ROTATED ODOMETERS

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Abstract. We describe the infinite interval exchange transformations, called the rotated odometers, that are obtained as compositions of finite interval exchange transformations and the von Neumann-Kakutani map. We show that with respect to Lebesgue measure on the unit interval, every such transformation is measurably isomorphic to the first return map of a rational parallel flow on a translation surface of finite area with infinite genus and a finite number of ends. We describe the dynamics of rotated odometers by means of Bratteli-Vershik systems, derive several of their topological and ergodic properties, and investigate in detail a range of specific examples of rotated odometers.

1. Introduction

In this paper, we consider a family of infinite interval exchange transformations (IETs) which arise as perturbations of the von Neumann-Kakutani map of the unit interval, and as the first return maps of flows of rational slope on certain flat surfaces of infinite genus. We study the dynamical and ergodic properties of the maps in this family.

Each map in this family has a unique aperiodic minimal subsystem, and thus this family presents a class of naturally arising systems with this property. We show that each aperiodic (not necessarily minimal) subsystem is measurably isomorphic to a Bratteli-Vershik system on a Cantor set, and study its ergodic invariant measures and the spectrum of its Koopman operator. We construct infinitely many examples, where each minimal set has the dyadic odometer as a factor, and infinitely many examples where each minimal set does not have the dyadic odometer as a factor, and it is not weakly mixing.

Rotated odometers. The von Neumann-Kakutani map $\mathfrak{a} : I \to I$, where is $I = [0, 1)$ is the half-open unit interval, is given by the formula

$$\mathfrak{a}(x) = x - (1 - 3 \cdot 2^{1-n}) \qquad \text{if } x \in I_n = [1 - 2^{1-n}, 1 - 2^{-n}),\ n \geq 1. \tag{1}$$

In words, it re-arranges the partition of I into the countable number of intervals $I_1, I_2, \dots$ in the opposite order, see Figure 3(a). We divide the interval $I = [0, 1)$ into q half-open subintervals of length $\frac{1}{q}$, and we let π be any permutation of q symbols. Let $R_\pi : I \to I$ to be the map which permutes the q subintervals of equal length according to π. Then

$$F_\pi = \mathfrak{a} \circ R_\pi : I \to I \tag{2}$$

is an infinite IET called a *rotated odometer*. The term 'rotated odometer' was introduced since for some permutations π, the map $R_\pi : I \to I$ may be seen as a rotation on the unit circle.

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Infinite genus surfaces. Let S be the unit square, and identify its vertical edges by a single translation (as if creating a cylinder), and its horizontal edges by the von Neumann-Kakutani map $\mathfrak{a}$, to obtain the surface L, see Section 2 for details. In order to make the identifications work we must remove a countable number of points from the horizontal edges, and as a result the surface L is non-compact. The removed points are identified into a single point, therefore the resulting surface has a single *end*, i.e., a distinct way to go to infinity, see Figure 1. The surface L has unit area, and infinite genus. Topological surfaces of this type (with a different metric) are called *Loch Ness monsters*, and they have appeared in the literature as leaves in foliations by surfaces [26, 16]. Loch Ness monsters also appear as translation surfaces with infinite angle or wild singularities, such as the *Chamanara* or the *baker's surface* [8, 28] or the *infinite staircase* [8]. An interesting family of infinite-type translation surfaces was constructed in [22]. The constructions of the families of surfaces in [22] and in our paper are reminiscent of that of the Chamanara surface in [8]. However, in our paper the Loch Ness monsters lack certain metric symmetries which are present in [8], and so the methods used to study the properties of the latter in [8], are not applicable in our case.

Consider the flow lines on the square S which are at the constant angle $\theta = \tan^{-1}(q/p)$ with the horizontal, where $p, q \in \mathbb{Z} \setminus \{0\}$. When the flow lines traverse the square from the bottom to the top, they travel through the horizontal distance p/q. Therefore, the first return map to the horizontal section in the surface L, corresponding to the horizontal edges of S, is the composition of a translation by p/q and the von Neumann-Kakutani map $\mathfrak{a}$, i.e., a rotated odometer. Then a natural question is, can any rotated odometer (2), i.e., for an arbitrary permutation π, be realized as the first return map of a flow on a Loch Ness monster? Our first Theorem 1.1 below states that the answer is yes, provided we can make mild modifications to the topology of the surface.

We denote one-dimensional Lebesgue measure by λ.

Theorem 1.1. *Let $q \geq 2$ and let π be a permutation of q symbols, and let $p \geq q$ be an integer. Then there exists a translation surface $L_{\pi,p}$ obtained by identifying by translations the sides of the unit square with countable number of boundary points removed, which has the following properties:*

(1) *The surface $L_{\pi,p}$ has finite area, one non-planar end and at most a finite number of planar ends.*
(2) *The metric completion of $L_{\pi,p}$ contains a single wild singularity and at most a finite number of cone angle singularities.*
(3) *There exists a section $P \subset L_{\pi,p}$ parallel to the horizontal edge of the unit square with Poincaré map $F : P \to P$ of the flow of rational slope q/p, such that (P, F, λ) is measurably isomorphic to the rotated odometer (I, F_π, λ).*

The technical notions in the statement of Theorem 1.1 are explained rigorously in Section 2, where this theorem is proved. Intuitively, singularities in this theorem correspond to the points we must remove from S when identifying edges in order to obtain a surface where every point has a Euclidean neighborhood. Distinct removed points may be identified into a single singularity. Each singularity results in a puncture in the surface L, and each puncture corresponds to an end of L. The notions of a planar or a non-planar end describe the topology of a neighborhood of an end, namely, if an end is non-planar, then every such neighborhood has infinite genus.

A finite area surface with infinite genus, one non-planar end and two planar ends is depicted in Figure 2. We call a Loch Ness monster with additional planar ends a *Loch Ness monster with whiskers*. A surface of this type is described in Example 2.3.

Dynamical properties of rotated odometers. We now study in detail the dynamics of the rotated odometer (I, F_π, λ) for any $q \geq 2$ and any permutation π on q symbols. Such a map can be considered as a perturbation of the von Neumann-Kakutani map $\mathfrak{a}$. From this point of view, it

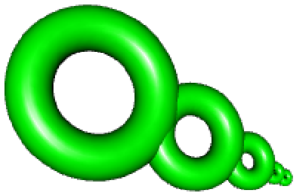

FIGURE 1. The finite area Loch Ness monster

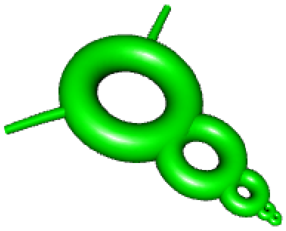

FIGURE 2. The Loch Ness monster with two whiskers

is natural to ask, which properties of the von Neumann-Kakutani map are preserved under such perturbation. We show that even in this highly controlled setting, much of the inner structure of the von Neumann-Kakutani system can be destroyed by a perturbation, although some features are preserved.

The first basic result is that the minimality of $\mathfrak{a}$ may be destroyed, but the minimal subset of the aperiodic subsystem is always unique. Recall that $I = [0, 1)$.

Theorem 1.2. *There exists a decomposition $I = I_{per} \cup I_{np}$ with the following properties:*

(1) *Every point in I_{per} is periodic, the restriction $F_\pi : I_{per} \to I_{per}$ is well-defined and invertible.*
(2) *If I_{per} is non-empty, then I_{per} is a (possibly infinite) union of half-open intervals $[x, y)$.*
(3) *The set I_{np} contains 0, and $F_\pi : I_{np} \to I_{np}$ is well-defined and is invertible at every point except 0.*
(4) *There is a unique minimal subsystem (I_{min}, F_π) of (I_{np}, F_π), and $0 \in I_{min}$.*

Theorem 1.2 is proved in Section 3. Examples 5.17, 5.21 and 5.22 show that I_{per} may be an empty set, or a finite or infinite union of half-open intervals. An infinite IET which contains an infinite collection of intervals of periodic points was also considered in [18], see Example 3.5.

Remark 1.3. The existence of a unique minimal aperiodic set imposes strong restrictions on the behavior of the system. For instance, as shown in [14], certain C^*-algebras associated to systems with unique minimal sets on zero-dimensional spaces can be exhibited as cross products of an abelian C^*-algebra by a single homeomorphism. We refer the reader to [14] for more on C^*-algebras and dimension groups in this setting. The rotated odometers considered in this paper, provide a naturally arising family of examples of dynamical systems with unique minimal sets; systems with this property are not readily found in the literature. This is another motivation to study rotated odometers.

Ergodic properties of rotated odometers. The rotated odometer map F_π acts on I by piecewise translations and so preserves Lebesgue measure λ on I. Moreover, in Section 7 we prove:

Theorem 1.4. *Lebesgue measure is ergodic for (I, F_π) if and only if there are no periodic points.*

One implication in Theorem 1.4 is immediate, the other one requires work. Another natural property of all rotated odometers is that they have zero topological entropy. The proof of Theorem 1.5 below can be found in Section 6.

Theorem 1.5. *For any $q \geq 1$ and any permutation π of q symbols, the topological entropy $h_{top}(F_\pi) = 0$.*

To further study the dynamics of the aperiodic subsystem (I_{np}, F_π) we use the standard technique of doubling points in the orbits of discontinuities to embed (I_{np}, F_π) into a dynamical system (I^*_{np}, F^*_π) given by a homeomorphism F^*_π of a Cantor set I^*_{np}. The procedure is described in detail in Section 4, with main results summarized in Theorem 4.1. Since the Cantor set I^*_{np} is obtained by adding to I_{np} a countable collection of points, there is a correspondence of invariant measures on the Cantor system and on (I_{np}, F_π). We show in Section 5 that the Cantor system (I^*_{np}, F^*_π) is conjugate to the Bratteli-Vershik system on an eventually stationary Bratteli diagram, see Theorem 5.11.

Bratteli-Vershik systems are a powerful tool to study the dynamics of maps of Cantor sets, described in many sources, see for instance [2, 1, 14] and references therein. In the rest of the paper, we use Bratteli-Vershik systems to study the number of ergodic invariant measures on (I_{np}, F_π), and the discrete spectrum of the Koopman operator for different ergodic measures.

Since the Bratteli-Vershik diagram conjugate to (I^*_{np}, F^*_π) is associated to an eventually periodic sequence of substitutions on q letters (see Section 5), we have the following result.

Theorem 1.6. *For any $q \geq 1$ and any permutation π of q symbols, the aperiodic subsystem (I_{np}, F_π) admits at most q ergodic invariant measures, and its unique minimal subsystem (I^*_{min}, F^*_π) is uniquely ergodic.*

Factors of rotated odometers. The next series of results is motivated by considering rotated odometers as permutations of the von Neumann-Kakutani map, which is known to be measurably isomorphic to the dyadic odometer. We know from Theorem 1.2 that the dynamical characteristics of the dyadic odometer, such as, for instance, minimality, may be destroyed by a perturbation to a rotated odometer. Therefore, it is natural to ask, whether they are preserved at least at the level of factors, i.e., whether the dyadic odometer is still a measure-theoretical or a topological factor of the rotated odometer. We ask this question for both the aperiodic system (I^*_{np}, F^*_π) and for its unique minimal set (I^*_{min}, F^*_π).

Theorem 1.7. *Let (I^*_{min}, F^*_π) be the minimal subset of (I^*_{np}, F^*_π). Then:*

(1) *There exist infinitely many $q \geq 3$, and permutations π of q symbols, such that the minimal system (I^*_{min}, F^*_π) has a dyadic odometer as a factor.*
(2) *There exist infinitely many $q \geq 3$, and permutations π of q symbols, such that the minimal system (I^*_{min}, F^*_π) does not factor to a dyadic odometer, and is not weakly mixing.*

Theorem 1.7 is proved in Section 8.5.

Host [19] proved that for substitution shifts, the measure-theoretical and topological rotational factors coincide, and so in Theorem 1.7 by a factor we mean either of them. The measure implicitly used in this theorem is the unique ergodic measure supported on the minimal set.

In both statements of Theorem 1.8 below, Lebesgue measure on I is ergodic for the rotated odometer (I, F_π), and $I_{np} = I$. The measure λ is the pushforward of the Lebesgue measure on I_{np} to I^*_{np} by the embedding map.

Theorem 1.8. *Let (I^*_{np}, F^*_π) be the aperiodic subsystem of a rotated odometer. Then:*

(1) *If $q = 5$ and $\pi = (01234)$, then the rotated odometer $(I^*_{np}, F^*_\pi, \lambda)$ has the dyadic odometer as the maximal equicontinuous factor, and the factor map is continuous.*
(2) *If $q = 5$ and $\pi = (02431)$, then the rotated odometer $(I^*_{np}, F^*_\pi, \lambda)$ has the cyclic group of order 4 as the maximal equicontinuous factor, but the factor map is not continuous.*

Theorem 1.8 is proved in Section 8.5.

Open problems. We show in Theorem 1.1 that such systems can be considered as first return maps of flows of rational slope, on certain translation surfaces of finite area and infinite genus with finite number of ends. The following question is natural.

Problem 1.9. *Find a Bratteli-Vershik system that models the Poincaré map of a flow of irrational slope on a translation surface of finite area with infinite genus and finite number of ends.*

The next open problem stems from Theorem 1.7 whose proof is constructive. At the moment we are not aware of a general condition which would ensure that a rotated odometer or its minimal set has, or does not have, the dyadic odometer as a factor. We pose this as an open question.

Problem 1.10. *Let $F_\pi = \mathfrak{a} \circ R_\pi : I \to I$ be a rotated odometer. Find necessary and sufficient conditions under which (I^*_{min}, F^*_π) has a dyadic odometer as a factor.*

In the system described in Statement 2 of Theorem 1.8, neither the minimal subsystem with respect to the unique ergodic measure, nor the aperiodic system (I^*_{np}, F^*_π) with respect to Lebesgue measure, have the dyadic odometer as factor. Instead, the minimal subsystem has an irrational eigenvalue, while the full rotated odometer factors onto the cyclic group with four elements. Therefore, the following question is natural.

Problem 1.11. *Are there any examples in our class of rotated odometers for which the minimal subsystem (I^*_{min}, F^*_π), or the aperiodic system (I^*_{np}, F^*_π) is weakly mixing?*

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2. Rotated odometers and flows on translation surfaces

In this section we recall the basic properties of infinite translation surfaces and prove Theorem 1.1.

2.1. **Loch Ness monsters.** Consider the unit square S without corner points. The upper and the lower sides of S are identified with the interior $(0,1)$ of $I = [0,1)$. The vertical sides of S are identified with $J = (0,1)$. We make the identification

$$(x, 1) \sim_h (\mathfrak{a}(x), 0),$$

see Figure 3(a), where $I_k = \left[1 - \frac{1}{2^{k-1}}, 1 - \frac{1}{2^k}\right)$, $k \geq 1$, are the intervals of continuity of $\mathfrak{a}$. We identify the vertical sides using the equivalence relation $(1, y) \sim_v (0, y)$ as in the standard torus.

Consider the set of discontinuity points of $\mathfrak{a}$ in the upper horizontal edge, and of their images under $\mathfrak{a}$ on the lower horizontal edge, given by

$$D = \{(1 - 2^{-k}, 1), (\mathfrak{a}(1 - 2^{-k}), 0) : k \geq 1\}.$$

Define the non-compact surface L by applying the equivalence relations,

$$L = (S \setminus D)/\sim_h, \sim_v . \tag{3}$$

The construction of L above is similar to that of the *Chamanara surface* in the literature [8, 28], except that the identification of the vertical sides of the square S in the Chamanara surface is done using the von Neumann-Kakutani map.

