ON DOUBLE SPIRALS IN FIBONACCI-LIKE UNIMODAL INVERSE LIMIT SPACES.

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Abstract. In this paper, we represent composants of unimodal inverse limit spaces as walks on a tree, related to the Hofbauer tower of the unimodal map. The goal is, for Fibonacci-like unimodal maps, to study the possibility of asymptotic composants. Whereas the question of their existence is still open, we show that Fibonacci-like inverse limit spaces possess so-called double spirals, which shows that points with different symbolic tails can still be on the same arc in such an inverse limit space. This shows that the converse of a result by Brucks & Diamond doesn't hold.

1. Introduction

The topological structure of inverse limit spaces is a central theme in continuum theory. The full classification of inverse limit spaces of a fixed unimodal bonding map revolves around the so-called Ingram conjecture, which was recently solved in [2]. Thus we know that the inverse limit spaces of tent maps $T : [0,1] \to [0,1]$, $x \mapsto \min\{sx, s(1-x)\}$ are non-homeomorphic for every two slopes $1 \leqslant s < s' \leqslant 2$. This result came after a long string of papers with partial answers and addressing various aspects of the fine-structure of such unimodal inverse limit spaces.

In [3], the existence of asymptotic arc-components for periodic unimodal inverse limit spaces was discovered (and an upper bound of their cardinality was given), using the substitutive nature of the symbolic dynamics of such inverse limit spaces. Two arc-components C and C' (*i.e.,* continuous images of $\mathbb{R}$ in the inverse limit space) are called *asymptotic* if they can be parametrised by $\varphi : \mathbb{R} \to C$ and $\varphi' : \mathbb{R} \to C'$ such that $d(\varphi(t), \varphi'(t)) \to 0$ as $t \to \infty$. In [11], a different algorithm for finding asymptotic arc-components and their cluster structure was presented, based on the symbolic approach of Brucks & Diamond [8], and it was also shown that unimodal maps with non-recurrent critical point have no asymptotic arc-components in their inverse limit spaces. In this paper, we extend these ideas to unimodal maps with non-periodic recurrent critical points (in particular unimodal maps of Fibonacci-like type) and present a new viewpoint on the algorithm from [11], which we call "walks through the Hofbauer tree (or tower)". Although we do not answer here the question whether Fibonacci-like

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unimodal inverse limit spaces possess asymptotic composants, we demonstrate the existence of (uncountably many) so-called double spirals in Fibonacci-like unimodal inverse limit spaces. This is a degenerate pair of asymptotic arc-components C and C', in the sense that $d(\varphi(t), \varphi'(t)) \to 0$ as $t \to \infty$ only because $\varphi(t)$ and $\varphi'(t)$ converge to the same point x in the inverse limit space $\varprojlim([c_2, c_1], T)$. Hence C and C' are joint at x, forming a single arc-component, but showing that the converse of a result of [8], namely that points in $\varprojlim([c_2, c_1], T)$ with the same tails of their backward itineraries belong to the same arc-component, is false. Indeed, points in C and C' have different tails, yet belong to the same (doubly spiraling) arc-component.

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2. Preliminaries

Let $T : [0,1] \to [0,1]$ be a tent map with slope s and critical point $c = \frac{1}{2}$: $T(x) = \min\{sx, s(-x)\}$. The sequence of its cutting times, *i.e.,* iterates such that the image of the central branch contains c, is denoted as $\{S_k\}_{k \geqslant 0}$ and kneading map is $Q : \mathbb{N} \to \mathbb{N}$, so (cf. [9])

$$S_0 = 1 \text{ and } S_k = S_{k-1} + S_{Q(k)}.$$

It sometimes shortens formulas if we use $R(k) := Q(k+1)$, so

$$S_{k+1} = S_k + S_{R(k)}.$$

Each unimodal map therefore is characterized by its kneading map. Conversely, each map $Q : \mathbb{N} \to \mathbb{N} \cup \{0\}$ satisfying $Q(k) < k$ and the *admissibility condition*

$$\{Q(k+j)\}_{j \geq 1} \succeq \{Q(Q^2(k)+j)\}_{j \geq 1} \tag{1}$$

(where $\succeq$ denotes the lexicographical ordering) is the kneading map of some unimodal map.

Let $\nu = \nu_1\nu_2\nu_3 \cdots \in \{0,1\}^{\mathbb{N}}$ be the *kneading sequence* of T. The cutting times relate to ν as

$$S_0 = 1, \quad S_k = \min\{i > S_{k-1} : \nu_i \neq \nu_{i-S_{k-1}}\}.$$

Lemma 1. *Let $S_{k-1} \leqslant n < S_k$ and let $\rho(n) = \min\{i > n : \nu_i \neq \nu_{i-n}\}$. Then the number of ones in $\nu_{n+1} \dots \nu_{\rho(n)}$ is even. If $\rho(n) \geqslant S_k$, then*

$$S_k - n = S_{Q^j(k)} \text{ where } j \text{ is } \begin{cases} \text{even} & \text{if } \rho(n) = S_k, \\ \text{odd} & \text{if } \rho(n) > S_k. \end{cases}$$

Proof. See [9]. □

For $e \in \{0,1\}$, let $\hat{e} = 1 - e$ denote the opposite symbol.

Lemma 2. *If $\nu_1 \dots \nu_{S_v}\nu_1 \dots \hat{\nu}_{S_u}$ is admissible, then $u \geq Q(v+1)$.*

Proof. If $u < Q(v+1)$, then $n := S_v + S_u < S_{v+1}$ is not a cutting time, so $\nu_1 \dots \hat{\nu}_n$ is not an admissible word. □

Let $\beta(n) = n - \sup\{S_k < n\}$ for $n \geq 2$ and find recursively the images of the central branch of T^n (the levels in the Hofbauer tower, see *e.g.* [9, 7]) as

$$D_1 = [0, c_1] \text{ and } D_n = [c_n, c_{\beta(n)}].$$

In [9] it is also shown (and this is not hard to see) that

$$D_n \subset D_{\beta(n)} \text{ for each } n, \tag{2}$$

and that if $J \subset [0,1]$ is a maximal interval on which T^n is monotone, then $T^n(J) = D_m$ for some $m \leqslant n$. Later in this paper we will put conditions on the kneading map, such as $Q(k) \to \infty$ or Q being monotone.

