BERNOULLI MEASURE OF ADMISSIBLE COMPLEX KNEADING SEQUENCES

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Abstract. Iterated quadratic polynomials give rise to a rich collection of different dynamical systems that are parametrized by a simple complex parameter c. The different dynamical features are encoded by the *kneading sequence* which is an infinite sequence over $\{0, 1\}$. Not every such sequence actually occurs in complex dynamics. The set of admissible kneading sequences was described by Milnor and Thurston for real quadratic polynomials, and by the authors in the complex case. We prove that the set of admissible kneading sequences has positive Bernoulli measure within the set of sequences over $\{0, 1\}$.

1. Introduction

One of the reasons why holomorphic dynamics is a successful subject is because the rigidity of the complex structure makes it possible to describe many properties in terms of symbolic dynamics. One of the most important concepts is that of the *kneading sequence*. In its simplest form, for real quadratic polynomials, it is a sequence of symbols "left" and "right" describing the location of the critical orbit with respect to the point of symmetry (the critical point). Milnor and Thurston [MT] gave a precise criterion which possible left/right-sequences occur as kneading sequences of real quadratic polynomials, or equivalently of any unimodal real map (but they write their combinatorics in a slightly different equivalent way, keeping track of whether the map is locally increasing or decreasing).

Kneading sequences have a natural generalization to complex quadratic polynomials, normalized as $z \mapsto z^2 + c$. To see this, suppose the dynamic ray at angle ϑ lands at the critical value, so that the rays at angles $\vartheta/2$ and $(1 + \vartheta)/2$ land at the critical point. Then the kneading sequence $\nu(\vartheta) = \nu_1\nu_2\nu_3 \cdots \in \{0, 1\}^{\mathbb{N}^*}$ of the angle ϑ can be defined as follows:

$$\nu_k \in \{0, 1\}^{\mathbb{N}^*}, \quad \nu_i = \begin{cases} 1 & \text{if } 2^{(i-1)}\vartheta \in (\,\vartheta/2, (\vartheta + 1)/2\,)\,; \\ 0 & \text{if } 2^{(i-1)}\vartheta \in (\,(\vartheta + 1)/2, \vartheta/2\,)\,; \\ \star & \text{if } 2^{(i-1)}(\vartheta) \in \{\vartheta/2, (\vartheta + 1)/2\}\,. \end{cases}$$

Since our quadratic polynomials are monic (have leading coefficient 1), we define dynamic rays in the canonical way so that every ray at angle ϑ approaches ∞ at angle

$2\pi\vartheta$. Then it is easy to see that the Milnor-Thurston kneading sequence of a real quadratic polynomial with the ϑ-ray landing at the critical value equals the kneading sequence $\nu(\vartheta)$ (identifying "left" by `1` and "right" by `0`). But this latter definition applies to all angles $\vartheta \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and thus to all complex polynomials (at least those for which some dynamic ray lands at the critical value; but this question does not matter from a combinatorial point of view).

Some technical remarks: The entry $\star$ occurs (at position n) if and only ϑ is periodic of period n; this boundary case happens in the Milnor-Thurston case also for periodic critical points. And it may happen that several dynamic rays land at the critical value, defining a kneading sequence for each. It turns out that the resulting kneading sequence is independent of this choice (this is related to the fact that, at least for postcritically finite polynomials, the critical value is always an endpoint of the connected hull of the critical orbit within the Julia set).

If the filled-in Julia set is locally connected and has no interior (so the Julia set is a dendrite), then it turns out that the kneading sequence alone allows one to give a complete topological description of the Julia set and its dynamics. It may happen that different complex quadratic polynomials have the same kneading sequence. In the dendrite case, this happens if and only if different Julia sets are topologically conjugate (by a conjugacy *not* necessarily respecting the embedding into the complex plane), and this is related to certain symmetries of the Mandelbrot set that are best described in terms of *internal addresses* (compare [S]). (If the main conjecture about quadratic polynomials holds — local connectivity of the Mandelbrot set, or equivalently, combinatorial rigidity — then the only conjugacy that respects the embedding into the plane is the identity.)