Non-compact surfaces are classified up to a homeomorphism by their genus and the number of ends. Intuitively, an *end* of a surface is a distinct way to go to infinity in the surface. Adding ends to a surface can be considered as its compactification [8, 5]. A surface with one end and infinite genus can be pictured as the Euclidean plane with an infinite number of handles attached, and this is the reason it was called the *Loch Ness monster* in [26]. The Euclidean plane has infinite area. The

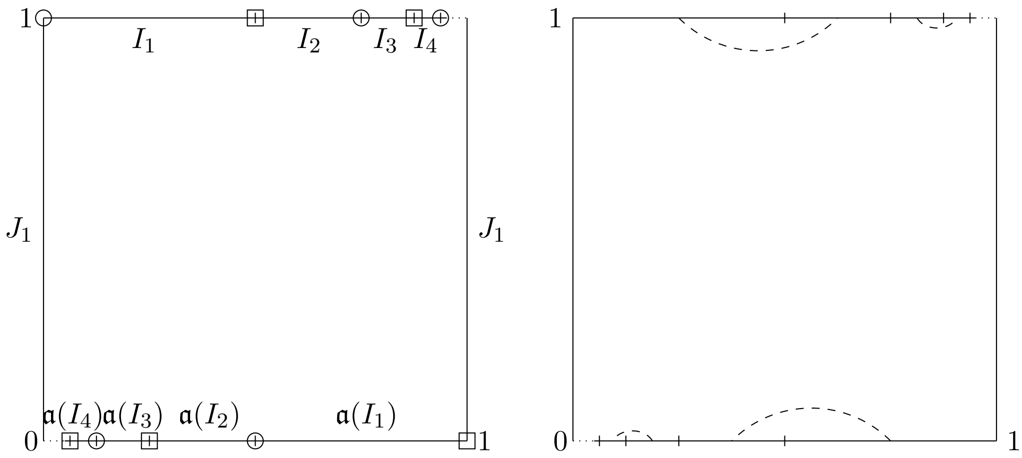

FIGURE 3. (a) Identifications of the horizontal sides of the unit square by the von Neumann-Kakutani map, and of the vertical sides by translations. Circles and squares represent identifications of limit points in $\overline{L}$. (b) Non-separating curves in L are represented by dashed lines.

surface in Figure 1 is homeomorphic to the plane with an infinite number of handles attached, and it has unit area.

An open neighborhood of an end is a surface in its own right, and so one can talk about the genus of this surface. An end e is *planar* if its has a neighborhood of genus zero, and e is called *non-planar* otherwise. The single end of the Loch Ness monster in Figure 1 is non-planar.

Arguments similar to the one for the Chamanara surface in [28] show that L is a translation surface of finite area and infinite genus with one non-planar end. We sketch the proof in Proposition 2.1 for completeness.

Let $\sigma \in \overline{L}$ be a singularity, and let B_ϵ be an open neighborhood of σ in $\overline{L}$ of radius $\epsilon > 0$. A singularity σ is *wild*, if there is no finite or infinite cyclic translation covering from $B_\epsilon \setminus \sigma$, to a once-punctured disc $B(0, \epsilon) \setminus \{0\} \subset \mathbb{R}^2$ for any $\epsilon > 0$. Recall that a *saddle connection* is a geodesic in $\overline{L}$ which joins two not necessarily distinct singularities in $\overline{L}$, and which does not contain a singularity in its interior. In particular, σ is wild if for any $\epsilon > 0$ the neighborhood B_ϵ contains an infinite number of saddle connections.

A closed curve $\gamma : [0, 1] \to L$ is *non-separating* if $L \setminus \gamma([0, 1])$ is connected. A surface L has genus g, if the maximum cardinality of a set of disjoint non-separating curves in L is g. If L admits an infinite number of disjoint non-separating curves, then it has infinite genus.

Proposition 2.1. *The surface L is a translation surface of finite area and infinite genus, and the metric completion $\overline{L}$ of L contains a single wild singularity. Thus L is a Loch Ness monster, that is, L is an infinite genus surface with one non-planar end.*

Proof. Recall from [28] that a translation surface is a surface which admits an atlas where the transition functions are locally translations. The surface L in (3) satisfies this definition since $\mathfrak{a}$ is an IET and so identifications are by translations. The metric completion $\overline{L}$ of L is clearly compact, and we claim that $\overline{L} \setminus L$ is a single point. To see that we use a similar argument to the one in [28, Example 1.15] for the Chamanara surface. Namely, the identifications of the horizontal edges by $\mathfrak{a}$ induce the identification between the limit points in the metric completion marked by circles and squares in Figure 3(a), and we conclude that there is no more than two points $\overline{L} \setminus L$, one marked by circles and another one by squares. Since the distance between these two points is not bounded away from zero, they are the same point, and $\overline{L} \setminus L$ consists of a single singularity denoted by σ. We notice that for every $\epsilon > 0$, the part of the neighborhood B_ϵ of σ near the corner points of S contains an infinite number of saddle connections (horizontal segments on the upper and the lower edges of the square). Thus σ is a wild singularity.

To see that L has infinite genus, for $k \geq 1$, let γ_k be a closed curve joining the middle point of the interval I_{2k} to the middle point of the interval I_{2k-1} lying below the upper horizontal edge of L, and then joining the middle point of $\mathfrak{a}(I_{2k-1})$ with the middle point of $\mathfrak{a}(I_{2k})$ and passing above the lower horizontal edge of L, see Figure 3(b). All such curves are disjoint. To see that the complement of γ_k in L is connected, note that every point in $L \setminus \gamma_k([0,1])$ is connected to the middle point of the square by a continuous path. This shows that L admits an infinite number of non-separating curves, and so has infinite genus. To show that L has one end we refer to [28, Proposition 3.10], where it is shown that if L is a translation surface such that the metric completion $\overline{L}$ is compact, then the space of ends of L is finite and it is in one-to-one correspondence with the set of singularities. Since L has a single singularity, it has one end. It is clear from the picture and the arguments above that every ϵ-neighborhood of σ contains an infinite number of non-separating curves, which implies that the single end of the surface L is non-planar. ☐

2.2. **Flows on the Loch Ness monster.** Every point $x \in L$ is contained in a chart of a maximal atlas whose transition maps are translations. Thus the tangent bundle $TL = \bigcup_{x \in L} T_xL$ carries a flat connection. For any $x \in L$ the exponential map $\exp_x : T_xL \to L$ is well-defined on an open neighborhood of 0 in T_xL depending on x. For any angle $\theta \in \mathbb{S}^1$ there is a vector field $\mathcal{X}_\theta$ on L, whose flow lines are straight lines which make the angle θ with the horizontal. Some flow lines of $\mathcal{X}_\theta$ reach the singular point σ in finite time, and so they are not defined for all $t \in \mathbb{R}$. Since the map $\mathfrak{a}$ has a countable discontinuity set, there is at most a countable number of such flow lines. Let L_θ be the union of flow lines that are defined for all $t \in \mathbb{R}$, and denote the flow by $\varphi_\theta : \mathbb{R} \times L_\theta \to L_\theta$.

Let $q \geq 2$ and $p \geq 1$ be integers, and let $\theta = \tan^{-1}(q/p)$. Let P be the image of the lower horizontal edge of the unit square under (3) in L_θ. Then P is a Poincaré section for the flow lines of the vector field $\mathcal{X}_\theta$, and we denote the Poincaré map of the flow by $F : P \to P$.

Consider the lift of the flow to the unit square S, and identify a copy of P with the lower edge, and another copy of P with the upper edge. For any $x \in P$ in the lower edge, consider the flow line through x. While traveling from the lower edge to the upper edge, the flow line through x also moves in the horizontal direction by distance p/q, possibly traversing S a few times. Divide the unit interval I into q subintervals of equal length, this induces the subdivision of P. Since p is an integer, the flow maps the subintervals in P onto subintervals in P, inducing a permutation π of a set of q symbols. For example, if $q = 3$ and $p = 1$, then the corresponding permutation is $\pi = (012)$ and if $p = 2$ then the corresponding permutation is $\pi = (021)$. It follows that the return map F can be described as the composition $\mathfrak{a} \circ R_\pi$ of a permutation of q intervals and the von Neumann-Kakutani map. To illustrate this visually, in Figure 3(a) choose a point on the horizontal edge of the square, apply R_π obtaining another point on the lower horizontal edge, and then traverse the square in the vertical direction. When the vertical flow line reaches the upper horizontal edge of the square, apply the von Neumann-Kakutani map $\mathfrak{a}$ and return to P using the identification (3).

We give the section P a measure λ induced from the Lebesgue measure on the interval $I = [0, 1)$. Since only a countable number of flow lines reach the singularity σ in $\overline{L}$, the discussion above results in the following statement. Denote by $j : P \to I = [0, 1)$ the inclusion map.

Proposition 2.2. *Let $q \geq 2$, $p \geq 1$ be integers, and let $F : P \to P$ be the Poincaré map of the flow φ_θ, where $\theta = \tan^{-1}(q/p)$. Then there exists a permutation π of q symbols, such that $j : P \to I$*

induces a measure-theoretical isomorphism of the dynamical systems (P, F, λ) *and* (I, F_π, λ)*, where* $F_\pi = \mathfrak{a} \circ R_\pi$ *is a rotated odometer and* λ *is the Lebesgue measure.*

2.3. **Rotated odometers and flows.** We now prove Theorem 1.1, that is, we show that any rotated odometer is measure-theoretically isomorphic to a Poincaré map of a flow of rational slope on an infinite-type translation surface of finite area. The topology of this surface may be slightly more complicated than that of L, since in order to obtain the given permutation π we may have to apply additional identifications on the vertical sides. As a result, a finite number of cone angle singularities may arise, which we will have to remove from L, creating additional planar ends.

We call the surface with infinite genus and one non-planar and a finite number of planar ends the *Loch Ness monster with whiskers*, see Figure 2.

Proof of Theorem 1.1. To obtain a given permutation of the intervals $\{I_i\}_{i=0}^{q-1}$ we divide the vertical sides of the square S into subintervals of equal length and permute them. For us to be able to do that, all flow lines starting at the horizontal edge of S must intersect the vertical side, so the angle θ must be less than $\frac{\pi}{4}$, so $\tan(\theta) < 1$. Take any $p = mq + r$ with $m \geq 1$ and $0 \leq r \leq q-1$ and let $\theta = \tan^{-1}(\frac{q}{p})$.

Consider $m+2$ copies of S such that for $1 \leq i \leq m+1$ the right vertical edge of the i-th copy is identified with the left vertical edge of the $i+1$-st copy. Let $\{I_k\}_{k=0}^{q-1}$ be the subdivision of the lower horizontal edge into intervals of length $\frac{1}{q}$, and $\{J_k\}_{k=0}^{p-1}$ and $\{J'_k\}_{k=0}^{p-1}$ be the subdivisions of respectively the left and the right vertical edges of the first copy of S into intervals of length $\frac{1}{p}$. The numbering of the intervals in the vertical edges increases from bottom to top.

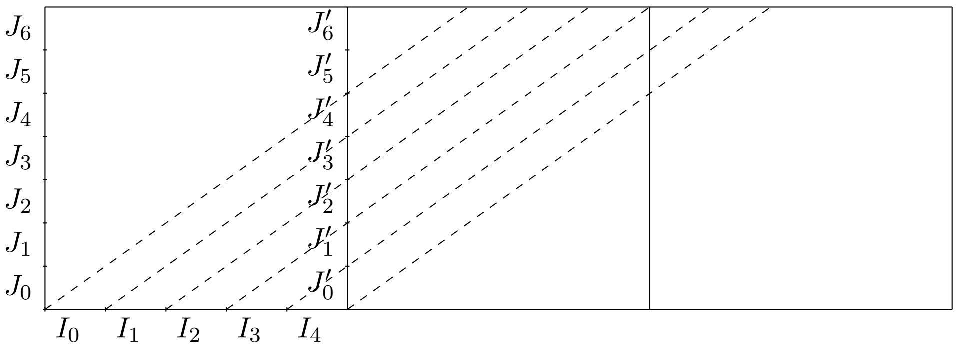

FIGURE 4. The flow for the rotation by $\frac{7}{5}$, so $q = 5$, $p = 7$ and $r = 2$; if the vertical edges are identified by a single translation, like in the torus, then the corresponding permutation of $\{I_k\}_{k=0}^{4}$ is $\pi = (02413)$. Identifications of the vertical sides using a non-trivial permutation π' of $\{J'_k\}_{k=0}^{6}$ may result in a different π. When glueing the vertical edges using a non-trivial π', we have to remove the endpoints of the intervals in $\{J'_k\}_{k=0}^{6}$ from S, possibly creating planar ends in the resulting surface.

Each flow line of φ_θ intersects at least $m+1$ and at most $m+2$ copies of S before reaching the upper horizontal edge. For example, in Figure 4 the flow lines of the points in the intervals I_3 and I_4 intersect three copies of S, while the flow lines of the points in I_0, I_1, I_2 intersect only two copies of S. The flow lines intersect only the first q elements $\{J'_k\}_{k=0}^{p-1}$, and so the intersection of the flow lines with the right vertical edge of the first copy of S defines a map

$$s : \{0, 1, \ldots, q-1\} \to \{0, 1, \ldots, p-1\}, \quad i \mapsto k = q - 1 - i$$

with range $\{0, \ldots, q-1\}$. There is the partial inverse $s^{-1} : \{0, 1, \ldots, q-1\} \to \{0, 1, \ldots, q-1\}$. The intersection of the flow lines with the upper horizontal edge followed by the identification of the copies of S as in the standard torus defines the map

$$t : \{0, 1, \dots, q-1\} \to \{0, 1, \dots, q-1\}, \quad i \mapsto i + r \bmod q.$$

We set

$$\pi' = s \circ t^{-1} \circ \pi \circ s^{-1}$$

and define the identifications of the intervals of the partitions $\{J_k\}_{k=0}^{p-1}$ and $\{J'_k\}_{k=0}^{p-1}$ by

$$J_k \sim \begin{cases} J_{\pi'(k)} & \text{if } 0 \leq k \leq q-1, \\ J_k & \text{if } q \leq k \leq p-1. \end{cases} \tag{4}$$

In words, we permute the lowest q intervals in the partition $\{J_k\}_{k=0}^{p-1}$ of the vertical side of S, and we keep the top $p-q$ intervals not permuted.

Build a surface $L_{\pi,p}$ as in Section 2.1, but identify the vertical sides of S using (4). Then flowing from the lower to the upper horizontal edge in S produces the permutation π on the intervals of the subdivision of the horizontal edge, and it follows that the Poincaré map $F : P \to P$ of the flow φ_θ constructed similarly to the one in Section 2.2, is measure-theoretically isomorphic to the rotated odometer (I, F_π).

It remains to check that the surface $L_{\pi,p}$ which is obtained from L by applying the identifications of subintervals on vertical edges is still a translation surface with a wild singularity. Some of the upper endpoints of the intervals $J_0, \dots, J_{p-2}$ may be identified with the singularity σ, the remaining ones are identified with each other. Since there is a finite number of them, this results in at most a finite number of additional cone angle singularities of the surface L, which are isolated from σ. The number of saddle connections and the number of non-separating curves that the surface can admit remains infinite, so the resulting surface is a translation surface with a wild singularity and of infinite genus. By [28, Proposition 3.10] the number of ends of $L_{\pi,p}$ is in one-to-one correspondence with the set of singularities of $L_{\pi,p}$, and by [28, Proposition 3.6] every cone angle singularity corresponds to a planar end. Therefore, $L_{\pi,p}$ is a Loch Ness monster with "whiskers", see Figure 2, where the number of whiskers corresponds to the number of cone angle singularities. □

Example 2.3. Let $q = 5$, and $\pi = (0)(1)(23)(4)$. Let $p = q$ then $\theta = \pi/4$, $r = 0$ and so t is the trivial permutation. Applying Theorem 1.1 we obtain $\pi_p = (0)(12)(3)(4)$. Considering the identification of vertical edges of the unit square given by π_p, one obtains that the metric completion of the surface $L_{\pi,p}$ has one wild singularity, one single cone angle singularity of multiplicity 3 and one removable cone angle singularity (i.e., of multiplicity 1). Thus $L_{\pi,p}$ has one non-planar and one planar end (we are not counting the removable singularity).

3. Periodic points and the unique minimal set

In this section we prove Theorem 1.2 which gives a basic description of the dynamics of periodic and non-periodic points of the rotated odometer (I, F_π), where $F_\pi = \mathfrak{a} \circ R_\pi$ is as in (2). Given an integer $q \geq 2$ and a permutation π of q symbols, $R_\pi : I \to I$ is a finite IET of q subintervals of I of equal length, determined by π.

Lemma 3.1. *The map $F_\pi = \mathfrak{a} \circ R_\pi : I \to I$ is invertible at every point in I except 0.*

Proof. The statement follows from the fact that the range of $\mathfrak{a}$ is $(0, 1)$, the range of R_π is $[0, 1)$, and both $\mathfrak{a}$ and R_π are translations on their intervals of continuity. □

We introduce a few notations which we use throughout the paper.