The core inverse limit space $\varprojlim([c_2, c_1], T)$ is

$$\varprojlim([c_2, c_1], T) = \{x = (\dots, x_{-2}, x_{-1}, x_0) : [0,1] \ni x_i = T(x_{i-1}) \text{ for all } i \leqslant 0\},$$

equipped with metric $d(x,y) = \sum_{i \leqslant 0} |x_i - y_i| 2^i$ and induced (or shift) homeomorphism

$$\hat{T}(\dots, x_{-2}, x_{-1}, x_0) = (\dots, x_{-2}, x_{-1}, x_0, T(x_0)).$$

Let $\pi_i : \varprojlim([c_2, c_1], T) \to [0,1]$, $\pi_i(x) = x_i$ be the i-th projection map. The *composant* of a point $x \in \varprojlim([c_2, c_1], T)$ is the union of all proper subcontinua of $\varprojlim([c_2, c_1], T)$ containing x; if $s > \sqrt{2}$ and unless x is contained in a non-arc subcontinuum, the composant of x coincides with the arc-connected component of x.

3. Folding Patterns and Treewalks

Assume that composant C of $\varprojlim([c_2, c_1], T)$ is a ray, parametrised as $\varphi : \mathbb{R} \to C$. Then for each $p \in \mathbb{N}_0$, we can find the sequence $\{t_n\}_{n \in \mathbb{Z}}$ such that $\varphi(t_n)$ is a p-turning point (or p-point for short), *i.e.*, there is $i > p$ such that $\pi_{-i}(\varphi(t_n)) = c$. The p-level is

$$\alpha_n := L_p(\varphi(t_n)) := i - p.$$

Unless otherwise stated, we will set $p = 0$ in the sequel and leave out p from the notation.

The sequence $\{\alpha_n\}_{n \in \mathbb{Z}}$ is called the *folding pattern* of C, and depending on the position of $\varphi(0) \in C$ and the orientation of φ, shifted and/or reversing versions of $\{\alpha_n\}_{n \in \mathbb{Z}}$ indicate folding pattern of the same composant. We will study folding patterns to determine whether C contains endpoints, and whether two composants C and C' are asymptotic, *i.e.*, they allow parametrizations φ and φ' such that $d(\varphi(t), \varphi'(t)) \to 0$ as $t \to \infty$.

Given $x = \varphi(t) \in C$, let $e(x) = \{e_i(x)\}_{i \in \mathbb{Z}}$ be the two-sided itinerary of x defined by

$$e_i(x) = \begin{cases} 0 & \text{if } i \leqslant 0, x_i \leqslant c \text{ or } i > 0, T^i(x_0) \leqslant c. \\ 1 & \text{if } i \leqslant 0, x_i \geqslant c \text{ or } i > 0, T^i(x_0) \geqslant c. \end{cases}$$

Hence the i such that $x_i = c$ (for $i \leqslant 0$) or $T^i(x_0) = c$ (for $i > 0$)leaves an ambiguity in $e(x)$, but if it occurs at position i then $e_{i+1}(x)e_{i+2}(x)\cdots = \nu$. For any word $w_1 \dots w_n$, define

$$\vartheta(w_1 \dots w_n) := \#\{1 \leqslant i \leqslant n : w_i = 1\}.$$

Assume that φ parametrises C such that $\varphi(0) = x$ and $\pi_0 \circ \varphi$ preserves orientation at 0. Then

$$\begin{cases} \alpha_1 = \sup_i\{e_{-i+1} \dots e_{-1} = \nu_1 \dots \nu_{i-1} \text{ and } \vartheta(\nu_1 \dots \nu_{i-1}) \text{ is even}\}; \\ \alpha_0 = \sup_i\{e_{-i+1} \dots e_{-1} = \nu_1 \dots \nu_{i-1} \text{ and } \vartheta(\nu_1 \dots \nu_{i-1}) \text{ is odd}\}. \end{cases} \tag{3}$$

This means that if A, $x \in A \subset C$, is the largest arc on which $\pi_0 \circ \varphi$ is homeomorphic, then $\pi_0(A) = [c_{\alpha_0}, c_{\alpha_1}]$ and the similar arcs $A_\pm$ adjacent to A are such that $\pi_0(A \cap A_-) = c_{\alpha_0}$ and $\pi_0(A \cap A_+) = c_{\alpha_1}$. The role of α_0 and α_1 is interchanged if φ reverses orientation.

As shown in [10], if the sup in (3) is infinite, then C contains an endpoint $p \in A$, and $\pi_0(p)$ is the right or left boundary point of $\pi_0(A)$ according to whether $\sup_i = \infty$ is achieved for the case that ϑ is even or odd. It can happen that both occur for the same x, and the entire arc-component of x consists of A alone, which can, but need not necessarily, be a singleton.

