THE CORE INGRAM CONJECTURE FOR NON-RECURRENT CRITICAL POINTS

ANA ANUŠIĆ, HENK BRUIN, JERNEJ ČINČ

ABSTRACT. We study inverse limit spaces of tent maps, and the Ingram Conjecture, which states that the inverse limit spaces of tent maps with different slopes are non-homeomorphic. When the tent map is restricted to its core, so there is no ray compactifying on the inverse limit space, this result is referred to as the Core Ingram Conjecture. We prove the Core Ingram Conjecture when the critical point is non-recurrent and not preperiodic.

1. INTRODUCTION

Inverse limit spaces made their first appearance in dynamical systems in 1967 when Williams [16, 17] showed that hyperbolic one-dimensional attractors can be represented as inverse limit spaces. The study of inverse limit spaces with the goal to describe complicated structures in strange attractors gained significance in the last two decades. For instance, the work of Barge & Holte [3] shows that for a wide range of parameters, attracting sets for maps in Hénon family are homeomorphic to inverse limit spaces of unimodal interval maps.

The *tent map family* $T_s : [0,1] \to [0,1]$ is defined as $T_s := \min\{sx, s(1-x)\}$ where $x \in [0,1]$ and $s \in (0,2]$. Let $c := 1/2$ denote the *critical point* and let $c_i := T_s^i(c)$ for every $i \in \mathbb{N}$. In this paper we are concerned with the inverse limit spaces $\varprojlim([0,1], T_s)$ using a single tent map from the parametrized family as a bonding map. It is not difficult to see that for $c \geq c_1$, $\varprojlim([0,1], T_s)$ is a point or an arc and thus not interesting. For the case $c \leq c_1 < 1$ it follows from Bennett's Theorem in [5] from 1962 that we can decompose $\varprojlim([0,1], T_s) = \varprojlim([c_2, c_1], T_s) \cup \mathfrak{C}$, where $\bar{0} := (\dots, 0, 0, 0) \in \mathfrak{C}$ and $\mathfrak{C}$ is a continuous image of $[0, \infty)$ which compactifies on $\varprojlim([c_2, c_1], T_s)$. Inverse limit space $\varprojlim([c_2, c_1], T_s)$ obtained from the forward invariant interval $[c_2, c_1]$ is called the *core of the inverse limit space*.

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In the early 90's a classification problem that became known as the *Ingram Conjecture* was posed:

If $1 \le s < \tilde{s} \le 2$ then the inverse limit spaces $\varprojlim([0,1],T_s)$ and $\varprojlim([0,1],T_{\tilde{s}})$ are not homeomorphic.

After partial results [11, 6, 15, 13], the Ingram Conjecture was finally answered in affirmative by Barge, Bruin & Štimac in [1]. However, the proof presented in [1] crucially depends on the ray $\mathfrak{C}$, so the core version of the Ingram Conjecture still remains open. For Hénon maps, $\mathfrak{C}$ plays the role of the unstable manifold of the saddle point outside the Hénon attractor; it compactifies on the attractor, but it is somewhat unsatisfactory to have to use this (and the embedding in the plane that it presupposes) for the topological classification. It is also not possible to derive the core version directly from the non-core version, because it is impossible to reconstruct $\mathfrak{C}$ from the core inverse limit space. This is for instance illustrated by the work of Minc [12] showing that in general there are $2^{\aleph_0}$ mutually non-homeomorphic rays which compactify on an arbitrary nondegenerate continuum.

In this paper we partially solve in the affirmative the classification problem called the *Core Ingram Conjecture*.

Theorem 1.1. *If $1 \le s < \tilde{s} \le 2$ and critical points of T_s and $T_{\tilde{s}}$ are non-recurrent, then the inverse limit spaces $\varprojlim([c_2,c_1],T_s)$ and $\varprojlim([\tilde{c}_2,\tilde{c}_1],T_{\tilde{s}})$ are not homeomorphic.*

If T_s has a non-recurrent critical orbit, then $\varprojlim([c_2,c_1],T_s)$ has no endpoints and if the critical orbit is recurrent, $\varprojlim([c_2,c_1],T_s)$ has endpoints (finitely many if the critical point is periodic and infinitely many if the critical orbit is infinite). For details see [4]. Thus, the recurrent and non-recurrent case are topologically different.

The solution of the Core Ingram Conjecture for tent maps leads to the similar conclusion for the "fuller" family of unimodal maps; see [2] for details. It also turns out that Theorem 1.1 can be reduced to the case where the slopes $s, \tilde{s} \in (\sqrt{2}, 2]$, rather than $(1,2]$; see [1] for details.

There exist two fixed points of T_s: 0 and $r := \frac{s}{s+1} \in [c_2,c_1]$. The *arc-component* of a point $u \in \varprojlim([0,1],T_s)$ is defined as the union of all arcs in $\varprojlim([0,1],T_s)$ containing u. Let us denote the arc-component from $\varprojlim([c_2,c_1],T_s)$ that contains $\rho := (\dots,r,r,r)$ by $\mathfrak{R}$. It is a continuous image of the real line and is dense in both directions, see e.g. [8]. Analogously, let $\tilde{\mathfrak{R}} \subset \varprojlim([\tilde{c}_2,\tilde{c}_1],T_{\tilde{s}})$ be the arc-component of the point $\tilde{\rho} = (\dots,\tilde{r},\tilde{r},\tilde{r})$, where $\tilde{r} := \frac{\tilde{s}}{\tilde{s}+1}$ is a fixed point of $T_{\tilde{s}}$. The main new ingredient in this paper is:

Theorem 1.2. *Let $\sqrt{2}<s \le \tilde{s} \le 2$ and assume that the critical points of T_s and $T_{\tilde{s}}$ are not recurrent. Let $\mathfrak{R} \subset \varprojlim([c_2,c_1],T_s)$ and $\tilde{\mathfrak{R}} \subset \varprojlim([\tilde{c}_2,\tilde{c}_1],T_{\tilde{s}})$ be as above. If $h : \varprojlim([c_2,c_1],T_s) \to \varprojlim([\tilde{c}_2,\tilde{c}_1],T_{\tilde{s}})$ is a homeomorphism, then $h(\mathfrak{R}) = \tilde{\mathfrak{R}}$.*

The main observation in the proof of this result is Lemma 3.17 which fails without the non-recurrence assumption. This presents the main obstacle in proving the Core Ingram Conjecture in general.

In the process of proving the Ingram Conjecture, partial solutions of the Core Ingram Conjecture were obtained as well. The first result is due to Kailhofer [11] in 2003, who proved the Core Ingram Conjecture for the case when critical point is periodic. In 2006, Good & Raines [10] proved that there exists a set $S \subset [\sqrt{2}, 2]$ of cardinality $2^{\aleph_0}$ such that for every $s \in S$ the map T_s has non-recurrent critical point c, omega limit set $\omega(c)$ is a Cantor set and $\varprojlim([c_2, c_1], T_s)$ is non-homeomorphic to $\varprojlim([\tilde{c}_2, \tilde{c}_1], T_{\tilde{s}})$ for any other $\tilde{s} \in S$. In 2007, Štimac [15] extended the result of Kailhofer and proved the Core Ingram Conjecture in the case when critical orbit is finite. Both Kailhofer and Štimac make use of a dense arc-component inside the core of the inverse limit space, but not the arc-component $\mathfrak{R}$. The most recent result regarding the Core Ingram Conjecture was obtained in 2015 by Bruin & Štimac [9] who proved that the conjecture holds for a set of parameters where the critical point is "extremely" (or persistently) recurrent and not periodic. The last result was obtained from observations on the arc-component $\mathfrak{R}$.

In this paper we prove the Core Ingram Conjecture when the critical point is not recurrent. The main line of the proof is similar as in the proof of the Ingram Conjecture in [1]. There, the authors first assume by contradiction that there exists a homeomorphism between $\varprojlim([0,1], T_s)$ and $\varprojlim([0,1], T_{\tilde{s}})$, where $s \neq \tilde{s} \in (\sqrt{2}, 2]$ and then it clearly follows that the arc-component $\mathfrak{C} \subset \varprojlim([0,1], T_s)$ maps to the arc-component $\tilde{\mathfrak{C}} \subset \varprojlim([0,1], T_{\tilde{s}})$, where $\bar{0} \in \tilde{\mathfrak{C}}$. Then they show that the structure of maximal link-symmetric arcs uniquely determines $\mathfrak{C}$ and thus the whole $\varprojlim([0,1], T_s)$.

The following result about the group of self-homeomorphisms extends as well. The proof requires only minor adjustments: one needs to replace the arc-component $\mathfrak{C}$ with $\mathfrak{R}$ in the proof of [7, Theorem 1.3].

Theorem 1.3. *Assume that T_s has a non-recurrent critical point. Then for every self-homeomorphism $h : \varprojlim([c_2, c_1], T_s) \to \varprojlim([c_2, c_1], T_s)$ there is $R \in \mathbb{Z}$ such that h and σ^R are isotopic.*

Let us give a short outline of the structure of the paper. In Section 2 we provide a basic set-up for tent maps, their inverse limit spaces and chainability. In Section 3 we study the structure of the arc-component $\mathfrak{R}$. In Section 4 we prove Theorem 1.2. In Section 5 we prove that $\mathfrak{R}$ is uniquely determined by the structure of long link-symmetric arcs which proves the Theorem 1.1.

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2. Preliminaries

2.1. **Tent maps.** Let $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$. We define a *tent map* $T_s : [0, 1] \to [0, 1]$ with slope $\pm s$ as $T_s(x) := \min\{sx, s(1 - x)\}$ and we restrict to $s \in (\sqrt{2}, 2]$. Thus in particular $c_1 = s/2$ and $c_2 = s(1-s/2)$. We call the interval $[c_2, c_1]$ the *core* of T_s. Throughout the paper we will assume that T_s has an infinite critical orbit, because the Core Ingram Conjecture has already been proven for the case when c is (pre)periodic, see [11, 14].

We say that $x \in [0, 1]$ is a *turning point* of T_s^j, if there exists $m < j \in \mathbb{N}$ such that $T_s^m(x) = c$. Two turning points $x, y \in [0, 1]$ of T_s^j are *adjacent* if $T_s^j|_{[x,y]}$ is monotone. The critical point of T_s is called *recurrent*, if for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $|c - c_n| < \varepsilon$.

For $b \in [0, 1]$ let $\hat{b} := 1 - b$ denote the *symmetric point around* c.

Lemma 2.1. *Let $a < b < d < e \in [0, 1]$, where b and d are turning points of T_s^j for some $j \in \mathbb{N}$, and T_s^j has no other turning point in (a, e). Then $T_s^j(a) \in [T_s^j(b), T_s^j(d)]$ or $T_s^j(e) \in [T_s^j(b), T_s^j(d)]$.*

Proof. Assume that $T_s^j(a) < T_s^j(d) < T_s^j(b) < T_s^j(e)$, see Figure 1.

Case I: Let $m < n < j$ such that $T_s^m(b) = c = T_s^n(d)$. We consider the image of $[a, e]$ under T_s^m.

a) Let $|T_s^m(a) - c| \geq |T_s^m(e) - c|$. This means that $\widehat{T_s^m(e)} \in [c, T_s^m(a)]$. Consequently, there is a point $x \in (a, b)$ such that $T_s^m(x) = \widehat{T_s^m(d)}$, but then $T_s^n(x) = T_s^n(d) = c$, contradicting that (a, b) contains no turning point of T_s^j.

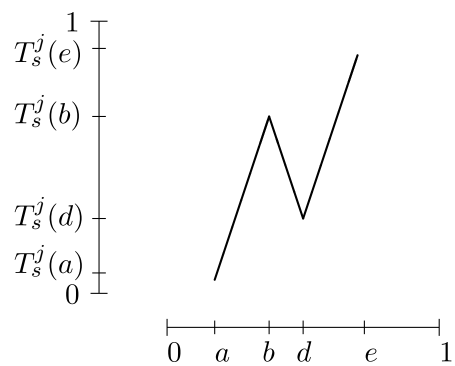

Figure 1. Example of a pattern that is not allowed by Lemma 2.1.

b) Let $|T_s^m(a) - c| < |T_s^m(e) - c|$. This means that $\widehat{T_s^m(a)} \in [c, T_s^m(e)]$. Consequently, there exists a point $y \in (b, e)$ such that $T^m(y) = \widehat{T^m(a)}$. It follows that $T_s^j(y) = T_s^j(a) < T_s^j(d)$ which contradicts that d is the minimum of T_s^j in (b, e).

Case II: Let $m < n < j$ such that $T_s^m(d) = c = T_s^n(b)$. Again we consider $T_s^m|_{[a,e]}$.

a) Let $|T_s^m(a)-c| \geq |T_s^m(e)-c|$. This means that $\widehat{T_s^m(e)} \in [c, T_s^m(a)]$. Therefore there exists a point $y \in (a,d)$ such that $T_s^j(y) > T_s^j(b)$, which contradicts that b is the maximum of T_s^j in (a,d).

b) Let $|T_s^m(a)-c| < |T_s^m(e)-c|$. This means that $\widehat{T_s^m(a)} \in [c, T_s^m(e)]$. Thus there exists $x \in (d,e)$ such that $T_s^n(x) = T_s^n(b) = c$, contradicting that (d,e) contains no turning point of T_s^j.