Definition 3.2. *Define*

$$N = \min\{n \in \mathbb{N} : 2^n > q\} \quad \textit{and} \quad L_k = [0, 2^{-kN}).$$

For fixed q and N, and any $k \geq 0$, we denote the partition of I into $q2^{kN}$ half-open subintervals of equal length by

$$\mathcal{P}_{kN,q} = \left\{ I_{k,i} := \left[\frac{i}{q2^{kN}}, \frac{i+1}{q2^{kN}} \right) : 0 \leq i \leq q2^{kN} - 1 \right\}.$$

In particular, $\mathcal{P}_{0,q}$ is the partition of I into q subintervals, and $I_{0,i} = I_i$, $i = 0, \ldots, q-1$, where I_i are the subintervals in Section 2.

The discontinuities set of $F_\pi = \mathfrak{a} \circ R_\pi$ is

$$D_0 = \{R_\pi^{-1}(1 - 2^{-n}) : n \geq 0\}. \tag{5}$$

Denote by D^+ the set of forward orbits of the points in D_0, and by D^- the set of backward orbits of $D_0 \setminus \{0\}$. Set $D = D_0 \cup D^+ \cup D^-$.

Lemma 3.3. *The set D is contained in the set $\left\{ \frac{p}{q2^n} : n \in \mathbb{N}, 0 \leq p \leq q2^n - 1 \right\}$.*

Proof. Note that the restriction of R_π to each interval in $\mathcal{P}_{0,q}$ is a translation by a rational number with denominator q, and $\mathfrak{a}$ acts by translations by multiples of 2^{-n}, $n \geq 1$, with n depending on a point in I. □

For a point $x \in I$, let $\mathrm{orb}^+(x)$, $\mathrm{orb}^-(x)$ and $\mathrm{orb}(x)$ be the forward, backward and two-sided orbit of x under F_π respectively. Clearly if $\mathrm{orb}^+(x)$ is periodic, then F_π is invertible at every point of $\mathrm{orb}^+(x)$ and so x has a two-sided periodic orbit.

As in the Introduction, we denote by I_{per} the set of points in I whose orbits under F_π are periodic, and by I_{np} the complement of I_{per} in I.

Proposition 3.4. *Consider the dynamical system (I, F_π). We have the following.*

(1) *For every $x \in I$, the forward orbit $\mathrm{orb}^+(x)$ is either periodic or accumulates at 0.*
(2) *The system (I_{np}, F_π) has the unique minimal set, denoted by I_{min}.*
(3) *If non-empty, the set of periodic points I_{per} is at most a countable union of half-open intervals with left and right endpoints in D.*

Proof. Let $x \in I$ and suppose $\mathrm{orb}^+(x) \cap L_k = \emptyset$ for some $k \geq 1$. We show that $\mathrm{orb}(x)$ is periodic.

Consider the partition $\mathcal{P}_{kN,q}$ of I from Definition 3.2. Let $L'_k = R_\pi^{-1}([1 - 2^{-kN}, 1))$ and note that $\mathrm{orb}^+(x)$ visits L_k if and only if it visits L'_k, since $\mathfrak{a}$ maps the interval $[1 - 2^{-kN}, 1)$ into L_k. Therefore, since by assumption $\mathrm{orb}^+(x)$ does not visit L_k, then it is contained in $I \setminus L_k \cup L'_k$. Note that the discontinuities of F_π in $I \setminus L_k \cup L'_k$ can only be at the left endpoints of the sets in $\mathcal{P}_{kN,q}$, and denote by

$$\mathcal{P}'_k = \{I_j \in \mathcal{P}_{kN,q} : I_j \cap \mathrm{orb}^+(x) \neq \emptyset\}$$

the collection of sets in $\mathcal{P}_{kN,q}$ visited by the orbit of x. Then the restriction $F_\pi|_{I_j}$ is continuous for each $I_j \in \mathcal{P}'_k$. Thus F_π permutes the intervals in $\mathcal{P}'_k$, and since there are only finitely many of them and F_π is injective, the orbit of x and so of each $I_j \in \mathcal{P}'_k$ is periodic. It follows that I_{per} is the union of half-open intervals.

If $\mathrm{orb}^+(x)$ is not periodic, then it must visit every L_k, $k \geq 1$, so $\mathrm{orb}^+(x)$ accumulates at 0. This proves parts (1) and (2).

We have seen in the proof above that I_{per} is the union of the intervals in the partitions $\mathcal{P}_{kN,q}$, for $k \geq 1$. We now claim that these intervals can assemble into larger half-open intervals of periodic points, so that the endpoints of these intervals are in D. Indeed, let x be such an endpoint, and suppose $x \notin D$. Then F_π is continuous at every point in $\mathrm{orb}(x)$, and since F_π is a translation on its intervals of continuity, x has an open neighborhood which consists of points periodic with the same period as x. This proves part (3). □

Examples 5.21 and 5.22 are rotated odometers with respectively countable and finite numbers of non-empty intervals of periodic points.

Proof of Theorem 1.2. Follows from Lemma 3.1 and Proposition 3.4. □

Example 3.5. We note the property of having an infinite collections of intervals of periodic points is also exhibited by infinite IETs which are not rotated odometers. For instance, in [18] half-open subintervals of I are rearranged in the manner similar to the von Neumann-Kakutani map, but the lengths of the subintervals are different the lengths in (1). Namely, $b : I \to I$ is given by

$$b(x) = x - 1 + k^{-1} + (k+1)^{-1} \quad \text{for } 1 - k^{-1} \leq x < 1 - (k+1)^{-1},\ k \in \mathbb{N}.$$

In this system, I_{per} is an infinite union of intervals where each point is periodic, and the complement of I_{per} is minimal.

4. Compactification to a Cantor system

In this section, we compactify the rotated odometer (I, F_π) to a dynamical system (I^*, F_π^*) given by a homeomorphism F_π^* of a Cantor set I^*, where $I = [0, 1)$ and F_π is defined by (2). The goal of this procedure is to get rid of discontinuities, and to do that we employ the well-known procedure of doubling the discontinuity points. Since I^* is obtained by adding to I a countable number of points, (I_{np}, F_π) and (I^*_{np}, F^*_π) are measurably isomorphic. We summarize the results of this section in the following theorem.

Theorem 4.1. *Let π be a permutation of $q \geq 2$ symbols, and let (I, F_π) be a rotated odometer. Let $I = I_{per} \cup I_{np}$ be a decomposition of I into the sets of periodic and aperiodic points.*

Then there exists an inclusion $\iota : I \to I^*$ into a compact space $I^* = I^*_{per} \cup I^*_{np}$, and a homeomorphism $F^*_\pi : I^* \to I^*$ with the following properties:

1. *The complement of I in I^* is countable, and $\iota(I_{per})$ and $\iota(I_{np})$ and contained in I^*_{per} and I^*_{np} respectively.*
2. *$F^*_\pi \circ \iota = \iota \circ F_\pi$.*
3. *I^*_{np} is a Cantor set, and every point in I^*_{np} is aperiodic under F^*_π.*
4. *There exists a measure μ on I^* such that (I, F_π, λ), where λ is Lebesgue measure, and (I^*, F^*_π, μ) are measurably isomorphic via the map ι.*

4.1. **Doubling the points.** Recall from Section 3 that we introduced the set $D = D_0 \cup D^- \cup D^+$, where D_0, defined by (5), is the set of discontinuities of the map F_π, and D^+ and D^- are the set of forward and backward orbits of the points in D_0 respectively. For each point in D we add a point as its double a point by setting

$$D^* = \{x^- : x \in D \setminus \{0\}\} \cup \{1\}.$$

Next, we consider the accumulation point of the discontinuity points. In the von Neumann-Kakutani map, the discontinuity points accumulate at 1. In the rotated odometer, the accumulation point is shifted by the inverse of R_π, so we denote

$$\widehat{x} := \lim_{y \nearrow 1} R_\pi^{-1}(y). \tag{6}$$

The point $\widehat{x}$ is either in the interior of I, or $\widehat{x} = 1$. The latter happens if and only if R_π fixes the last interval in the partition $\mathcal{P}_{0,q}$. If $\widehat{x}$ is in the interior of I, then $\widehat{x}$ is the double of the left endpoint of one of the intervals $\{R_\pi^{-1}(1-2^{-n}) : n \geq 0\}$. In both cases $\widehat{x}$ is in D^*.

Let $I^* = I \cup D^*$. To underline that the points in D are left endpoints of intervals of continuity, for all $x \in D$ denote $x^+ := x \in I^*$. The points in D^* are the added right limit points at the points of discontinuity. If $\widehat{x} = 1$, then set $\widehat{x}^- = \widehat{x} = 1$, and $\widehat{x}^+ = 0$.

4.2. **Order topology on I^*.** The interval $I \cup \{1\}$ has order $\leq$ induced from $\mathbb{R}$. We extend this order to an order on $I^* = I \cup D^*$ by defining:

(1) $x^- \leq x^+$ for all $x \in D$,
(2) for all $y \in I \cup \{1\}$ and all $x \in D$, if $y \leq x$ then $y \leq x^-$.

For all $x^+ \in D$ there are no points between x^- and x^+, so adding x^- to I can be thought of as creating a gap. We equip I^* with the order topology with open sets given by

$$\mathcal{B} = \{(a,b) : a,b \in I^*\} \bigsqcup \{[0,b) : b \in I^*\} \bigsqcup \{(a,1] : a \in I^*\}.$$

Lemma 4.2. *Let $x^+ \in D$ and $y^- \in D^*$ with $x^+ < y^-$. Then the subset $[x^+, y^-]$ of I^* is closed and open.*

4.3. **Extension to a homeomorphism of I^*.** We define an extension $F_\pi^* : I^* \to I^*$ to coincide with F_π on the points in I, and so by definition $F_\pi^* : D \to D$. It remains to define F_π^* on D^*.

For every $x^- \in D^* \setminus \{1\}$ there is $x^+ \in D$, such that $x^- = \lim_{y \nearrow x^+} y$. If $x^- \neq \widehat{x}^-$, that is, if x^- is not the accumulation point of discontinuities $\widehat{x}^-$ defined in (6), then x^+ has a half-neighborhood on the left where F_π^* is continuous, and we define

$$F_\pi^*(x^-) = \lim_{y \nearrow x^+} F_\pi(y).$$

We also set $F_\pi^*(\widehat{x}^-) = 0$, so we have for the forward orbit of $\widehat{x}^-$

$$\mathrm{orb}^+(\widehat{x}^-) = \{\widehat{x}^-\} \cup \mathrm{orb}^+(0).$$

If $\widehat{x}^- \neq 1$, then also set

$$F_\pi(1) = \lim_{y \to 1} F_\pi(y).$$

The above remarks are summarized in the following statement.

Proposition 4.3. *For the map $F_\pi^* : I^* \to I^*$ the following hold:*

(1) *F_π^* maps points of $D^* \setminus \{\widehat{x}^-\}$ to D^*, and points of D to D.*
(2) *F_π^* is a bijection, and $(F_\pi^*)^{-1}(D) \subset D$.*
(3) *$\widehat{x}^-$ is aperiodic under F_π^*.*

4.4. **Periodic and non-periodic points of the compactified system.** By Theorem 1.2 we have the decomposition $I = I_{per} \cup I_{np}$ of the unit interval into the F_π-invariant sets of periodic and non-periodic points respectively, and I_{per} is a countable (possibly empty) union of half-open intervals with endpoints in D.

Let $[x,y) \subset I_{per}$ with $x, y \in D$. Then there is $y^- \in D^*$ corresponding to y, and y^- is periodic under the extension of F_π to I^*. Define

$$I_{per}^* = \{[x^+, y^-] \subset I^* : [x,y) \subset I_{per},\ x,y \in D\}. \tag{7}$$

The following lemma is a direct consequence of the definitions.

Lemma 4.4. *We have the following properties:*

(1) I_{per} *is an open subset of* I^*_{per}.
(2) I^*_{per} *is an open subset of* I^*.
(3) I^*_{per} *is invariant under the map* F^*_π.

Let $I^*_{np} = I^* \setminus I^*_{per}$ be the complement, so $I^* = I^*_{np} \cup I^*_{per}$. Clearly $I_{np} \subset I^*_{np}$.

Proposition 4.5. *We have the following properties:*

(1) I^*_{np} *consists of aperiodic points.*
(2) I^* *is compact, and* I^*_{np} *is closed in* I^*. *Moreover,* $I^*_{np} = \overline{I_{np}}$, *the topological closure of* I_{np} *in* I^*.
(3) I^*_{np} *is invariant under the map* F^*_π.

Proof. To show (1), note that if $z \in I^* \setminus I_{per}$ is periodic, then it must be in D^*. Then $z \neq \widehat{x}^-$ and must be the right endpoint of an interval of continuity. Since F_π is a translation on its intervals of continuity, z is the right endpoint of an interval of periodic points, and so $z \in I^*_{per}$, and (1) and (3) follow. To show item (2), we note that I^* is a totally ordered set with order topology, so it is compact, and I^*_{np} is closed since I^*_{per} is open. □

4.5. Properties of the aperiodic subsystem (I^*_{np}, F^*_π).

Proposition 4.6. *We have the following.*

(1) *The set* D *is dense in* I_{np}.
(2) *The set* I^*_{np} *is a Cantor set.*
(3) *The restriction* $F^*_\pi : I^*_{np} \to I^*_{np}$ *is a homeomorphism.*

Proof. We have to show that every non-empty intersection $(x, y) \cap I_{np}$ contains a point of D. If (x, y) contains an interval of periodic points, then obviously $(x, y) \cap I_{np}$ contains a discontinuity, so assume that there are no periodic points in (x, y).

By item (1) of Proposition 3.4 every orbit that is not periodic accumulates at 0. If F^n_π is continuous on (x, y) for all $n \geq 1$, then the orbit of every point in (x, y) gets arbitrarily close to 0, which is impossible since F^n_π is a translation and so it preserves distances between the points in (x, y). Therefore, $F^n_\pi(x, y)$ must contain a discontinuity for some $n \geq 1$, and D is dense in I^*. It follows by standard arguments that I^*_{np} is a Cantor set and $F_\pi : I^*_{np} \to I^*_{np}$ is a homeomorphism. □

Define a measure μ on I^* by setting $\mu([x^+, y^-]) = y - x$ for every clopen set $[x^+, y^-] \subset I^*$, $x, y \in D$. Since D^* is countable, the following conclusion is straightforward.

Lemma 4.7. *The inclusion* $\iota : I \to I^*$ *induces a measurable isomorphism of dynamical systems* (I, F_π, λ) *and* (I^*, F^*_π, μ), *where* λ *is Lebesgue measure.*

Remark 4.8. Since the map $F_\pi : I \to I$ is not invertible at 0, the set I_{np} contains orb(0). A natural question for which we do not know the answer is if it is possible that the orbits of all discontinuity points in D_0, except for the orbit of 0, are periodic.

5. Bratteli-Vershik systems

In this section we construct a Bratteli-Vershik system which is conjugate to the aperiodic system (I^*_{np}, F^*_π) defined in Section 3. We start by recalling in Sections 5.1 and 5.2 the background on Bratteli-Vershik systems. The main technical result is stated in Theorem 5.11, which is then proved in Sections 5.4– 5.7.

5.1. **Substitutions.** We recall some standard constructions in symbolic dynamics, a good reference for which is the survey of Durand [11].

Let $\mathcal{A} = \{0, \dots, q-1\}$ be a finite alphabet, and let $\mathcal{A}^*$ be the set of all words of finite length in this alphabet, and let $\Sigma = \mathcal{A}^{\mathbb{N}}$ be the set of infinite sequences in this alphabet.

Definition 5.1. *A* substitution $\chi : \mathcal{A} \to \mathcal{A}^*$ *is a map which assigns to every $a \in \mathcal{A}$ a single word $\chi(a) \in \mathcal{A}^*$, and which extends to $\mathcal{A}^*$ and Σ by concatenation:*

$$\chi(b_1 b_2 \dots b_r) = \chi(b_1)\chi(b_2)\dots\chi(b_r),\ r \geq 1.$$

The $q \times q$ matrix M, where the i,j-th entry is the number of letters j in $\chi(i)$, is called the associated matrix *of χ.*

Definition 5.2. *We say that a substitution $\chi : \mathcal{A} \to \mathcal{A}^*$ is:*

- primitive*, if there is $r \geq 1$ such that for all $i \in \mathcal{A}$, $\chi^r(i)$ contains every letter in $\mathcal{A}$.*
- proper *if there exist two letters $a, b \in \{0, \dots, q-1\}$ such that for all $i \in \{0, \dots, q-1\}$, the first letter of $\chi(i)$ is a and the last letter of $\chi(i)$ is b.*

The substitutions we consider will sometimes be *primitive*, and they will always be *proper*.