Lemma 3. *If x and x' have the same left tail or eventually the same folding pattern, then they belong to the same arc-component.*

Proof. To clarify, $e(x)$ and $e(x')$ have the same left tail if there is N such that $e_i(x) = e_i(x')$ for all $i < -N$. The first statement was proved by Brucks & Diamond [8]. For the second statement, the folding pattern $\{\alpha_n\}_{n \in \mathbb{Z}}$ of x defines the itinerary $e(x)$ uniquely as follows: Let

$$k_0 = 1 \text{ and } k_i = \min\{k > k_{i-1} : \alpha_k > \alpha_{k_{i-1}}\}.$$

Then set

$$e_{-k_i} \dots e_{-1} = \nu_1 \dots \nu_{k_i - k_{i-1} - 2} \hat{\nu}_{k_i - k_{i-1} - 1} \nu_{k_i - k_{i-1}} \dots \nu_{k_i - k_{i-2} - 2} \hat{\nu}_{k_i - k_{i-2} - 1} \nu_{k_i - k_{i-2}} \dots \dots \nu_{k_i}.$$

Repeating this for k_{i+1}, we see that $e_{-k_i} \dots e_0$ will not change, so this construction is consistent, and it will lead to the unique left itinerary that allows for the given folding pattern. If x and x' have eventually the same folding pattern, then $e(x)$ and $e(x')$ have the same tail and hence belong to the same arc-component. □

Let us now give further rules for the folding pattern relating to cutting times.

$$\text{For each } n \in \mathbb{Z}, \text{ there is } k_n \text{ such that } S_{k_n} = |\alpha_n - \alpha_{n-1}|. \tag{4}$$

Proof. Assume $l := \alpha_n > \alpha_{n-1}$ and let $J_{-l} \ni x_{-l}$ be the maximal interval on which T^l is monotone. By the definition of α_n, $c \in \partial J_{-l}$, so J_{-l} is a central domain of monotonicity of T^l. Furthermore $T^{\alpha_n - \alpha_{n-1}}(J_{-l}) \ni c$ by the definition of α_{n-1}, and therefore $\alpha_n - \alpha_{n-1}$ is indeed a cutting time. □

Rule (4) allows the visualisation of C and its folding pattern as an infinite walk on a tree, constructed as follows:

(1) Start with D_2, that is $\bullet_2$ ——— $\bullet_1$
(2) Attach D_n, that is $\bullet_n$ ——— $\bullet_{\beta(n)}$ to vertex $\bullet_{\beta(n)}$ for $n = 3, 4, 5, \dots$.
(3) The arc $\bullet_n$ ——— $\bullet_m$ is shorter as $|n - m|$ is larger (in analogy to the p-adic topology).

Figure 1 gives the tree for the Fibonacci map, that is the map T with cutting times equal to the Fibonacci numbers and kneading map $Q(k) = \max\{0, k - 2\}$. The walk

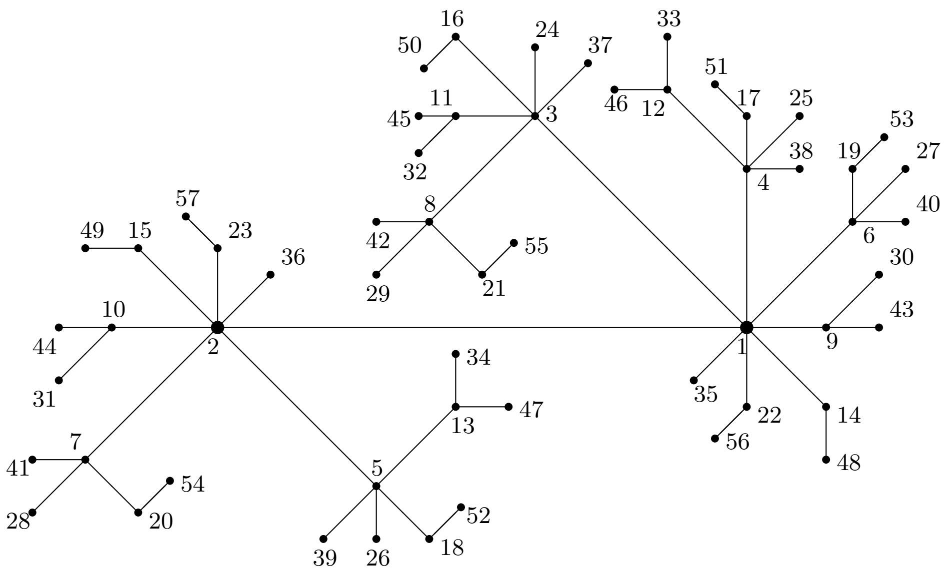

FIGURE 1. Tree of the Fibonacci map up to node 57.

on this tree is carried out as follows:

- ν is the public key (the same for all arc-components C) and $e = \dots e_{-3}e_{-2}e_{-1}e_0$ is key for a specific arc-components C.
- Let

$$R/L = \sup\{i > 0 : e_{-i+1} \dots e_{-1} = \nu_1 \dots \nu_{i-1}, \vartheta(\nu_1 \dots \nu_{i-1}) \text{ is even/odd}\}, \tag{5}$$

  cf. (3).
- (1) Compute R, move to node R and swap entry e_{-R}.
  (2) Compute L, move to node L and swap entry e_{-L}.
  (3) Goto 1.
- R or $L = \infty$ corresponds to endpoints of C.

Example 1. *The kneading sequence of the Fibonacci map T is*

$$\nu = 100111011001010011100100111011001 1\cdots$$

and starting with the backward itinerary in the first row, we perform the above algorithm for a few steps:

$$
\begin{array}{l|c}
\dots 1\ 1\ \underline{1}\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0 & R=17 \\
\dots 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ \underline{0}\ 1\ 0\ 0 & L=4 \\
\dots 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ \underline{0} & R=1 \\
\dots 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ \underline{1}\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 1 & L=9 \\
\dots 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ \underline{1} & R=1 \\
\dots 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ \underline{1}\ 1\ 0\ 0 & L=4 \\
\dots 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ \underline{0} & R=1 \\
\dots 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ \underline{0}\ 1 & L=2 \\
\dots 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ \underline{1} & R=1 \\
\dots 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ \underline{1}\ 1\ 0 & L=3 \\
\dots 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ \underline{0} & R=1 \\
\dots 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ \underline{0}\ 1\ 0\ 0\ 1\ 1 & L=6
\end{array}
$$

where the underlined entry is swapped at the next line. Extending this pattern we get a walk:

$$
\begin{gathered}
17-4-1-9-1-4-1-2-1-3-1-6-1-3-1-2-1-4-1- \\
2-1-3-1-2-5-2-1-3-1-2-1-4-1-2-7-2-1-4-12
\end{gathered}
$$

which in fact represents the graph of T^9 on D_8.