In [LS] the question was raised which sequences in $\{\mathtt{0},\mathtt{1}\}^{\mathbb{N}^*}$ occur as kneading sequences of complex quadratic polynomials or equivalently as kneading sequences of angles; we call such kneading sequences *complex admissible.* This extension of the Milnor-Thurston characterization from the real to the complex case was answered in [BS, Theorem 4.2] in terms of a necessary and sufficient combinatorial condition involving internal addresses: see Definition 2.3.

In this note we prove the following result.

1.1. Theorem (Positive Measure for Admissible Kneading Sequences)
The set of admissible kneading sequences has positive $(\frac{1}{2}\text{-}\frac{1}{2})$*-product measure as subset of* $\{\mathtt{0},\mathtt{1}\}^{\mathbb{N}^*}$.

In fact, the same result holds for any $(p, 1-p)$-product measure with $p \in (0,1)$.

The admissibility condition is so that the non-complex admissible kneading sequences form a countable union of *cylinders* in $\{\mathtt{0},\mathtt{1}\}^{\mathbb{N}^*}$ (a n-cylinder is the set of sequences with common first n entries). It thus suffices to discuss admissibility of periodic sequences.

The fundamental tool for characterizing complex admissible kneading sequences is the fact (shown in [BKS]) that every periodic kneading sequence ν has an associated *abstract Hubbard tree* which is a finite topological tree that satisfies all properties of Hubbard trees of quadratic polynomials, except that it may possibly not have an embedding into the plane that is respected by the dynamics. But it is a finite tree T with a continuous surjective map $f\colon T \to T$ of degree at most two, with a single critical point which has a finite orbit that contains all endpoints of T, and subject to a certain expansivity condition. Such a tree gives much more structure to a kneading sequence, and it can be decided relatively easily whether or not it has an embedding compatible with the dynamics, and thus whether or not the given kneading sequence is complex admissible. The only obstruction is a periodic branch point where the first return map does not permute all local branches transitively (and not even with the same period); we call such a branch point *evil.* We have a precise condition if a given kneading sequence ν has an evil branch points of any period m: see Definition 2.3; if this happens, we say that ν *fails the admissibility condition for period* m. A kneading sequence ν (periodic or not) is complex admissible if and only if it does not fail the kneading sequence for any period m.

The simplest $\star$-periodic kneading sequence that is not complex admissible is $\nu = \overline{\texttt{10110}\star}$; the associated Hubbard is shown in Figure 1.1. The sequence ν fails the admissibility condition for period 3, corresponding to an evil branch point of period 3 in the tree. The entire 6-cylinder of sequences starting with $\texttt{101100}\ldots$ fails the admissibility condition for period $m = 3$.

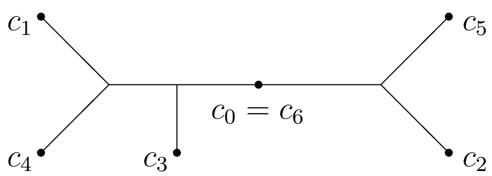

FIGURE 1.1. The Hubbard tree with kneading sequence $\overline{\texttt{10110}\star}$ contains an orbit of evil branch points of period 3: the first return map interchanges two arms and fixes the third. We write $c_i = f^{\circ i}(0)$.

Further properties on Hubbard trees, including the precise construction, properties and connections to external rays, quadratic laminations etc., can be found in [DH, M, Po, BS, BKS, K].