The case when $T_s^j(a) > T_s^j(d) > T_s^j(b) > T_s^j(e)$ follows analogously. □

2.2. **Inverse limits and chainability.** The *inverse limit space* $\varprojlim([0,1],T_s)$ is the collection of all backward orbits

$$\varprojlim([0,1],T_s) := \{(\dots, u_{-2}, u_{-1}, u_0) : T_s(u_{i-1}) = u_i \in [0,1] \text{ for all } i \leq 0\},$$

equipped with the product topology and the *shift homeomorphism*

$$\sigma((\dots, u_{-2}, u_{-1}, u_0)) = (\dots, u_{-2}, u_{-1}, u_0, T_s(u_0))$$

for every $u := (\dots, u_{-2}, u_{-1}, u_0) \in \varprojlim([0,1],T_s)$. We call $\varprojlim([c_2,c_1],T_s)$ the *core of the inverse limit space*. For $k \in \mathbb{N}_0$, define the *k-th projection map* as $\pi_k : \varprojlim([0,1],T_s) \to [0,1]$, $\pi_k(u) = u_{-k}$. We denote an arbitrary arc-component by $\mathfrak{U}$ and the arc-component that contains a fixed point $\bar{0} = (\dots, 0, 0, 0)$ by $\mathfrak{C}$. In this paper we mostly study the arc-component $\mathfrak{R}$ associated with the other fixed point $r := \frac{s}{s+1} \in [c_2, c_1]$ of T_s, so $\rho := (\dots, r, r, r) \in \mathfrak{R}$. Observe that $\rho \in \mathfrak{R} \subset \varprojlim([c_2,c_1],T_s)$, while $\mathfrak{C} \not\subseteq \varprojlim([c_2,c_1],T_s)$.

Definition 2.2. *The* arc-length *of two points $u, v \in \mathfrak{U}$ is defined as*

$$d(u,v) := s^k |u_{-k} - v_{-k}|,$$

where $k \in \mathbb{N}_0$ is such that $\pi_k \colon [u,v] \to [c_2,c_1]$ is injective. Note that this definition does not depend on $k \in \mathbb{N}_0$.

Definition 2.3. *A space X is* chainable *if there exist finite open covers $\mathcal{C} := \{\ell_i\}_{i=1}^n$ of X, called* chains, *of arbitrarily small* mesh $(\mathcal{C}) := \max_{i\in\{1,\dots,n\}} \operatorname{diam}(\ell_i)$ *such that the* links *ℓ_i satisfy $\ell_i \cap \ell_j \neq \emptyset$ if and only if $|i-j| \leq 1$. Clearly the interval $[c_2,c_1]$ is chainable. We call $\mathcal{C}_k$ a* natural chain *of $\varprojlim([c_2,c_1],T_s)$ if for $j \in \{1,\dots,n\}$ the following is true:*

1. *There exists a chain $\{I_k^1, I_k^2, \dots, I_k^n\}$ of $[c_2,c_1]$ such that $\ell_k^j := \pi_k^{-1}(I_k^j)$ are links of $\mathcal{C}_k$.*
2. *Each point $x \in \bigcup_{i=0}^k T_s^{-i}(c)$ in $[c_2,c_1]$ is a boundary point of some link I_k^j.*
3. *For each i there is j such that $T_s(I_{k+1}^i) \subset I_k^j$.*

We say that chain $\mathcal{C}'$ *refines* chain $\mathcal{C}$ (written $\mathcal{C}' \preceq \mathcal{C}$) if for every link $\ell' \in \mathcal{C}'$ there is a link $\ell \in \mathcal{C}$ such that $\ell' \subset \ell$. Condition (3) ensures that $\mathcal{C}_{k+1} \preceq \mathcal{C}_k$.

2.3. Patterns and symmetry.

Definition 2.4. *A point $u \in \varprojlim([c_2, c_1], T_s)$ is called a* k-point *(with respect to the chain $\mathcal{C}_k$) if there exists $n \geq 1$ such that $\pi_{k+n}(u) = c$. Note that if c is not periodic, such n is unique and we call it the* k-level *of u and denote it by $L_k(u)$.*

Definition 2.5. *Let $A \subset \varprojlim([c_2, c_1], T_s)$ be an arc and let $u_0, \dots, u_N \in A$ be a complete list of k-points such that $u_0 \prec u_1 \prec \dots \prec u_N$, where the order $\prec$ is inherited from the standard order on $[0,1]$ via a parametrization $\varphi : [0,1] \to A$. Then the list of levels $L_k(u_0), L_k(u_1), \dots, L_k(u_N)$ is called* k-pattern *of A.*

Remark 2.6. *From the Definition 2.5 it follows that A is the concatenation of arcs $[u_{j-1}, u_j]$ with pairwise disjoint interiors and π_k maps $[u_{j-1}, u_j]$ bijectively onto $[c_{L_k(u_{j-1})}, c_{L_k(u_j)}]$. Equivalently, if $i \in \mathbb{N}_0$ is such that $\pi_{k+i} : A \to \pi_{k+i}(A)$ is injective, then the graph $T^i|_{\pi_{k+i}(A)}$ has the same k-pattern as A. That is, T^i has turning points $\pi_{k+i}(u_0) < \dots < \pi_{k+i}(u_N)$ in $\pi_{k+i}(A)$ and $T^i(\pi_{k+i}(u_j)) = c_{L_k(u_j)}$ for $0 \leqslant j \leqslant N$. We will call this list* k-pattern *of $T^i|_{\pi_{k+i}(A)}$ as well.*

Example 2.7. *The arc A as in Figure 2 has a k-pattern* 312 *or* 213*, depending on the choice of orientation.*

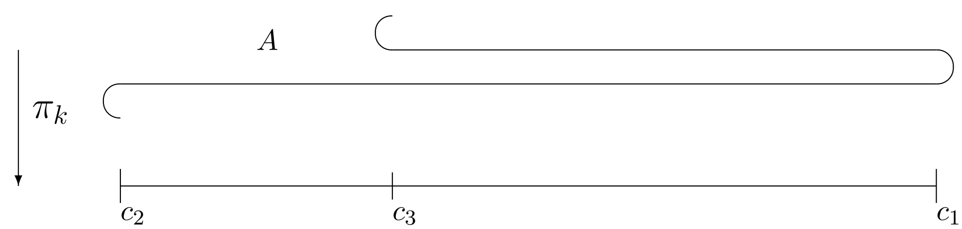

FIGURE 2. Arc A with k-pattern 312

Definition 2.8. *We say that an arc $A := [e, e'] \subset \varprojlim([c_2, c_1], T_s)$ is* k-symmetric *if $\pi_k(e) = \pi_k(e')$ and the k-pattern of the open arc (e, e') is a palindrome.*

Remark 2.9. *Definition 2.8 implies that the k-pattern of (e, e'), where A is a k-symmetric arc, is of odd length and the letter in the middle is the largest. This can be easily seen by considering the smallest $j > k$ such that $\pi_j : A \to [c_2, c_1]$ is injective.*

Lemma 2.10. *Let P be a k-pattern that appears somewhere in $\varprojlim([c_2, c_1], T_s)$,* i.e., *there is an arc $A \subset \varprojlim([c_2, c_1], T_s)$ with k-pattern P. If arc-component $\mathfrak{U}$ contains no arc with k-pattern P, then $\mathfrak{U}$ is not dense in $\varprojlim([c_2, c_1], T_s)$.*

Proof. For every pattern P that appears in $\varprojlim([c_2, c_1], T_s)$ there exist $n \in \mathbb{N}$ and $J \subset [c_2, c_1]$ such that the graph of $T_s^n|_J$ has pattern P. This means that if there is a subarc $A \subset \mathfrak{U}$ such that π_{k+n} maps A injectively onto J, then A has k-pattern P. If $\mathfrak{U}$ is dense in $\varprojlim([c_2, c_1], T_s)$, then $\pi_{k+n}(\mathfrak{U}) = [c_2, c_1]$ for every $n \in \mathbb{N}_0$. By Lemma 2.1 there exists an arc $B \subset \mathfrak{U}$ such that $\pi_{k+n}(B)$ maps injectively onto $[c_2, c_1]$. Thus there indeed exists $A \subset B \subset \mathfrak{U}$ such that π_{k+n} maps A injectively onto J. This finishes the proof. $\square$

Definition 2.11. *Take an arc-component $\mathfrak{U}$ and a natural chain $\mathcal{C}_k$ of $\varprojlim([c_2, c_1], T_s)$ for some $k \in \mathbb{N}$. For a point $u \in \mathfrak{U}$ such that $u \in \ell \in \mathcal{C}_k$ we denote the arc-component of $\ell \cap \mathfrak{U}$ containing u by A^u_ℓ, or simply A^u if the link $\ell \in \mathcal{C}_k$ is clear from the context.*

Definition 2.12. *Given a chain $\mathcal{C}$ and an arc $A \subset \varprojlim([c_2, c_1], T_s)$, the* link-pattern *$LP(A)$ is the list of links $\ell \in \mathcal{C}$ that A goes through consecutively.*

Remark 2.13. *Definition 2.12 is somewhat ambiguous, because we do not indicate which linear order we take, and more importantly, whether we include the first/last link that A intersects if already the list without the first/last link covers A. We allow each of these lists to serve as a link-pattern.*

Definition 2.14. *Arc A is called k-*link-symmetric *(with respect to the chain $\mathcal{C}_k$) if A has a link-pattern which is a palindrome. This link-pattern then automatically has odd length, and there is a unique link in the middle, the* midlink *ℓ, which contains a unique arc-component $A^m \subset A \cap \ell$ corresponding to the middle letter of the palindrome. We call the point m in A^m with the largest k-level the* midpoint *of A.*

Remark 2.15. *The definition of the* midpoint *m above is just for completeness; since we can not topologically distinguish points in the same arc component A^m, any point $u \in A^m$ would serve equally well. If A is contained in a single link $\ell \in \mathcal{C}_k$, then it is k-link symmetric by default, but $\ell \cap A$ does not need to contain a k-point. In that case, any point in $\ell \cap A$ can serve as midpoint of A.*

Remark 2.16. *Every k-symmetric arc is also k-link-symmetric but the converse does not hold. This is one of the main obstacles in the proof of the Core Ingram Conjecture.*

Definition 2.17. *We define the* reflection of $v \in \mathfrak{U}$ around $u \in \mathfrak{U}$ *as a point $R_u(v) \in \mathfrak{U}$ such that $[v, R_u(v)]$ is k-link symmetric with midpoint u. If possible, we choose $R_u(v)$ so that $[v, R_u(v)]$ is a k-symmetric arc.*

Definition 2.18. *We define the* reflection *around $a \in \mathbb{R}$ by $\bar{R}_a(x) := 2a - x$ for all $x \in \mathbb{R}$.*

3. The structure of the arc-component $\mathfrak{R}$

3.1. **The arcs A_i.** Recall that $\mathfrak{R}$ is the arc-component in $\varprojlim([c_2, c_1], T_s)$ containing the point $\rho = (\dots, r, r, r)$.

Definition 3.1. *Let $\mathcal{C}_k$ be a natural chain of $\varprojlim([c_2, c_1], T_s)$. For every $i \in \mathbb{N}$ we define $A_i \subset \mathfrak{R}$ to be the arc-component of $\pi^{-1}_{k+i}([c_2, \hat{c}_2])$ which contains ρ and let $m_i := \pi^{-1}_{k+i}(c) \in A_i$.*

Lemma 3.2. *The arc $A_i \subset \mathfrak{R}$ is k-symmetric with midpoint m_i for every $i \in \mathbb{N}$.*

Proof. For every $i \in \mathbb{N}$ we obtain that $\pi_{k+i}(A_i) = [c_2, \hat{c}_2]$ injectively on $[c_2, c_1]$ which is symmetric around c and so A_i is k-symmetric. $\square$

Define

$$\xi := \min\{i \geq 3 : c_i \leq c\}.$$

Note that ξ always exists and $\xi - 3$ has to be an even number or 0, otherwise the tent map T_s is renormalizable (which we excluded by taking the slope $s > \sqrt{2}$).

Now we explain some basic facts that we often use in the following lemmas.

Assume that T_s is such that $\xi > 3$. Because ξ is the smallest natural number so that $c_2 < c_\xi \leq c$ it follows that $c_i > c$ for $i \in \{3, \dots, \xi - 1\}$. Furthermore, since $s > 1$ and we restrict to non-renormalizable maps, it follows that $c < c_{\xi-2} < \dots < c_5 < c_3 < r < c_4 < c_6 < \dots < c_{\xi-1} < c_1$. Because $T_s|_{[c,c_1]}$ reverses orientation, we obtain $r < \hat{c}_2 < c_4$.