Lemma 4. *Further rules on folding patterns $\{\alpha_n\}_{n\in\mathbb{Z}}$ are as follows:*

(6) *If $\alpha_n < \alpha_{n-1} < \alpha_{n+1}$ then $k_n = Q(k_{n+1})$.*

(7) *If $\alpha_{n-1} < \alpha_n < \alpha_{n+1}$ then $Q(Q(k_{n+1})+1) \leqslant k_n < Q(k_{n+1}+1)$.*

(8) *If $\alpha_n = \alpha_m$ for $m < n$, then there exists $m < l < n$ such that $\alpha_l > \alpha_m$.*

If $\alpha_m = \alpha_n \neq \alpha_l$ for $m < l < n$, then $\alpha_{m+i} = \alpha_{n-i}$ for all

(9) *$0 \leqslant i \leqslant n-m$ and $\alpha_{(m+n)/2} > \alpha_l$ for all $m \leqslant l \leqslant n, l \neq (m+n)/2$.*

Proof. Let $A = [\varphi(t_{n-1}), \varphi(t_{n+1})] \subset C$, and let $J := \pi_{-\alpha_n}(A)$. First assume that $\alpha_n < \alpha_{n-1} < \alpha_{n+1}$. Then J is the arc

$$J = [c_{\alpha_{n-1}-\alpha_n}, c_{\alpha_{n+1}-\alpha_n}] = [c_{S_{k_n}}, c_{S_{k_{n+1}}}] \ni c = \pi_{-\alpha_n}\varphi(t_n).$$

In fact, since $\alpha_{n+1} > \alpha_{n-1}$, $J = D_{S_{k_{n+1}}}$ is the image of the central branch of $T^{S_{k_{n+1}}}$. Therefore $k_n = Q(k_{n+1})$, proving (6) and in fact $\alpha_{n+1} - \alpha_{n-1} = S_{k_{n+1}} - S_{k_n} = S_{k_{n+1}-1}$.

Next assume that $\alpha_{n-1} < \alpha_n < \alpha_{n+1}$, then (recalling that $z_k, \hat{z}_k \in T^{-S_k}(c)$ are closest precritical points)

$$J = [c_{\alpha_{n-1}-\alpha_n}, c_{\alpha_{n+1}-\alpha_n}] = [z_{k_n}, c_{S_{k_{n+1}}}] \ni c = \pi_{-\alpha_n}\varphi(t_n).$$

Since $T^{\alpha_n}|_J$ has only one turning point, namely c, and $\alpha_{n+1} > \alpha_n - \alpha_{n-1} = S_{k_n}$, we see that $c_{S_{k_{n+1}}} \in [z_{k_n}, \hat{z}_{k_n}]$, and therefore $Q(k_{n+1}+1) \geqslant k_n + 1$. Also $c_{S_{Q(k_{n+1})}} \notin [z_{k_n}, \hat{z}_{k_n}]$ because otherwise there would have been a node $T^{\alpha_n}(c_{S_{Q(k_{n+1})}})$ between α_{n-1} and α_n. Therefore $Q(Q(k_{n+1})+1) \leqslant k_n$, proving (7).

Rules (8) and (9) definitely not new, see *e.g.* [13, 16]. To prove (8), take the non-degenerate arc $A = [\varphi(t_m), \varphi(t_n)] \subset C$, and assume by contradiction that $\alpha_l \leqslant \alpha_m$ for all $l \in \{m+1, \dots, n-1\}$. Then $J := \pi_{-\alpha_m}(A)$ is an arc such that $T^{\alpha_m}|_J$ is monotone. But also both endpoints of J map to c under T^{α_m}, a contradiction.

To prove (9), take the non-degenerate arc $A = [\varphi(t_m), \varphi(t_n)] \subset C$, and let $m < l < n$ be such that α_l is maximal. Then $\pi_{-l} : A \to \pi_{-l}(A)$ is monotone and $\pi_{-l}(\varphi(t_l)) = c$. The pattern of precritical points is symmetric about c, and since $\pi_{-l}(\varphi(t_m))$ and $\pi_{-l}(\varphi(t_n))$ are precritical points of the same order, $\varphi_{-l}(A)$ is a symmetric arc around c, implying (9). This completes the proof. □

Lemma 5. *Assume that the kneading map $Q(k) \to \infty$ as $k \to \infty$. Then for every $K \in \mathbb{N}$ there exists $L \in \mathbb{N}$ such that $\alpha_{n-2}, \alpha_{n-1} \leqslant K$ implies that $\alpha_n \leqslant L$.*

Proof. Let L be so large that $S_{Q(Q(l)+1)} \geqslant K$ for all $l \in \mathbb{N}$ with $S_l > L - K$. Assume by contradiction that $\alpha_n > L$, so $S_{k_n} = \alpha_n - \alpha_{n-1} > L - K$. Therefore

$$S_{Q(Q(k_n)+1)} \leqslant \begin{cases} S_{k_{n-1}} & \text{by (7) if } \alpha_{n-2} < \alpha_{n-1}, \\ S_{Q(k_n)} = S_{k_{n-1}} & \text{by (6) if } \alpha_{n-1} < \alpha_{n-2}. \end{cases}$$

But $S_{k_{n-1}} = |\alpha_{n-1} - \alpha_{n-2}| < K$, so this contradicts that $S_{Q(Q(k_n)+1)} \geqslant K$, completing the proof. □

4. Asymptotic Composants

Definition 1. *Two distinct arc-components C and C' of $\varprojlim([c_2, c_1], T)$ are* asymptotic *if there are parametrizations φ and φ' such that $d(\varphi(t), \varphi'(t)) \to 0$ as $t \to \infty$.*

This implies that when viewed as walks on the tree, their courses are getting closer, and we express this in terms of their folding patterns as follows.