2. Basic definitions and properties

By convention, a kneading sequence starts with $\mathtt{1}$. We say that a sequence ν is *$\star$-periodic of period n* if $\nu = \overline{\nu_1 \dots \nu_{n-1}\star}$ with $\nu_1 = \mathtt{1}$ and $\nu_i \in \{\mathtt{0},\mathtt{1}\}$ for $1 < i < n$. Let

$$\begin{aligned} \Sigma &:= \{\mathtt{0},\mathtt{1}\}^{\mathbb{N}^*} , \\ \Sigma^1 &:= \{\nu \in \Sigma \colon \text{the first entry in } \nu \text{ is } \mathtt{1}\} , \\ \Sigma^\star &:= \Sigma^1 \cup \{\text{all } \star\text{-periodic sequences except } \overline{\star}\} , \end{aligned}$$

In order to avoid silly counterexamples, $\overline{\star}$ is not considered to belong to $\Sigma^\star$. All sequences in $\Sigma^\star$ will be called *kneading sequences*.

While kneading sequences are just binary sequences, they have a "human-readable" recoding in terms of *internal addresses*, which are strictly increasing sequences of integers starting with 1 (and which give an elegant description of the corresponding parameters within the Mandelbrot set $\mathcal{M}$: taking a path in $\mathcal{M}$ from the main cardioid to some parameter c, the successive entries of the internal address give the lowest periods of the hyperbolic components that one encounters on this path, see [LS]). The conversion algorithm is bijective and is based on the following function.

2.1. Definition (ρ-Function and Internal Address)
For a sequence $\nu \in \Sigma^\star$, define

$$\rho_\nu : \mathbb{N}^* \to \mathbb{N}^* \cup \{\infty\}, \quad \rho_\nu(n) = \inf\{s > n : \nu_s \neq \nu_{s-n}\}.$$

We usually write ρ for ρ_ν and call $\operatorname{orb}_\rho(s) = \{\rho^{\circ i}(s)\}_{i \geq 0}$ the ρ-orbit of s. The case $s = 1$ is the most important one; this is the internal address *of ν and we denote it as*

$$1 = S_0 \to S_1 \to S_2 \to \dots$$

with $S_{k+1} = \rho(S_k)$. If $\rho^{\circ k+1}(1) = \infty$, then we say that the internal address is finite: $1 \to \rho(1) \to \dots \to \rho^{\circ k}(1)$.

The map from kneading sequences in Σ^1 to internal addresses is injective. In fact, the algorithm of this map (originally from [LS, Algorithm 6.2]) can easily be inverted:

2.2. Algorithm (From Internal Address to Kneading Sequence)
The following inductive algorithm turns internal addresses into kneading sequences in Σ^1: the internal address $S_0 = 1$ has kneading sequence $\overline{\mathtt{1}}$, and given the kneading sequence ν^k associated to $1 \to S_1 \to \dots \to S_k$, the kneading sequence associated to $1 \to S_1 \to \dots \to S_k \to S_{k+1}$ consists of the first $S_{k+1} - 1$ entries of ν^k, then the opposite to the entry S_{k+1} in ν (switching $\mathtt{0}$ and $\mathtt{1}$), and then repeating these S_{k+1} entries periodically.

Proof. The kneading sequence $\overline{\mathtt{1}}$ has internal address 1. If ν^k has internal address $1 \to S_1 \dots \to S_k$ and ν is the internal address of period S_{k+1} as constructed in the algorithm, then the internal address of ν clearly starts with $1 \to S_1 \to \dots \to S_k$, and $\rho_\nu(S_k) = S_{k+1}$, so the internal address of ν is $1 \to S_1 \dots \to S_k \to S_{k+1}$. $\square$

The ρ-function is of fundamental importance in the work of Penrose [Pe] under the name of *non-periodicity function*; the internal address is called *principal non-periodicity function.*

It is by means of the ρ-function that evil periodic branch points can be detected. This leads to the condition given in [BS, Definition 4.1]:

2.3. Definition (The Admissibility Condition)
A kneading sequence $\nu \in \Sigma^\star$ fails the admissibility condition for period m *(which is the period of the corresponding evil branch point) if the following three conditions hold:*

(1) *the internal address of ν does not contain m;*
(2) *if $s < m$ divides m, then $\rho(s) \leq m$;*
(3) *$\rho(m) < \infty$ and if $r \in \{1, \ldots, m\}$ is congruent to $\rho(m)$ modulo m, then* $\mathrm{orb}_\rho(r)$ *contains m.*

A kneading sequence fails the admissibility condition *if it does so for some $m \geq 1$.*
An internal address fails the admissibility condition *if its associated kneading sequence does.*

A kneading sequence (periodic or not) is called *admissible* unless it fails this condition for any period m.