Next we argue that $\pi_{k+i} \circ \sigma = T_s \circ \pi_{k+i}$. Take a point $u = (\dots, u_{-2}, u_{-1}, u_0) \in \varprojlim([0,1], T_s)$. Then $\pi_{k+i}(\sigma((\dots, u_{-2}, u_{-1}, u_0))) = \pi_{k+i}((\dots, u_{-1}, u_0, T_s(u_0))) = u_{-(k+i)+1} = T_s(u_{-(k+i)}) = T_s(\pi_{k+i}(u))$.

Lemma 3.3. *$A_i \subset A_{i+2}$ for all $i \in \mathbb{N}$.*

Proof. Note that $\pi_{k+i}(A_i) = \pi_{k+i+2}(A_{i+2}) = [c_2, \hat{c}_2]$. We distinguish two cases:

Case I: Let $c_3 < c$. Then $c \in \pi_{k+i+1}(A_{i+2})$ and there exists an arc $B \subset A_{i+2}$ such that $\rho \in B$ and $\pi_{k+i}(B) = [c_2, c_1]$ injectively, see Figure 3. Since $A_i \subset B$ it follows that $A_i \subset A_{i+2}$.

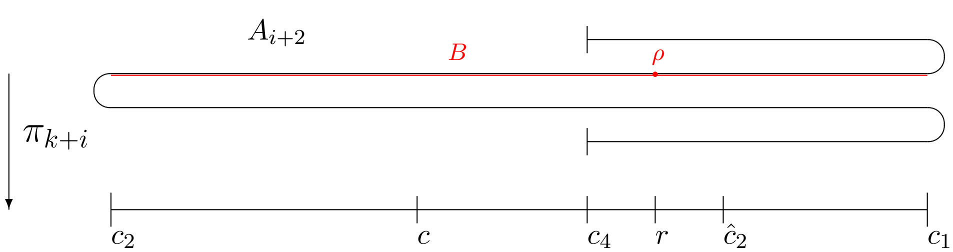

FIGURE 3. The arc A_{i+2} as in Case I.

Case II: Let $c_3 \geq c$. Because $c_3 = T_s(\hat{c}_2) < r < \hat{c}_2 < c_4$, T_s maps $\pi_{k+i}(A_{i+2})$ in a 2-to-1 fashion onto the interval $[c_2, c_4]$, see Figure 4. We find an arc $B \subset A_{i+2}$ such that $\rho \in B$ and $\pi_{k+i}(B) = [c_2, c_4]$ injectively. Because $c_4 > \hat{c}_2$ also $A_i \subset A_{i+2}$. $\square$

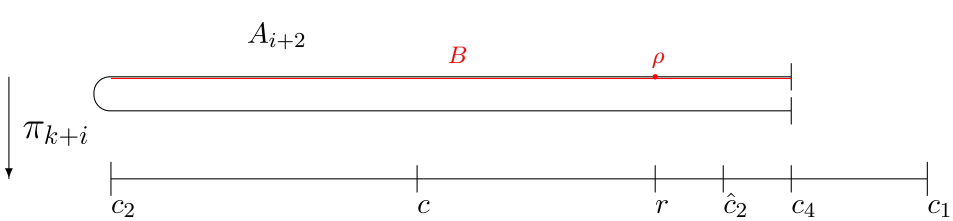

FIGURE 4. The arc A_{i+2} as in Case II.

In the following lemma let $A_{i,j} \subset A_i \subset \mathfrak{R}$ denote the longest arc (in arc-length) such that $\rho \in A_{i,j}$ and $\pi_{k+j} : A_{i,j} \to [c_2, c_1]$ is injective for some $j \leq i$, see Figure 5. Note that $A_{i,i} = A_i$.

Lemma 3.4. *$A_i \subset A_{i+\xi}$ and $A_i \not\subseteq A_{i+l}$ for every $i \in \mathbb{N}$ and every odd $l < \xi$.*

Proof. We distinguish two cases:

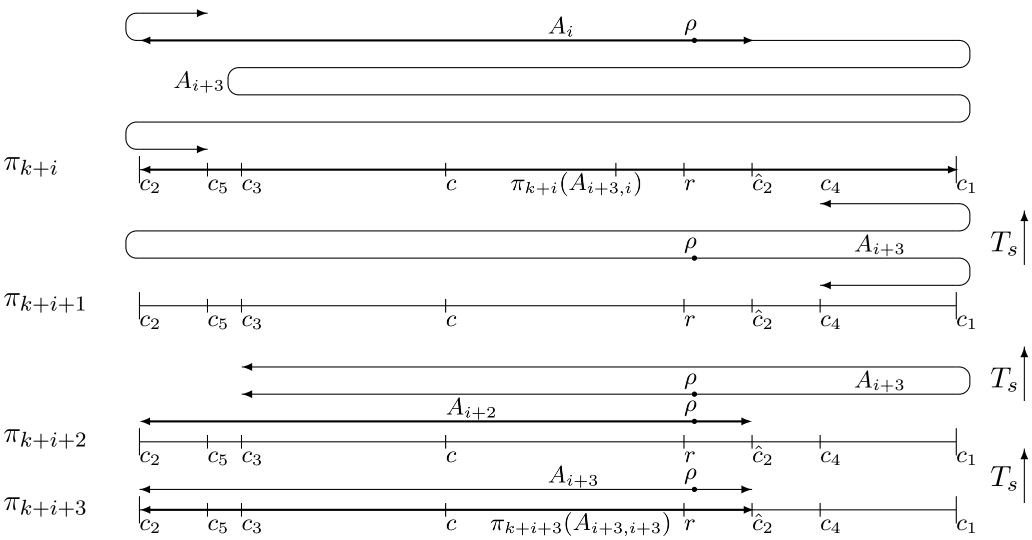

FIGURE 5. Arcs $A_i, A_{i+1}, A_{i+2}, A_{i+3}$ in mentioned projections as in Case I.

Case I: First assume that $\xi = 3$. Since $T_s(c_2) = c_3 > c_2$ it follows that $\pi_{k+i}(A_i) \not\subseteq \pi_{k+i}(A_{i+1})$ and thus $A_i \not\subseteq A_{i+1}$. However, $[c, c_1] \subset \pi_{k+i+2}(A_{i+3,i+2}) = T_s([c_2, \hat{c}_2])$. This means that $\pi_{k+i+1}(A_{i+3,i+1}) = [c_2, c_1]$ and hence $\pi_{k+i}(A_{i+3,i}) = [c_2, c_1]$, see Figure 5. we conclude that $A_i \subset A_{i+3}$. This finishes the proof for $\xi = 3$.

Case II: Let $\xi \geq 5$. Note that $c_\xi < c < c_i$ for every $i \in \{3, \dots, \xi - 1\}$. Thus we observe that $[c_2, \hat{c}_2] \not\subseteq \pi_{k+i+\xi-l}(A_{i+\xi,i+\xi-l}) = [c_{2+l}, c_1]$ for every odd $l \in \{1, \dots, \xi - 4\}$. It follows that $A_i \not\subseteq A_{i+l}$ for every odd $l \in \{1, \dots, \xi - 4\}$.
Because $T_s^{\xi-2}(c_2) = c_\xi$ it follows that $[c, c_1] \subset \pi_{k+i+2}(A_{i+\xi,i+2}) = [c_\xi, c_1]$ and $k + i + 2$ is the smallest such index. However, because $c_2 < c_\xi$ and $\pi_{k+i+2}(A_{i+2}) = [c_2, \hat{c}_2] \not\subseteq \pi_{k+i+2}(A_{i+\xi,i+2})$ it also holds that $\pi_{k+i}(A_i) \not\subseteq \pi_{k+i}(A_{i+\xi-2,i})$, so $A_i \not\subseteq A_{i+\xi-2}$. As above we observe that $\pi_{k+i}(A_{i+\xi,i}) = [c_2, c_1]$ and thus $A_i \subset A_{i+\xi}$. □

Recall that m_i denotes the midpoint of the arc A_i.

Lemma 3.5. *It holds that $m_{i+2} \in \partial A_i$ and $m_{i+1} \notin A_i$. Furthermore, if $\xi = 3$, then $m_i \in A_{i+1}$ and if $\xi > 3$, then $m_i \notin A_{i+1}$.*

Proof. Since $\pi_{k+i}(m_i) = c$, also $\pi_{k+i}(m_{i+2}) = c_2$. Because $A_i \subset A_{i+2}$ it follows that $m_{i+2} \in \partial A_i$.

To prove the second statement, observe that $\pi_{k+i}(m_{i+1}) = c_1 \notin [c_2, \hat{c}_2]$ for $s < 2$.

For the third statement first assume that $\xi = 3$; it follows that $c \in [c_3, c_1]$ and so $m_i \in A_{i+1}$. If $\xi > 3$ then $c_3 > c$ and thus $c \notin \pi_{k+i}(A_{i+1})$, so it follows that $m_i \notin A_{i+1}$. □

Lemma 3.6. *It holds that $\rho \in [m_i, m_{i+1}]$ and $\rho \notin [m_i, m_{i+2}]$ for all $i \in \mathbb{N}$.*

Proof. By definition $\pi_{k+i}(A_i) = [c_2, \hat{c}_2]$ and $\pi_{k+i}(m_i) = c$, so $\pi_{k+i}(m_{i+1}) = T_s(\pi_{k+i}(m_i))$ $= c_1$. Since $r \in [\pi_{k+i}(m_i), \pi_{k+i}(m_{i+1})]$ it follows that $\rho \in [m_i, m_{i+1}]$.

The second statement of the lemma follows directly from Lemma 3.5. □

Lemma 3.5 states that $m_{i+2} \in \partial A_i$. We denote the other boundary point of A_i by $\hat{m}_{i+2}$.

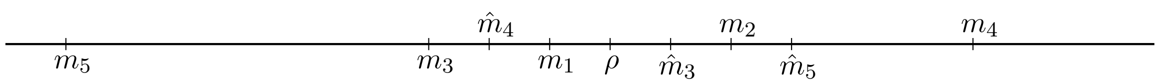

FIGURE 6. The structure of the arc-component $\mathfrak{R}$; $A_i = [m_{i+2}, \hat{m}_{i+2}]$ has midpoint m_i.

Note that all properties of (k-link symmetric) arcs $\{A_i\}_{i\in\mathbb{N}}$ proved in this section are topological, meaning they are preserved under a homeomorphism.

3.2. ε-symmetry.

Definition 3.7. *Let $J := [a, b] \subset [c_2, c_1]$ be an interval. The map $f : J \to \mathbb{R}$ is called* ε-symmetric *if there is a continuous bijection $x \mapsto i(x) =: \hat{x}$ swapping a and b, such that $|f(x) - f(\hat{x})| < \varepsilon$ for all $x \in J$. Since $i : J \to J$ has a unique fixed point m, we say that f is ε-symmetric with* center m *(or just ε-symmetric* around m*).*

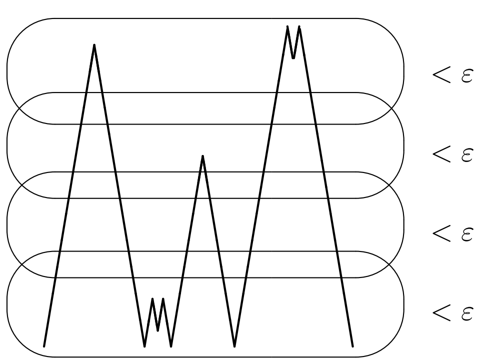

FIGURE 7. A graph of an ε-symmetric map.

Remark 3.8. *Let $A \subset \mathfrak{R}$ be a k-link symmetric arc with midpoint m_A, where* mesh $(\mathcal{C}_k) < \varepsilon$. *If i is such that $\pi_{k+i}|_A \colon A \to \pi_{k+i}(A)$ is injective in $[c_2, c_1]$, then $T^i|_{\pi_{k+i}(A)}$ is ε-symmetric around $\pi_{k+i}(m_A)$, see Figure 7.*

Next we restate Proposition 3.6. from [1] although the definition of ε-symmetry is slightly generalized here. However, all arguments in the proof of Proposition 3.6. from [1] with the new definition of ε-symmetry remain the same.

Proposition 3.9. *For every $\delta > 0$ there exists $\varepsilon > 0$ such that for every $n \geq 0$ and every interval $H = [a, b] \ni m$ such that $|m - c|, |c - a|, |c - b| > \delta$, $T^n|_H$ is not ε-symmetric around m.*

3.3. Completeness of the sequence $\{A_i\}_{i\in\mathbb{N}}$.

Definition 3.10. *Let $k \in \mathbb{N}$, $u \in \mathfrak{U}$, and let $\{G_i\}_{i\in\mathbb{N}} \subset \mathfrak{U}$ be a sequence of k-link symmetric arcs with midpoints m_i respectively and $u \in G_i$, for all $i \in \mathbb{N}$. The sequence $\{G_i\}_{i\in\mathbb{N}}$ is called a* complete sequence of k-link symmetric arcs *with respect to u if every k-link symmetric arc $G \subset \mathfrak{U}$ such that $u \in G$ (not contained in a single link of a chain $\mathcal{C}_k$) has midpoint in $\{m_i\}_{i\in\mathbb{N}}$.*

Proposition 3.11. *There exists $\varepsilon > 0$ such that $\{A_i\}_{i\in\mathbb{N}}$ is a complete sequence of k-link symmetric arcs with respect to ρ for every $k \in \mathbb{N}$ with* mesh $(\mathcal{C}_k) < \varepsilon$.