If $\chi(a)$ starts with a, we get a fixed point of χ which is (unless $\chi(a) = a$) an infinite sequence

$$\rho = \rho_1\rho_2\rho_3 \dots = \lim_{j\to\infty} \chi^j(a) \in \Sigma. \tag{8}$$

For sequences $s = (s_k) \in \Sigma$, define the *left shift* by

$$\sigma : \Sigma \to \Sigma, \qquad s_1 s_2 \dots \mapsto s_2 s_3 \dots. \tag{9}$$

Consider the topological closure $X_\rho = \overline{\mathrm{orb}_\sigma(\rho)}$, where orb_σ is the orbit of ρ under σ. The dynamical system (X_ρ, σ) is called a *subshift.*

Definition 5.3. *The subshift (X_ρ, σ) is* linearly recurrent *if there is $L \geq 1$ such that for every sequence $s \in X_\rho$ and $k \geq 1$, every finite word $w = w_1 \dots w_k$ in s reoccurs within $L|w|$ entries, where $|w|$ denotes the length of w.*

If the substitution χ is primitive, then (X_ρ, σ) is linearly recurrent and minimal, see for instance [11]. Then we can use, for instance, the results of [7] to compute eigenvalues and eigenfunctions of the Koopman operator of this dynamical system.

5.2. **Bratteli-Vershik systems.** We now define Bratteli diagrams and Bratteli-Vershik systems.

Definition 5.4. *A* Bratteli diagram (V, E) *is an infinite graph with the set of vertices $V = \bigsqcup_{k\geq 0} V_k$ and the set of edges $E = \bigsqcup_{k\geq 0} E_k$ with the following properties:*

- *V_0 consists of a single vertex v_0, called the* root *of the Bratteli diagram.*
- *For $k \geq 0$, V_k is a finite set.*
- *For $k \geq 0$, each edge $e \in E_k$ connects the vertex $\boldsymbol{s}(e) \in V_k$ to the vertex $\boldsymbol{t}(e) \in V_{k+1}$, where $\boldsymbol{s}, \boldsymbol{t} : E \to V$ are called the* source *and the* target *maps respectively.*

In addition, the Bratteli diagram $(V, E, <)$ is ordered *if for each $k \geq 2$ and $v \in V_k$, there is a total order $<$ on the incoming edges e with $\boldsymbol{t}(e) = v$.*

We assume that for every $v \in V_k$ there exists at least one outgoing edge $e \in E_k$ with $v = \boldsymbol{s}(e)$, and for every $v \in V_{k+1}$ there exists at least one incoming edge $e \in E_k$ with $v = \boldsymbol{t}(e)$, for $k \geq 1$. We assume that every $v \in V_1$ is connected to the root v_0 by a single edge, so $\#E_0 = \#V_1$.

Recall that a square matrix M is *primitive* if it has a power with strictly positive entries.

Definition 5.5. *Let (V, E) be a Bratteli diagram. For $k \geq 1$, the* associated matrix M_k *to* E_k *has* i, j*-entries equal to the number of edges from* $j \in V_k$ *to* $i \in V_{k+1}$.

A Bratteli diagram (V, E) is stationary *if for all $k \geq 1$ the associated matrices satisfy $M_k = M_1$. A stationary Bratteli diagram (V, E) is* simple *if the associated matrix is primitive.*

Here is an example which is fundamental for the rest of the paper.

Example 5.6. Suppose we are given a substitution χ with the alphabet $\mathcal{A}$ as in Definition 5.1.

We construct a Bratteli diagram (V, E) by taking $V_k = \mathcal{A}$, and defining E_k so that there is an edge from $j \in V_k$ to $i \in V_{k+1}$ for each appearance of the letter j in $\chi(i)$, for $k \geq 1$. Then $M_k = M_1$ for all $k \geq 1$, and (V, E) is stationary. If the substitution χ is primitive, then (V, E) is simple.

We define the order $<$ on the incoming edges to $i \in V_{k+1}$ as the order of the corresponding letters in the word $\chi(i)$.

Next we define the Vershik map on the path space of an ordered Bratteli diagram $(V, E, <)$.

Bratteli diagrams emerged in the area of C^*-algebras [3], and they were given a dynamical interpretation, when Vershik equipped them with an order and a successor map, described below and now called the Vershik map [30]. It was shown in [14] that every minimal homeomorphism on the Cantor set can be represented as a Bratteli-Vershik system. Later Medynets [24] extended this to all aperiodic homeomorphisms on the Cantor set. For a general survey, we refer to [11].

Definition 5.7. *A finite (resp. infinite)* path *in the Bratteli diagram $(V, E, <)$ is a finite (resp. infinite) sequence of edges $(e_k)_{k=0}^m$ (resp. $(e_k)_{k\geq 0}$), such that for all $1 \leq k < m$ (resp. $k \geq 1$) we have $e_k \in V_k$ and $\boldsymbol{s}(e_k) = \boldsymbol{t}(e_{k-1})$.*

Definition 5.8. *Let $v_0 \in V_0$ be the root of the Bratteli diagram. For $k \geq 1$, define the* height *of the vertex $i \in V_k$ as the number of finite paths from v_0 to i:*

$$h_i^{(k)} = \#\{e_0 \dots e_{k-1} : \boldsymbol{s}(e_0) = v_0,\ \boldsymbol{t}(e_{k-1}) = i \in V_k\}.$$

Let $h^{(k)}$ be the vector with entries $h_i^{(k)}$, for $0 \leq i \leq |V_k| - 1$, and $k \geq 1$.

We define the space of infinite paths by

$$X_{(V,E,<)} = \{(e_k)_{k\geq 0} : e_k \in E_k,\, \boldsymbol{t}(e_k) = \boldsymbol{s}(e_{k+1}) \text{ for all } k \geq 0\},$$

and, to make it a topological space, we give each finite edge set E_k discrete topology, and equip the space $X_{(V,E,<)}$ with the product topology.

We now define the *Vershik map* $\tau : X_{(V,E,<)} \to X_{(V,E,<)}$. For each $v \in V_{k+1}$, $k \geq 1$, there is a total order $<$ on the set of edges $e \in E_k$ such that $\boldsymbol{t}(e) = v$, and so this set has unique minimal and maximal edges. If E_0 contains a single incoming edge to a vertex i in V_k, then this edge is both minimal and maximal. Let $X^{\min}_{(V,E,<)}$ (resp. $X^{\max}_{(V,E,<)}$) be the subsets of $X_{(V,E,<)}$ consisting of paths with only minimal (resp. only maximal) edges.

Given a path $e = (e_k)_{k\geq 0} \in X_{(V,E,<)}$, let $k \geq 0$ be smallest index such that $e_k \in E_k$ is not the maximal incoming edge at $v_k \in V_k$ with respect to $<$. Then put

$$\begin{cases} \tau(e)_j = e_j \quad \text{for } j > k, \\ \tau(e)_k \text{ is the successor of } e_j \text{ among all incoming edges at } v_k, \\ \tau(e)_0 \dots \tau(e)_{k-1} \text{ is the minimal path connecting } v_0 \text{ with } \boldsymbol{s}(\tau(e)_k). \end{cases}$$

If no such k exists, then $e \in X^{\max}_{(V,E,<)}$, and we have to choose $e' \in X^{\min}_{(V,E,<)}$ to define $\tau(e) = e'$. For the rest of the paper we assume that there is a unique minimal sequence $e^{\min}$ and unique maximal sequence $e^{\max}$, so we can define $\tau(e^{\max}) = e^{\min}$ which makes τ into a homeomorphism.

Definition 5.9. *Let $(V, E, <)$ be an ordered Bratteli diagram with unique maximal and unique minimal paths. This diagram together with the Vershik map $\tau : X_{(V,E,<)} \to X_{(V,E,<)}$, defined as above, is called the* Bratteli-Vershik system.

Lemma 5.10. *For a Bratteli-Vershik system defined in Definition 5.9, assume in addition that the diagram $(V, E, <)$ is simple. Then the Vershik map is minimal.*

5.3. **Main result of the section.**

Theorem 5.11. *Consider the aperiodic Cantor system (I^*_{np}, F^*_π), and let $\mathcal{A} = \{0, 1, \dots, q-1\}$ be a finite alphabet. There exists a sequence $(\chi_k)_{k\geq 1}$ of substitutions*

$$\chi_k : \mathcal{A} \to \mathcal{A}^*, \quad i \mapsto \chi_k(i),$$

and an ordered Bratteli diagram $(V, E, <)$ with the following properties:

1. *The sequence $(\chi_k)_{k\geq 1}$ is eventually periodic, that is, there exists $k_0 \geq 1$ and $p_0 \geq 1$ such that $\chi_k = \chi_{k_0+sp_0}$ for all $k \geq k_0$ and $s \geq 0$.*
2. *The set $V_0 = \{v_0\}$ is a singleton, and for $k \geq 1$ the vertex set V_k is identified with a non-empty subset of $\mathcal{A}$.*
3. *The edge set E_k and the order on the subset of incoming edges for $i \in V_k$ is determined by the substitution χ_k.*
4. *The path space $X_{(V,E,<)}$ of the diagram (V, E) has a unique maximal and a unique minimal path, and the Vershik map $\tau : X_{(V,E,<)} \to X_{(V,E,<)}$ is a homeomorphism.*
5. *There is a homeomorphism $\psi : (I^*_{np}, F^*_\pi) \to (X_{(V,E,<)}, \tau)$, such that $\psi \circ F^*_\pi = \tau \circ \psi$.*

In the rest of this section we prove Theorem 5.11.

Given $q \in \mathbb{N}$, recall that by Definition 3.2

$$N = \min\{n \in \mathbb{N} : 2^n \geq q\} \text{ and } L_k = [0, 2^{-kN}) \text{ for } k \geq 1.$$

Our analysis of the infinite IET $F_\pi = \mathfrak{a} \circ R_\pi$ is by means of the successive first return maps to the sections L_k.

5.4. **First return maps.** Denote by $\mathcal{P}^{cod}_{kN,q} = \{I_{k,i} : 0 \leq i \leq q-1\}$ the partition of L_k into q intervals of equal lengths of the partition $\mathcal{P}_{kN,q}$ given by Definition 3.2.

Recall from the introduction that $F_\pi = \mathfrak{a} \circ R_\pi$, where π is a permutation of q symbols and $R_\pi : I \to I$ is an IET with finite number of intervals of equal length, induced by π. We will prove that for $k \geq 1$, the return map $F_{\pi,k}$ has a similar property, as described by the following proposition.

Proposition 5.12. *Let $F_{\pi,k} : L_k \to L_k$, $k \geq 1$ be the first return maps. Then there exist permutations π_k of q symbols, and finite IETs $R_{\pi,k} : L_k \to L_k$ of the partitions $\mathcal{P}^{cod}_{kN,q}$ such that*

1. *$F_{\pi,k} = \mathfrak{a}_k \circ R_{\pi,k}$, where $\mathfrak{a}_k$ is a scaled copy of the von Neumann-Kakutani map $\mathfrak{a}$, given by*
$$\mathfrak{a}_k(x) = \frac{1}{2^{kN}}\, \mathfrak{a}\left(2^{kN}(x)\right).$$
2. *The sequence $(\pi_k)_{k\geq 1}$ is eventually periodic.*
3. *If $(\pi_k)_{k\geq 1}$ is strictly pre-periodic, and $S = \{\pi_1, \dots, \pi_{k_0}\}$ is the pre-periodic part of the sequence, then none of the permutations in S occurs in the periodic part.*

Proof. For $j \geq 1$, denote by $C_j = [1 - 2^{-(j-1)}, 1 - 2^{-j})$ the half-open intervals on which $\mathfrak{a}$ is continuous. We argue by induction.

For $k = 0$ we have $F_{\pi,0} = F_\pi$, $R_{\pi,0} = R_\pi$ and $\mathfrak{a}_0 = \mathfrak{a}$ by definition.

For the induction step, assume that the statement of the proposition holds for $F_{\pi,k-1}$. We know that the rotated odometer map F_π maps the interval $H_1 = R_\pi^{-1}([1-2^{-N},1))$ onto $L_1 = [0, 2^{-N})$ discontinuously. We now note that $F_{\pi,k-1}$ maps the interval $H_k := R^{-1}_{\pi,k-1}\left(\left[\frac{2^N-1}{2^{kN}}, \frac{2^N}{2^{kN}}\right)\right)$ onto $L_k = [0, \frac{1}{2^{kN}})$ discontinuously. This follows from the fact that L_{k-1} is subdivided into $q2^N$ intervals of the partition $\mathcal{P}_{kN,q}$, and the multiplication by $2^{(k-1)N}$ maps these intervals onto $q2^N$ intervals of the partition $\mathcal{P}_{N,q}$, in particular, L_k onto $L_1 = [0, 2^{-N})$ and H_k onto $H_1 = R^{-1}_{\pi,1}([1-2^{-N},1))$.

We note that for $1 \leq j \leq N$, the intervals C_j are partitioned into the intervals of $\mathcal{P}_{N,q}$, and so $\mathfrak{a}$ is continuous intervals of $\mathcal{P}_{N,q}$ contained in $C_1 \cup \cdots \cup C_N = I \setminus L_1'$. Then $F_{\pi,k-1}$ is continuous on any interval of $\mathcal{P}_{kN,q}$ contained in $L_{k-1} \setminus H_k$, and maps any such interval onto another interval of $\mathcal{P}_{kN,q}$. The first return map $F_{\pi,k-1}$ is discontinuous on the intervals of $\mathcal{P}_{kN,q}$ contained in H_k.

Denote by $t_{k,i} \geq 1$ the smallest integer such that $F^{t_{k,i}-1}_{\pi,k-1}(I_{k,i}) \subset H_k$, $I_{k,i} \in \mathcal{P}^{cod}_{kN,q}$ and define

$$R_{\pi,k} : L_k \to L_k, \qquad I_{k,i} \mapsto R_{\pi,k-1} \circ F^{t_{k,i}-1}_{\pi,k-1}(I_{k,i}) + 2^{kN} - 1. \tag{10}$$

Then the first return map $F^{t_{k,i}}_{\pi,k} = \mathfrak{a}_k \circ R_{\pi,k}$, which proves (1).

The q intervals in the coding partition $\mathcal{P}^{cod}_{kN,q}$ of L_k have a natural order, induced by the order in which they cover L_k. The map $R_{\pi,k} : L_k \to L_k$ is a permutation of the sets in $\mathcal{P}^{cod}_{kN,q}$, and so it defines a permutation π_k of q symbols. This permutation need not be the same permutation as π, however, since the group of permutations on q symbols is finite, either $\pi_k \neq \pi_j$ for $j > k$, or there exists a smallest index $1 \leq k < j$ such that $\pi_k = \pi_j$, and then $\pi_k = \pi_{s(j-k)}$ for all $s \geq 1$. This finishes the proof of the proposition. □

5.5. **Periodic points and the first return maps.** Since every interval $I_{k,i}$ in $\mathcal{P}^{cod}_{kN,q}$ in Proposition 5.12 returns to L_k, and an orbit visits L_k if and only if it visits M_k, we have $M_k = \bigcup_{i=0}^{q-1} F^{t_{k,i}-1}_{\pi,k-1}(I_{k,i})$. However, the orbits of the sets in $\mathcal{P}^{cod}_{kN,q}$ may miss some of the intervals in the partition of L_{k-1} into intervals of $\mathcal{P}_{kN,q}$. Such intervals contain points with periodic orbits.

Definition 5.13. *The first return map $F_{\pi,k}$* covering *if the orbits of the sets in $\mathcal{P}^{cod}_{kN,q}$ visit every element of $\mathcal{P}_{kN,q}$ in L_{k-1}, that is, if*

$$\bigcup_{i=0}^{q-1} \bigcup_{t=0}^{t_{k,i}-1} F^t_{\pi,k-1}(I_{k,i}) = L_{k-1}. \tag{11}$$

The set in (11) is a disjoint union of intervals in $\mathcal{P}_{kN,q}$, and it follows by induction that if the first return maps $F_{\pi,j}$ are covering for all $1 \leq j \leq k$, then

$$\bigcup_{i=0}^{q-1} \bigcup_{t=0}^{h_{k,i}-1} F^t_\pi(I_{k,i}) = I, \tag{12}$$

where $h_{k,i}$ is the first return time of $I_{k,i}$ to L_k under iterations of F_π.

Definition 5.14. *The rotated odometer $F_\pi : I \to I$ is* covering *if for every $k \geq 1$, the first return map $F_{\pi,k}$ is covering, that is, $F_{\pi,k}$ satisfies* (11).

