathsf{Ret}_\mathsf{B}(f, x) \colon B \in \mathcal{B}\},$$

where $\mathsf{Ret}_\mathsf{B}(f, x) = \{n \in \mathbb{Z} \colon f^n(x) \in B\}$. It can be verified that the mapping Φ defined as

$$\Phi(f, x) = (\mathsf{Ret}_\mathsf{B}(f, x) \colon B \in \mathcal{B}) \in \mathcal{P}(\mathbb{Z})^\mathcal{B}$$

is a reduction of pointed transitive Cantor maps to the equality of countable sets in $\mathcal{P}(\mathbb{Z})^\mathcal{B}$, i.e., (f, x) is conjugate to (g, y) if and only if $\mathsf{Ret}(f, x) = \mathsf{Ret}(g, y)$.

The following question is the missing part to complete Table 1.

Question 26. What is the complexity of conjugacy of transitive pointed Hilbert cube homeomorphisms (or maps)?

The most natural candidate for the complexity of conjugacy of transitive pointed Hilbert cube homeomorphisms (or maps) is the universal orbit equivalence relation.

Since triangular maps i.e., maps $f \colon I^2 \to I^2$ of the form $f(x, y) = (g(x), h(x, y))$ for continuous maps $g \colon I \to I$ and $h \colon I^2 \to I$, lie in between one-dimensional and two dimensional and there is a gap in the complexity of the last two mentioned equivalence relations, the following question is natural.

Question 27. What is the complexity of conjugacy of triangular maps? Is it Borel bireducible to E_{S_∞} or to E_{G_∞}?

Positive answer to the next question would provide a strengthening of Theorem 16.

Question 28. Is conjugacy of closed binary relations on the closed interval Borel reducible to the S_∞-universal orbit equivalence relation?

The answer to the preceding question is affirmative if Hjorth's conjecture is true.

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