Proof. Fix $\delta := \frac{1}{2}\min\{|c-r|, |c-x| : x \in T^{-2}(c)\}$, take $\varepsilon > 0$ as in Proposition 3.9 and a chain $\mathcal{C}_k$ such that mesh $(\mathcal{C}_k) < \varepsilon$. Assume that there exists a k-link symmetric arc $B \ni \rho$ in $\mathfrak{R}$ which is not contained in a single link of $\mathcal{C}_k$ and its midpoint $m \neq m_i$ for every $i \in \mathbb{N}$. Without loss of generality we can assume that

(1) m is the closest to ρ (in arc-length) among all midpoints of such arcs.

Since m is a k-point and there are no k-points in (m_1, m_2), we obtain that $m \notin (m_1, m_2)$. Thus by Lemma 3.6 there exists $i \in \mathbb{N}$ such that $m \in (m_{i+2}, m_i)$.

Case I: Assume that $R_m(m_i) \in [m_{i+2}, m_i] \cup A^{m_{i+2}}$ (recall Definitions 2.11 and 2.17).

Let $A \subset B \cap [m_{i+2}, \hat{m}_{i+2}]$ be the maximal k-link symmetric arc with midpoint m. Let a, b be the boundary points of A such that $m_{i+2} \preceq b \prec m_i \prec a \preceq \hat{m}_{i+2}$ (recall that $\prec$ denotes linear order on $\mathfrak{R}$).

Denote by $a' := R_{m_i}(a)$, $b' := R_{m_i}(b)$, $m' = R_{m_i}(m)$ and $\rho' := R_{m_i}(\rho)$, so that the arcs $[a, a'], [b, b']$ and $[\rho, \rho']$ are k-symmetric arcs with midpoint m_i. Define an arc $A' := [a', b']$, see Figure 8.

Since A is k-link symmetric and $A \subset [m_{i+2}, \hat{m}_{i+2}]$, A' is k-link symmetric with midpoint m'.

Note that either $b = m_{i+2}$ or $b \prec m_{i+2}$.
If $b = m_{i+2}$ it holds by the assumption of Case I that $[m_{i+2}, m_i] \subset A$ and thus $\rho' \in A$. Therefore $\rho \in A'$.

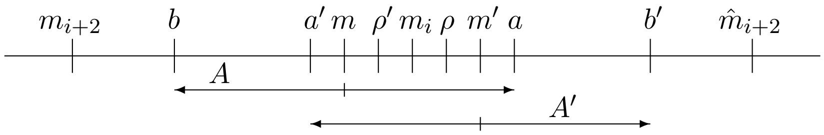

FIGURE 8. Case I of the proof.

If $b \prec m_{i+2}$, then by assumption of Case I it follows that $B = A$ and thus $\rho \in A$. Because $m \prec m_i \prec \rho$, it follows that $\rho' \in A$ and thus again $\rho \in A'$.

By (1) there exists $j < i$ such that $m' = m_j$.

Now we study $\pi_{k+i}(A)$, see Figure 9. Since $m \in (m_{i+2}, m_i)$ it follows that $\pi_{k+i}(m) \in (c_2, c)$.

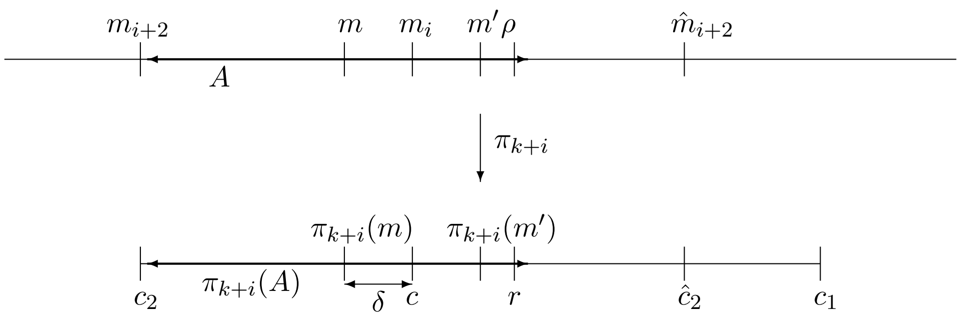

FIGURE 9. Arc A in projection π_{k+i} as in Case I of the proof.

If $|\pi_{k+i}(m) - c| > \delta$, we use Proposition 3.9 to conclude that A is not k-link symmetric, a contradiction. Assume that $|\pi_{k+i}(m) - c| \leq \delta$. Since $m' = m_j$ for some $j < i$ it follows that $m' \notin (m_{i-2}, m_i)$. Thus $|\pi_{k+i}(m) - c| = |\pi_{k+i}(m') - c| = |\pi_{k+i}(m_j) - c| \geq |\pi_{k+i}(m_{i-2}) - c| > \delta$, a contradiction. The last inequality follows from the fact that $\pi_{k+i}(m_{i-2}) \in T^{-2}(c)$ and the definition of δ.

Case II: Assume that $R_m(m_i) \notin [m_{i+2}, m_i] \cup A^{m_{i+2}}$.

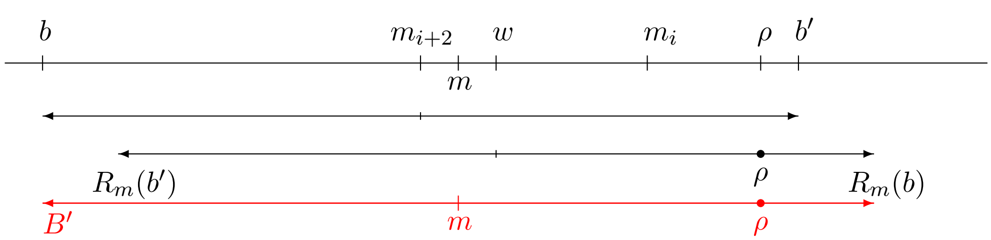

FIGURE 10. Reflections as in Case II of the proof.

Let b be the endpoint of $B \cap [m_{i+4}, m_{i+2}]$ that is the furthest away from ρ. Take $w := R_m(m_{i+2})$ (see Definition 2.17) and note that $w \in (m, m_i)$ by the assumption for this case. Denote by $b' := R_{m_{i+2}}(b)$ so that the arc $[b, b'] \subset [m_{i+4}, \hat{m}_{i+4}]$ is

k-symmetric with midpoint m_{i+2}. We reflect the arc $[b, b']$ over m and obtain an arc $[R_m(b'), R_m(b)]$ (see Figure 10) which is k-link symmetric with midpoint w. Denote by B' the maximal k-link symmetric arc around m such that $B' \subset B \cap [m_{i+4}, \hat{m}_{i+4}]$. Since $[m_{i+4}, \hat{m}_{i+4}]$ is k-link symmetric around m_{i+2} and $\rho \in [m_{i+2}, \hat{m}_{i+4}]$, counting the links through which the arcs $[m_{i+2}, \rho] \supset [m, \rho]$ pass consecutively, we conclude that $\rho \in B'$, see Figure 10. So $R_m(\rho)$ is well-defined and $R_m(\rho) \in [b, m]$. We conclude that $\rho \in [R_m(b'), R_m(b)]$ which contradicts the minimality of m, because we found a k-link symmetric arc $[R_m(b'), R_m(b)]$ with midpoint w such that $[R_m(b'), R_m(b)] \ni \rho$, $[w, \rho] \subset [m, \rho]$ and $w \in (m_{i+2}, m_i)$. ☐

3.4. **ε-symmetry and ε-closeness in the infinite non-recurrent case.** The results in this section rely on the non-recurrence of the critical point.

Proposition 3.12. *Assume that $s \in (\sqrt{2}, 2]$ is such that c is not recurrent and not preperiodic. For every $\delta > 0$ there exists $\varepsilon > 0$ with the following property: if $c \in J \subset [c_2, c_1]$ is an interval with midpoint m, and $|c - \partial J| \geqslant 5\delta$, then for each $n \geqslant 0$, either $T_s^n|_J$ is not ε-symmetric or $|c - m| \leqslant \varepsilon s^{-n}$.*

Proof. Fix $\delta > 0$ and let $\varepsilon = \varepsilon(\delta) > 0$ be chosen as in Proposition 3.9 and additionally such that $T_s^n(c) \notin (c - \varepsilon, c + \varepsilon)$ for all $n \geqslant 1$, which is possible because c is not recurrent.

If $|c - m| > \delta$, then we use Proposition 3.9 to conclude that $T_s^n|_J$ is not ε-symmetric. Assume by contradiction that $T_s^n|_J$ is ε-symmetric and $\varepsilon s^{-n} < |c - m| \leqslant \delta$. The choice of ε implies that T_s^n is monotone on a one-sided neighbourhood of c of length εs^{-n}, and maps it therefore onto an interval of length ε. This means that $T_s^n([c, m])$ has length at least ε, so that c and m must be distinct centres of ε-symmetry of T_s^n. Therefore $c^1 := \bar{R}_m(c) \in J$ is another center of ε-symmetry, and so is $c^2 := \bar{R}_{c^1}(c)$. Let $c^{i+2} := \bar{R}_{c^i}(c^{i+1})$ for every $i \in \mathbb{N}$ so that $c^{i+2} \in [c_2, c_1]$. Take the smallest $N \in \mathbb{N}$ so that the center of ε-symmetry c^N of T_s^n is satisfying $|c - c^N| > \delta$. Since $|c - c^{N-1}| < \delta$, it follows that $|c - c^N| < 2\delta$. We conclude that $c^N \in J$, because we assumed that $|c - \partial J| \geqslant 5\delta$. Define the interval $J' := [a, b]$, where $a, b \in J$ are chosen such that $|a - c^N| = |b - c^N|$ is the largest such that $[a, b] \subset J$. Therefore, c^N is the center of ε-symmetry of $T_s^n|_{J'}$ and $c \in J'$. Since $|c - c^N| < 2\delta$ and $|c - \partial J| \geqslant 5\delta$, we conclude that $|c - \partial J'| > \delta$. Applying Proposition 3.9 for the interval J' we conclude that $T_s^n|_{J'}$ is not ε-symmetric, which is a contradiction. ☐

Proposition 3.13. *Assume that $T_s^n|_J$ is ε-symmetric around $m \in J \subset [c_2, c_1]$ and $\operatorname{diam}(T_s^n(J)) \geq \varepsilon$ for some $\varepsilon > 0$. Then there exists $i < n$ such that $|c - T_s^i(m)| < \varepsilon s^{i-n}$.*

Proof. Assume $T_s^n|_J$ is ε-symmetric around m and $|c - T_s^i(m)| \geq \varepsilon s^{i-n}$ for every $i < n$. Specifically, $T_s^{-i}(c) \cap (m - \varepsilon s^{-n}, m + \varepsilon s^{-n}) = \emptyset$ for every $i < n$. Therefore $T_s^n|_{(m - \frac{\varepsilon}{2}s^{-n}, m + \frac{\varepsilon}{2}s^{-n})}$ is injective and $\operatorname{diam}(T_s^n((m - \frac{\varepsilon}{2}s^{-n}, m + \frac{\varepsilon}{2}s^{-n}))) = \varepsilon$. Since $\operatorname{diam}(T_s^n(J)) \geq \varepsilon$ and $T_s^n|_J$ is ε-symmetric around m, it follows that $(m - \frac{\varepsilon}{2}s^{-n}, m + \frac{\varepsilon}{2}s^{-n}) \subset J$. Thus we get a contradiction with ε-symmetry of the interval J around m. ☐

Corollary 3.14. *Suppose that c is not recurrent. Then there exists $\varepsilon > 0$ with the following property: if $j \geqslant 1$ and $J \subset [c_2, c_1]$ such that $J \supset (c_j - \varepsilon, c_j + \varepsilon)$, then $T_s^n|_J$ is not ε-symmetric with midpoint c_j for any $n \geqslant 0$.*

Proof. Let $\varepsilon > 0$ be such that $c_i \notin (c - \varepsilon, c + \varepsilon)$ for every $i \in \mathbb{N}$. By Proposition 3.13, if $T^n|_J$ is ε-symmetric around c_j, then there exists $N < n$ such that $|c - c_{j+N}| < \varepsilon s^{N-n} < \varepsilon$, which is a contradiction with the definition of ε. □

Definition 3.15. *We say that the maps $f : J \to \mathbb{R}$ and $g : K \to \mathbb{R}$ for intervals $J, K \subset [c_2, c_1]$ are* ε-close *if there exists a homeomorphism $\psi : J \to K$ such that $|f(x) - g \circ \psi(x)| < \varepsilon$ for all $x \in J$, see Figure 11.*

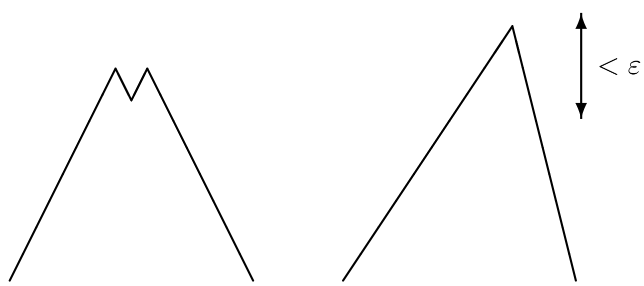

FIGURE 11. Graphs of ε-close maps.