As a consequence of this argument, we deduce the following general lemma, without the assumption that all $F_{\pi,k}$ are covering:

Lemma 5.15. *The set of non-periodic points in I satisfies*

$$I_{np} = \bigcap_{k\geq 1} \bigcup_{i=0}^{q-1} \bigcup_{j=0}^{h_{k,i}-1} F^j(I_{k,i}).$$

We now use the collection $(L_k, F_{\pi,k})$ to build a symbolic representation of this system.

5.6. **The construction of the Bratteli-Vershik system.** In this section we build an ordered Bratteli diagram $(V', E', <)$, determined by the sequence $(L_k, F_{\pi,k})$ of first return maps of Section 5.4, for $k \geq 1$. Note that $(L_k, F_{\pi,k})$ may have periodic points, so to prove the theorem we will have to restrict to a subdiagram of $(V', E', <)$, which is described further in Section 5.7 .

We use the partitions $\mathcal{P}^{cod}_{q,(k-1)N}$ as coding partitions to define the substitutions χ_k for $k \geq 1$. Indeed, the sets of $\mathcal{P}^{cod}_{q,(k-1)N}$ partition L_{k-1}, and the first return map $F_{\pi,k-1}$ maps the intervals from $\mathcal{P}^{cod}_{q,kN}$ onto the intervals of $\mathcal{P}_{q,kN}$ in L_{k-1}. The latter are contained in the intervals of $\mathcal{P}^{cod}_{q,(k-1)N}$ which are ordered naturally from 0 to $q-1$. So for $k \geq 1$ we define a substitution

$$\chi_k(i) = e_0 \dots e_{t_{k,i}-1}, \qquad \text{where } F^t_{\pi,k-1}(I_{k,i}) \subset I_{k-1,e_t} \in \mathcal{P}^{cod}_{q,(k-1)N},\ 0 \leq t < t_{k,i}.$$

That is, we track what sets of the partition $\mathcal{P}^{cod}_{q,(k-1)N}$ of L_{k-1} the orbit of $I_{k,i}$ under $F_{\pi,k-1}$ visits before returning to L_k. The associated matrix is given by

$$M_k = (m_{i,j})_{i,j=0}^{q-1} \qquad m_{i,j} = \#\{0 \leq t < t_{k,i} : e_t = j\}.$$

This proves item (1) of Theorem 5.11. The properties of $(\chi_k)_{k\geq 1}$ described in Theorem 5.11 follow from item (2) in Proposition 5.12, and the fact that χ_k is determined by the permutation π_k.

The following property of the sequence $(\chi_k)_{k\geq 1}$ follows from the fact that every set $I_{k,i}$, contained in L_k, visits M_k before returning to L_k.

Lemma 5.16. *Every substitution in the sequence $(\chi_k)_{k\geq 1}$ is proper, namely, every word $\chi_k(i)$ starts with 0, and ends with b_k, depending only on k and not on $i \in \{0, \dots, q-1\}$.*

We now can proceed similarly to the case of a single substitution in Section 5.1. By Lemma 5.16 the sequence

$$\rho = \lim_{k\to\infty} \chi_1 \circ \cdots \circ \chi_k(0)$$

is well-defined (it is actually the itinerary of 0 in I). We define the *S-adic subshift* (X_ρ, σ) similarly to Section 5.1, formula (9), and the paragraph below (9).

Since the sequence $(\chi_k)_{k\geq 1}$ is eventually periodic, the properties of the corresponding subshift are effectively the same as for the case of a single substitution. Namely, we can take χ to be the composition of the substitutions χ_k in one period of the sequence $(\chi_k)_{k\geq 1}$, with associated matrix B obtained as the product of the matrices associated to the substitutions in one period.

We now adapt the procedure of Example 5.6 to the case of a sequence of substitutions $(\chi_k)_{k\geq 1}$.

Set $V'_0 = \{v_0\}$ and $V'_k = \{0, \dots, q-1\}$ for $k \geq 1$, so the vertices in V'_k correspond to the sets in $\mathcal{P}^{cod}_{(k-1)N,q}$ for $k \geq 1$. Define the set E'_k of edges from V'_k, and the order on the edges, using the substitution χ_k as in Example 5.6. Then the number of incoming edges to $i \in V'_{k+1}$ is equal to $|\chi_k(i)| = t_{k,i}$.

Example 5.17. Let $q = 3$ and $\pi = (012)$, then R_π is the rotation over $1/3$. The first return map $F_{\pi,1}$ corresponds to the following substitution and associated matrix M_1:

$$\chi_1 : \begin{cases} 0 \to 0221 \\ 1 \to 0221 \\ 2 \to 0011 \end{cases} \qquad M_1 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

Since $F_{\pi,1}$ is conjugate to F_π, we find that $\chi_k = \chi_1$ and $M_k = M_1$ for all $k \geq 1$, generating a substitution shift (X_ρ, σ), where X_ρ is the shift-orbit closure of the fixed point

$$\rho = 0 \cdot 221 \cdot 001100110221 \cdot 0221001100110221 \ldots$$

of the substitution χ_1. The corresponding Bratteli diagram is given in Figure 5.

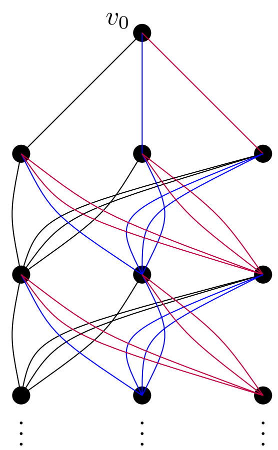

FIGURE 5. The Bratteli diagram of $F_{(012)}$ with incoming edges ordered left to right.

Reinterpreting Definition 5.13, we can say that χ_k is *covering* if $\sum_{i=0}^{q-1} |\chi_k(i)| = q2^N$, where $|\chi_k(i)|$ denotes the length of the word $\chi_k(i)$. We also obtain an alternative expression for the number of paths from the root v_0 to $i \in V_k$, namely $h_i^{(k)} = |\chi_1 \circ \cdots \circ \chi_k(i)|$.

Remark 5.18. It is useful to point out that the notions of primitive and covering do not imply each other. In Example 5.22 below χ_1 is primitive but not covering, whereas in Example 7.1 for every $\chi_k = \chi_1$, $k \geq 1$, the substitution is covering but not primitive.

Lemma 5.19. *The ordered Bratteli diagram $(V', E', <)$ constructed above has a unique minimal and unique maximal path, and so the Vershik map $\tau' : X_{(V',E',<)} \to X_{(V',E',<)}$, constructed as in Section 5.2, is a homeomorphism.*

Proof. By Lemma 5.16 $e^{\min} = 0^\infty$ is the unique minimal path, and the maximal path is $e^{\max} = b_1 b_2 b_3 \ldots$, where b_k is the last letter of any word of the substitution χ_k. □

5.7. **Conjugacy to the rotated odometer.** In this section we show that the aperiodic system (I^*_{np}, F^*_π) is conjugate to a subsystem of the Bratteli-Vershik system $(X_{(V',E',<)}, \tau)$, constructed in Section 5.6.

Theorem 5.20. *Consider the ordered Bratteli diagram $(V', E', <)$ and the Bratteli-Vershik system $(X_{(V',E',<)}, \tau')$, constructed in Section 5.6. There exists a subdiagram $(V, E, <)$ of $(V', E', <)$ with associated Bratteli-Vershik system $(X_{(V,E,<)}, \tau)$, such that there is a homeomorphism $\psi : I^*_{np} \to X_{(V,E,<)}$, which satisfies $\psi \circ F^*_\pi(x) = \tau \circ \psi(x)$ for all $x \in I^*_{np}$.*

Proof. Consider the sequence of first return maps $(L_k, F_{\pi,k})$, $k \geq 1$. Recall that for $k \geq 1$ the interval L_k has a partition $\mathcal{P}^{cod}_{kN,q} = \{I_{k,i} : 0 \leq i \leq q-1\}$ into q sets, and $|V'_k| = |\mathcal{P}^{cod}_{kN,q}| = q$.

Let $V_0 = V'_0$, and for $k \geq 1$ let $i \in V_k$ if and only if $I_{k,i} \cap I_{np} \neq \emptyset$. Let $e \in E_k$ if and only if $\mathbf{s}(e) \in V_k$ and $\mathbf{t}(e) \in V_{k+1}$ for some $k \geq 0$. Give the edges in $E = \bigsqcup_{k \geq 0} E_k$ the order which is the restriction of the order in E'. Let $X_{(V,E,<)}$ be the path space of the subdiagram with vertex set $V = \bigsqcup_{k\geq 0} V_k$ and edge set E. Then the Vershik map τ' on $X_{(V',E',<)}$ induces the Vershik map τ on $X_{(V,E,<)}$. More precisely, for the paths $e, e' \in X_{(V',E',<)}$ we have $\tau(e) = e'$ if and only if there is $n \geq 0$ such that $(\tau')^n(e) = e'$ in $X_{(V',E',<)}$.

We now define the map $\psi : I_{np} \to X_{(V,E,<)}$.

Let $x \in I_{np}$, then for any $k \geq 1$ there is a unique $i_k(x) \in \{0, \ldots, q-1\}$ such that $F^j(I_{k,i_k(x)}) \ni x$ for some $0 \leq j < h_{k,i_k(x)}$. Then $i(x) := (i_k(x))_{k\geq 0}$ is the sequence of vertices in the Bratteli-Vershik diagram corresponding to $\psi(x)$.

We determine the sequence of edges $(e_k(x))_{k\geq 0}$ inductively. There is a single edge to each $i_1(x) \in V_1$ from v_0, so $e_0(x)$ is determined. Next, there is a unique $0 \leq j_1 < |\chi_1(i_2(x))|$ such that $x \in F^{j_1}_\pi(I_{1,i_1(x)})$, so we take $e_1(x)$ to be the j_1-th incoming edge to $i_2(x) \in V_2$. If $e_0(x), \ldots, e_{k-1}(x)$, and so $j_1, \ldots, j_{k-1}$, are determined, there is a unique $0 \leq j_k < |\chi_k(i_{k+1}(x))|$ such that $F^{j_1}_{\pi,1} \circ F^{j_2}_{\pi,2} \circ \cdots \circ F^{j_k}_{\pi,k}(I_{k+1,i_{k+1}(x)}) \ni x$, and we take e_k to be the j_k-th incoming edge to $i_{k+1}(x) \in V_{k+1}$. This defines $\psi(x) = (e_k)_{k\geq 0}$. We then extend ψ from I_{np} to its closure I^*_{np} in the standard way. Then ψ is a continuous surjective map. Injectivity follows from the fact that the lengths of the sets in the partitions $\mathcal{P}_{q,kN}$ tend to zero with k, and so the orbits of any two distinct points eventually visit distinct sets of the coding partitions $\mathcal{P}^{cod}_{kN,q}$.

By construction $\psi(\widehat{x}^-)$, where $\widehat{x}^-$ is defined in (6), is the maximal path of both $X_{(V,E,<)}$ and $X_{(V',E',<)}$, while $\psi(0)$ is the minimal path in both diagrams. We have $F^*_\pi(\widehat{x}^-) = 0$, and, since edges in E are ordered according to the order of symbols in the substitutions $(\chi_k)_{k\geq 1}$, it follows that $\psi \circ F_\pi = \tau \circ \psi$. □

We now give examples which illustrate that the set of periodic points I_{per} may be non-empty, and so restricting to a subdiagram in Theorem 5.20 may be necessary.

Example 5.21. Let $q = 7$ and $\pi = (0654321)$. Then:

$\pi \to \pi_1$	Substitution χ_1	Associated Matrix
$(0654321) \to (0654321)$	$\begin{cases} 0 \to 01461360 & \sum_{i=0}^{6} \lvert\chi_1(i)\rvert = 14 \\ 1 \to 0 \\ 2 \to 0 \\ 3 \to 0 \\ 4 \to 0 \\ 5 \to 0 \\ 6 \to 0 \end{cases}$	$\begin{pmatrix} 2 & 2 & 0 & 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

The substitution χ_1 is not covering, because $\sum_{i=0}^{6} |\chi_1(i)| = 14$ while $q2^N = 7 * 2^3 = 56$. Since $\pi = \pi_1 = \pi_k$ for $k \geq 1$, every section L_k contains a subinterval of periodic points, and I_{per} is a countable union of intervals.

Example 5.22. Let $q = 5$ and $\pi = (02413)$. Then:

$\pi \to \pi_1$	Substitution χ_1	Associated Matrix
$(02413) \to (01234)$	$\begin{cases} 0 \to 044332 & \sum_{i=0}^{4} \lvert\chi_1(i)\rvert = 30 \\ 1 \to 044332 & \\ 2 \to 044332 & \\ 3 \to 044332 & \\ 4 \to 012012 & \end{cases}$	$\begin{pmatrix} 1 & 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 2 & 2 & 0 & 0 \end{pmatrix}$

The substitution χ_1 is not covering, because $\sum_{i=0}^{4} |\chi_1(i)| = 30$ while $q2^N = 5*2^3 = 40$. However, for the first return system $(L_1, F_{\pi,1})$ the map $F_{\pi,1} = F_{\pi_1}$ is determined by the substitution $\pi_1 = (01234)$, which is studied in detail in Example 7.1 and Proposition 8.5. The associated substitution χ_2 is covering, and we have $\chi_k = \chi_2$ for $k \geq 2$. Therefore, for $k \geq 2$ the first return systems $(L_k, F_{\pi,k})$ have no periodic points, and I_{per} is a finite union of intervals.

Since by Theorem 1.2 the system (I_{np}, F_π) has a unique minimal subsystem, then (I^*_{np}, F^*_π) also has a unique minimal set (I^*_{min}, F^*_π). The set I^*_{min} corresponds to a simple subdiagram $(\widehat{V}, \widehat{E})$ of (V, E) with associated Bratteli-Vershik system $(X_{(\widehat{V},\widehat{E},<)}, \widehat{\tau})$, where $\widehat{\tau}$ is a restriction of τ to $X_{(\widehat{V},\widehat{E},<)}$.

We now can prove Theorem 1.6 as a consequence of Theorem 5.11.

Proof of Theorem 1.6. Since the number of vertices at each level of the subdiagram (V, E) is bounded by q, by [1, Theorem 4.3] (I^*_{np}, F^*_π) has at most q ergodic measures. The minimal system (I^*_{min}, F^*_π) is uniquely ergodic, because its stationary part is a primitive substitution shift. □

6. Entropy of rotated odometers

In this section we prove Theorem 1.5. For this we use the formula for the upper bound for entropy of an infinite IET from [8], as well as the conjugacy of the aperiodic system (I^*_{np}, F^*_π) to a Bratteli-Vershik system obtained in Theorem 5.11.

Let $h(I, S, \lambda)$ denote the entropy of the dynamical system (I, S) on an interval $I = [0, 1)$ with respect to Lebesgue measure λ.

Theorem 6.1. [8] *Let $S : I \to I$ be any infinite IET, and let $\ell_1 \geq \ell_2 \geq \ell_3 \geq \dots$ be the lengths of the subintervals on which S is continuous. For $m \geq 1$ define $\Lambda_m = \sum_{i=1}^{\infty} \ell_{m+i}$. Then*

$$h(I, S, \lambda) \leq \liminf_{m\to\infty} \Lambda_m \cdot \log m.$$

Proposition 6.2. *Let π be a permutation on $q \geq 2$ symbols, and let (I, F_π, λ) be a rotated odometer. Then $h(I, F_\pi, \lambda) = 0$.*

Proof. Recall that $N = \min\{n \in \mathbb{N} : 2^n \geq q\}$ and consider the partitions $\mathcal{P}_{kN,q}$ of I given by Definition 3.2. Define $J_k = R_\pi^{-1}([1 - 2^{-kN}, 1))$, and consider the complement of J_k in J_{k-1}, $k \geq 1$, with $J_0 = I$. The complement $J_{k-1} \setminus J_k$ is the union of $(2^N - 1)q$ intervals of $\mathcal{P}_{kN,q}$, and we set $m_k = k(2^N - 1)q$ and $\Lambda_{m_k} = 2^{-kN}$. Then

$$\Lambda_{m_k} \log m_k = 2^{-kN} \log\left(k(2^N - 1)q\right) = \frac{\log k + \log((2^N - 1)q)}{2^{kN}}.$$

As q and N are fixed, $\Lambda_{m_k} \log m_k \to 0$ as $k \to \infty$, and the statement follows. □

Proof of Theorem 1.5. By Theorem 5.11 the aperiodic system (I^*_{np}, F^*_π) of the rotated odometer is conjugate to a Bratteli-Vershik system representing an eventually periodic S-adic transformation. Every ergodic invariant measure of such S-adic transformations, just like for any substitution shift, has zero entropy. Since by Proposition 6.2, Lebesgue measure also has zero entropy (and naturally all equidistributions on periodic orbits, if there are any, have zero entropy), then every measure has zero entropy. By the variational principle, (I^*_{np}, F^*_π) and hence (I^*, F^*_π) has zero topological entropy, and because (I, F_π) is a factor of (I^*, F^*_π), it has zero topological entropy too. $\square$

7. Ergodicity of Lebesgue measure

In this section we prove Theorem 1.4, that is, we show that Lebesgue measure λ on $I = [0,1)$ is ergodic if and only if the rotated odometer (I, F_π) has no periodic points.