Proposition 1. *Assume that T is a tent map with slope $s > 1$ and kneading map $Q(k) \to \infty$. Let C and C' be rays with distinct backward tails. Then C and C' are asymptotic if and only if they have parametrizations whose folding patterns $\{\alpha_n\}_{n\in\mathbb{Z}}$ and $\{\alpha'_n\}_{n\in\mathbb{Z}}$ satisfy the following properties:*

- *There are non-decreasing subsequences $n_i \to \infty$ and $n'_i \to \infty$ such that $\alpha_{n_i} = \alpha'_{n'_i}$ for all i.*
- *For every $\delta > 0$ there is $N \in \mathbb{N}$ such that for all $i \geqslant N$,*
  (a) *if $n_i < n < n_{i+1}$ then $\alpha_{n_i} = \alpha_{n_{i+1}} \geqslant 1/\delta$, $\alpha_{n_i} = \beta^k(\alpha_n)$ for some $k \geqslant 1$ and $|c_{\beta^{k-1}(\alpha_n)} - c_{\beta^k(\alpha_n)}| < \delta$;*
  (a') *if $n'_i < n < n'_{i+1}$ then $\alpha'_{n'_i} = \alpha'_{n'_{i+1}} \geqslant 1/\delta$, $\alpha'_{n'_i} = \beta^k(\alpha'_n)$ for some $k \geqslant 1$ and $|c_{\beta^{k-1}(\alpha'_n)} - c_{\beta^k(\alpha'_n)}| < \delta$.*

In other words, the walks of C and C' visit the same nodes along the subsequences $\{n_i\}_{i\in\mathbb{N}}$ and $\{n'_i\}_{i\in\mathbb{N}}$, but between the node $\alpha_{n_i} = \alpha_{n_{i+1}} = \alpha'_{n'_i} = \alpha'_{n'_{i+1}} \geqslant 1/\delta$ there can be excursions that are different for C and C'. The diameters of these excursion are eventually smaller than any prescribed δ. Indeed, as $\alpha_{n_i} = \beta^k(\alpha_n)$, we have by (2) that $c_{\alpha_n} \in D_{\beta^{k-1}(\alpha_n)}$ and $|D_{\beta^{k-1}(\alpha_n)}| = |c_{\beta^{k-1}(\alpha_n)} - c_{\beta^k(\alpha_n)}| < \delta$, and the same for the primed sequences. Note also that the end nodes of the excursions satisfy $\alpha_{n_i} = \alpha_{n_{i+1}} = \alpha'_{n'_i} = \alpha'_{n'_{i+1}}$ because they are paths from this node to itself. On the other hand, C and C' must exhibit infinitely many different excursions, because otherwise they eventually have the same folding pattern, and by Lemma 3, we would have $C = C'$.

Remark 1. *The asymptotic tails of C and C' are dense in $\varprojlim([c_2, c_1], T)$ if and only if $\liminf_{n\to\infty} \alpha_n = \liminf_{n\to\infty} \alpha'_n = 1$. A priori, asymptotic tails of C and C' need not be dense, but in the extreme case that $\liminf_{t\to\infty} \varphi(t) = \liminf_{t\to\infty} \varphi'(t)$ is a single point ω, then C and C' belong actually to the same arc-component; they are the two halves of a double spiral, see Section 5. The parametrizations φ and φ' still satisfy the conclusions of Proposition 1, though.*

Proof. $\Leftarrow$: Let s be the slope of T. Let $\varepsilon > 0$ be arbitrary, and choose $\delta > 0$ such that

$$2\,\frac{\delta s + (1-\delta)s^{-M}}{s-1} + \frac{s^{-M}}{s-1} < \varepsilon \qquad \text{for } M = \lfloor 1/\delta \rfloor.$$

Take N as in the hypothesis. For $N \leqslant n_i < n_{i+1}$, C has an excursion from the node α_{n_i} to itself. Take $n_i < n < n_{i+1}$ arbitrary, then

$$|\pi_0(\varphi(t)) - \pi_0(\varphi(t_{n_i}))| \leqslant |\pi_0(\varphi(t_{n_i+1})) - \pi_0(\varphi(t_{n_i}))| \leqslant \delta \qquad \text{for all } t \in [t_{n_i}, t_{n_{i+1}}].$$

Since $M \leqslant \alpha_{n_i} = \beta^k(\alpha_n) \leqslant \alpha_n$, we find that

$$e(\varphi(t))_{i=-M+1}^{0} = e(\varphi(t_{n_i}))_{i=-M+1}^{0} \qquad \text{for all } t \in [t_{n_i}, t_{n_{i+1}}].$$

and therefore $|\pi_{-i}(\varphi(t)) - \pi_{-i}(\varphi(t_{n_i}))| \leqslant s^{-i}\delta$ for all $i \leqslant M$. Hence we can compute the distance

$$d(\varphi(t), \varphi(t_{n_i})) \leqslant \sum_{0\leqslant i\leqslant M} \delta s^{-i} + \sum_{i>M} s^{-i} = \frac{\delta s + (1-\delta)s^{-M}}{s-1} \qquad \text{for all } t \in [t_{n_i}, t_{n_{i+1}}].$$

The same estimate holds for the excursion of C' between $\varphi'(t_{n'_i})$ and $\varphi'(t_{n'_{i+1}})$. Furthermore,

$$d(\varphi(t_{n_i}), \varphi'(t_{n'_i})) \leqslant \sum_{i>M} s^{-i} = \frac{s^{-M}}{s-1}.$$