Remark 3.16. *Maps that are ε-close can have different number of branches in general. However, in the non-recurrent case and for $\varepsilon > 0$ small enough, the number of branches must be the same, disregarding branches of the diameter less than ε that may appear at the ends of the interval. Note also that ε-closeness is not an equivalence relation because it is not transitive.*

Lemma 3.17. *Assume that the critical point c of the map T_s is not recurrent. Then there is $\varepsilon > 0$ such that, whenever $T_s^i|_{[c_2,c_1]}$ and $T_s^j|_{[a,b]}$ are ε-close for some interval $[a, b] \subset [c_2, c_1]$, there is a closed interval $J' := [a', b'] \subset [c_2, c_1]$ such that $|a' - a|, |b' - b| < \varepsilon$ so that T_s^{j-i} maps J' homeomorphically onto $[c_2, c_1]$.*

Remark 3.18. *The closed interval J' addresses the technicality that if e.g. $i = j = 0$ and $a = c_2 + \varepsilon/2$, $b = c_1 - \varepsilon/2$, then $T_s^i|_{[c_2,c_1]}$ and $T_s^j|_{[a,b]}$ are ε-close, but without the adjustment of $J' = [c_2, c_1]$, the lemma would fail.*

Proof. Take $\varepsilon = \frac{1}{100} \inf\{|c - c_n| : n \geq 1\}$. Assume that $T_s^i|_{[c_2,c_1]}$ and $T_s^j|_{[a,b]}$ are ε-close, with a homeomorphism $\psi : [a, b] \to [c_2, c_1]$ as in Definition 3.15.

If $i > j$, then $T_s^i|_{[c_2,c_1]}$ has more branches than $T_s^j|_{[a,b]}$, so they cannot be ε-close. If $i = j$, then there is nothing to prove. Therefore $i < j$.

Suppose that $T_s^{j-i}|_{[a,b]}$ is a homeomorphism onto a subinterval of $[c_2, c_1]$. If $T_s^{j-i}([a, b]) \supset [c_2 + \varepsilon s^{-i}, c_1 - \varepsilon s^{-i}]$, then (since by non-recurrence of c, the point c_n cannot be εs^{-i}-close to c_2 or c_1 for every $n > 2$) we can adjust the interval $[a, b]$ to $[a', b']$ so that $T_s^{j-i}([a', b']) \supset$

$[c_2, c_1]$. In this case, the lemma is proved. If on the other hand $T_s^{j-i}([a,b]) \not\supset [c_2 + \varepsilon s^{-i}, c_1 - \varepsilon s^{-i}]$, then $T_s^j|_{[a,b]}$ cannot be ε-close to $T_s^i|_{[c_2,c_1]}$.

Now we assume that $T_s^{j-i}|_{[a,b]}$ is not a homeomorphism onto a subinterval of $[c_2, c_1]$. Since $T_s^{j-i}([a,b]) \subset [c_2, c_1]$, there is $t \in [a,b]$ such that $x := \psi(t) = T_s^{j-i}(t) \in [c_2, c_1]$; let $U \ni t$ be the maximal closed interval in $[a,b]$ such that $T_s^{j-i}|_U$ is monotone.

Take $t' \in \partial U \setminus \{a,b\}$ closest to t, so that $T_s^{j-i}(t') = c_n$ for some $n \geqslant 1$. Let U' be the maximal neighbourhood of t' such that $T_s^{j-i}(U')$ is contained in an ε-neighbourhood V of c_n. It follows that $T_s^j|_{U'}$ is ε-symmetric (see Figure 12).

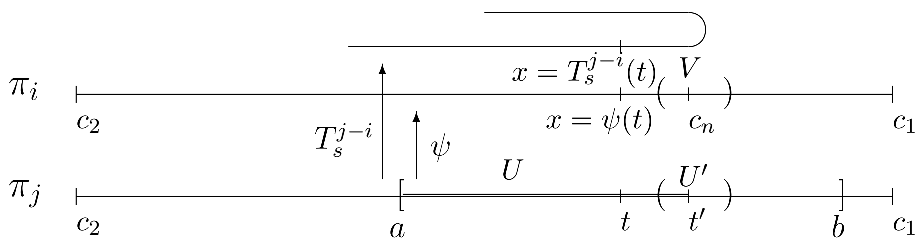

FIGURE 12. Step in the proof of Lemma 3.17.

If $\psi|_U$ and $T_s^{j-i}|_U$ have the same orientation, then, by ε-closeness, $|T_s^i(\psi(y)) - T_s^j(y)| < \varepsilon$ for all $y \in U$, which means that $|T_s^{j-i}(y) - \psi(y)| < \varepsilon s^{-i}$ for all $y \in U$. However, $T_s^i|_V$ is not ε-symmetric due to Corollary 3.14, and therefore the ε-closeness is violated on the set U'.

On the other hand, if $\psi|_U$ and $T_s^{j-i}|_U$ have opposite orientation, then T_s^i is ε-symmetric on a neighbourhood of x, with c_n in its closure. Let V' be the mirror image of V when reflected in x. Then by the ε-symmetry of T_s^i around x and around c_n, T_s^i has to be ε-symmetric on V' as well. But this contradicts Corollary 3.14 again, completing the proof. □

Definition 3.19. *Let $A, B \subset \varprojlim([c_2,c_1], T_s)$ be arcs with k-patterns P_A and P_B respectively,* mesh$(\mathcal{C}_k) < \varepsilon$. *We say that* k-patterns P_A and P_B are ε-close *if there exist $i, j \in \mathbb{N}$ such that $\pi_{k+i}|_A$ and $\pi_{k+j}|_B$ are injective on $[c_2, c_1]$ and $T^i|_{\pi_{k+i}(A)}$ and $T^j|_{\pi_{k+j}(B)}$ are ε-close maps.*

In the following remark we paraphrase the statement of Lemma 3.17 in the context of $\varprojlim([c_2,c_1], T_s)$ as it is going to be used in the proof of Theorem 1.2.

Remark 3.20. *Fix $\varepsilon > 0$ as in the proof of Lemma 3.17 and take $k \in \mathbb{N}$ such that* mesh$(\mathcal{C}_k) < \varepsilon$. *Assume that $n \in \mathbb{N}$ and $Q, Q' \subset \varprojlim([c_2,c_1], T_s)$ are arcs with $(k+n)$-patterns P and P' respectively where $P = 12$. Lemma 3.17 claims that if the k-patterns of Q and Q' are ε-close, then $P' = 12$ (or 21 depending on the orientation).*

4. Arc-component $\mathfrak{R}$ is fixed under homeomorphisms

From now on assume that $\sqrt{2} < s \neq \tilde{s} \leqslant 2$ and that tent maps T_s and $T_{\tilde{s}}$ have non-recurrent infinite critical orbits.

Assume by contradiction that there exists a homeomorphism $h : \varprojlim([c_2, c_1], T_s) \to \varprojlim([\tilde{c}_2, \tilde{c}_1], T_{\tilde{s}})$. Our goal in this section is to prove Theorem 1.2 (which holds also if $s = \tilde{s}$).

Set

$$\delta_0 := \frac{1}{100} \inf\{|c - c_n|,\ |\tilde{c} - \tilde{c}_n| : n \geq 1\}. \tag{2}$$

Take $\varepsilon = \varepsilon(\delta_0) > 0$ such that Proposition 3.9, Proposition 3.11, Proposition 3.12, Corollary 3.14 and Lemma 3.17 all apply for both $\varprojlim([c_2, c_1], T_s)$ and $\varprojlim([\tilde{c}_2, \tilde{c}_1], T_{\tilde{s}})$.

Choose integers k', l, k so large that mesh $(\mathcal{C}_{k'})$, mesh $(\tilde{\mathcal{C}}_l)$, mesh $(\mathcal{C}_k) < \varepsilon$ and

$$h^{-1}(\tilde{\mathcal{C}}_l) \preceq \mathcal{C}_{k'} \quad \text{and} \quad h(\mathcal{C}_k) \preceq \tilde{\mathcal{C}}_l. \tag{3}$$

Let $B_i := h(A_i) = [h(\hat{m}_{i+2}), h(m_{i+2})]$ for every $i \in \mathbb{N}_0$; since $h(\mathcal{C}_k)$ refines $\tilde{\mathcal{C}}_l$, B_i is link-symmetric in $\tilde{\mathcal{C}}_l$. If B_i is not contained in a single link of $\tilde{\mathcal{C}}_l$, we denote its midpoint by n_i. Note that $h(m_i) \in A^{n_i}$ for every $i \in \mathbb{N}$. We denote $R_{n_i}(n_{i+2})$ by $\hat{n}_{i+2}$, see Figure 16. Note also that $h(\hat{m}_{i+2}) \in A^{\hat{n}_{i+2}}$. Let $(\dots q_{-2}, q_{-1}, q_0) = q := h(\rho)$.

Lemma 4.1. *The sequence $\{B_i\}_{i\in\mathbb{N}} \subset \varprojlim([c_2, c_1], T_{\tilde{s}})$ is an* eventual *complete sequence of l-link symmetric arcs with respect to q, in the sense that for every l-link symmetric arc with midpoint n and containing q, either $n = n_i$ for some $i \in \mathbb{N}$ or $n \in (n_1, n_2)$.*

Proof. Assume by contradiction that there exists an l-link-symmetric arc $B \ni q$ with midpoint $n \in h(\mathfrak{R})$ such that $n \neq n_i$ for every $i \in \mathbb{N}$, $n \notin (n_1, n_2)$ and B is not contained in a single link of the chain $\tilde{\mathcal{C}}_l$. Take B such that n is the closest to q (in arc-length) with the above properties. There exists $j \in \mathbb{N}$ such that $n \in (n_j, n_{j+2})$.

Because we chose chains such that $h^{-1}(\tilde{\mathcal{C}}_l) \preceq \mathcal{C}_{k'}$, the arc $A := h^{-1}(B)$ is k'-link symmetric and $\rho \in A$.

Throughout this proof we use A^u to denote the arc-component of u in ℓ for a k'-point $u \in \ell \in \mathcal{C}_{k'}$.

Assume that the midpoint m of A is not contained in A^{m_j} or $A^{m_{j+2}}$, thus $m \in (m_j, m_{j+2})$ and $m_j \in A$. Note that $\mathcal{C}_k = h^{-1} \circ h(\mathcal{C}_k) \preceq h^{-1}(\tilde{\mathcal{C}}_l) \preceq \mathcal{C}_{k'}$ and thus $k \geq k'$. We conclude that every k-point is a k'-point. Specifically, m_j is a k'-point and since A^ρ contains no k'-points, it is easy to see that A is not contained in a single link of $\mathcal{C}_{k'}$. Since $\{A_i\}_{i\in\mathbb{N}}$ is a complete sequence of k-link symmetric arcs with respect to ρ, we get that $\{A_{i+k-k'}\}_{i\in\mathbb{N}}$ is a complete sequence of k'-link symmetric arcs with respect to ρ. Thus $m = m_{i+k-k'}$ for some $i \geq 1$. But $m = m_{i+k-k'} \in (m_j, m_{j+2})$ gives a contradiction.

If $m \in A^{m_j}$, then by the definition of a midpoint we conclude that $m = m_j$ and thus $A^n \ni h(m) = h(m_j) \in A^{n_j}$. Since n and n_j are both midpoints and $A^n = A^{n_j}$, we conclude that $n = n_j$. An analogous argument shows that if $m \in A^{m_{j+2}}$ then $n = n_{j+2}$. $\square$

Recall that $\tilde{\mathfrak{R}}$ is the arc-component in $\varprojlim([\tilde{c}_2, \tilde{c}_1], T_{\tilde{s}})$ containing $\tilde{\rho} = (\dots, \tilde{r}, \tilde{r}, \tilde{r})$, where $\tilde{r} = \frac{\tilde{s}}{\tilde{s}+1}$ is a fixed point of $T_{\tilde{s}}$.

Assume by contradiction that $h(\mathfrak{R}) \neq \tilde{\mathfrak{R}}$, so in particular $h(\mathfrak{R}) \not\ni \tilde{\rho}$.