Our argument is based on the discussion of the *covering* property of the rotated odometer, defined in Definition 5.14. To recall, let (I, F_π) be a rotated odometer, and $(\chi_k)_{k\geq 1}$ be the sequence of substitutions given by Theorem 5.11. The integer N, the sections L_k and the partitions $\mathcal{P}_{kN,q}$ of I into $q2^{kN}$ subintervals, for $k \geq 1$, used in the arguments below, were defined in Definition 3.2.

Recall from Definition 5.14 that F_π is covering if for all $k \geq 1$ we have $\sum_{i=0}^{q-1} |\chi_k(i)| = q2^N$ and so, as in (11),

$$\bigcup_{i=0}^{q-1} \bigcup_{j=0}^{|\chi_k(i)|-1} F^j_{\pi,k-1}\Big(\Big[\frac{i}{q2^{kN}}, \frac{i+1}{q2^{Nk}}\Big)\Big) = \Big[0, \frac{1}{2^{(k-1)N}}\Big) = L_{k-1}. \tag{13}$$

We will see below that if the rotated odometer is covering, then Lebesgue measure is ergodic, and therefore a.e. orbit (although not necessarily every) is dense in I.

On the other hand, if (13) fails for some k, then there is a half-open subinterval of I that is not visited by the orbit of any $x \in \big[0, \frac{1}{2^{kN}}\big)$, for k sufficiently large. Since all aperiodic orbits accumulate at 0, this shows that no orbit is dense in I. We have seen in Theorem 1.5 that the minimal subsystem (I^*_{min}, F^*_π) is strictly ergodic, and therefore every orbit is dense in I^*_{min}, but this does not need to hold for I^*_{np}.

Example 7.1. Let $q = 5$ and $\pi = (01234)$. Applying the algorithm of Section 5.6 we obtain:

$\pi \to \pi_1$	Substitution χ_1	Associated Matrix
$(01234) \to (01234)$	$\begin{cases} 0 \to 03 & \sum_{i=0}^4 \lvert\chi_1(i)\rvert = q2^N = 40 \\ 1 \to 03 & \\ 2 \to 03 & \\ 3 \to 03 & \\ 4 \to 04222111431431430420420422211143 & \end{cases}$	$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 4 & 8 & 8 & 4 & 8 \end{pmatrix}$

The sequence $(\chi_k)_{k\geq 1}$ is constant with $\chi_k = \chi_1$ for $k \geq 1$, see also Proposition 8.5 for further properties of this example. The substitution χ_1 is not primitive because 4 does not occur in $\chi_1(i)$, $i = 0, \dots, 3$, and the minimal system (I^*_{min}, F^*_π) is easily seen to be the dyadic odometer. Note also that since χ_1 is covering, there is a dense orbit by Theorem 1.4, and also $X_{(V,E,<)} = X_{(V',E',<)}$. However, not every orbit is forward recurrent. Indeed, consider the path $x \in X_{(V,E,<)}$ which passes through vertices labeled $(4, 4, 4, \dots)$, and where each edge is the maximal incoming edge from 4 to 4. The orbit of the path x is not recurrent under the forward iterations of the Vershik map τ. Indeed, as soon as the orbit of x moves away from the maximal edge e_1 from 4 to 4 at level k to an edge e_2 joining 3 and 4, it never returns to e_1, and therefore to the cylinder set corresponding to the part of the path x from the root v_0 to the vertex labelled by 4 at level k. Hence the orbit of x converges to the minimal Cantor subset of $X_{(V,E,<)}$ corresponding to I^*_{min} in forward time; its backward orbit, however, is dense in $X_{(V,E,<)}$.

A stationary Bratteli-Vershik system $(X_{(V,E,<)}, \tau)$ with primitive associated matrix is minimal, see for instance [27]. The statement below concerns the rotated odometer (I, F_π) which may have periodic points, and for which the sequence of substitutions $(\chi_k)_{k\geq 1}$ is eventually periodic, but not necessarily stationary.

Lemma 7.2. *Let π be a permutation of $q \geq 2$ symbols, and let (I, F_π) be a rotated odometer. Let $(\chi_k)_{k\geq 1}$ be the associated sequence of substitutions given by Theorem 5.11. If the matrix associated to the periodic part of $(\chi_k)_{k\geq 1}$ is primitive, then every point in $[0,1)$ is recurrent under F_π.*

Proof. Consider the partitions $\mathcal{P}_{kN,q}$, given by Definition 3.2, and the sets $I_{k,i} = \left[\frac{i}{q2^{kN}}, \frac{i+1}{q2^{kN}}\right)$ for $0 \leq i < q-1$, of these partions, which subdivide the sections L_k, $k \geq 1$. Applying (13), for each k and i there is a positive integer $h_{k,i} \in \mathbb{N}$ such that $F_\pi^{h_{k,i}}(I_{k,i}) \subset \left[0, \frac{1}{2^{kN}}\right)$, and $h_{k,i}$ is the smallest positive integer with this property. These numbers are in fact the heights of the Bratteli diagram, introduced in Definition 5.8. Define

$$U_k := \bigcup_{i=0}^{q-1} \bigcup_{j=0}^{h_{k,i}-1} F_\pi^j(I_{k,i}), \qquad U = \bigcap_{k\geq 1} U_k.$$

If the substitution χ_k is not covering for some $k \geq 1$, then there are intervals $\left[\frac{l}{q2^{kN}}, \frac{l+1}{q2^{kN}}\right)$ disjoint from U_k, for some $0 \leq l \leq q2^{kN} - 1$. Points in such intervals have orbits not accumulating on 0, and therefore by Proposition 3.4 they are periodic, and in particular recurrent. Hence it remains to consider points in U.

By construction U contains only non-periodic orbits, and so there is an embedding $\iota : U \to I^*_{np}$. Let $x \in U$, and let $e(x) \in X_{(V,E,<)}$ given by Theorem 5.20.

Let $\varepsilon > 0$ be arbitrary and take k_0 so large that $1/(q2^{k_0 N}) < \varepsilon$. Because $x \in U_{k_0-1}$, there is $j \geq 0$ such that F_π^j maps L_{k_0} isometrically onto a neighborhood of x. On the other hand, by Proposition 3.4, $\mathrm{orb}(x)$ accumulates on 0, so there is $m \geq 1$ such that $F_\pi^m(x) \in L_{k_0}$. Thus x returns to an ε-neighborhood of itself after $j + m$ iterates. □

Proof of Theorem 1.4. We note that the system (I^*_{np}, F^*_π) has no periodic points if and only if it is covering in the sense of Definition 5.14. If there are periodic points, then Lebesgue measure is not ergodic.

Assume that (I^*_{np}, F^*_π) is covering. Let B be the transition matrix associated to the periodic part of the sequence $(\chi_k)_{k\geq 1}$, that is, B is the product of the matrices associated to the substitutions χ_k in one period of the sequence. Recall that (I^*_{np}, F^*_π) has a unique minimal set. Then by renaming the symbols $\{0, \dots, q-1\}$ and telescoping we can put B into the standard block-matrix (Frobenius) form used in [1, 2]:

$$B = \left(\begin{array}{c|c|c|c} F_1 & 0 & \dots & 0 \\ \hline X_{2,1} & F_2 & \dots & 0 \\ \hline \vdots & & \ddots & \vdots \\ \hline X_{t,1} & X_{t,2} & \dots & F_t \end{array}\right), \tag{14}$$

where every non-zero submatrix F_i is primitive, and for every $2 \leq i \leq t$, at least one of $X_{i,j}$ is a non-zero matrix.

For each of non-zero diagonal blocks F_i (say of size $d_i \times d_i$), there is one ergodic measure μ_i, namely provided the leading eigenvalue of F_i is greater than 1, and the associated left eigenvector of B can be chosen to be non-negative, see [2, Section 3]. There are no other ergodic measures. Note that F_t is non-zero, since otherwise the symbol $q-1$ does not occur in any substitution words, which contradicts the fact that (I^*_{np}, F^*_π) is covering. Thus μ_t is non-zero.

Denote also by μ_i be the ergodic measures lifted along the inclusion $\iota : I \to I^*$ to (I, F_π). By the ergodic decomposition, Lebesgue measure $\lambda = \sum_{i=1}^t a_i \, \mu_i$ for some choice of $a_i \in [0,1]$. But for all $i < t$, the substitution associated to the first $D = \sum_{j \le i} d_j$ symbols leaves out the remaining symbols, and hence cannot be covering. This means that μ_i is supported on a Cantor set of Hausdorff dimension $0 \le \frac{\log D}{\log q} < 1$, and hence μ_i is not absolutely continuous with respect to Lebesgue measure. This in turn means that $a_i = 0$ for $i < t$, so $\lambda = \mu_t$ is ergodic. $\square$

Example 7.3. Consider again the rotated odometer with $q = 5$ and $\pi = (01234)$ from Example 7.1. The system of substitutions $(\chi_k)_{k \ge 1}$ is constant. Interchanging the labels for the symbols 1 and 3, we obtain the matrix B in the form (14), namely

$$B = \left(\begin{array}{cc|cc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \hline 4 & 4 & 8 & 8 & 8 \end{array}\right) \text{ with three blocks } F_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ F_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ F_3 = (8),$$

so we expect two ergodic measures. Following the algorithm in [2] and computing the eigenvalues and the left eigenvectors of B, we obtain an eigenvalue $\lambda_1 = 2$ with non-negative eigenvector $\vec{v}_1^\ell = (1,1,0,0,0)$, and $\lambda_2 = 8$ with eigenvector $\vec{v}_2^\ell = (1,1,1,1,1)$. The first eigenvalue corresponds to the ergodic measure μ_1 supported on the minimal set I^*_{min}, and the second to the ergodic measure μ_2 supported on $I^*_{np} = I^*$. By Theorem 1.4, μ_2 lifts to Lebesgue measure on I and, in particular, (I, F_π) has dense orbits.

The question whether the aperiodic part (I^*_{np}, F^*_π) of the rotated odometer always has a dense forward orbit, without the assumption that the Bratteli-Vershik system is simple, remains open. The following sample substitutions illustrate why it is hard to answer this question.

$$\chi : \begin{cases} 0 \to 01 \\ 1 \to 01 \\ 2 \to 021 \end{cases} \quad \text{with } B = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \hline 1 & 1 & 1 \end{array}\right) \quad \text{or} \quad \chi : \begin{cases} 0 \to 01 \\ 1 \to 01 \\ 2 \to 0221 \\ 3 \to 0331 \end{cases} \quad \text{with } B = \left(\begin{array}{cc|c|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 2 & 0 \\ \hline 1 & 1 & 0 & 2 \end{array}\right).$$

Both substitutions are not primitive, and the first gives an isolated path passing through the vertices $(2,2,2,\dots)$ in the Bratteli-Vershik diagram. Since for the rotated odometer systems I^*_{np} is a Cantor set, this substitution cannot occur in rotated odometers. However, for the second substitution, the path space is a Cantor set and every path in the Bratteli diagram is recurrent under the Vershik map, but since symbols 2 and 3 do not communicate, there is no dense orbit.

8. Equicontinuous factors of rotated odometers

In this section we prove Theorems 1.7 and 1.8. More precisely, with the aim of classifying the dynamical systems (I^*_{np}, F^*_π) and more specifically (I^*_{min}, F^*_π) up to isomorphism, we consider the spectrum of the Koopman operator $U_{F_\pi} g = g \circ F_\pi$, where $g : I \to \mathbb{R}$ is a measurable function. The theory of eigenvalues of the Koopman operator of Bratteli-Vershik systems was developed in multiple papers, see for instance [7, 12, 13, 29, 2].

8.1. **Stationary diagrams.** First let us assume that the sequence $(\chi_k)_{k\geq 1}$ is constant, that is, for all $k \geq 1$ we have $\chi_k = \chi_1$. Then the corresponding Bratteli-Vershik system is stationary with associated matrix B. Denote $h^{(1)} = (1, 1, \dots, 1)^T$, and consider the sequence of integer vectors

$$h^{(n+1)} = B^n \cdot h^{(1)}. \tag{15}$$

Then the component $h_j^{(n)}$, $0 \leq j \leq q-1$, is equal to the number of paths in the Bratteli diagram from v_0 to the j-th vertex of V_n, or, equivalently, to the height of the j-th stack at level n in the cutting-and-stacking representation of the system.

Consider the Bratteli-Vershik system $(X_{(\widehat{V},\widehat{E},<)}, \widehat{\tau})$, which corresponds to a simple subdiagram of the diagram $(V, E, <)$, and which is conjugate to (I^*_{min}, F^*_π). For simplicity assume that B is in the Frobenius form. Then the submatrix F_1 of B is primitive. Since all our Bratteli diagrams are eventually periodic (or even stationary), we can use Host's results [19] (see also [29]) on substitution shifts to obtain a condition for the eigenvalues of the Koopman operator, expressed in the language of the Bratteli-Vershik systems in Theorem 8.1 below.

We say that ζ is a *measurable* (resp. *continuous*) *eigenvalue* of the Koopman operator, if the corresponding eigenfunction is measurable (resp. continuous).

Theorem 8.1. *The number $\zeta = e^{2\pi i\alpha}$ is an eigenvalue of the simple Bratteli-Vershik system $(X_{(\widehat{V},\widehat{E},<)}, \widehat{\tau})$ if and only if*

$$\lim_{n\to\infty} \zeta^{h_i^{(n)}} = 1, \quad \text{for } i = 0, \dots, q-1, \tag{16}$$

where $q = \#V_k$, $k \geq 1$, and the corresponding eigenfunction is continuous.

For the rational eigenvalues of the Koopman operator, we have the following.

Lemma 8.2. [12, Proposition 2] *Let $k/d \in \mathbb{Q}$. Then $e^{2\pi ik/d}$ is a continuous eigenvalue of the simple Bratteli-Vershik system $(X_{(\widehat{V},\widehat{E},<)}, \widehat{\tau})$ if and only if d divides $h_i^{(n)}$ for all $0 \leq i \leq q-1$ and all sufficiently large n.*

Thus the rational spectrum of the Koopman operator of a Bratteli diagram consists of two parts: the eigenvalues $\zeta = e^{2\pi i\alpha}$, where α is a common divisor of the eigenvalues of the matrix B, and so determined by the matrix B, and the combinatorial eigenvalues which depend on the decomposition of $h^{(1)}$ over the basis of right eigenvectors of B. We present examples of both types in this section.

Example 8.3. Consider a Bratteli diagram where for $k \geq 0$ the vertex set V_k consists of a single vertex, and the edge set E_k consists of two edges, ordered from 0 to 1. The space path of this diagram can be identified with the space of infinite sequences $\Omega = \{0, 1\}^{\mathbb{N}}$, and there are unique minimal and maximal paths, consisting only of 0's and 1's respectively. It is well-known that the von Neumann-Kakutani map $(I, \mathfrak{a})$ is measurably isomorphic to the dynamical system on Ω induced by the Vershik map of the diagram, so we denote the induced map on Ω also by $\mathfrak{a}$. The system $(\Omega, \mathfrak{a})$ is called the *dyadic odometer*.

The matrix associated to this Bratteli diagram is the matrix

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{with eigenvalues 0 and 2.} \tag{17}$$

Thus the rational spectrum of the Koopman operator for $(\Omega, \mathfrak{a})$ consists of continuous eigenvalues $\{e^{2\pi ip/2^n} \mid p, n \geq 1\}$.

Recall for instance from [13] that if $\psi : (Y, g) \to (X, f)$ is a factor map of dynamical systems, then (continuous) eigenvalues of the Koopman operator of (X, f) must be contained in the set of (continuous) eigenvalues of (Y, g). Thus the aperiodic system (I^*_{np}, F^*_π) (resp. its minimal subsystem

(I^*_{min}, F^*_π)) of the rotated odometer factors onto the dyadic odometer $(\Omega, \mathfrak{a})$ if and only if the rational spectrum of the Koopman operator of (I^*_{np}, F^*_π) (resp. of its minimal subsystem (I^*_{min}, F^*_π)) contains the set $\{e^{2\pi i p/2^n} \mid p, n \geq 1\}$.