Using the triangle inequality, we find for every $t \in [t_{n_i}, t_{n_{i+1}}]$ and $t' \in [t_{n'_i}, t_{n'_{i+1}}]$ that

$$d(\varphi(t), \varphi'(t')) \leqslant 2\,\frac{\delta s + (1-\delta)s^{-M}}{s-1} + \frac{s^{-M}}{s-1} \leqslant \varepsilon.$$

This proves that C and C' are asymptotic.
$\Rightarrow$: Now assume that C and C' are asymptotic. Take $\delta > 0$ arbitrary and take $K > 1/\delta$ such that $|D_k| < \delta$ for all $k \geqslant K$. Then take $L = L(K)$ as in Lemma 5. This means in particular that if k is minimal such that $S_k \geqslant K$, then $Q(l) \geqslant k$ for each $l \in \mathbb{N}$ with $S_l \geqslant L$. Let δ_0 be so small that $|c_k - c_l| < \delta_0$ for all $k < l \leqslant L$. Let $\varphi : \mathbb{R} \to C$ and $\varphi' : \mathbb{R} \to C'$ be parametrizations such that $\lim_{t\to\infty} d(\varphi(t), \varphi'(t)) = 0$, and take τ such

that $d(\varphi(t), \varphi'(t)) < \delta_0$ for all $t \geqslant \tau$. Finally let $N \in \mathbb{Z}$ be such that the Nth folding points of C and C' occur at parameters $t_N, t'_N \geqslant \tau$.

Now we will construct the non-decreasing subsequences $n_i \to \infty$ and $n'_i \to \infty$ inductively, such that C can make excursions between nodes $\alpha_{n_i} = \alpha_{n_{i+1}} \leqslant L$ only if $n_i < n_{i+1}$ (and similarly C' can make excursions between nodes $\alpha'_{n'_i} = \alpha'_{n'_{i+1}} \leqslant L$ only if $n'_i < n'_{i+1}$); see Figure 2.

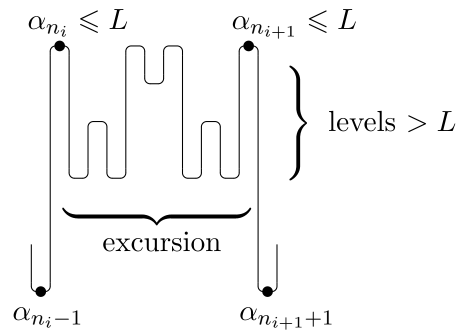

FIGURE 2. Impression of an excursion from and to the node $\alpha_{n_i} = \alpha_{n_{i+1}} \leqslant L$. Within an excursion there may be n such that $\alpha_n \leqslant L$ is small, but only both α_{n-1} and $\alpha_{n+1} > L$.

Let $n_1 \geqslant N$ be minimal such that $\alpha_{n_1} \leqslant K$. Then there is n'_1 such that $|t_{n_1} - t_{n'_1}| < \delta_0$ and $\alpha'_{n'_1} = \alpha_{n_1}$. Assuming n_i and n'_i are found, we set

$$\begin{cases} m & := & \min\{n > n_i : \alpha_n \leqslant L \text{ and } \min\{\alpha_{n-1}, \alpha_{n+1}\} \leqslant L\}; \\ m' & := & \min\{n' > n'_i : \alpha'_{n'} \leqslant L \text{ and } \min\{\alpha'_{n'-1}, \alpha'_{n'+1}\} \leqslant L\}. \end{cases}$$

We distinguish four cases:
(i) $\alpha_m \neq \alpha_{n_i}$ and $\alpha'_m \neq \alpha'_{n'_i}$. Then set $n_{i+1} = m$ and $n'_{i+1} = m'$ (No excursions.)
(ii) $\alpha_m = \alpha_{n_i}$ and $\alpha'_{m'} \neq \alpha'_{n'_i}$. Then set $n_{i+1} = m$ and $n'_{i+1} = n'_i$ (C has an excursion.)
(iii) $\alpha_m \neq \alpha_{n_i}$ and $\alpha'_{m'} = \alpha'_{n'_i}$. Then set $n_{i+1} = n_i$ and $n'_{i+1} = m'$ (C' has an excursion.)
(iv) $\alpha_m = \alpha_{n_i}$ and $\alpha'_{m'} = \alpha'_{n'_i}$. Then set $n_{i+1} = m$ and $n'_{i+1} = m'$ (C and C' both have excursions.)

Now suppose that $n_i < n < n_{i+1}$, so C has an excursion from and to node $\alpha := \alpha_{n_i} = \alpha_{n_{i+1}}$. This means in particular that for $n = n_i + 1$, $\alpha_n > L$ because otherwise n would be included in the sequence $\{n_j\}_{j\geqslant 1}$, and not in an excursion.

If $\alpha \geqslant K$ then $\alpha > 1/\delta$, and $|D_{\alpha_{n_i+1}}| = |c_{\alpha_{n_i+1}} - c_{\alpha_{n_i}}| \leqslant \delta$. But $c_{\alpha_n} \in D_{\alpha_{n_i+1}}$ by (2), so the conclusion (a) holds.

If on the other hand $\alpha \leqslant K$, and since we also have $\alpha_{n_i-1} \leqslant L$, there are two possibilities:
(i) $\alpha_{n_i-1} \leqslant K$: Then $\alpha_{n_i+1} \leqslant L$ by Lemma 5, and so $n_i + 1$ is included in the sequence $\{n_j\}_{j\geqslant 1}$.
(ii)$K < \alpha_{n_i-1} \leqslant L$: Then $\alpha_{n_i} = \beta(\alpha_{n_i-1}) = \beta(\alpha_{n_i+1})$ and by rule (6), $k_{n_i} = Q(k_{n_i+1})$ or $k_{n_i+1} = Q(k_{n_i})$. By the choice of L this means in either case that $\alpha_{n_i+1} \leqslant L$. So again $n_i + 1$ is included in the sequence $\{n_j\}_{j\geqslant 1}$.