Assume also by contradiction that there exists $N \in \mathbb{N}$ such that the projections $q_{-i} > \tilde{c}$ for all $i \geq N$. Then the coordinates of $\sigma^{-N}(q)$ and $\tilde{\rho}$ are all obtained using the same right inverse branch of $T_{\tilde{s}}$. Thus $\sigma^{-N}(q)$ and $\tilde{\rho}$ are connected by an arc, and hence $h(\mathfrak{R}) \ni \tilde{\rho}$, a contradiction. Therefore $q_{-i} < c$ infinitely often. Analogously as above it follows that $q_{-i} > \tilde{c}$ infinitely often, because a fixed point $\bar{0} \notin h(\mathfrak{R})$.
Therefore, there is $l' > l+1$ such that $q_{-(l'+1)} < \tilde{c} < q_{-l'}$ and $\pi_{l'} : B_1 \to [\tilde{c}_2, \tilde{c}_1]$ is injective, where $B_1 = h(A_1)$.

The crux of the proof is to show that $h(\mathfrak{R})$ cannot contain the l'-pattern 12, and therefore by Lemma 2.10 cannot be dense in $\varprojlim([\tilde{c}_2, \tilde{c}_1], T_{\tilde{s}})$, which contradicts the fact that $\mathfrak{R}$ is dense in $\varprojlim([c_2, c_1], T_s)$.

Let $B \ni q$ be the maximal arc such that $\pi_{l'} : B \to [\tilde{c}_2, \tilde{c}_1]$ is injective. Therefore, $\pi_{l'+1}(B) \not\ni c$. Since $q_{-(l'+1)} \in \pi_{l'+1}(B)$ and $q_{-(l'+1)} < c$, it follows that $\pi_{l'+1}(B) \subset [\tilde{c}_2, \tilde{c}]$. Hence $\pi_{l'}(B) \subset T_{\tilde{s}}([\tilde{c}_2, \tilde{c}]) = [\tilde{c}_3, \tilde{c}_1]$, see Figure 13.

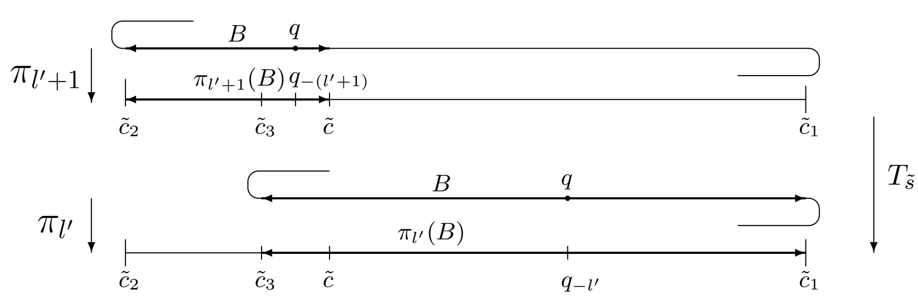

FIGURE 13. Arc B in projections $\pi_{l'+1}$ and $\pi_{l'}$.

Let $Q \subset h(\mathfrak{R})$ be the closest (in the arc-length distance) arc to q such that $\pi_{l'} : Q \to [\tilde{c}_2, \tilde{c}_1]$ is a bijection. It follows that $q \notin Q$.

Lemma 4.2. *Assume that an arc $Q \subset h(\mathfrak{R})$ is such that $\pi_{l'} : Q \to [\tilde{c}_2, \tilde{c}_1]$ is a bijection. If $Q \subset B_j$ for $j \in \mathbb{N}$ minimal, then $Q \subset [n_j, n_{j+2}]$.*

Proof. Let us assume by contradiction that n_j is in the interior of Q. Note that $\pi_{l'}(Q) = [\tilde{c}_2, \tilde{c}_1]$ and $\pi_{l'}(n_j) \neq \tilde{c}_1, \tilde{c}_2$. Let δ_0 be chosen as in the equation (2).

Note that $q \notin Q$, but $q \in B_j$.

Take the largest (in arc-length) arc $Q' \subset Q \subset B_j$ which is l-link symmetric with midpoint n_j and note that $\pi_{l'}|_{Q'}$ is injective. Let $[a,b] := \pi_{l'}(Q')$ and note that either $a = \tilde{c}_2$ or $b = \tilde{c}_1$.

Assume that $b = \tilde{c}_1$. Let us study $T_{\tilde{s}}^{-1}([a,b])$. Note that $T_{\tilde{s}}^{-1}(b) = T_{\tilde{s}}^{-1}(\tilde{c}_1) = \tilde{c}$ and denote by a_{-1} the endpoint of $T_{\tilde{s}}^{-1}([a,b])$ such that $a_{-1} \in (\tilde{c}, \tilde{c}_1]$. Let $Q'' \subset B_j$ be the largest (in arc-length) l-link symmetric arc with midpoint n_j and such that $\pi_{l'+1}|_{Q''}$ is injective. Note that $[\tilde{c}, a_{-1}] \subset \pi_{l'+1}(Q'')$. Since $Q \subset B_j$ and $|\tilde{c} - \tilde{c}_n| > 100\delta_0$ for every $n \in \mathbb{N}$, the interval $J := [\tilde{c} - 5\delta_0, a_{-1}]$ is also contained in $\pi_{l'+1}(Q'')$. See Figure 14.

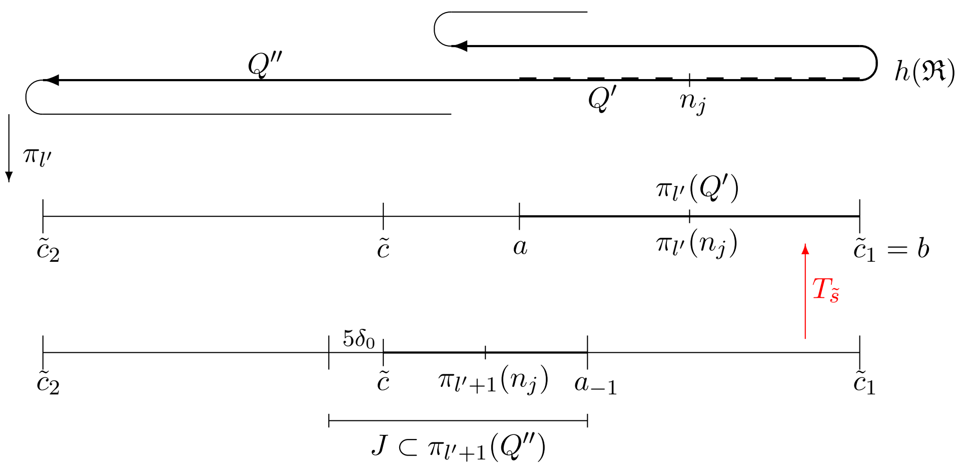

FIGURE 14. Projections of arcs from the proof of Lemma 4.2; arc Q' is denoted with dashed line and is contained in arc Q'' which is denoted with thick line.

Case I: Assume that $|a_{-1} - \tilde{c}_1| > 5\delta_0$.

- If $|\pi_{l'+1}(n_j) - \tilde{c}| > \varepsilon s^{-(l'-l+1)}$, then there exists an interval $J' \supset J$ with midpoint $\pi_{l'+1}(n_j)$ which satisfies conditions of Proposition 3.12, so $T_{\tilde{s}}^{l'-l+1}|_{J'}$ is not ε-symmetric around $\pi_{l'+1}(n_j)$, which contradicts the l-link symmetry of Q''.
- If $|\pi_{l'+1}(n_j) - \tilde{c}| \le \varepsilon s^{-(l'-l+1)}$, then there is a point $u \in A^{n_j}$ such that $\pi_{l'+1}(u) = \tilde{c}$. Since $\pi_{l'}(n_j) \neq \tilde{c}_1$, we have $\pi_{l'+1}(n_j) \neq \tilde{c}$ and thus $u \neq n_j$. Note that the l-level of u is greater than the l-level of n_j, which is a contradiction to the definition of midpoint.

Case II: Assume that $|a_{-1} - \tilde{c}_1| \le 5\delta_0$.
Then $|a - \tilde{c}_2| \le 5\delta_0 s \le 10\delta_0$, so $|a - \tilde{c}| > 5\delta_0$.

- If $|\pi_{l'}(n_j) - \tilde{c}| > \varepsilon s^{-(l'-l)}$, then by Proposition 3.12 we get a contradiction with the l-link symmetry of Q'.
- If $|\pi_{l'}(n_j) - \tilde{c}| \le \varepsilon s^{-(l'-l)}$, then $\pi_{l'}(n_j) = \tilde{c}$, because otherwise we get a contradiction with the definition of midpoint as above. But if $\pi_{l'}(n_j) = \tilde{c}$, then $T_{\tilde{s}}^{l'-l}|_{[a,\tilde{c}_1]}$

is ε-symmetric around $\tilde{c}$. However, since $|\tilde{c}_2 - T_{\tilde{s}}(a)| > |\tilde{c}_2 - \tilde{c}_3| > 100\delta_0 > \varepsilon$ it follows that $\operatorname{diam}(T_{\tilde{s}}^{l'-l-1}([\tilde{c}_2, T_{\tilde{s}}(a)])) > \varepsilon$ (see Figure 15). Because $T_{\tilde{s}}|_{[a,\tilde{c}_1]}$ maps 2-to-1 on the interval $[T_{\tilde{s}}(a), \tilde{c}_1]$ and bijectively onto $[\tilde{c}_2, T_{\tilde{s}}(a)]$ we get a contradiction with $T_{\tilde{s}}^{l'-l}|_{[a,\tilde{c}_1]}$ being ε-symmetric around $\tilde{c}$. See Figure 15.

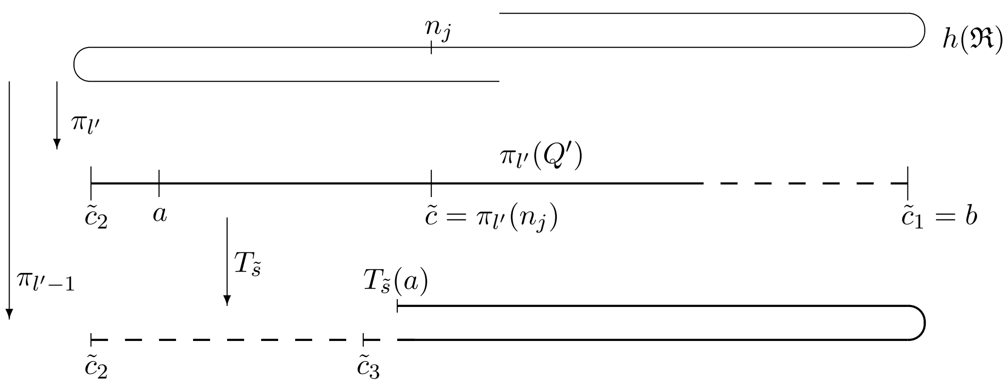

FIGURE 15. Step in the proof of Lemma 4.2, case $\pi_{l'}(n_j) = \tilde{c}$. Since the dashed interval is long, its image under $T^{l'-l-1}$ is longer than ε.

If $a = \tilde{c}_2$ we study $T_{\tilde{s}}^{-2}([a, b])$ and proceed similarly as in the preceding paragraphs. $\square$

Proof of Theorem 1.2. Let $Q \subset h(\mathfrak{R})$ be an arc with l'-pattern 12. Such Q exists by Lemma 2.10. Without loss of generality, we can assume that $\pi_{l'} : Q \to [\tilde{c}_2, \tilde{c}_1]$ is bijective. As we already observed, $q \notin Q$. Assume without loss of generality that Q is closest to q, in the sense that there is no other arc with l'-pattern 12 closer to q in arc-length distance.

Let P be the l-pattern of Q; it is the $T_{\tilde{s}}^{l'-l}$-image of the l'-pattern 12.

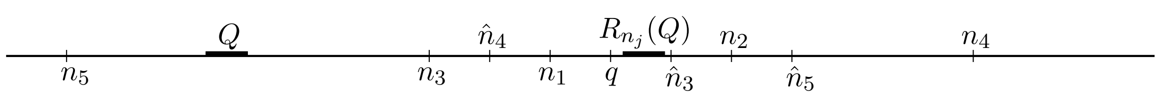

FIGURE 16. The midpoints and endpoints of l-link symmetric arcs $[n_i, \hat{n}_i]$.

Now let j be the minimal natural number such that $Q \subset B_j$. Then $Q \subset [n_j, n_{j+2}]$ by Lemma 4.2. Since B_j is l-link-symmetric around n_j, we can reflect Q in n_j, obtaining the arc $R_{n_j}(Q) \subset h(\mathfrak{R})$ which has l-pattern ε-close to P (see Figure 16). Lemma 3.17 implies that $R_{n_j}(Q)$ has l'-pattern 12, contradicting the choice of Q. Thus there exists no arc $Q \subset h(\mathfrak{R})$ with l-pattern P, which contradicts that $h(\mathfrak{R})$ is dense in $\varprojlim([\tilde{c}_2, \tilde{c}_1], T_{\tilde{s}})$ (Lemma 2.10). $\square$

5. The Core Ingram Conjecture

Note that once we know that the arc-component $\mathfrak{R}$ is fixed under homeomorphisms, the results from this section follow without the non-recurrence assumption.

Recall that $p \in \varprojlim([c_2, c_1], T_s)$ is called a *k-point* if there exists $n > 0$ such that $\pi_{k+n}(p) = c$, and we write $L_k(p) = n$. Note that if c is not periodic, k-level n is unique.