8.2. **Eventually stationary diagrams.** Recall that the sequence of substitutions $(\chi_k)_{k\geq 1}$, associated to the first return maps $(L_k, F_{\pi,k})$ in Theorem 5.11 is eventually periodic. Denote by k_0 the length of the pre-periodic part of $(\chi_k)_{k\geq 1}$, and by p_0 the length of one period. Then set

$$B = M_{k_0+p_0} \cdot \dots \cdot M_{k_0+1}, \tag{18}$$

and

$$\vec{w} = M_{k_0} \cdots M_1 \cdot h^{(1)}, \tag{19}$$

where $\cdot$ is the matrix multiplication. The stationary Bratteli diagram with associated matrix (18) and the vector $h^{(1)} = \vec{w}$ can be obtained from the diagram $(V, E, <)$ by telescoping. Thus for eventually pre-periodic systems with non-trivial pre-periodic part the theory described in Section 8.1 holds with B and $h^{(1)}$ in (18) and (19).

The pre-periodic part represented by $M_{k_0} \cdots M_1$ corresponds to the first return map of F_{π,k_0} to L_{k_0}, and the entire system (I^*_{np}, F^*_π) is Kakutani equivalent but not necessarily measurably isomorphic to the first return map (L_{k_0}, F_{π,k_0}). In general, the spectrum of a system and its first return map can be very different (as the example from [25, Section 4.5] shows rather spectacularly). The part of the rational spectrum determined by the common divisors of eigenvalues of the matrix B is independent of $\vec{w}$, but the combinatorial part may depend on it. In fact, the system can have extra rational eigenvalues $e^{2\pi i k/d}$ if the entries in the matrix product $M_{k_0} \cdots M_1$ are multiples of d.

8.3. **Rotated odometers with dyadic odometer factors.** When $q = 2^n$, then the aperiodic subsystem (I^*_{np}, F^*_π) is always conjugate to the dyadic odometer, and we study this case in detail in paper [4]. In this section we concentrate on the case when $q \geq 3$ and $q \neq 2^n$, for any $n \geq 1$.

A detailed study of examples for $q = 3, 5, 7$ shows that the minimal subsystem (I^*_{min}, F^*_π) of the aperiodic subsystem (I^*_{np}, F^*_π) may have the dyadic odometer as a factor. In some cases the dyadic eigenvalues of (I^*_{min}, F^*_π) are determined by the matrix B, and in some cases they arise in the combinatorial part of the spectrum. In order to prove Theorems 1.7 and 1.8 we first describe several such examples, and then build on them to prove the theorems.

Proposition 8.4. *Let $q = 3$ and let $\pi = (012)$ or $\pi = (021)$. Then (I^*_{np}, F^*_π) is conjugate to the dyadic odometer, and $e^{2\pi i k/2^n}$ are continuous eigenvalues for all $k, n \in \mathbb{N}$.*

Proof. Applying the algorithm of Section 5.6 to the systems in question we obtain:

$\pi \to \pi_1$	Substitution χ_1	Associated Matrix	char. polynomial
$(012) \to (012)$	$\begin{cases} 0 \to 0221 \\ 1 \to 0221 \\ 2 \to 0011 \end{cases}$	$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix}$	$x^3 - 2x^2 - 8x$ with eigenvalues $4, -2, 0$
$(021) \to (021)$	$\begin{cases} 0 \to 0112211220 \\ 1 \to 0 \\ 2 \to 0 \end{cases}$	$\begin{pmatrix} 2 & 4 & 4 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$x^3 - 2x^2 - 8x$ with eigenvalues $4, -2, 0$

In both cases the associated matrices are primitive, so the system is minimal. Since in both cases 4 and -2 are eigenvalues of the matrices, for any $p, n \geq 1$ the number $e^{2\pi i p/2^n}$ is a continuous eigenvalue of the Koopman operator. Consequently, the dyadic odometer is a factor of (I^*_{np}, F^*_π).

In both cases the Bratteli diagram has the equal incoming edge property so it is Toeplitz [17], and invertible Toeplitz shifts are odometers [9, below Theorem 5.1]. Thus (I^*_{np}, F^*_π) it is conjugate to the dyadic odometer. $\square$

Proposition 8.5. *Let $q = 5$ and let $\pi = (01234)$. Then the following is true for the aperiodic system (I^*_{np}, F^*_π):*

1. *The substitutions $\chi_k = \chi_1$ for all $k \geq 1$.*
2. *The minimal set I^*_{min} is a proper subset of I^*_{np}.*
3. *For any $p, n \geq 1$, the number $e^{2\pi i p/2^n}$ is a continuous eigenvalue of (I^*_{min}, F^*_π), and (I^*_{min}, F^*_π) is conjugate to the dyadic odometer.*
4. *The dyadic odometer is the maximal equicontinuous factor of (I^*_{np}, F^*_π).*

Proof. Applying the algorithm of Section 5.6 to the systems in question, we obtain:

$\pi \to \pi_1$	Substitution χ_1	Associated Matrix
$(01234) \to (01234)$	$\begin{cases} 0 \to 03 & \sum_{i=0}^4 \lvert\chi_1(i)\rvert = 40 \\ 1 \to 03 & \\ 2 \to 03 & \\ 3 \to 03 & \\ 4 \to 0422211143143143042042042221143 & \end{cases}$	$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 4 & 8 & 8 & 4 & 8 \end{pmatrix}$

The system (I^*_{np}, F^*_π) is not minimal, since $\chi_1(i) = 03$ for $i = 0, 1, 2, 3$, and so the associated matrix of the substitution is not primitive. The minimal subdiagram has vertices 0 and 3 at each level, and the associated matrix is (17) with eigenvalues 0 and 2. It follows that every $e^{2\pi i p/2^m}$, $p, m \geq 1$ is a continuous eigenvalue of the Koopman operator of the minimal subsystem, and by [13, Lemma 1.6.7] the dyadic odometer $(\Omega, \mathfrak{a})$ is a continuous factor of (I^*_{min}, F^*_π), say with factor map ψ. Since the minimal subsystem has no other eigenvalues, by [13, Theorem 1.5.6], the restriction $\psi|_{I^*_{min}}$ to the minimal subsystem is a conjugacy to the dyadic odometer. This proves items (1) - (3).

By Example 7.3 (I^*_{np}, F^*_π) has a measure μ_2 which lifts to the ergodic measure on I, and the eigenvalues of B are 2 and 8. By the algorithms in [2] for $m \geq 1$ the number $e^{2\pi i 2^{-m}}$ is a measurable eigenvalue of $(I^*_{np}, F^*_\pi, \mu_2)$, and so $(I^*_{np}, F^*_\pi, \mu_2)$ factors on the dyadic odometer. By [2, Remark 6.6] the factor map is continuous.

Next we show that the dyadic odometer $(\Omega, \mathfrak{a})$ is the maximal equicontinuous factor of (I^*_{np}, F^*_π). Note that (I^*, F^*_π) has no periodic orbits, so $I^*_{np} = I^*$ and $X_{(V',E,'<)} = X_{(V,E,<)}$ in the notation of Section 5.7. The minimal subsystem $(X_{(\widehat{V},\widehat{E},<)}, \widehat{\tau})$ of $(X_{(V,E,<)}, \tau)$ is conjugate via ψ to $(\Omega, \mathfrak{a})$, in particular, every point in $(\Omega, \mathfrak{a})$ has a preimage in $(X_{(\widehat{V},\widehat{E},<)}, \widehat{\tau})$ and possibly more preimages in $X_{(V,E,<)} \setminus X_{(\widehat{V},\widehat{E},<)}$. Recall from Example 7.1 that the forward orbit of the path $z \in X_{(V,E,<)}$, that consists of only maximal edges $\bar{e}_i \in E_i$ from $4 \in V_i$ to $4 \in V_{i+1}$, is non-recurrent under the Vershik map τ, and its forward orbit $\mathrm{orb}_\tau(z)$ converges on the minimal subset $X_{(\widehat{V},\widehat{E},<)}$.

Now take any path $x = (x_i) \in X_{(V,E,<)}$ and any $K \in \mathbb{N}$, and find another path $x'_K = (x'_{K,i}) \in X_{(V,E,<)}$ such that $x'_{K,i} = x_i$ for $1 \leq i \leq K$ and $x'_{K,i} = z_i = \bar{e}_i$ for all sufficiently large i. Such a path always exists since $\chi_1(4)$ contains all symbols in $\mathcal{A}$. Hence $\mathrm{orb}_\tau(x'_K)$ accumulates on the minimal subset $X_{(\widehat{V},\widehat{E},<)}$. Let $y, y'_K \in X_{(\widehat{V},\widehat{E},<)}$ be paths such that $\psi(y) = \psi(x)$ and $\psi(y'_K) = \psi(x'_K)$. By continuity of ψ, $y'_K \to y$ and of course also $x'_K \to x$ as $K \to \infty$. Since $\tau^k(x')$ accumulates on the minimal set $X_{(\widehat{V},\widehat{E},<)}$ as $k \to \infty$, and $\psi(\tau^k(x')) = \psi(\tau^k(y'))$ for all k, uniform continuity of ψ shows that $d(\tau^k(x'_K), \tau^k(y'_K)) \to 0$ as $k \to \infty$.

But since K is arbitrary, x and y are *regionally proximal.* Regionally proximal points must have the same image under the factor map ψ_{MEF} onto the maximal equicontinuous factor, see [21, Proposition 2.47]. It follows that since the dyadic odometer is an equicontinuous factor, it has to be the maximal one. Indeed, suppose that $(X_{(V,E,<)}, \tau)$ has a larger equicontinuous factor, with factor map ψ_{MEF}. If $x \in X_{(V,E,<)} \setminus X_{(\widehat{V},\widehat{E},<)}$ and $y \in X_{(\widehat{V},\widehat{E},<)}$ are points such that $\psi(x) = \psi(y)$, then they are regionally proximal. But then $\psi_{MEF}(x)$ and $\psi_{MEF}(y)$ are also regionally proximal. Since equicontinuous systems cannot have distinct regionally proximal points, $\psi_{MEF} = \psi$ and $(\Omega, \mathfrak{a})$ is indeed the maximal equicontinuous factor. □

Proposition 8.6. *Let $q = 7$ and $\pi = (0516234)$. Then the following is true for the aperiodic system (I^*_{np}, F^*_π):*

1. *The substitutions $\chi_k = \chi_1$ for all $k \geq 1$.*
2. *The minimal set I^*_{min} is a proper subset of I^*_{np}.*
3. *For any $p, m \geq 1$, the number $e^{2\pi i p/2^m}$ is a continuous eigenvalue of (I^*_{min}, F^*_π), and so (I^*_{min}, F^*_π) has the dyadic odometer as a factor. Every rational eigenvalue $e^{2\pi i 2^{-n}}$ belongs to the combinatorial part of the spectrum of (I^*_{min}, F^*_π).*
4. *The minimal subsystem (I^*_{min}, F^*_π) has no eigenvalues $e^{2\pi i\alpha}$ for irrational α.*
5. *The system (I^*_{np}, F^*_π) has a single ergodic invariant measure μ_1 supported on the minimal subset (I^*_{min}, F^*_π).*

Proof. Applying the algorithm of Section 5.6 we obtain:

$\pi \to \pi_1$	Substitution χ_1	Associated Matrix	char. polynomial
$(0516234) \to (0516234)$	$\begin{cases} 0 \to 0321 & \sum_{i=0}^{6} \lvert\chi_1(i)\rvert \\ 1 \to 0321 & \quad = 20 \\ 2 \to 001 & \\ 3 \to 011 & \\ 4 \to 01 & \\ 5 \to 01 & \\ 6 \to 01 & \end{cases}$	$\begin{pmatrix} 1&1&1&1&0&0&0 \\ 1&1&1&1&0&0&0 \\ 2&1&0&0&0&0&0 \\ 1&2&0&0&0&0&0 \\ 1&1&0&0&0&0&0 \\ 1&1&0&0&0&0&0 \\ 1&1&0&0&0&0&0 \end{pmatrix}$	$x^7 - 2x^6 - 6x^5$ with eigenvalues $1 \pm \sqrt{7}$ and 0 (multiplicity 5)

Restricting to the minimal subset we obtain the matrix

$$B = \begin{pmatrix} 1&1&1&1 \\ 1&1&1&1 \\ 2&1&0&0 \\ 1&2&0&0 \end{pmatrix}$$

which has eigenvalues 0 (with multiplicity 2) and $1 \pm \sqrt{7}$.

Although B does not have eigenvalues that are multiples of 2, the dyadic odometer is still a factor of the minimal subsystem by the following argument.

Lemma 8.7. *For any $m \geq 1$, there exists $n_m \geq 1$ such that 2^m divides of every component in $h^{(n)}$ for all $n \geq n_m$.*

Proof. Note that since the components of $h^{(1)}$ are equal, we have $h^{(n)} = (a_n, a_n, b_n, b_n)$, for $a_n = 2(a_{n-1} + b_{n-1})$, $b_n = 3a_{n-1}$. In particular, a_n is always even and b_n is even for $n \geq 2$, so the lemma holds for $m = 1$ with $n_1 = 2$. Suppose there is n_m such that 2^m divides a_n and b_n for $n \geq n_m$. Then

2^{m+1} divides a_{n_m+1}, and b_{n_m+2}. Also, 2^{m+1} divides a_{n_m+2}. The statement follows by induction, and we obtain $n_{m+1} = n_m + 2$. □

We now look for eigenvalues $e^{2\pi i\alpha}$ with α irrational. Both irrational eigenvalues of B are outside of the unit circle, and they are algebraic conjugates with minimal polynomial $x^2 - 7$. According to [12, Corollary 1], if $e^{2\pi i\alpha}$ is an eigenvalue, then for each algebraic conjugacy class of the eigenvalues of B, there is a polynomial $g(x) \in \mathbb{Q}[x]$ such that $g(x)$ takes the value α on each element in the conjugacy class which is outside of the unit circle, and such that the vector $h^{(1)}$ has non-trivial projection on the eigenspace corresponding to this element.

In our case, since $1 \pm \sqrt{7}$ are the only non-zero eigenvalues of B and $h^{(n)}$, $n \geq 1$, are vectors with integer components, $h^{(1)}$ must have non-zero projection on both eigenspaces. Therefore, if $e^{2\pi i\alpha}$ is a continuous eigenvalue, then there is a polynomial $g(x)$ with rational coefficients such that

$$g(1+\sqrt{7}) = g(1-\sqrt{7}) = \alpha \notin \mathbb{Q}.$$

Then also the polynomial $\widetilde{g}(x) = g(x+1)$ has integer coefficients $\tilde{g}_i$, and

$$\alpha = g(1 \pm \sqrt{7}) = \widetilde{g}(\pm\sqrt{7}) = \sum_i \widetilde{g}_i (\pm\sqrt{7})^i = a \pm b\sqrt{7}$$

for $a = \sum_{i \text{ even}} \widetilde{g}_i 7^{i/2} \in \mathbb{Q}$ and $b = \sum_{i \text{ odd}} \widetilde{g}_i 7^{(i-1)/2} \in \mathbb{Q}$. It follows that $b = 0$, and so $\alpha = a$ must be rational, which is a contradiction. Therefore, such polynomial $g(x)$ does not exist, and there are no eigenvalues of the form $e^{2\pi i\alpha}$ with α irrational.

Finally, since the associated matrix of the substitution has a single non-zero diagonal block, (I^*_{np}, F^*_π) has a single ergodic measure, corresponding to the minimal subsystem. □

A similar set of arguments gives the following.