3. The measure of admissible kneading sequences

We equip the space Σ^1 of all $\mathtt{0}$-$\mathtt{1}$-sequences starting with $\mathtt{1}$ with the product topology and the $(\frac{1}{2}$-$\frac{1}{2})$-product measure μ, normalized so that $\mu(\Sigma^1) = 1$.

3.1. Definition (n-Admissible Cylinders and Kneading Sequences)
Let $E \subset \Sigma^1$ be the set of all sequences which satisfy the Admissibility Condition 2.3 for every m (the set of admissible kneading sequences*). For $n \geq 1$, a finite word $\nu_1 \ldots \nu_n$ with $\nu_i \in \{\mathtt{0}, \mathtt{1}\}$ and $\nu_1 = \mathtt{1}$ is called an* admissible n-word *if there is a $\nu \in E$ which begins with $\nu_1 \ldots \nu_n$. An* admissible n-cylinder *is an n-cylinder that contains an admissible sequence. Finally, let E_n be the union of the admissible n-cylinders; this is the* set of n-admissible kneading sequences.

Whether or not $\nu \in \Sigma^1$ fails the admissibility condition for period m depends only on the first $\rho_\nu(m)$ entries in ν, so if ν fails the condition for period m, then an entire $\rho_\nu(m)$-cylinder is non-admissible. Thus $E = \cap_{n \geq 1} E_n$ is a decreasing intersection of sets which are simultaneously open and compact.

3.2. Lemma (Admissible Kneading Sequences Form Cantor Set)
The set $E \subset \Sigma^1$ of admissible kneading sequences forms a Cantor set so that $\Sigma^1 \setminus E$ is dense in Σ^1. In particular, E is measurable with respect to μ.

Proof. Any violation of the admissibility condition for period m discards an entire cylinder subset of Σ^1. The set of non-admissible kneading sequences is a union of such cylinder sets and hence open. As a closed subset of the compact set Σ^1, the set E is compact. Clearly, E is totally disconnected because $\Sigma^1 \supset E$ is.

Recall from Algorithm 2.2 that if $S_k = \rho_\nu^{\circ k}(1)$ is an element of the internal address of ν then ν^k is the S_k-periodic kneading sequence that coincides with ν for S_k entries. If $\nu \in E$, then $\nu^k \in E$. We can always extend admissible kneading sequences via direct bifurcations, so for any admissible internal address $1 \to \ldots \to S_k$ and any $p > 1$, the internal address $1 \to S_1 \to \ldots \to S_k \to pS_k$ is admissible. Since k and p are arbitrary, ν is not an isolated point. Finally, non-admissible sequences are dense in Σ^1 because any finite word can be continued into a non-admissible one. Indeed, given any $\nu = \nu_1\nu_2\nu_3 \cdots \in \Sigma^1$ and any prime $m > \rho_\nu(1)$, define a sequence $\nu' = \nu'_1\nu'_2 \ldots$ with $\nu'_k = \nu_k$ unless m divides k, choosing ν'_m so that m is not in the internal address of ν', and $\nu'_{2m} \neq \nu'_m$. Then $\rho_{\nu'}(m) = 2m$ and ν' fails the admissibility condition for period m: the first and third conditions are clearly satisfied, and the second one is void because m is prime. □