Definition 5.1. *Let $\mathfrak{U} \subset \varprojlim([c_2, c_1], T_s)$ be an arc-component and $u \in \mathfrak{U}$. We say that a k-point $p \in \mathfrak{U}$ such that $L_k(p) = n$ is a* salient k-point with respect to u *if $\pi_{k+n}|_{[u,p]}$ is injective.*

Remark 5.2. *The above definition says that a salient point p is a k-point of level n and that there are no k-points between u and p with k-level greater than n. Thus it corresponds to the existent definition of a salient point (for example in* [1]*). We will work with salient k-points with respect to $\rho, \tilde{\rho}$ or q but because it is clear with respect to which point we work we refer to them only as salient k(or l)-points.*

Lemma 5.3. *For any $i \in \mathbb{N}$, the midpoint m_i of $A_i \subset \mathfrak{R}$ is a salient k-point with respect to ρ and its k-level is i.*

Proof. By the definition of A_i, we obtain that $\rho \in A_i$, $\pi_{k+i}(m_i) = c$ and $\pi_{k+i}|_{A_i}$ is injective onto $[c_2, \hat{c}_2]$. This proves the claim. $\square$

We consider l-link symmetric arcs $B_i = h(A_i) \subset \tilde{\mathfrak{R}}$, where the chains $\mathcal{C}_k$ and $\tilde{\mathcal{C}}_l$ satisfy (3). Let $\tilde{A}_i \subset \tilde{\mathfrak{R}} \subset \varprojlim([\tilde{c}_2, \tilde{c}_1], T_{\tilde{s}})$ be the arc-component of $\pi^{-1}_{l+i}([\tilde{c}_2, \hat{\tilde{c}}_2])$ containing $\tilde{\rho}$ for every $i \in \mathbb{N}$. The arcs $\tilde{A}_i$ are all l-symmetric with salient l-points $\tilde{m}_i$ of level i and they form a complete sequence with respect to $\tilde{\rho} = (\dots, \tilde{r}, \tilde{r})$.

Let n_i and $\tilde{m}_i$ be the midpoints of arcs B_i and $\tilde{A}_i$ respectively. In the next two lemmas we show how B_i and $\tilde{A}_i$ relate to each other.

Lemma 5.4. *There exists $N \in \mathbb{N}$ such that for every $j \geq N$ there exists $j' \in \mathbb{N}$ such that $\tilde{\rho} \in B_j$, $q = h(\rho) \in \tilde{A}_{j'}$ and $n_j = \tilde{m}_{j'} \notin [\tilde{\rho}, q]$ for every $j \geq N + 2$.*

Proof. By Lemma 3.3 and applying h we obtain that $\cup_{i \text{ odd}} B_i = \cup_{i \text{ even}} B_i = \tilde{\mathfrak{R}}$ and $B_i \subset B_{i+2}$ for every $i \in \mathbb{N}$, so there exists N such that $[\tilde{\rho}, q] \subset B_j$ for all $j \geqslant N$. Lemma 3.5 implies that $n_j \notin [\tilde{\rho}, q]$ for $j \geqslant N + 2$. The argument for the arcs $\tilde{A}_i$ is analogous. Because $\{\tilde{A}_i\}$ is the complete sequence for $\tilde{\mathfrak{R}}$ with respect to $\tilde{\rho}$ it follows that $n_j = \tilde{m}_{j'}$. $\square$

Definition 5.5. *Given an arc $A = [u, v] \subset \varprojlim([c_2, c_1], T_s)$, we call the arc-component of $A \cap \ell$ of u (where the link $\ell \ni u$) the* link-tip *of A at u. Similarly for v. Let $A^{-\ell} = A \setminus \{$link-tips$\}$. We say that two arcs A and B are* close *if $A^{-\ell} = B^{-\ell}$ and denote it by $A \approx B$, .*

Lemma 5.6. *There exists $N \in \mathbb{N}$ such that for every $j \geq N$ there exists $j' \in \mathbb{N}$ such that $B_j \approx \tilde{A}_{j'}$.*

Proof. Take N from Lemma 5.4. Assume by contradiction that there exists $j \geq N$ such that $B_j \not\approx \tilde{A}_{j'}$ for every $j' \in \mathbb{N}$. By the completeness of $\{\tilde{A}_i\}_{i\in\mathbb{N}}$, there exists some $j' \in \mathbb{N}$ such that $n_j = \tilde{m}_{j'}$. As B_j and $\tilde{A}_{j'}$ are both l-link symmetric with the same midpoint, either $B_j^{-\ell} \subsetneq \tilde{A}_{j'}^{-\ell}$ or $\tilde{A}_{j'}^{-\ell} \subsetneq B_j^{-\ell}$, where $A^{-\ell}$ and $B^{-\ell}$ are as in the Definition 5.5. Assume that $B_j^{-\ell} \subsetneq \tilde{A}_{j'}^{-\ell}$. Note that since $n_j = \tilde{m}_{j'} \notin [\tilde{\rho}, q]$ we obtain that $n_{j+2} \in (\tilde{m}_{j'}, \tilde{m}_{j'+2})$. But then B_{j+2} would be l-link-symmetric and contain q and $\tilde{\rho}$, and since the midpoint of B_{j+2} lies in $(\tilde{m}_{j'}, \tilde{m}_{j'+2})$, this contradicts the completeness of $\{\tilde{A}_i\}_{i\in\mathbb{N}}$. The second case follows similarly, but instead of the completeness of $\{\tilde{A}_i\}_{i\in\mathbb{N}}$ we use the eventual completeness of sequence $\{B_i\}_{i\in\mathbb{N}}$, see Lemma 4.1. $\square$

Proposition 5.7. *There exist $N, M \in \mathbb{N}$ such that $L_l(n_{N+i}) = i + M$ for every $i \in \mathbb{N}_0$.*

Proof. Take N from Lemma 5.4. There exist $j', j'' \in \mathbb{N}_0$ such that:

$$B_N \approx \tilde{A}_{j'}, B_{N+2} \approx \tilde{A}_{j'+2}, B_{N+4} \approx \tilde{A}_{j'+4}, \ldots$$

$$B_{N+1} \approx \tilde{A}_{j''}, B_{N+3} \approx \tilde{A}_{j''+2}, B_{N+5} \approx \tilde{A}_{j''+4}, \ldots$$

or in terms of l-levels $L_l(n_{N+2i}) = j' + 2i$, $L_l(n_{N+2i+1}) = j'' + 2i$ for all $i \in \mathbb{N}_0$. So far we only know that j' and j'' must be of different parity. Assume $j'' > j'$, so there exists an odd natural number $j \geq 1$ such that $j'' = j' + j$. Since $B_N = h(A_N) \not\subset h(A_{N+1}) = B_{N+1}$, we conclude from Lemma 3.4 that $j < \xi$. Assume by contradiction that $j > 1$ and take $i = \xi - j$. From Lemma 3.4 we obtain that $\tilde{A}_{j'} \subseteq \tilde{A}_{j'+\xi}$. But $\tilde{A}_{j'} \approx B_N$, $\tilde{A}_{j'+\xi} = \tilde{A}_{j'+j+i} = \tilde{A}_{j''+i} \approx B_{N+i+1}$. Thus we get $h(A_N) = B_N \subseteq B_{N+i+1} = h(A_{N+i+1})$ which is a contradiction because $i + 1 < \xi$ and $i + 1$ odd. We conclude that $j = 1$.
The other possibility is that $j'' < j'$. Since also $B_{N+1} = h(A_{N+1}) \not\subseteq h(A_N) = B_N$, we conclude that $j' = j'' + j$, where $j < \xi$ odd. Recall that $\tilde{A}_{j''} \subseteq \tilde{A}_{j''+\xi}$, but $\tilde{A}_{j''} \approx B_{N+1}$ and $\tilde{A}_{j''+\xi} \approx B_{N+\xi-j}$, where $\xi - j \in \mathbb{N}$ is even. Thus there exists an even natural number $0 < i < \xi$ such that $B_{N+1} \subseteq B_{N+i}$, which is again a contradiction.
So the only possibility is $j'' = j' + 1$, which gives $B_{N+i} \approx \tilde{A}_{j'+i}$ for every $i \in \mathbb{N}_0$ and this finishes the proof. $\square$

So far we have shown that there exist $N, M \in \mathbb{N}$ such that a homeomorphism h maps the salient point of k-level $i + N$ *close to* the salient point of l-level $i + M$ for every $i \in \mathbb{N}_0$. Here *close to* means that $h(m_{i+N})$ is in the same link of $\tilde{\mathcal{C}}_l$ as $\tilde{m}_{i+M}$ and the arc-component of the link containing point $\tilde{m}_{i+M}$ also contains the point $h(m_{i+N})$. Note that this works for any k and l such that $h(\mathcal{C}_k) \preceq \tilde{\mathcal{C}}_l$. The salient $(k+N)$-point of $(k+N)$-level i is the salient k-point of k-level $i + N$. Therefore, if we consider $\mathcal{C}_{k+N}$ instead of $\mathcal{C}_k$, then $h(m_i)$ is close to $\tilde{m}_{i+M}$ for every $i \geq 1$.

The proof of the Core Ingram Conjecture now follows analogously as in [9]. We first need to prove that a homeomorphism h preserves the sequence of k-points and then argue that the sequences of k-points and l-points of $\mathfrak{R}$ and $\tilde{\mathfrak{R}}$ respectively are never the same, unless $s = \tilde{s}$.

Proposition 5.8. *Let $n \in \mathbb{N}$ and $u \in \mathfrak{R}$ be a k-point with k-level n. Then $h(u) \in \tilde{\mathfrak{R}}$ is in the link of $\tilde{\mathcal{C}}_l$ that contains $\tilde{m}_{n+M}$ and the arc-component of the link that contains $h(u)$ also contains an l-point v with l-level $n+M$ (see Figure 17).*

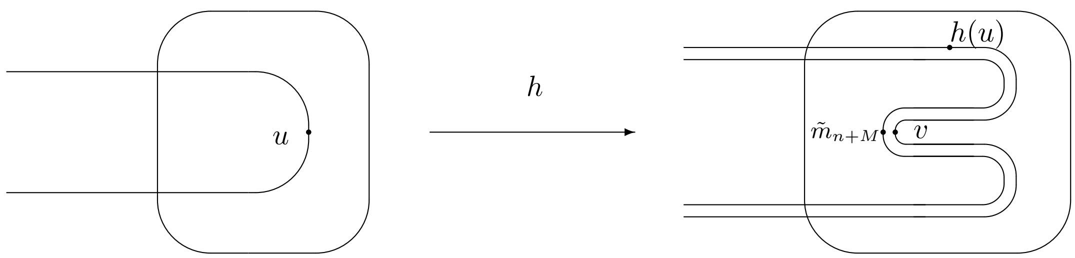

FIGURE 17. Claim of the Proposition 5.8.

Proof. For $i \in \mathbb{N}$ denote by S_i the longest arc in $\mathfrak{R}$ containing m_i such that $\pi_{k+i}|_{S_i}$ is injective. Note that S_i is exactly the arc-component of $\pi_{k+i}^{-1}([c_2, c_1])$ which contains m_i and that $\pi_{k+i}(S_i) = [c_2, c_1]$. Also note that $A_i \subset S_i$ and the endpoint of A_i projecting with π_{k+i} to c_2 agrees with one endpoint of S_i. Let $S_i^{\rho}, S_i^{\neg\rho} \subset S_i$ be the arc-components of $\pi_{k+i}^{-1}([c, c_1])$ and of $\pi_{k+i}^{-1}([c_2, c])$ respectively, with m_i as the common boundary point. Note that $\rho \in S_i^{\rho}$ and its endpoints are m_i and m_{i+1}. Also, $\rho \notin S_i^{\neg\rho}$ and its endpoints are m_i and m_{i+2}. Also note that $S_i^{\neg\rho}$ is shorter (in arc-length) than S_i^{ρ} and that $S_{i+1}^{\rho} = S_i$. We will prove the proposition for k-points in S_i by induction on i.
Note that all k-points in S_1 are salient, and by the remarks preceding this proposition it follows that the proposition holds for salient points. Assume that the proposition holds for all k-points in $S_i (= S_{i+1}^{\rho})$. Take a k-point $u \in S_{i+1}^{\neg\rho} \setminus \{m_{i+1}, m_{i+3}\}$ with k-level n. Note that $n < i+1$ by the definition of S_{i+1}. Also, since $S_{i+1}^{\neg\rho}$ is shorter than S_{i+1}^{ρ} there exists a k-point $R_{m_{i+1}}(u) \in S_{i+1}^{\rho}$ such that $[u, R_{m_{i+1}}(u)]$ is k-symmetric with midpoint m_{i+1}. Observe that $h([u, R_{m_{i+1}}(u)])$ is l-link symmetric with midpoint $\tilde{m}_{i+1+M}$, because it is the point with the highest l-level in the link containing $h(m_{i+1})$. Since $R_{m_{i+1}}(u) \in S_{i+1}^{\rho} = S_i$, $h(R_{m_{i+1}}(u))$ is in the link containing $\tilde{m}_{n+M}$ and the arc-component of the link containing $h(R_{m_{i+1}}(u))$ contains an l-point v' such that $L_l(v') = n+M$. Take such v' closest (in arc-length) to $\tilde{m}_{i+1+M}$ such that there are no points of l-level greater or equal than $i+1+M$ in $(\tilde{m}_{i+1+M}, v')$. Since $n < i+1$, we obtain that $L_l(v') = n+M < i+1+M = L_l(\tilde{m}_{i+1+M})$. Note that there exists an l-point v such that the arc $[v, v']$ is l-symmetric with midpoint $\tilde{m}_{i+1+M}$. This implies that v and v' both have the same level $n+M$, and that they belong to the same link. Arc-component of the link containing v must also contain point $h(u)$.