Proposition 8.8. *Let $q = 7$ and $\pi = (0361425)$. Then the following is true for the aperiodic system (I^*_{np}, F^*_π):*

1. *The sequence $(\chi_k)_{k\geq 1}$ is constant, that is, $\chi_k = \chi_1$ for all $k \geq 1$.*
2. *The minimal set I^*_{min} is a proper subset of I^*_{np}.*
3. *For any $p, m \geq 1$, the number $e^{2\pi i p/2^m}$ is a continuous eigenvalue of (I^*_{min}, F^*_π), and so (I^*_{min}, F^*_π) has the dyadic odometer as a factor. Every rational eigenvalue $e^{2\pi i p/2^m}$ belongs to the combinatorial part of the spectrum of (I^*_{min}, F^*_π).*
4. *The minimal subsystem (I^*_{min}, F^*_π) has no eigenvalues of the form $e^{2\pi i\alpha}$, where α is irrational.*
5. *The system (I^*_{np}, F^*_π) has a single ergodic invariant measure μ_1 supported on the minimal subset (I^*_{min}, F^*_π).*

Proof. Applying the algorithm of Section 5.6 we obtain:

$\pi \to \pi_1$	Substitution χ_1	Associated Matrix	char. polynomial
(0361425) $\to$ (0361425)	$\begin{cases} 0 \to 0653 & \sum_{i=0}^{6} \lvert\chi_1(i)\rvert = 40 \\ 1 \to 0653 \\ 2 \to 0653 \\ 3 \to 0653 \\ 4 \to 013121212121212023 \\ 5 \to 013 \\ 6 \to 023 \end{cases}$	$\begin{pmatrix} 1&0&0&1&0&1&1 \\ 1&0&0&1&0&1&1 \\ 1&0&0&1&0&1&1 \\ 1&0&0&1&0&1&1 \\ 2&7&7&2&0&0&0 \\ 1&1&0&1&0&0&0 \\ 1&0&1&1&0&0&0 \end{pmatrix}$	$x^7 - 2x^6 - 6x^5$ with eigenvalues $1 \pm \sqrt{7}$ and 0 (multiplicity 5)

Restricting to the minimal set we obtain the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad \text{with eigenvalues } 1 \pm \sqrt{7}, 0\,(\text{multiplicity } 4).$$

By a similar argument as in Proposition 8.6 the system (I^*_{np}, F^*_π) has no irrational eigenvalues. Next note that, since the initial values $h^{(1)} = (1,1,1,1,1,1)^T$ are all equal, for $n \geq 1$

$$h^{(n)} = (a_n, a_n, a_n, a_n, b_n, b_n), \qquad \text{for } a_n = 2(a_{n-1} + b_{n-1}),\ b_n = 3a_{n-1}.$$

The argument proceeds as in Lemma 8.7 to show that for every $p, m \geq 1$ the number $e^{2\pi i p/2^m}$ is a (continuous) eigenvalue of the Koopman operator. $\square$

8.4. **A rotated odometer without the dyadic odometer factor.** In this section we exhibit an example of a rotated odometer which does not have the dyadic odometer as a factor.

Proposition 8.9. *Let $q = 5$ and let $\pi = (02431)$. Then the following is true for the aperiodic system (I^*_{np}, F^*_π):*

1. *The sequence $(\chi_k)_{k\geq 1}$ is constant, that is, $\chi_k = \chi_1$ for all $k \geq 1$.*
2. *The minimal set I^*_{min} is a proper subset of I^*_{np}.*
3. *For all integers $m \geq 1$ and $1 \leq p < 2^m$, the number $e^{2\pi i p/2^m}$ is not an eigenvalue of (I^*_{min}, F^*_π). So the dyadic odometer is not a factor of (I^*_{min}, F^*_π).*
4. *For any $a, b \in \mathbb{Q}$ and $\alpha = a + b\sqrt{5}$, there exists $s \in \mathbb{Z}$ such that the number $e^{2\pi i s\alpha}$ is an eigenvalue of the minimal subsystem (I^*_{min}, F^*_π), so (I^*_{min}, F^*_π) is not weakly mixing.*
5. *The aperiodic system (I^*_{np}, F^*_π) with Lebesgue measure has the cyclic group with four elements as the maximal equicontinuous factor, but the factor map is not continuous.*

Proof. Applying the algorithm of Section 5.6 we obtain item (1) of the Proposition:

$\pi \to \pi_1$	Substitution χ_1	Associated Matrix	char. polynomial
(02431) $\to$ (02431)	$0 \to 04212$ $\sum_{i=0}^4 \lvert\chi_1(i)\rvert = 40$ $1 \to 042$ $2 \to 04012$ $3 \to 0401334133420133413342 12$ $4 \to 012$	$\begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 5 & 3 & 8 & 5 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$	$x^5 - 10x^4 + 18x^3$ $+58x^2 + 47x + 8$ with eigenvalues $8, 2 \pm \sqrt{5}, -1, -1$

The system is covering, so $(X_{(V',E',<)}, \tau') = (X_{(V,E,<)}, \tau)$. The associated matrix has two eigenvalues of absolute value greater than 1, $\lambda_1 = 2 + \sqrt{5}$ with left eigenvector $v_1^\ell = (1, \frac{1}{2}(\sqrt{5}-1), 1, \frac{1}{2}(\sqrt{5}-1))$ and $\lambda_2 = 8$ with left eigenvector $v_2^\ell = (1,1,1,1,1)$. Thus there are two invariant measures, μ_1 supported on I^*_{min} and μ_2 supported on I^*_{np}.

Restricting to the minimal subset I^*_{min} we obtain the symmetric matrix

$$B = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \text{with eigenvalues } 2 \pm \sqrt{5}, -1, -1.$$

Computing the decomposition of the vector $h^{(1)}$ over the basis of right eigenvectors, we obtain that $h^{(1)}$ is a linear combination of the eigenvectors corresponding to $2 \pm \sqrt{5}$. The eigenvalue $2 - \sqrt{5}$ is inside the unit circle. Take $g_{a,b}(x) = a(x-2)+b = \alpha$ with $a, b \in \mathbb{Q}$, then $g(2+\sqrt{5}) = a\sqrt{5}+b$ is irrational. Then by [12, Corollary 1] there exists $s \in \mathbb{Z}$ such that $e^{2\pi i s(a\sqrt{5}+b)}$ is an eigenvalue of the Koopman operator of $(I^*_{min}, F^*_\pi, \mu_1)$. We showed item (4) of the proposition.

If $e^{2\pi i/2^m}$ is an eigenvalue of the Koopman operator of the minimal subsystem, then by Lemma 8.2, 2^m must divide $h_i^{(n)}$ for $0 \le i \le 4$, and n large enough. Let $h^{(n)} = (a_n, b_n, a_n, b_n, c_n)$, then

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \left(\begin{array}{cc|c} 3 & 2 & 0 \\ 2 & 1 & 0 \\ \hline 6 & 10 & 8 \end{array}\right) \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}. \tag{20}$$

Inverting the first block, we find

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix},$$

so $gcd(a_{n+1}, b_{n+1}) = gcd(a_n, b_n) = \cdots = gcd(a_0, b_0) = 1$. This shows that there cannot be a combinatorial rational eigenvalue $e^{2\pi i\alpha}$ other than $\alpha = 0$, and the minimal subsystem (I^*_{min}, F^*_π) has no rational eigenvalues. This shows item (3).

Now consider $(I^*_{np}, F^*_\pi, \mu_2)$ with Lebesgue measure μ_2. We show that $e^{2\pi i/2^m}$ is an eigenvalue of $(I^*_{np}, F^*_\pi, \mu_2)$ if and only if $m \in \{1, 2\}$. It follows that the cyclic group with 4 elements is a measurable factor of $(I^*_{np}, F^*_\pi, \mu_2)$. We show also that for any other rational or irrational α, the number $e^{2\pi i\alpha}$ is not an eigenvalue of $(I^*_{np}, F^*_\pi, \mu_2)$.

By [2, Theorem 6.3] to determine eigenvalues of $(I^*_{np}, F^*_\pi, \mu_2)$ we have to consider so-called 'diamonds', which for our situation are pairs of paths in $(X_{(V,E,<)}, \tau)$ of the same finite length $r \ge 1$ which start at the vertex marked by 3 in V_1 and end at the vertex marked by 3 in V_{r+1}. Since 3 does not occur in any $\chi_1(i)$ for $i \ne 3$, every edge in such a path joins vertices marked by 3 at consecutive levels, and it is sufficient to consider paths of length 1 which are just pairs of edges (j, j') between the vertices marked by 3 in V_1 and V_2. Let κ and κ' be the orders of j and j' in the set of edges incoming to 3 in V_2. Then by [2, Lemma 6.4] if 2^m divides $(\kappa - \kappa')h_3^{(n)}$ for all sufficiently large $n \ge 1$ and for all diamonds (j, j'), then $e^{2\pi i 2^{-m}}$ is an eigenvalue for $(I^*_{np}, F^*_\pi, \mu_2)$, where $h_3^{(n)}$ is the height of the 3-rd stack in V_n. Since the word $\chi_1(3)$ contains a subword of two consecutive 3's, we conclude that $e^{2\pi i 2^{-m}}$ is an eigenvalue if and only if 2^m divides $h_3^{(n)} = c_n$ for all sufficiently large n.

Since a_n and b_n are always odd, we can write $a_n = 2u_n + 1$ and $b_n = 2v_n + 1$. Then

$$c_{n+1} = 6a_n + 10b_n + 8c_n = 2(6u_n + 3 + 10v_n + 5 + 4c_n) = 4(3u_n + 5v_n + 2c_n + 4),$$

which shows that $e^{2\pi i/2}$ and $e^{2\pi i/4}$ are measurable eigenvalues of the Koopman operator for $(I^*_{np}, F^*_\pi, \mu_2)$; they are not continuous because $(I^*_{\min}, F^*_\pi)$ doesn't have these eigenvalues. We note that c_{n+1} is divisible by 8 if and only if the expression in the parenthesis in the formula for c_{n+1} is even. This can happen only if u_n and v_n are both odd, or they are both even.

We have that

$$a_{n+1} = 3a_n + 2b_n = 3(2u_n + 1) + 2(2v_n + 1) = 6u_n + 3 + 4v_n + 2,$$

so $u_{n+1} = 3u_n + 2v_n + 2$, which shows that u_{n+1} is even if and only if u_n is even. Since $a_1 = 1$ and so $u_1 = 0$ is even, we conclude that u_n is always even. Similarly,

$$b_{n+1} = 2a_n + b_n = 2(2u_n + 1) + 2v_n + 1 = 4u_n + 2 + 2v_n + 1,$$

so $v_{n+1} = 2u_n + 1 + v_n$, which shows that v_{n+1} is even if v_n is odd, and v_{n+1} is odd if v_n is even. Since $v_1 = 0$ is even, it follows that for $k \ge 1$ the height c_{2k} is not divisible by 8, and so $e^{2\pi i/2^m}$ for $m \ge 3$ is not an eigenvalue of $(I^*_{np}, F^*_\pi, \mu_2)$.

To show that there are no other rational eigenvalues, let $p \geq 3$ be a prime. Suppose by contradiction that $e^{2\pi i/p} \to 1$, so $p|c_n$ for n sufficiently large.

Now note that for $n \geq 1$ we have $a_n = F_{3n}$ and $b_n = F_{3n-1}$, where F_n is the n-th Fibonacci number. The sequence of Fibonacci numbers $(F_n \bmod p)_n$ is periodic, therefore $(6a_n + 10b_n) \bmod p$ is also periodic, and there are infinitely many n's such that

$$(6a_n + 10b_n) \bmod p \equiv (6a_0 + 10b_0) \mod p = 16 \bmod p \not\equiv 0 \bmod p.$$

Recalling that $c_{n+1} = 6a_n + 10b_n + 8c_n$, we find that c_n and c_{n+1} cannot be simultaneously divisible by p, so $e^{2\pi i/p}$ cannot be an eigenvalue.

To show that $\alpha = u + v\sqrt{5}$, $u \in \mathbb{Z}$, $v \in \mathbb{N}$, is not an eigenvalue of $(I^*_{np}, F^*_\pi, \mu_2)$, first note that by subtracting $u-2v$, it suffices to verify $\alpha = v(2+\sqrt{5})$. The first block in formula (20) is a Pisot matrix (in fact, it is the third power of $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, with eigenvalues $\delta_+ = 2 + \sqrt{5}$ and $\delta_- = 2 - \sqrt{5} \in (-\frac{1}{2}, 0)$.

By standard computation, we find

$$\begin{aligned} a_n &= \frac{5+3\sqrt{5}}{10}\delta_+^n + \frac{5-3\sqrt{5}}{10}\delta_-^n \\ b_n &= \frac{5+\sqrt{5}}{10}\delta_+^n + \frac{5-\sqrt{5}}{10}\delta_-^n \\ c_n &= 5 \cdot 8^n - \frac{10+4\sqrt{5}}{5}\delta_+^n - \frac{10-4\sqrt{5}}{5}\delta_-^n. \end{aligned} \tag{21}$$

Therefore

$$\begin{aligned} \alpha c_n &= v\delta_+ \left(5 \cdot 8^n - \frac{10+4\sqrt{5}}{5}\delta_+^n - \frac{10-4\sqrt{5}}{5}\delta_-^n \right) \\ &= v \left(5 \cdot 8^n(\delta_+ - 8) + 5 \cdot 8^{n+1} - \frac{10+4\sqrt{5}}{5}\delta_+^{n+1} - \frac{10-4\sqrt{5}}{5}\delta_-^{n+1} - \frac{10-4\sqrt{5}}{5}(\delta_+ - \delta_-)\delta_-^n \right) \\ &= v \left(5 \cdot 8^n(\sqrt{5} - 6) + c_{n+1} + 4(2-\sqrt{5})\delta_-^n \right). \end{aligned}$$

Thus the distance to the nearest integer is

$$\|\alpha c_n\| = \|5 \cdot 8^n v\sqrt{5} + 4v(2-\sqrt{5})\delta_-^n\| \geq \|5v \cdot 8^n\sqrt{5}\| - \|4v\delta_-^{n+1}\|.$$

Let $\varepsilon_n \in (-\frac{1}{2}, \frac{1}{2})$ be the fractional part of $5v \cdot 8^n\sqrt{5}$. Then for $|\varepsilon_n| \leq \frac{1}{16}$ we have $\varepsilon_{n+1} = 8\varepsilon_n$, so $|\varepsilon_n|$ increases in n until $|\varepsilon_n| > \frac{1}{16}$. Therefore $\|5v \cdot 8^n\sqrt{5}\| \not\to 0$ and hence neither does $\|\alpha c_n\| \to 0$, so $e^{2\pi i\alpha}$ is not an eigenvalue of the global system.

Finally, we show that there is no other eigenvalue for μ_2. Say $e^{2\pi i\alpha}$ for $\alpha \notin \mathbb{Q}[\sqrt{5}]$ is an eigenvalue. Then,

$$\|\alpha(6a_n + 10b_n)\|^2 = \|\alpha c_{n+1} - 8\alpha c_n\|^2 \leq 81 \max\{\|\alpha c_{n+1}\|^2, \|\alpha c_n\|^2\}$$

is summable in n. Using $a_n = F_{3n}$ and $b_n = F_{3n-1}$, where F_n are the Fibonacci numbers,

$$\begin{aligned} \alpha(6a_n + 10b_n) &= 2\alpha a_n(3 + 5\frac{b_n}{a_n})) \\ &= 2\alpha a_n(3 + 5(\sqrt{5} - 1)) + 5\alpha\sqrt{5}(2-\sqrt{5})^n \\ &= \beta a_n + o(2^{-n}) \qquad \text{for } \beta = 2\alpha(5\sqrt{5} - 2). \end{aligned}$$

Therefore $\|\beta(a_{n+1} + a_n)\|$ is summable. By (21) we have

$$\beta(a_{n+1} + a_n) = \beta(3+\sqrt{5})a_n + o(2^{-n}) =: \gamma a_n + o(2^{-n}),$$

and we know (by (16) and since $a_n = F_{3n}$ for the Fibonacci numbers F_n) that $\|\gamma a_n\|$ is only square summable if $\gamma \in \mathbb{Q}[\sqrt{5}]$. Hence (I^*_{np}, F^*_π) has no irrational eigenvalues. This shows item (5). $\square$

8.5. **Proofs of Theorems 1.7 and 1.8.** *Proof of Theorem 1.7.* The minimal subsystems in Propositions 8.5 and 8.9 provide examples for items (1) and (2) of Theorem 1.7. From one example, it is always possible to construct infinitely many examples by doubling q and changing the permutation π to

$$\pi' : \{0, \dots, 2q-1\} \to \{0, \dots, 2q-1\}, \quad \pi'(i) = \begin{cases} \pi(i-q) & \text{if } i \geq q \\ i+q & \text{if } i < q \end{cases}$$

because then the first return map of $F_{\pi'}$ to $[0, 1/2)$ is conjugate to F_π on $[0, 1)$ via the scaling $h(x) = 2x$. This proves the theorem. $\square$

Proof of Theorem 1.8. Item (1) is proved in item (4) of Proposition 8.5, and item (2) of the theorem is proved in (5) of Proposition 8.9. $\square$

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