The argument for excursions in C' is the same, which proves condition (a'), and also that $\alpha'_{n_{i+1}} = \alpha'_{n'_{i+1}}$ (since they equal $\alpha_{n_i} = \alpha'_{n'_i}$), if C and/or C' have an excursion from $\alpha_{n_i} = \alpha'_{n'_i}$.

Finally, if C and C' have no excursion from $\alpha_{n_i} = \alpha'_{n'_i}$, then $\alpha_{n_{i+1}} \leqslant L$ and $\alpha'_{n'_{i+1}} \leqslant L$ must be equal, because otherwise $|c_{\alpha_{n_{i+1}}} - c_{\alpha'_{n'_{i+1}}}| > \delta_0$, contradicting the choice of N. This concludes the proof. □

Remark 2. *It is only for the implication '⇒' that the assumption $Q(k) \to \infty$ is used, and only via Lemma 5.*

Remark 3. *In* [11, Corollary 2] *it was shown that there are no asymptotic arc-components if c is strictly preperiodic. The same proof goes through if c is non-recurrent. Non-recurrence of the critical point implies that T is* long-branched, i.e., *there is $\delta > 0$ such that the images of all branches of T^n for all $n \geqslant 1$ have length $\geqslant \delta$. There are long-branched maps with a recurrent critical point. However, long-branchedness disallows excursions of short diameter, and therefore it seems unlikely that there are asymptotic composants in the inverse limit space of a long-branch maps with recurrent, but non-periodic branch-point.*

5. Double Spirals

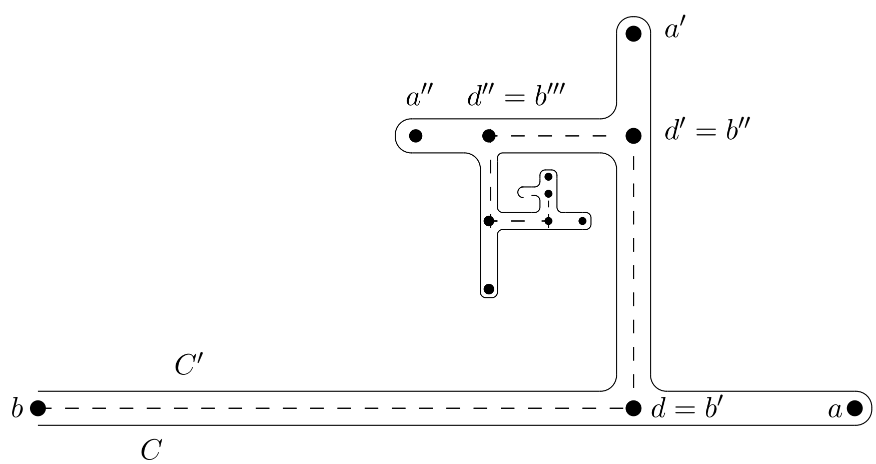

Figure 3. A double spiral composed of C and C' represented as tree-walks. This doesn't use any excursions of types (i)-(v) as the rays never go down from nodes $b, b', b'', \dots$ anymore

In this section we show that the converse of Brucks & Diamond's result (namely that points with the same symbolic tail belong the same arc-component) is false for maps with unbounded kneading map due to the existence of double spirals. Let C and C' be two different continuous images C and C' of the real line that are characterised by their left tails. So for all $x, y \in C$ there is $n = n(x,y)$ such that $e_{-i}(x) = e_{-i}(y)$ for all $i \geqslant n$, and similarly for $x', y' \in C'$, but the tail of C is different from the tail of C'. Then C and C' form a *double spiral* if they are asymptotic in the sense that $\lim_{t\to\infty}\varphi(t) = \lim_{t\to\infty}\varphi'(t) = \omega$ for some $\omega \in \varprojlim([c_2,c_1],T)$, see Figure 3. So in fact, C and C' are two halves of the same arc-component connected at ω. Clearly, if $C \cup C'$ form a double spiral, so do $\hat{T}(C \cup C')$, and they are joined at $\hat{T}(\omega)$. From the below proofs it will become clear that $\alpha_1(\omega)$ or $\alpha_0(\omega)$ is infinite, so ω cannot be a periodic point under $\hat{T}$. This implies that if there are double spirals, then there are infinitely many of them. Since $\varprojlim([c_2,c_1],T)$ is chainable, and hence atriodic, a triple spiral is impossible.

> Question: Are there two-sided double spirals? That is: can the ray $C = \varphi(\mathbb{R})$ have a spiral companion at both ends $t \to \infty$ and $t \to -\infty$?

We prove

Theorem 1. *If the kneading map Q allows a sequence $(k_i)_{i\in\mathbb{N}}$ such that*

$$k_i + 1 = Q(Q(k_{i+1}) + 1) \qquad \text{for all } i \geqslant 1, \tag{10}$$

then $\varprojlim([c_2,c_1],T)$ has double spirals.

Proof. First note that condition (10) implies k_i is strictly increasing, so in particular $\limsup_k Q(k) = \infty$.

Let us describe possible tree walks that rays C and C' could perform, see Figure 3. Starting from b, the ray C visits d, loops around a, visits d again and then d', loops around a' and then visits d' again, and then d'' etc. In terms of the backward itinerary this reads as follows. (Here we underline the positions that are to be changed in the next step, and we start off with $\vartheta(e_1 \dots e_b)$ even.)