This concludes the proof for every k-point in S_{i+1}. Since $\cup_i S_i = \mathfrak{R}$, this concludes the proof. $\square$

Proposition 5.9. *Let k, l, k' be such that $h(\mathcal{C}_k) \preceq \tilde{\mathcal{C}}_l \preceq h(\mathcal{C}_{k'})$ holds as in (3). Take $M, M' \in \mathbb{N}$ such that h maps every k-point with k-level n close to l-point with l-level $n+M$ and h^{-1} maps every l-point with l-level n close to k'-point with k'-level $n+M'$. Then for every $K \in \mathbb{N}$, there is an orientation preserving bijection between*

$$\{u \in [m_K, m_{K+1}] : L_k(u) = n\} \textit{ and } \{v \in [\tilde{m}_{K+M}, \tilde{m}_{K+1+M}] : L_l(v) = n+M\}.$$

Proof. First we claim that $M+M' = k-k'$. Take the salient k-point m_i with $L_k(m_i) = i$ and note that it is also a salient k'-point with $L_{k'}(m_i) = k+i-k'$. Note that by remarks before Proposition 5.8, homeomorphism h maps the salient k-point with level i close to the salient l-point with l-level $i+M$, which is mapped by h^{-1} close to the salient k'-point with k'-level $i+M+M'$. This means that the salient k'-point with k'-level $k+i-k'$ belongs to the same arc-component of the same link of the chain $\mathcal{C}_{k'}$ that contains the salient k'-point with k'-level $i+M+M'$. But this is only possible if the points are equal which implies that $M+M' = k-k'$.
Denote by z_i, $i = 1, \dots, a$, all k-points with k-level n in $[m_K, m_{K+1}]$ such that $m_K \prec z_1 \prec \cdots \prec z_a \prec m_{K+1}$ (where $x \prec y \prec z$ if $[x,y] \subset [x,z]$ for $x,y,z \in \mathfrak{U} \subset \varprojlim([c_2,c_1],T_s))$. Similarly, denote by $\tilde{z}_j$, $j = 1, \dots, b$, all l-points with l-level $n+M$ in $[\tilde{m}_{K+M}, \tilde{m}_{K+1+M}]$ such that $\tilde{m}_{K+M} \prec \tilde{z}_1 \prec \cdots \prec \tilde{z}_b \prec \tilde{m}_{K+1+M}$. We will first prove that $a \le b$.
Recall that for an l-point u such that $u \in \ell \in \tilde{\mathcal{C}}_l$ we denote the arc-component of u in ℓ by A^u. We can find $N > 0$ such that $A^{\sigma^N(\tilde{m}_{K+M})}, A^{\sigma^N(\tilde{z}_1)}, \dots, A^{\sigma^N(\tilde{z}_b)}, A^{\sigma^N(\tilde{m}_{K+1+M})}$ are all different. Also, every point $u \in \{\sigma^N(\tilde{m}_{K+M}), \sigma^N(\tilde{z}_1), \dots, \sigma^N(\tilde{z}_b), \sigma^N(\tilde{m}_{K+1+M})\}$ has to be a midpoint of A^u. Otherwise, there would exist another l-point with l-level $n+M+N$ in the same arc-component which is impossible since we separated them. Since $\sigma^N(m_K) = m_{K+N}$ and $\sigma^N(\tilde{m}_{K+M}) = \tilde{m}_{K+M+N}$, we get from Proposition 5.8 that for every $i \in \{1, \dots, a\}$ there exists unique $j \in \{1, \dots, b\}$ such that $h(\sigma^N(z_i)) \in A^{\sigma^N(\tilde{z}_j)}$. This defines a function $x \mapsto \tilde{x}$ for every k-point $x \in [m_K, m_{K+1}]$ with $L_k(x) = n$ to an l-point $\tilde{x} \in [\tilde{m}_{K+M}, \tilde{m}_{K+M+1}]$ with $L_l(\tilde{x}) = n+M$. Note that we can take N such that σ^N preserves orientation and so $x \prec y$ implies $\tilde{x} \prec \tilde{y}$.
Next we want to prove that $x \mapsto \tilde{x}$ is injective. Assume there are $i_1, i_2 \in \{1, \dots, a\}$ such that $h(\sigma^N(z_{i_1})), h(\sigma^N(z_{i_2})) \in A^{\sigma^N(\tilde{z}_j)}$, for some $j \in \{1, \dots, b\}$. There exists a k-point w such that $\sigma^N(z_{i_1}) \prec w \prec \sigma^N(z_{i_2})$ and such that $L_k(w) > n+N$. Note that $h(w) \in A^{\sigma^N(\tilde{z}_j)}$. But then there exists an l-point $\tilde{w} \in A^{\sigma^N(\tilde{z}_j)}$ with l-level strictly greater than $n+N+M$ which is in contradiction with $\sigma^N(\tilde{z}_j)$ being the center of the link. This proves that the above function $x \mapsto \tilde{x}$ is injective, *i.e.,* $a \le b$.
It follows that

$$\begin{aligned}
&\#\{k\text{-points in } [m_K, m_{K+1}] \text{ with } k\text{-level } n\} \\
&\quad \leqslant \#\{l\text{-points in } [\tilde{m}_{K+M}, \tilde{m}_{K+1+M}] \text{ with } l\text{-level } n+M\} \\
&\quad \leqslant \#\{k'\text{-points in } [m_{K+M+M'}, m_{K+1+M+M'}] \text{ with } k'\text{-level } n+M+M'\}.
\end{aligned}$$

We proved that $M+M' = k-k'$ so the last number is equal to the number of k'-points in $[m_{K+k-k'}, m_{K+1+k-k'}]$ with k'-level $n+k-k'$. But this is actually equal to the number of k-points in $[m_K, m_{K+1}]$ with k-level n. This proves that $a = b$. □

Proof of Theorem 1.1. We claim that the k-pattern of $[m_{n-1}, m_n]$ is equal to the $(l+M)$-pattern of $[\tilde{m}_{n-1}, \tilde{m}_n]$ and that $T_s^n(c) > c$ if and only if $T_{\tilde{s}}^n(\tilde{c}) > \tilde{c}$ for every $n \ge 2$. This gives $s = \tilde{s}$.

The claim is obviously true for $n = 2$. For the inductive step, assume that it is true for all positive integers $< n$.

Specifically, the k-pattern of $[m_{n-2}, m_{n-1}]$ is the $(l+M)$-pattern of $[\tilde{m}_{n-2}, \tilde{m}_{n-1}]$. Denote all k-points in $[m_{n-2}, m_{n-1}]$ by $m_{n-2} = z_0 \prec z_1 \prec \ldots \prec z_i \prec z_{i+1} = m_{n-1}$. Denote all $(l+M)$-points in $[\tilde{m}_{n-2}, \tilde{m}_{n-1}]$ analogously by $\tilde{m}_{n-2} = z_0 \prec \tilde{z}_1 \prec \ldots \prec \tilde{z}_i \prec \tilde{z}_{i+1} = \tilde{m}_{n-1}$. Since patterns are the same, $L_k(z_j) = L_{l+M}(\tilde{z}_j)$ for all $j \in \{0, \ldots, i+1\}$. By the inductive assumption it follows that $c \in \pi_k(z_j, z_{j+1})$ if and only if $\tilde{c} \in \pi_{l+M}(\tilde{z}_j, \tilde{z}_{j+1})$ for all $j \in \{0, \ldots, i\}$. Since $\sigma([m_{n-2}, m_{n-1}]) = [m_{n-1}, m_n]$ and every subarc $[z_j, z_{j+1}]$ is mapped to the subarc $[\sigma(z_j), \sigma(z_{j+1})]$ with k-pattern $L_k(z_j)+1, 1, L_k(z_{j+1})+1$ or $L_k(z_j)+1, L_k(z_{j+1})+1$ according to whether $\pi_k([z_{j-1}, z_j])$ contains c or not, inductive hypothesis for $n-1$ completely determines the k-pattern of $[m_{n-1}, m_n]$. The same holds for the arc $[\tilde{m}_{n-2}, \tilde{m}_{n-1}]$. Since we assumed that $T_s^{n'}(c) > c$ if and only if $T_{\tilde{s}}^{n'}(\tilde{c}) > \tilde{c}$ for all $n' < n$, this gives that the k-pattern of $[m_{n-1}, m_n]$ is the same as the $(l+M)$-pattern of $[\tilde{m}_{n-1}, \tilde{m}_n]$.

From now on we study $[m_{n-1}, m_n]$ and $[\tilde{m}_{n-1}, \tilde{m}_n]$. Write $m_{n-1} \prec z_1 \prec \cdots \prec z_i \prec m_n$ and $\tilde{m}_{n-1} \prec \tilde{z}_1 \prec \cdots \prec \tilde{z}_i \prec \tilde{m}_n$, where $\{z_1, \ldots, z_i\} \subset \mathfrak{R}$ is the set of all k-points in $[m_{n-1}, m_n]$ and $\{\tilde{z}_1, \ldots, \tilde{z}_i\} \subset \tilde{\mathfrak{R}}$ is the set of all $(l+M)$-points in $[\tilde{m}_{n-1}, \tilde{m}_n]$. From the previous paragraph we obtain that $L_k(z_j) = L_{l+M}(\tilde{z}_j)$ for every $j \in \{1, \ldots, i\}$.

Assume by contradiction that $T_s^n(c)$ and $T_{\tilde{s}}^n(\tilde{c})$ are on the different sides of c in $[c_2, c_1]$ and $\tilde{c}$ in $[\tilde{c}_2, \tilde{c}_1]$ respectively. Since $\pi_k(m_n) = T_s^n(c)$ and $\pi_{l+M}(\tilde{m}_n) = T_{\tilde{s}}^n(\tilde{c})$, by assumption $c \in \pi_k((z_i, m_n))$ and $\tilde{c} \notin \pi_{l+M}((\tilde{z}_i, \tilde{m}_n))$ or the opposite. The inductive hypothesis gives $c \in \pi_k((z_j, z_{j+1}))$ if and only if $\tilde{c} \in \pi_{l+M}((\tilde{z}_j, \tilde{z}_{j+1}))$ for all $j \in \{1, \ldots, i-1\}$. Apply σ to $[m_{n-1}, m_n]$ and $[\tilde{m}_{n-1}, \tilde{m}_n]$ and count the number of k-points in $\sigma([m_{n-1}, m_n]) = [m_n, m_{n+1}]$ and the number of $(l+M)$-points in $\sigma([\tilde{m}_{n-1}, \tilde{m}_n]) = [\tilde{m}_n, \tilde{m}_{n+1}]$. Every point of k-level strictly greater than 1 in $[m_n, m_{n+1}]$ is a shift of some z_j and every point of $(l+M)$-level greater than 1 in $[\tilde{m}_n, \tilde{m}_{n+1}]$ is a shift of some $\tilde{z}_j$. So it suffices to count the k-points of k-level 1 in $[m_n, m_{n+1}]$ and the $(l+M)$-points of $(l+M)$-level 1 in $[\tilde{m}_n, \tilde{m}_{n+1}]$. Such points are obtained as shifts of points in $[m_{n-1}, m_n]$ (respectively $[\tilde{m}_{n-1}, \tilde{m}_n]$) which are projected to c by π_k (respectively to $\tilde{c}$ by π_{l+M}). The number of such points in $[m_{n-1}, m_n]$ differs by one from the number of points in $[\tilde{m}_{n-1}, \tilde{m}_n]$, because by our assumption either $c \in \pi_k((z_i, m_n))$ or $\tilde{c} \in \pi_{l+M}((\tilde{z}_i, \tilde{m}_n))$, but not both. That is, the number of k-points of k-level 1 in $[m_n, m_{n+1}] = \sigma([m_{n-1}, m_n])$ is different from the number of $(l+M)$-points of $(l+M)$-level 1 in $[\tilde{m}_n, \tilde{m}_{n+1}] = \sigma([\tilde{m}_{n-1}, \tilde{m}_n])$ which contradicts Proposition 5.9. $\square$

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(A. Anušić) Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, 10000 Zagreb, Croatia

E-mail address: ana.anusic@fer.hr

(H. Bruin) Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria

E-mail address: henk.bruin@univie.ac.at

(J. Činč) Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria

E-mail address: jernej.cinc@univie.ac.at