3.3. Corollary (Admissible Periodic Sequences)
Let R_k be the number of admissible periodic kneading sequences of period k in Σ^1. Then

$$\lim_{k\to\infty} \frac{R_k}{2^{k-1}} = \mu(E) \ .$$

Proof. We have $\mu(E_k) = R_k 2^{-(k-1)}$ and $\mu(E) = \lim_{k\to\infty} \mu(E_k)$. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Fix $n_1 := 100$ and define a sequence of integers $\{n_i\}_{i\geq 1}$ so that n_{i+1} is the largest integer less than $\frac{3}{2}n_i$ for which $n_{i+1} - n_i$ is divisible by $2i$ (this gives 100, 148, 220, 328, 488,...). Let $m_i := (n_{i+1} - n_i)/2$. Then one can check that $(1.45)^i < m_i$. Clearly, $n_{i+1} < \frac{3}{2}n_i < n_i + \frac{3}{4}n_{i-1} < n_i + n_{i-1}$ and $m_i < n_i/4$.

Let F_1 be some cylinder of length n_1 containing an admissible sequence so that no block of 40 consecutive $\mathtt{0}'s$ appears among the first n_1 entries. For $i \geq 1$, define

$$F_{i+1} := \left\{ \begin{array}{ll} \nu \in F_i : & \text{every } k \leq n_i + m_i \text{ has an } s \in \mathrm{orb}_{\rho_\nu}(k) \cap \mathrm{orb}_{\rho_\nu}(1) \\ & \text{with } n_i + m_i < s \leq n_{i+1} \\ & \text{and the entries } n_i,\ n_i+1,\ \ldots,\ n_{i+1}-1 \text{ in } \nu \text{ do} \\ & \text{not contain a block of } \lfloor n_i/8 \rfloor \text{ zeroes.} \end{array} \right\}$$

For every $\nu \in F_{i+1}$, every $k \leq n_i + m_i$ satisfies $\rho_\nu(k) \leq n_{i+1}$, so every F_{i+1} is a union of cylinder sets of length n_{i+1}.

Claim. *The second condition for F_{i+1} implies that for all $N \leq n_{i+1}$, the first N entries in any $\nu \in F_{i+1}$ do not contain a contiguous block of $\lfloor N/4 \rfloor$ zeroes.*

Indeed, for any $n_j < N$, the longest block of consecutive zeroes among entries $n_j \ldots n_{j+1} - 1$ has length less than $n_j/8$; if these are near the end, they can be continued by less than $n_{j+1}/8$ further zeroes, yielding a total number of consecutive zeroes of less than $n_j/8 + n_{j+1}/8 < (5/16)n_j$, and they end at entry number $(9/8)n_{j+1} < (27/16)n_j$, so among the first $(27/16)n_j$ entries there are no more than $(5/16)n_j < (27/16)n_j/4$ consecutive zeroes.

The two main steps in the proof are $E \supset \cap_i F_i$ and $\mu(F_{i+1}) \geq c_i \mu(F_i)$ for numbers $c_i > 0$ with $\prod_i c_i > 0$, from which the conclusion will follow.

(1) Admissibility of cylinders F_i and $E \supset \bigcap_i F_i$. By induction over $i \geq 1$, we show that all $\nu \in \cap_j F_j$ are n_i-admissible for all i. They are n_1-admissible by hypothesis. Suppose $\nu \in \cap_j F_j$ is n_i-admissible but not n_{i+1}-admissible; then ν fails the Admissibility Condition 2.3 for some period m with $n_i < \rho(m) \leq n_{i+1}$. Let $r \in \{1, 2, \dots, m\}$ be congruent to $\rho(m)$ modulo m.

If $m \leq n_{i-1} + m_{i-1}$, then by definition of F_{i-1}, there is an $s \in \operatorname{orb}_\rho(m) \cap \operatorname{orb}_\rho(1)$ with $s \leq n_i$; since $m \notin \operatorname{orb}_\rho(1)$, we have $\rho(m) \leq s \leq n_i$, which is a contradiction. Hence $m > n_{i-1} + m_{i-1}$.