C	Conditions
$\dots e_{S_x} e_1 \dots \hat{e}_{S_{w-1}} \dots e_{S_w} e_1 \dots \hat{e}_{S_{v-1}} \dots e_{S_v} e_1 \dots \underline{\hat{e}_{S_u}} e_1 \dots e_b$	$b < S_{Q(u+1)}$
$\downarrow$ even	
$\dots e_{S_x} e_1 \dots \hat{e}_{S_{w-1}} \dots e_{S_w} e_1 \dots \hat{e}_{S_{v-1}} \dots \underline{e_{S_v}} e_1 \dots e_{S_u} e_1 \dots e_b$	$S_{Q(Q^2(v)+1)} < b + S_u < S_{Q(v+1)}$
$\downarrow$ odd	
$\dots e_{S_x} e_1 \dots \hat{e}_{S_{w-1}} \dots e_{S_w} e_1 \dots \underline{\hat{e}_{S_{v-1}}} \dots \hat{e}_{S_v} e_1 \dots e_{S_u} e_1 \dots e_b$	$b + S_u < S_{Q(Q(v)+1)}$
$\downarrow$ even	
$\dots e_{S_x} e_1 \dots \hat{e}_{S_{w-1}} \dots e_{S_w} e_1 \dots e_{S_{v-1}} \dots \underline{\hat{e}_{S_v}} e_1 \dots e_{S_u} e_1 \dots e_b$	
$\downarrow$ odd	
$\dots e_{S_x} e_1 \dots \hat{e}_{S_{w-1}} \dots \underline{e_{S_w}} e_1 \dots e_{S_{v-1}} \dots e_{S_v} e_1 \dots e_{S_u} e_1 \dots e_b$	$b + S_u + S_v < S_{Q(Q(w)+1)}$
$\downarrow$ even	
$\dots e_{S_x} e_1 \dots \underline{\hat{e}_{S_{w-1}}} \dots \hat{e}_{S_w} e_1 \dots e_{S_{v-1}} \dots e_{S_v} e_1 \dots e_{S_u} e_1 \dots e_b$	$S_{Q(Q(v)+1)} < b + S_u + S_v < S_{Q(Q(w)+1)}$
$\downarrow$ odd	
$\dots \underline{e_{S_x}} e_1 \dots e_{S_{w-1}} \dots \underline{\hat{e}_{S_w}} e_1 \dots e_{S_{v-1}} \dots e_{S_v} e_1 \dots e_{S_u} e_1 \dots e_b$	
$\downarrow$ even	
$\dots e_{S_x} e_1 \dots e_{S_{w-1}} \dots e_{S_w} e_1 \dots e_{S_{v-1}} \dots e_{S_v} e_1 \dots e_{S_u} e_1 \dots e_b$	$b + S_u + S_v + S_w < S_{Q(Q(x)+1)}$
$\downarrow$ odd	

Starting from b, the ray C' visits d, d', d'', etc., so we get

C'	Conditions
$\dots \hat{e}_{S_x} e_1 \dots\dots \hat{e}_{S_w} e_1 \dots\dots \hat{e}_{S_v} e_1 \dots \underline{\hat{e}_{S_u}} e_1 \dots e_b$	$b < S_{Q(u+1)}$
$\downarrow$ even	
$\dots \hat{e}_{S_x} e_1 \dots\dots \hat{e}_{S_w} e_1 \dots\dots \underline{\hat{e}_{S_v}} e_1 \dots e_{S_u} e_1 \dots e_b$	$S_{Q(Q^2(v)+1)} < S_u + b < S_{Q(Q(v)+1)}$
$\downarrow$ odd	
$\dots \hat{e}_{S_x} e_1 \dots\dots \underline{\hat{e}_{S_w}} e_1 \dots\dots e_{S_v} e_1 \dots e_{S_u} e_1 \dots e_b$	$S_{Q(Q^2(w)+1)} < S_v + S_u + b$
$\downarrow$ even	$< S_{Q(Q(w)+1)}$
$\dots \underline{\hat{e}_{S_x}} e_1 \dots\dots e_{S_w} e_1 \dots\dots e_{S_v} e_1 \dots e_{S_u} e_1 \dots e_b$	$S_{Q(Q^2(x)+1)} < S_w + S_v + S_u + b$
$\downarrow$ odd	$< S_{Q(Q(x)+1)}$

In the condition column, the upper bounds to b, $b + S_u$, $b + S_u + S_v$, etc. show that for the indicated values of α_n we indeed have $e_{-\alpha_n+1} \dots e_{-1} = \nu_1 \dots \nu_{\alpha_n - 1}$ (and similar for C' and values of α'_n). The lower bounds assure that no greater value than α_n can be found. If we assume that

$$u + 1 = Q(Q(v+1)), \quad v + 1 = Q(Q(w+1)), \quad w + 1 = Q(Q(x+1)), \text{ etc.} \tag{11}$$

then the conditions in the right column follow immediately. Thus whenever we have an infinite sequence $(k_i)_{i\in\mathbb{N}}$ satisfying (10), we can continue this construction indefinitely, so the double spiral emerges. $\square$

Remark 4. *This theorem shows that if Q is non-decreasing and surjective (such as the Fibonacci map), then $\varprojlim([c_2, c_1], T)$ has uncountably many double spirals. Indeed, suppose $Q(k) = \max\{0, k-2\}$. We need not set $u + 1 = Q(Q(v+1))$ in (11), but it suffices to take $Q(Q^2(v) + 1) \leqslant u < Q(Q(v+1))$. If u is found, this leaves two choices for v, which leaves again two choices for w, etc. This amounts to uncountably many sequences $(k_i)_{i\in\mathbb{N}}$ that correspond to a double spiral.*

Conversely, if there is only one sequence $(k_i)_{i\in\mathbb{N}}$ satisfying (10), and Q is bounded otherwise, then there are only countably many double spirals. Since a double spiral is still a ray, this difference in cardinality does not give a way to distinguish inverse limit spaces.

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