If $r \geq n_{i-1} + m_{i-1}$, then $\rho(m) \geq m + r > 2n_{i-1} + 2m_{i-1} = n_i + n_{i-1} > n_{i+1}$, again a contradiction. Thus $r < n_{i-1} + m_{i-1}$, and there is an $s \in \operatorname{orb}_\rho(r) \cap \operatorname{orb}_\rho(1)$ with $s \leq n_i$. By the admissibility condition, $m \in \operatorname{orb}_\rho(r)$ and $m \notin \operatorname{orb}_\rho(1)$, hence $s > m$. Thus $s \in \operatorname{orb}_\rho(m)$ and $\rho(m) \leq s \leq n_i$ in contradiction to n_i-admissibility of ν. This final contradiction shows that $E \supset \cap_i F_i$ as claimed.

(2) A bound on the number of jumps of ρ. Given ν and an integer r, let $J_\nu(r) = \{k \leq r : \rho(k) > r\}$. Obviously, $J_\nu(r)$ depends only on the first r entries of ν.

Claim. *For each $i \geq 1$,*

$$\text{if } \nu \in F_i \text{ and } n_i + m_i \leq r < n_{i+1} \text{ then } \#J_\nu(r) < i/2 + 150 \ . \qquad (\star)$$

Indeed, if $k \in J_\nu(r)$, then $\nu_{k+1} \dots \nu_r = \nu_1 \dots \nu_{r-k}$. Suppose that $r - k > n_2$. Then there is a unique $j \geq 1$ with $n_{j+2} \geq r - k > n_{j+1}$, and the hypothesis $r < n_{i+1}$ implies $j < i$. Since all $k' \in \{1, 2, \dots, n_j + m_j\}$ have $\rho(k') \leq n_{j+1} < r - k$ by definition of F_{j+1}, it follows that all $k' \in \{k+1, k+2, \dots, k + n_j + m_j\}$ have $\rho(k') \leq k + n_{j+1} < r$, hence $k' \notin J_\nu(r)$.

We turn this into an inductive argument: write $J_\nu(r) = \{k_1, k_2, k_3, \dots, k_S\}$ with $k_s < k_{s+1}$ for all s. Let j_s be so that $n_{j_s+2} \geq r - k_s > n_{j_s+1}$. Then $k_{s+1} > k_s + n_{j_s} + m_{j_s}$, hence

$$\begin{aligned} n_{j_{(s+1)}+1} < r - k_{s+1} &< r - k_s - n_{j_s} - m_{j_s} \leq n_{j_s+2} - n_{j_s} - m_{j_s} \\ &= n_{j_s} + 2m_{j_s} + 2m_{j_s+1} - n_{j_s} - m_{j_s} = m_{j_s} + 2m_{j_s+1} \\ &< n_{j_s}/4 + n_{j_s+1}/2 < n_{j_s} \ , \end{aligned}$$

so $j_{s+1} + 1 < j_s$ or $j_{s+1} \leq j_s - 2$. Since we started with $r - k_1 < r < n_{i+1}$, after s turns of this argument we have $r - k_{s+1} < n_{j_s}$ with $j_s \leq i + 1 - 2s$, and for $s \leq (i-1)/2$ we have $j_s \leq 2$, so $r - k_{s+1} < n_2$ or $k_{s+1} > r - n_2$, and there are at most n_2 such values of k_s. “ It follows that $\#J_\nu(r) \leq (i-1)/2 + 1 + n_2 < i/2 + 150$ as claimed.

(3) Separate treatment of sub-blocks. Now we begin the proof that $\mu(F_{i+1})$ is not too small compared to $\mu(F_i)$. Suppose $i \geq 40$, which implies $\lfloor \log_2(i/2+150) \rfloor + 1 \leq \lfloor i/4 \rfloor$. Let C be any $n_i + m_i$-cylinder in F_i and pick $\nu \in C$. Divide the integer interval $[n_i + m_i + 1, n_{i+1}]$ into m_i/i blocks $B_1, B_2, \dots$ of length i. By Equation $(\star)$, $\# J_\nu(n_i+m_i) < i/2+150$, and there are 2^i different possibilities for B_1 which extend C. We claim that at least $2^{i/2}$ of them are *barriers* in the sense that for all $k \in J_\nu(n_i+m_i)$, we have $n_i + m_i + 2\lfloor i/4 \rfloor \in \mathrm{orb}_\rho(k)$. This is useful in view of the first condition of the definition of F_i. To construct these barriers, divide B_1 into three subblocks: the first two of length $\lfloor i/4 \rfloor$ each, the last of length $i - 2\lfloor i/4 \rfloor \geq i/2$.

In the second subblock, every entry is `0`. The third subblock is filled arbitrarily with `0` and `1`, for which there are at least $2^{i/2}$ possibilities. The first subblock needs more care.

In the first subblock of B_1, choose the first entry (at position $n_i + m_i + 1$) so that $n_i + m_i + 1 \in \mathrm{orb}_\rho(k)$ for at least half of the elements $k \in J_\nu(n_i + m_i)$; choose the second entry so that $n_i + m_i + 2 \in \mathrm{orb}_\rho(k)$ for at least half of the remaining elements in $J_\nu(n_i + m_i)$, and so on. The first $\lfloor \log_2(i/2 + 150) \rfloor + 1$ entries in B_1 suffice so that for every $k \in J_\nu(n_i + m_i)$, $\mathrm{orb}_\rho(k)$ contains at least one of these positions. Since $\lfloor \log_2(i/2 + 150) \rfloor + 1 \leq \lfloor i/4 \rfloor$, we have used only positions within the first subblock of B_1. Fill the remaining entries within the first subblock of B_1 arbitrarily.

Since the first N entries of ν do not contain a contiguous block of $\lfloor N/4 \rfloor$ zeroes, it follows that the orbit of every $k \in J_\nu(n_i + m_i)$ visits an entry in the second subblock of B_1, and $\mathrm{orb}_\rho(k)$ contains the last position in the second subblock of B_1, which is $n_i + m_i + 2\lfloor i/4 \rfloor \in \mathrm{orb}_\rho(k)$.

An analogous construction can be repeated for all the other blocks B_2, B_3,.... Suppose that at least a single block B_j has a barrier as above. Then the construction yields n_{i+1}-cylinders which satisfy the first condition for F_{i+1}: for $k \leq n_i+m_i$, we either have $\rho(k) \leq n_i + m_i$ and we consider $\rho(k)$ instead of k, or $k \in J_\nu(n_i + m_i + (j-1)i)$ and $\mathrm{orb}_\nu(k)$ visits $n_i+m_i+2\lfloor i/4 \rfloor+(j-1)i$. Therefore $n_i+m_i+2\lfloor i/4 \rfloor+(j-1)i \in \mathrm{orb}_\nu(k)$ for every $k \leq n_i + m_i + 2\lfloor i/4 \rfloor$, in particular for $k = 1$.

(4) Estimating the relative loss of measure. Given the $n_i + m_i$-cylinder C, there are 2^i continuations into $n_i + m_i + i$-cylinders in F_{i+1}, and at least $2^{i/2}$ of them have a barrier in B_1 as constructed above, so at most a relative proportion of $1 - 2^{-i/2}$ has no barrier within B_1. The same bound for the relative proportions holds for B_2, B_3, ... We find that the proportion of n_{i+1}-cylinders in C with no barrier at any block B_j for $j = 1, \dots, m_i/i$ is $(1 - 2^{-i/2})^{m_i/i}$. We have $m_i > (1.45)^i > 2^{i/2}$, so $m_i 2^{-i/2} > \alpha^i$ with $\alpha > 1$. Therefore

$$(1 - 2^{-i/2})^{m_i/i} < \left(\exp(-2^{-i/2})\right)^{m_i/i} = \exp\left(-2^{-i/2}\frac{m_i}{i}\right) < \exp(-\alpha^i/i) < i/\alpha^i,$$

and the relative proportion of n_{i+1}-cylinders in F_{i+1} satisfying the first condition is at least $1 - i/\alpha^i$.

We still have to take care of the second condition for F_{i+1}. Among the 2^{2m_i} continuations from any n_i-cylinder into an n_{i+1}-cylinder, there are $2^{2m_i-\lfloor n_i/8\rfloor}$ which have $\lfloor n_i/8\rfloor$ contiguous entries `0` beginning at any given position $n_i, n_i+1, \ldots, n_{i+1}-\lfloor n_i/8\rfloor$, and less than $2m_i 2^{2m_i-\lfloor n_i/8\rfloor}$ which have $\lfloor n_i/8\rfloor$ contiguous `0`'s between entries n_i and $n_{i+1}-1$ (inclusively). The proportion of n_{i+1}-cylinders in F_i which do not contain $\lfloor n_i/8\rfloor$ consecutive `0`'s is thus at least $1-2m_i2^{-\lfloor n_i/8\rfloor} > 1-\frac{1}{2}n_i2^{-\lfloor n_i/8\rfloor}$.

Therefore

$$\begin{aligned}\mu(F_{i+1}) &\geq \left[1-\left(1-2^{-i/2}\right)^{m_i/i}\right]\left[1-\frac{1}{2}n_i2^{-\lfloor n_i/8\rfloor}\right]\mu(F_i)\\ &> \left[1-\frac{i}{\alpha^i}\right]\left[1-\frac{n_i}{2\cdot 2^{\lfloor n_i/8\rfloor}}\right]\mu(F_i)\end{aligned}$$

for $i\geq 40$. Since $\sum_i i/\alpha^i<\infty$ and $\sum_i n_i/2^{\lfloor n_i/8\rfloor}<\infty$, it follows

$$\mu(E)\geq\mu\left(\bigcap_i F_i\right)\geq\mu(F_{40})\prod_{i\geq 40}\left[1-\frac{i}{\alpha^i}\right]\left[1-\frac{n_i}{2\cdot 2^{\lfloor n_i/8\rfloor}}\right]>0$$

because F_{40} contains at least one cylinder set. This proves the theorem. □

Remark. A similar proof shows that $\mu_p(E)>0$ for the $(p,1-p)$-product measure μ_p.

A computation (shown in Table 1) suggests that about 80 percent of Σ^1 is admissible. Unfortunately, the convergence is too slow to make precise estimates for $\mu(E)$. Let us discuss some examples from this table: for $n=6$, the 2^5 total cylinders of length 6 are the binary words of length 6 starting with `1`, and the only non-admissible cylinder is `101100` (the only non-admissible $\star$-periodic sequence is $\overline{\texttt{10110}\star}$, but the cylinder `101 101` is admissible). For $n=7$, the two non-admissible cylinders are the two continuations of `101100`; for $n=8$, we have four continuations, as well as the new cylinders `1001 1000`, `1101 1100` and `1011 1100`.

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n	non-admis.	total	non-admis./total
6	1	32	0.031250
7	2	64	0.031250
8	7	128	0.054688
9	17	256	0.066406
10	44	512	0.085938
11	96	1 024	0.093750
12	221	2 048	0.107910
13	473	4 096	0.115479
14	1 028	8 192	0.125488
15	2 160	16 384	0.131836
16	4 544	32 768	0.138672
17	9 408	65 536	0.143555
18	19 488	131 072	0.148682
19	39 984	262 144	0.152527
20	81 963	524 288	0.156332
21	167 138	1 048 576	0.159395
22	340 393	2 097 152	0.162312
23	691 104	4 194 304	0.164772
24	1 401 610	8 388 608	0.167085
25	2 836 989	16 777 216	0.169098
26	5 737 023	33 554 432	0.170977
27	11 586 451	67 108 864	0.172652
28	23 382 085	134 217 728	0.174210
29	47 143 911	268 43 5456	0.175625
30	94 995 724	536 870 912	0.176943

TABLE 1. The number of non-admissible cylinders of length n, among all the 2^{n-1} cylinders in Σ^1 of this length, for $n = 6, 7, \ldots, 30$.

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